SUPPLEMENTAL MATERIAL

Details of the calculation described in this work will be presented in Ref. [29]. Here, in subsection A we collect the main parameters of the simulations performed in the isosymmetric QCD theory in Ref. [20]. These simulations correspond to a prescription for the separation between QED and QCD corrections, which is different from that of Ref. [2] adopted in this work for the calculation of $\delta R_{K\pi}$. We show that the two prescriptions differ only by effects which are well within the uncertainties of the input parameters of Ref. [20].

In subsection B we sketch some of the key points of the numerical analysis and illustrate the quality of the results by showing the time-dependence of the most complicated diagrams, i.e. those in Fig. 4(a) and (b) in which a photon is exchanged between the quarks and the final-state charged lepton.

A. Simulation parameters

The main parameters of the simulations performed within isosymmetric QCD in Ref. [20] are collected in Table I.

ensemble	β	V/a^4	$a\mu_{ud}$	$a\mu_\sigma$	$a\mu_\delta$	N_{cf}	$a\mu_s$	M_π (MeV)	M_K (MeV)
$A40.40$	1.90	$40^3 \cdot 80$	0.0040	0.15	0.19	100	0.02363	317(12)	576(22)
$A30.32$		$32^3 \cdot 64$	0.0030			150		275(10)	568(22)
$A40.32$			0.0040			100		316(12)	578(22)
$A50.32$			0.0050			150		350(13)	586(22)
$A40.24$		$24^3 \cdot 48$	0.0040			150		322(13)	582(23)
$A60.24$			0.0060			150		386(15)	599(23)
$A80.24$			0.0080			150		442(17)	618(14)
$A100.24$			0.0100			150		495(19)	639(24)
$A40.20$		$20^3 \cdot 48$	0.0040			150		330(13)	586(23)
$B25.32$	1.95	$32^3 \cdot 64$	0.0025	0.135	0.170	150	0.02094	259 (9)	546(19)
$B35.32$			0.0035			150		302(10)	555(19)
$B55.32$			0.0055			150		375(13)	578(20)
$B75.32$			0.0075			80		436(15)	599(21)
$B85.24$		$24^3 \cdot 48$	0.0085			150		468(16)	613(21)
$D15.48$	2.10	$48^3 \cdot 96$	0.0015	0.1200	0.1385	100	0.01612	223 (6)	529(14)
$D20.48$			0.0020			100		256 (7)	535(14)
$D30.48$			0.0030			100		312 (8)	550(14)

TABLE I: *Values of the valence and sea bare quark masses (in lattice units), of the pion and kaon masses for the $N_f = 2+1+1$ ETMC gauge ensembles used in Ref. [20] and for the gauge ensemble, A40.40 added to improve the investigation of FVEs. A separation of 20 trajectories between each of the N_{cf} analysed configurations. The bare twisted masses μ_σ and μ_δ describe the strange and charm sea doublet according to Ref. [23]. The values of the strange quark bare mass $a\mu_s$, given for each β, correspond to the physical strange quark mass $m_s^{phys}(\overline{\rm MS}, 2\,GeV) =$ 99.6(4.3) MeV and to the mass renormalization constants determined in Ref. [20]. The central values and errors of pion and kaon masses are evaluated using the bootstrap procedure of Ref. [20].*

Three values of the inverse bare lattice coupling β and several lattice volumes have been considered. For the earlier investigation of FVEs ETMC had produced three dedicated ensembles, A40.20, A40.24 and A40.32, which share the same quark masses and lattice spacing and differ only in the lattice size L. To improve the present investigation we have generated a further gauge ensemble, A40.40, at a larger value of L.

At each lattice spacing different values of the light sea quark mass have been considered. The light valence and sea bare quark masses are always taken to be degenerate ($a\mu_{ud}^{sea} = a\mu_{ud}^{val} = a\mu_{ud}$).

In Ref. [20] the values of the physical u/d and strange quark masses, $m_{ud}^{phys}(\overline{\rm MS}, 2\,{\rm GeV}) = 3.70(17)\,{\rm MeV}$ and $m_s^{phys}(\overline{\rm MS}, 2\,{\rm GeV}) = 99.6(4.3)\,{\rm MeV}$, as well as the values of the lattice spacing, $a = 0.0885(36)$, $0.0815(30)$, $0.0619(18)$

fm at $\beta = 1.90$, 1.95 and 2.10, have been determined using the following inputs for the isosymmetric QCD theory: $M_\pi^{(0)} = M_{\pi^0} = 134.98\,\text{MeV}$, $M_K^{(0)} = 494.2\,\text{MeV}$ and $f_\pi^{(0)} = 130.41\,\text{MeV}$. The first two inputs correspond to the values suggested in the FLAG reviews [1], while the value of $f_\pi^{(0)}$ corresponds to the use of the experimental rate $\Gamma(\pi_{\mu 2})$, the value of $|V_{ud}|$ from Ref. [31] and the value $\delta R_\pi = 0.0176\,(21)$ obtained in ChPT [17, 18] and currently adopted by the PDG [19].

We will refer to the choice of the above three hadronic inputs as the FLAG/PDG prescription, which differs from that of Ref. [2] adopted in this work. We now show that the differences between the two prescriptions are well within the uncertainties of the input parameters of Ref. [20].

In Ref. [5] we have calculated the leading-order QED and QCD corrections to the pion and kaon masses within the prescription of Ref. [2]. Consequently, also the isosymmetric pion and kaon masses corresponding to the above prescription have been evaluated, obtaining $M_\pi^{(0)} = 134.9\,(2)\,\text{MeV}$, $M_K^{(0)} = 494.4\,(1)\,\text{MeV}$. These values differs only very slightly from the corresponding inputs used in the FLAG/PDG prescription.

In Ref. [29] we shall provide our result for δR_π, which differs slightly from the one obtained in ChPT [17, 18] and adopted by the PDG [19]. Since the quantity $[f_\pi^{(0)}]^2(1+\delta R_\pi)$ is prescription free, the resulting change of $[f_\pi^{(0)}]^2$ turns out to be less than 0.5%. Given that $[M_\pi^{(0)}/f_\pi^{(0)}]^2 \propto m_{ud}^{phys} + \mathcal{O}([m_{ud}^{phys}]^2)$, the change of the renormalized quark mass m_{ud}^{phys} is less than 0.02 MeV and, analogously, the change in m_s^{phys} is less than 0.5 MeV. Such variations of the physical u/d and strange quark masses produce changes in δR_π and $\delta R_{K\pi}$, which are, however, well within the statistical uncertainties, as can be easily inferred from Fig. 6 in the case of $\delta R_{K\pi}$.

The above findings indicate that the prescription of Ref. [2] and the FLAG/PDG one differ only by effects which are well within the uncertainties of the input parameters of Ref. [20]. This justifies the use of the FLAG average for the ratio $f_K^{(0)}/f_\pi^{(0)}$ to get Eq. (15) as well as the comparison of our result (14) with the ChPT prediction of Refs. [17, 18].

B. Evaluation of $\delta A_P^\mu/\delta A_P^{(0)}$

The evaluation of the diagrams 4(a) and (b), corresponding to the term δA_P^ℓ, starts from the correlator $\delta C_P^\ell(t)$ defined as

$$\delta C_\ell(t) = \sum_{\alpha\beta} \overline{u}_{\nu_\ell \alpha}(p_{\nu_\ell}) \overline{C}_1(t)_{\alpha\beta} v_{\ell\beta}(p_\ell) \ , \tag{19}$$

where $\overline{C}_1(t)_{\alpha\beta}$ is given by Eq. (35) of Ref. [6], while t is the time distance between the P-meson source and the insertion of the weak (V-A) current. At large time distances and for $T \to \infty$ one has

$$\delta C_\ell(t) \underset{t \gg a}{\longrightarrow} \frac{Z_P^{(0)} \delta A_P^\ell}{2M_P^{(0)}} X_P^\ell e^{-M_P^{(0)} t} \ , \tag{20}$$

where $X_P^\ell = Tr\left[\gamma_0(1-\gamma_5)\ell\overline{\ell}\gamma_0(1-\gamma_5)\nu_\ell\overline{\nu}_\ell\right]$ is the tree-level leptonic trace evaluated on the lattice [29]. Analogously, in the absence of the photon exchange between the quarks and the final-state lepton, the tree-level correlator $C_\ell^{(0)}(t)$ behaves at large time separations as

$$C_\ell^{(0)}(t) \underset{t \gg a}{\longrightarrow} \frac{Z_P^{(0)} A_P^{(0)}}{2M_P^{(0)}} X_P^\ell e^{-M_P^{(0)} t} \ . \tag{21}$$

Thus, the ratio

$$\delta R_P^\ell(t) \equiv \frac{\delta C_\ell(t)}{C_\ell^{(0)}(t)} \underset{t \gg a}{\longrightarrow} \frac{\delta A_P^\ell}{A_P^{(0)}} \tag{22}$$

should exhibit a plateau at the value $\delta A_P^\ell/A_P^{(0)}$. The quality of the signal for $\delta R_P^{\ell=\mu}(t)$ is illustrated in Fig. 1 for charged kaon and pion decays into muons in the case of the ensemble $D30.48$.

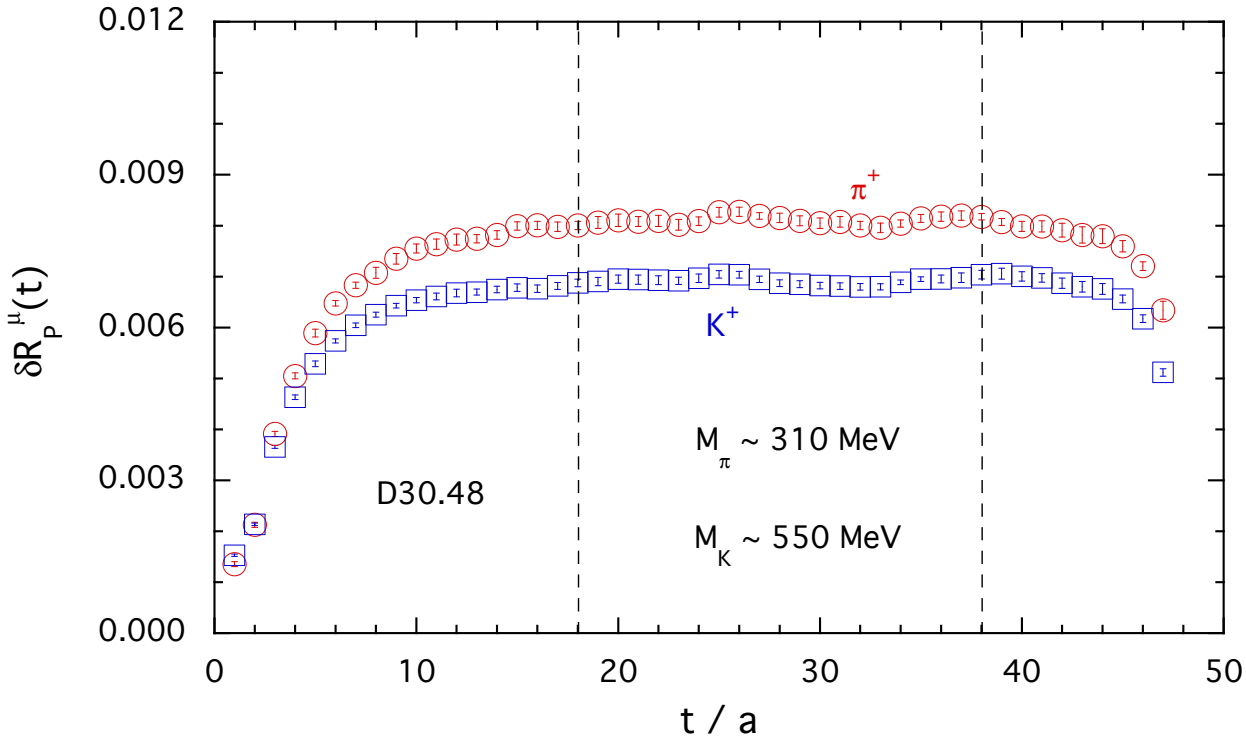

FIG. 1: *Results for the ratio* $\delta R_P^\mu(t)$*, given by Eq. (22), in the case of kaon and pion decays to muons for the gauge ensemble D30.48. The vertical dashed lines indicate the time region used for the extraction of the amplitude ratio* $\delta A_P^\mu / A_P^{(0)}$*. Errors are statistical only.*