MAT1193 – Notes on how to obtain functions from other functions.

In this class we will consider many functions. However all the functions we will study can be built by taking a handful of basic functions and then combining them using a handful of operations. For a given function, it is critical to understand how it's built. The reason is that in this class, we'll be taking the derivatives of functions. You don't need to know about that yet, but the way we'll study the derivative is to find the derivative of the basic functions, and then find out how the derivative acts when you combine functions using the operations that are presented here. If you know how functions are put together, then you can understand the derivative by understanding a few basic steps. If you DON"T understand how functions are put together, then taking the derivative will seem like a complicated mess. In other words, understanding the construction of functions is a key step in learning calculus.

So let's start by listing the set of basic functions that are used in this class. Each of these basic types of functions have their own set of notes explaining some of the details.

1. The simplest type of function is a **constant function**: $f(x) = b$, where b is a parameter. For example, with $b=-13$ we have $f(x) = -13$.

2. **Power functions** are defined by taking a power of x: $f(x) = x^a$ where a is a parameter. If $a=2$, we have $f(x) = x^2$.

3a. **Exponential functions** are defined by taking a number to the x power: $f(x) = a^x$. Note that this is different from the power function. For power functions, x is the base and the exponent is a constant power. For exponential functions, the base is a constant and the input variable is the exponent. The most important exponential function is the 'natural exponential function', $f(x) = e^x$ where is a constant value e

3b. **The logarithmic functions** are defined as the inverse of exponential functions. If $y = F(x) = \log(x)$ then y is the exponent that you raise e to to end up at x, that is $x = e^y$. See the notes on exponents and logs for more info.

4. Finally, there are the trigonometric functions. As you know, there are several trig functions, but we'll really only deal with the two basic trig functions $f(x)= \sin(x)$ (the sine function) and $F(x) = \cos(x)$ (the cosine function). Both of these are referred to as '**sinusoidal functions**.'

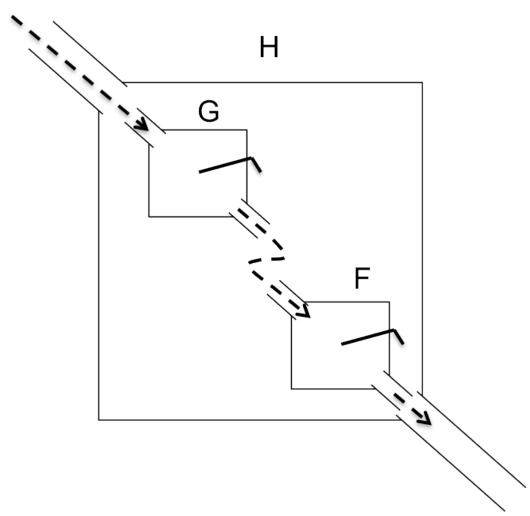
With the basic functions out of the way, we go on to consider several ways of combining two functions. For some of these combinations, describing the 'rule' for the combination seems overly complicated - things are so obvious that you can just write down the answers. However, it will be useful to be careful about how functions are created 'by the book' since later in the class we'll be transforming functions with the derivative and we'll need to know exactly how the function is put together.

1. The simplest way to combine two functions is by **function addition or subtraction**. Suppose we have a function H defined as the sum of the functions F and G, i.e. H = F+G. To find the value H(x) , we just add F(x) to G(x). For example, suppose $F(x) = x^3$ and $G(x) = e^x$. Then $H(x) = x^3+e^x$ and $H(2) = 2^3+e^2 = 15.3891$.
1a. Constant sum. Note that adding a constant in a function can be seen as a function addition where one of the two functions is a constant function. For example, if F(x) = x ($=x^1$, a power function with exponent = 1) and G(x) = -6, then H(x)=F(x)+G(x) = x-6.

2. The next simplest way to combine functions is by **function multiplication or division.** Like addition and subtraction, you just multiply the values of the two functions. So if H is the function F*G, then H(x) = F(x)*G(x). For example, if F(x) = x and G(x) = x+5, then H(x) = x(x+5). Division is defined in a similar way, but you have to be careful to never divide by zero!

2a. Constant product. Like the constant sum rule, one of the functions can be a constant function. So $H(x) = 6x^2$ can be seen as the product of the function F(x) = 6 and $G(x) = x^2$.

3. The third major way of combining two functions is by a process known as **function composition**. We write H = FoG, i.e. H equals F composed with G. To compose two functions you find the output from the second function as use it as the input to the first function. In our function machine picture this looks like this



Notice that H = FoG seems to put things in the wrong order since G is applied first, then the output of G is fed into F, but G comes later (to the right) in FoG. The reason is this. The function H(x) (where you feed x into the function H) can be written H(x) = F(G(x)). First you find the value G(x), then you plug that number into F. GoF looks similar to G(F(x)).

Basic problems with function composition are pretty easy: just plug things in. So if $G(x) = x^2$ and $F(y) = 4*y+5*y^2$, then $H(x) = (FoG)(x) = F(G(x)) = 4*(x^2)+5*(x^2)^2$

It's a bit more confusing, but F and G are just functions, so we could have written $F(x) = x^2$ and $G(x) = 4*x+5*x^2$. We can still find $G(F(x)) = 4*(x^2)+5*(x^2)^2$ but we're plugging x^2, interpreted as the output of the function F, in for x, interpreted as the input to the function G. Confusing yes, but names are just names. Another thing that helps here is to put ( ) around the output of F; that helps to keep it in your head that you're treating the entire output of F as one 'thing' and that 'thing' is what you substitute in for the input value to G.

Another place that function composition comes up is when we define $F(x) = 4*x+7$, and write $F(z^2-8)$. This means we just use z^2-8 as the input to F, so we just plug it in: $F(z^2-8) = 4*(z^2-8)+7$. Note that $F(z^2-8)$ is another way of writing the function FoG where F is the function $G(z) = z^2-8$.