

Algorithms – Randomization 21-1

Randomization

Design and Analysis of Algorithms
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Algorithms – Randomization 21-2

Randomization

Algorithmic design patterns.

- Greed.
- Divide-and-conquer.
- Dynamic programming.
- Network flow.
- Randomization. in practice, access to a pseudo-random number generator




Randomization. Allow fair coin flip in unit time.

Why randomize? Can lead to simplest, fastest, or only known algorithm for a particular problem.

Example.
Symmetry breaking protocols, graph algorithms, quicksort, hashing, load balancing, Monte Carlo integration, cryptography.

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Experiments and Outcomes

- Experiment: Tossing a coin
Outcomes: {heads, tails} 
- Experiment: Rolling a dice
Outcomes: {1,2,3,4,5,6} 
- Experiment: Rolling two dice
Outcomes: $\{1, \dots, 6\} \times \{1, \dots, 6\}$
or $\{A \subseteq \{1, \dots, 6\} : |A| \leq 2\}$ 
- Experiment: Buying 3 lottery tickets (out of 100,000)
Outcomes: 3-element subsets of $\{1, \dots, 100000\}$

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Sample Space and Events

- The set of all outcomes of an experiment is called the **sample space**
- Sometimes we are interested not in a single outcome, but an **event** that happens in several outcomes

Examples:

- Get heads at least 3 times when tossing 5 coins 
- Win a prize in lottery 
- Get 2 aces in a poker hand 

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Events

- Let S be the sample space of a certain experiment. An event is any subset of S

Examples:

- Experiment: Tossing 2 coins
Sample space: $S = \{\text{heads,tails}\} \times \{\text{heads,tails}\}$
Event: Get exactly 1 heads
 $A = \{(\text{heads,tails}), (\text{tails,heads})\}$
- Experiment: Rolling 2 dice
Sample space: $S = \{1, \dots, 6\} \times \{1, \dots, 6\}$
Event: The sum of the dice is 6
 $A = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$

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Probability

- In all our experiments each of the possible outcomes has the same likelihood of occurrence, or the same probability of occurrence
- If this is the case we can use the model of **classic** or **finite probability**
- Under the assumption of equal likelihood, let S be the sample space for an experiment. If $|S| = n$, $a \in S$, and $A \subseteq S$, then

$$\Pr(\{a\}) = \Pr(a) = \frac{1}{n} \quad \text{the probability that } a \text{ occurs}$$

$$\Pr(A) = \frac{|A|}{n} \quad \text{the probability that } A \text{ occurs}$$

Examples

- The probability of getting heads in the coin tossing experiment
 Sample space: $S = \{\text{heads,tails}\}$, Event: $A = \{\text{heads}\}$,

$$\Pr(A) = \frac{|A|}{|S|} = \frac{1}{2}$$
- The probability to get even number in the dice rolling experiment
 Sample space: $S = \{1,2,3,4,5,6\}$, Event: $A = \{2,4,6\}$

$$\Pr(A) = \frac{|A|}{|S|} = \frac{3}{6} = \frac{1}{2}$$
- 100 tickets, numbered 1,2,3,..., 100, are sold to 100 different people for a drawing. Four different prizes are awarded, including a grand prize (a trip to Tahiti). Find the probability that ticket 47 wins a prize while ticket 73 does not.

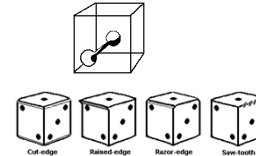
Equal Likelihood

- Equal likelihood of outcomes is a nontrivial property.
- It is not the case for flipping coins!
 See recent Persi Diaconis work



- One can make a crooked dice:

- loaded dice
- floaters
- tapping dice
- shapes
- bevells



More General Probability

- Sample space: Any set S
- Event: 'Any' subset of S
- Probability: A measure, that is a function $\Pr: P(S) \rightarrow [0,1]$, such that
 - $\Pr(\emptyset) = 0$
 - $\Pr(S) = 1$
 - $\Pr(A) \geq 0$ for all $A \subseteq S$
 - for any disjoint $A, B \subseteq S$, $\Pr(A \cup B) = \Pr(A) + \Pr(B)$

More General Probability: Crooked Dice

- Suppose we made a loaded dice
 $S = \{1,2,3,4,5,6\}$
 $\Pr(1) = 1/16$,
 $\Pr(2) = \Pr(3) = \Pr(4) = \Pr(5) = 1/8$
 $\Pr(6) = 7/16$
 $\Pr(\{i,j,\dots,m\}) = \Pr(i) + \Pr(j) + \dots + \Pr(m)$
- Find $\Pr(\{1,3,5\})$

More General Probability: Geometric Probability

- How to measure the area of an island?



- Draw a rectangle around the island and drop many random points
- Then
$$\frac{\text{area of the island}}{\text{area of the rectangle}} \approx \frac{\text{\# of points within the island}}{\text{total \# of points}}$$
- Sample space: Points in the rectangle
 Events: Measurable sets of points
 Probability: The area of an event

Properties of Probability

Theorem

Let S be the sample space of a certain experiment, A, B events.
 Then

- a) $\Pr(\bar{A}) = 1 - \Pr(A)$
- b) $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

Proof

$$\begin{aligned} \text{b) } \Pr(A \cup B) &= \Pr(A - B) + \Pr(B - A) + \Pr(A \cap B) \quad (\text{as these sets are disjoint}) \\ &= (\Pr(A - B) + \Pr(A \cap B)) + (\Pr(B - A) + \Pr(A \cap B)) - \Pr(A \cap B) \\ &= \Pr(A) + \Pr(B) - \Pr(A \cap B) \end{aligned}$$

Q. E. D.

Examples

- Two integers are selected, at random and without replacement, from $\{1, 2, \dots, 100\}$. What is the probability the integers are consecutive?
- If three integers are selected, at random and without replacement, from $\{1, 2, \dots, 100\}$, what is the probability their sum is even?

Probability Reminder

Discrete random variable:

A variable that takes values with certain probability

Example:

The amount of money you win buying a lottery ticket:

there are 1000 tickets, 1 wins \$10000, 10 win \$100, the rest win nothing

$\Pr[X = 10000] = 1/1000$, $\Pr[X = 100] = 1/100$, $\Pr[X = 0] = 989/1000$

Random Variables

Expectation

Let X be a discrete random variable with values v_1, \dots, v_k

Then $E[X] = v_1 \cdot \Pr[X = v_1] + \dots + v_k \cdot \Pr[X = v_k]$

Example:

$$\begin{aligned} E[\text{your win}] &= 10000 \cdot \Pr[X = 10000] + 100 \cdot \Pr[X = 100] + 0 \cdot \Pr[X = 0] \\ &= 10000 \cdot 1/1000 + 100 \cdot 1/100 + 0 \cdot 989/1000 \\ &= 11 \end{aligned}$$

One random variable interesting for us is the running time of some algorithm