Preface

This book arose from lecture notes that I began to develop in 2010–2011 for a first course in ordinary differential equations (ODEs). At Calvin College, the students in this course are primarily engineers. In our engineering program, the only (formal) linear algebra the students are required to see throughout their undergraduate career is what is presented in the ODE course. This is not unusual, as the ABET Accreditation Criteria of 2014–2015 do not explicitly require a course devoted to the study of linear algebra alone. Since, in my opinion, the amount of material on linear algebra covered in, e.g., the classical text of Boyce and DiPrima [**9**] is insufficient if that is all you will see in your academic career, I found it necessary to supplement the ODE course with my own notes on linear algebra. Eventually, it became clear that in order to have a seamless transition between the linear algebra and ODEs, there needed to be one text. This is not a new idea; for example, two recent texts that have a substantive linear algebra component are by Boelkins et al. [**6**] and Edwards and Penney [**15**].

Because there is a substantive linear algebra component in this text, I—and, more important, the students—find it much easier, when discussing the solutions of linear systems of ODEs, to focus more on the ODE aspects of the problems and less on the underlying algebraic manipulations. I have found that doing the linear algebra first allows for a more extensive exploration of linear systems of ODEs. Most important, it becomes possible to study and solve much more interesting examples and applications. The inclusion of more modeling and model analysis is extremely important; indeed, it is precisely what is recommended in the 2013 report by the National Academy of Sciences on the current state and future of the mathematical sciences.

The applications presented in this text are labeled "Case Studies." I chose this moniker because I want to convey to the reader that in solving particular problems, we were going to do more than simply find a solution; instead, we were going to take time to determine what the solution was telling us about the dynamical behavior for the given physical system. There are 20 case studies presented herein. Some are classical, e.g., damped mass-spring systems and mixing problems (compartment models), but several are not typically found in a text such as this. Nonclassical examples include a discrete susceptible-infected-recovered (SIR) model, a study of the effects on the body of lead ingestion, strongly damped systems (which can be recast as a singular perturbation problem), and a (simple) problem in the mathematics of climate. It is (probably) not possible to present all of these case studies in a one-semester course. On the other hand, the large number allows the instructor to choose a subset that will be of particular interest to his or her class.

The book is formatted as follows. In Chapter 1, we discuss not only the basics of linear algebra that will be needed for solving systems of linear ODEs, e.g., Gaussian elimination, matrix algebra, and eigenvalues/eigenvectors, but also such foundational material as subspaces, dimension, etc. While the latter material is not necessary to solve ODEs, I find that this is a natural time to introduce students to these more abstract linear algebra concepts. Moreover, since linear algebra is such important foundational material for a mathematical understanding of all of the sciences, I feel that it is essential that the students learn as much as they reasonably can in the short amount of time that is available. It is typically the case that the material in Chapter 1 can be covered in about 12–15 class periods. Primarily because of time constraints, when presenting this material, I focus primarily on the case of the vector space $\mathbb{R}^n$. The culminating section in the chapter is that on eigenvalues and eigenvectors. Here, I especially emphasize the utility of writing a given vector as a linear combination of eigenvectors. The closing section consists of four case studies: three determine the large "time" behavior associated with discrete dynamical systems, and one is a study of digital filters. If the reader and/or instructor wishes to have a supplementary text for this chapter, the free book by Hefferon [20][1] is an excellent companion.

Once the linear algebra has been mastered, in Chapter 2 we begin the study of ODEs by solving scalar first-order linear ODEs. We briefly discuss the general existence/uniqueness theory as well as the numerical solution. When solving ODEs numerically, we use the MATLAB programs `dfield8b.m` and `pplane8b.m` developed by J. Polking (provided on the book website at `http://www.siam.org/books/ot145`). These MATLAB programs have accompanying Java applets DFIELD and PPLANE.[2] My experience is that these software tools are more than sufficient to numerically solve the problems discussed in this chapter. We next analytically find the homogeneous and particular solutions to the linear problem. In this construction, we do three things:

(a) derive and write the homogeneous solution formula in such a way that the later notion of a homogeneous solution being thought of as the product of a matrix-valued solution and a constant vector is a natural extension,

(b) derive and write the variation-of-parameters solution formula in such a manner that the ideas easily generalize to systems, and

(c) develop the technique of undetermined coefficients.

The chapter closes with a careful analysis of, first, the one-tank mixing problem under the assumption that the incoming concentration varies periodically in time and, second, a mathematical finance problem. The idea here is to

(a) show students that understanding is not achieved with a solution formula; instead, it is necessary that the formula be written "correctly" so that as much physical information as possible can be gleaned from it;

(b) introduce students to the ideas of amplitude plots and phase plots; and

(c) set students up for the later analysis of the periodically forced mass spring.

As a final note, in many (if not almost all) texts, there is typically in this type of chapter an extensive discussion on nonlinear ODEs. I chose to provide only a cursory treatment of this topic at the end of this book because of

[1] `http://joshua.smcvt.edu/linearalgebra/`.

[2] `http://math.rice.edu/~dfield/dfpp.html`.

(a) my desire for my students to understand and focus on linearity and its consequences, and

(b) the fact that we at Calvin College teach a follow-up course on nonlinear dynamics using the wonderful text by Strogatz [**37**].

In Chapter 3, we study systems of linear ODEs. We start with five physical examples, three of which are mathematically equivalent in that they are modeled by a second-order scalar ODE. We show that nth-order scalar ODEs are equivalent to first-order systems and thus (we hope) convince the student that it is acceptable to skip (for the moment) a direct study of these higher-order scalar problems. We almost immediately focus on the case of the homogeneous problem being constant coefficient and derive the homogeneous solution via an expansion in terms of eigenvectors. From a pedagogical perspective, I find (and my students seem to agree) that this is a natural way to see how the eigenvalues and eigenvectors of a matrix play a key role in the construction of the homogeneous solution and in particular how using a particular basis may greatly simplify a given problem. Moreover, I find that this approach serves as an indirect introduction to the notion of Fourier expansions, which is, of course, used extensively in a successor course on linear partial differential equations (PDEs). After we construct the homogeneous solutions, we discuss the associated phase plane. As for the particular solution, we mimic the discussion of the previous chapter and simply show the few modifications that must be made in order for the previous scalar results to be valid for systems. My experience has been that the manner in which things were done in the previous chapter helps the student see that it is not the case that we are learning something entirely new and different; instead, we are just expanding on an already understood concept. The chapter closes with a careful analysis of three problems: a two-tank mixing problem in which the incoming concentration into one of the tanks is assumed to vary periodically in time, a study of the effect of lead ingestion, and an SIR model associated with zoonotic (animal-to-human) bacterial infections. As in the previous chapter, the goal is to not only construct the solution to the mathematical model but also understand how the solution helps us understand the dynamics of the given physical system.

In Chapter 4, we solve higher-order scalar ODEs. Because all of the theoretical work has already been done in the previous chapter, it is not necessary to spend too much time on this particular task. In particular, there is a relatively short presentation on how one can use the systems theory to solve the scalar problem. The variation of parameters formula is not re-derived; instead, it is presented just as a special case of the formula for systems. We conclude with a careful study of several problems: the undamped and damped mass-spring systems, a (linear) pendulum driven by a constant torque, and a coupled mass-spring system. Nice illustrative Java applets treat the forced and damped oscillations of a spring pendulum[3] and coupled oscillators.[4] There are also illustrative movies that are generated by MATLAB (provided on the book website, `http://www.siam.org/books/ot145`).

In Chapter 5, we solve scalar ODEs using the Laplace transform. The focus here is to solve only those problems for which the forcing term is a linear combination of Heaviside functions and delta functions. In my opinion, any other type of forcing term can be more easily handled with the method of either undetermined

[3]`http://www.walter-fendt.de/ph14e/resonance.htm`.

[4]`http://www.lon-capa.org/%7emmp/applist/coupled/osc2.htm`.

coefficients or variation of parameters. Moreover, we focus on using the Laplace transform as a method to find the particular solution with zero initial data. The understanding here is that we can find the homogeneous solution using the ideas and techniques from previous chapters. Because of the availability of SAGE (or WolframAlpha or any other CAS), we spend little time on partial fraction expansions and the inversion of the Laplace transform. The subsequent case studies are somewhat novel. We start by providing a physical interpretation of delta function forcing. We then find a way to stop the oscillations for an undamped mass-spring system. For another problem, we study a one-tank mixing problem in which the incoming concentration varies periodically in time. The injection strategy is modeled as an infinite sum of delta functions. The last case study involves the analysis of a strongly damped mass-spring problem. We show that this system can be thought of as a singular perturbation problem that is (formally) mathematically equivalent to a one-tank mixing problem. We next discuss how the Laplace transform generates transfer function and the manner in which this function is a model for a physical system. Finally, we show that the convolution integral leads to a variation-of-parameters formula for the particular solution.

In Chapter 6, we cover topics that are not infrequently discussed if time permits: separation of variables, phase-line analysis, and series solutions. Each topic is only briefly touched on, but enough material is presented herein for the student to get a good idea of what each one is about. For the latter two topics, I present case studies that could lead to a more detailed examination of the topic (using outside resources) if the student and/or instructor wishes to study it more.

Almost every section concludes with a set of homework problems. Moreover, there is a section at the end of each chapter that is labeled "Group Projects." The problems contained in these sections are more challenging, and I find it to be the case that students have a better chance of understanding and solving them if they work together in groups of 2–4 people. My experience is that students truly enjoy working on these problems and very much appreciate working collaboratively. I typically assign 1–3 of these types of problems per semester.

None of the homework problems have attached to them a solution that is included in this book. I suspect that many (if not most) students will find this lack of solved problems troubling. Two relatively cheap (potentially supplemental) texts that address this issue are Lipschutz and Lipson [**26**] for the linear algebra material and Bronson and Costa [**10**] for the ODE material. Of course, other books, e.g., [**5**, **12**, **18**, **32**], can be found simply by going to the library and looking through the (perhaps) dozens of appropriate books.

Throughout this text, we expect students to use a CAS to do some of the intermediate calculations. I generally use SAGE[5] and/or WolframAlpha.[6] There are several advantages to using these particular CAS:

(a) It is not necessary to learn a programming language to use WolframAlpha.

(b) The commands are intuitive.

(c) Online help is readily available.

(d) Both SAGE and WolframAlpha are easily accessible and free (as of June 2015).

[5]`http://www.sagemath.org/`.

[6]`http://www.wolframalpha.com/`.

Of course, one can use other CAS packages. Since there is currently no universal agreement (even within my department!) as to which package is best to use, I do not want to limit this text to a particular system. Consequently, I do not include output of any particular CAS in the text. Within the text, I use the moniker SAGE to indicate that I have done a calculation using a CAS. Moreover, to help the students become acquainted with how a CAS is used in the intermediate calculations, on the book website there is a SAGE worksheet that lists the commands used to do a particular referenced calculation. My expectations here are

(a) that the student will use whichever CAS he or she wants, and

(b) that interested students who have some experience with a particular CAS will quickly learn how to use it.

Finally, in this text, a CAS is *never* used to completely solve a given problem, as it is important that the student thoroughly understand what intermediate calculations are needed to find the solution. As seen in the Case Studies, having a mathematical solution to the problem (which is the purview of the CAS) solves only half the problem. More work, in which a CAS may or may not be helpful, is needed in order to extract from the solution the desired answer to the given problem. My goal is to help students learn how to provide a physical interpretation of the solution.

All figures appear in black and white in the print version of this book. Figures are reproduced in full color in the ebook version, available for purchase from `http://www.siam.org/books/ot145`.

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For the glory of the most high God alone,
And for my neighbour to learn from.
— *J. S. Bach*