Students's Solutions Manual

for

Fundamentals of Analysis

by Steven G. Krantz

Preface

This Manual contains the solutions to selected exercises in the book *Fundamentals of Analysis* by Steven G. Krantz, hereinafter referred to as "the text."

The problems solved here have been chosen with the intent of covering the most significant ones, the ones that might require techniques not explicitly presented in the text, or the ones that are not easily found elsewhere.

The solutions are usually presented in detail, following the pattern in the text. Where appropriate, only a sketch of a solution may be presented. Our goal is to illustrate the underlying ideas in order to help the student to develop his or her own mathematical intuition.

Notation and references as well as the results used to solve the problems are taken directly from the text.

Chapter 1

Number Systems

1.1 The Real Numbers

1. Since α is an upper bound for A, $-\alpha$ is a lower bound bound for B. If there were a number $\alpha' > -\alpha$ that is a lower bound for B, then $-\alpha'$ would be an upper bound for A that is less than α. That would be a contradiction. Hence $-\alpha$ is the infimum (or greatest lower bound) for B.

Now assume that A is bounded below and let β be the greatest lower bound. Define B as before to be $B = \{-a : a \in A\}$. Then $-\beta$ is an upper bound for B. If there were a number $\beta' < \beta$ that is an upper bound for B, then $-\beta'$ would be a lower bound for A that exceeds β. That would be a contradiction. Hence $-\beta$ is the supremum (or least upper bound) for B.

3. The least upper bound is $\sqrt{2}$. As we know from Theorem 1.7, this statement makes sense in the context of the real numbers (in fact, in the proof of that theorem, we essentially define $\sqrt{2}$ to be supremeum of S). But we know from Pythagoras's result that $\sqrt{2}$ does not exist in the rational numbers.

5. The set S is certainly bounded above by the number that is the circumference of C. The least upper bound p is the irrational number that we call π. All irrational numbers exists and are well defined in the real number system. But not in the rational number system.

7. The set $S = \{0, 1, 2, 3, \ldots\}$ is not bounded above and does not have a least upper bound. The set $T = \{0, -1, -2, -3, \ldots\}$ is not bounded below and does not have a greatest lower bound.

10. Any real number x is vacuously an upper bound for $\emptyset$. Thus no particular real number x can be the least upper bound (since $x - 1$ would also be an upper bound). Thus $-\infty$ is the least upper bound (i.e., it is less than or equal to all other upper bounds).

Any real number is vacuously a lower bound for $\emptyset$. Thus no particular real number can be the greatest lower bound (since $x + 1$ would also be a lower bound). Thus $+\infty$ is the greatest lower bound (i.e., it is greater than or equal to all other lower bounds).

13. Let α be a point in the image, with $\alpha = f(a)$. Then a is a local minimum for f, so there is an interval containing a on which f is a minumum. We may assume that that interval has rational endpoints. Thus we assign to each point in the image a pair of rational numbers. We conclude that the image is countable.

1.2 The Complex Numbers

1. If $z = x + iy$ and $w = u + iv$ are complex numbers then

$$\begin{aligned} z + w &= (x + iy) + (u + iv) = (x + u) + i(y + v) \\ &= (u + x) + i(v + y) = w + z\,. \end{aligned}$$

Here we used commutativity of real addition in the third equality.

If $z = x + iy$, $w = u + iv$, $r = m + in$ are complex numbers then

$$\begin{aligned} z + [w + r] &= (x + iy) + [(u + iv) + (m + in)] \\ &= (x + iy) + [(u + m) + i(v + n)] = [x + (u + m)] + i[y + (v + n)] \\ &= [(x + u) + m] + i[(y + v) + n] = [((x + u) + i(y + v)] + (m + in) \\ &= [(x + iy) + (u + iv)] + (m + in) = [z + w] + r\,. \end{aligned}$$

Here we used associativity of the reals in the fourth equality.

The proof of the distributive law is similar.

3. Let $z = x + iy$ and $w = u + iv$. We see that

$$\overline{z/w} = \overline{(x+iy)/(u+iv)} = \overline{(x+iy)\cdot(u-iv)/[(u+iv)(u-iv)]}$$
$$= \overline{[(xu+yv)+i(-xv+yu)]/(u^2+v^2)} = [(xu+yv)-i(-xv+yu)]/(u^2+v^2)$$
$$= (x-iy)(u+iv)/(u^2+v^2) = (x-iy)/(u-iv) = \overline{z}/\overline{w}\,.$$

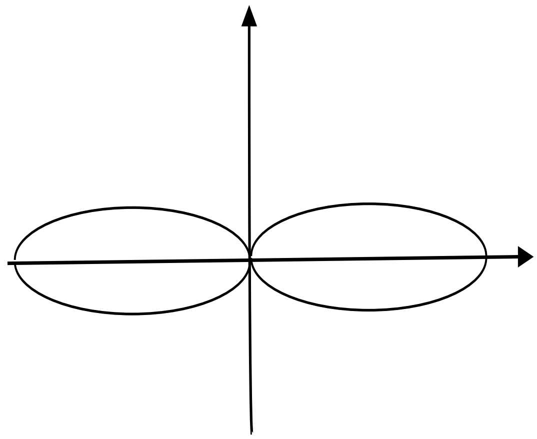
5. Let S be the set of all complex numbers with rational real part. The mapping

$$\begin{array}{ccc} \mathbb{R} & \rightarrow & S \\ y & \mapsto & 0+iy \end{array}$$

is one-to-one. Hence $\text{card}(\mathbb{R}) \leq \text{card}(S)$. Thus S is uncountable.

7. Let
$$S = \{z \in \mathbb{C} : |z| = 1\}\,.$$
Consider the function
$$\varphi : [0, 2\pi) \rightarrow S$$
given by
$$\varphi(t) = e^{it}\,.$$
Then φ is one-to-one and onto. So S has the same cardinality as the interval $[0, 2\pi)$. We conclude that S is uncountable.

9. Write $1 + i = \sqrt{2}e^{i\pi/4}$. Then the equation
$$r^3e^{3i\theta} = \sqrt{2}e^{i\pi/4}$$
leads to
$$re^{i\theta} = 2^{1/6}e^{i\pi/12}\,.$$
Next, the equation
$$r^3e^{3i\theta} = \sqrt{2}e^{i9\pi/4}$$
leads to
$$re^{i\theta} = 2^{1/6}e^{i9\pi/12}\,.$$
Finally, the equation
$$r^3e^{3i\theta} = \sqrt{2}e^{17i\pi/4}$$



leads to

$$re^{i\theta} = 2^{1/6}e^{i17\pi/12}\,.$$

Those are all the cube roots.

11. Let $p(z) = a_0 + a_1z + a_2z^2 + \cdots a_kz^k$ be a polynomial with all coefficients a_j real. Let α be a complex root of p. Then

$$p(\overline{\alpha}) = a_0 + a_1\overline{\alpha} + a_2\overline{\alpha}^2 + \cdots a_k\overline{\alpha}^k = \overline{a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_k\alpha^k} = \overline{0} = 0\,.$$

Hence $\overline{\alpha}$ is a root of p.

13. Squaring both sides, we see that the curve is equivalent to

$$|z^2| + |z^2 - 1| = 1\,.$$

So z^2 lies on an ellipse with major axis stretching from 0 to 1. The picture then is as in the figure.

15. Refer to the solution of Exercise 10. The kth roots of a complex number α are the roots of the polynomial equation $z^k - \alpha = 0$. There are k such roots.

Chapter 2

Sequences

2.1 Convergence of Sequences

3. Let $\epsilon > 0$. Choose $N_1 > 0$ such that, if $j > N_1$, then $|a_j - \alpha| < \epsilon$. Likewise, choose $N_2 > 0$ such that, if $j > N_2$, then $|c_j - \alpha| < \epsilon$. Let $N = \max\{N_1, N_2\}$. If $j > N$, then

$$b_j - \alpha \leq c_j - \alpha \leq |c_j - \alpha| < \epsilon \,.$$

Likewise,

$$b_j - \alpha > -\epsilon \,.$$

Thus

$$|b_j - \alpha| < \epsilon \,.$$

That proves the result.

5. The answer is no. We can even construct a sequence with arbitrarily long repetitive strings and with subsequences that converges to *any* real number α. Indeed, order $\mathbb{Q}$ into a sequence $\{q_n\}$. Consider the following sequence

$$\{q_1, q_2, q_2, q_1, q_1, q_1, q_2, q_2, q_2, q_2, q_3, q_3, q_3, q_3, q_3, q_1, q_1, q_1, q_1, q_1, q_1, \cdots\}.$$

In this way we have repeated each rational number infinitely many times, and with arbitrarily long strings. From the above sequence we can find subsequences that converge to any real number.

7. Notice that

$$\int_0^1 \frac{dt}{1+t^2} = \mathrm{Tan}^{-1}(t)\bigg|_0^1 = \frac{\pi}{4}\,.$$

Now approximate the integral by its Riemann sums.

9. The sequence is majorized by $e^j/e^{2j} = e^{-j} \to 0$. So the sequence converges to 0.

11. The sequence

$$3\,,\ 3.1\,,\ 3.14\,,\ 3.141\,,\ 3.1415\,,\ 3.14.159\ \ldots$$

consists of rational numbers that converge to π.

The sequence

$$a_j = 2 + \sqrt{2}/j$$

consists of irrational numbers that converge to 2.

13. Let $\beta > \alpha$ be irrational numbers. Let $\epsilon = \beta - \alpha$. Let $q > 0$ be a rational number that is smaller than ϵ. Consider the sequence $\{jq\}_{j=-\infty}^{\infty}$. Then some element jq must lie between α and β.

15. Consider the sequence

$$1\,,\ 3\,,\ 2\,,\ 4\,,\ 6\,,\ 5\,,\ 7\,,\ 9\,,\ 8\,,\ldots.$$

Then it is clear that the sequence tends to infinity, but it does not do so monotonically.

2.2 Subsequences

1. Clearly any increasing sequence $\{a_j\}$ that is bounded above is bounded. By Bolzano-Weierstrass it has a convergent subsequence $\{a_{j_k}\}$. But the same argument shows that any subsequence has a convergent subsequence with the same limit α. By Exercise 3 of the last section, the full sequence converges to α. In fact α is simply the least upper bound of the sequence.

5. The sequence $\{1, 2, 3, \ldots\}$ is bounded below but does not have a convergent subsequence.

The sequence $\{-1, -2, -3, \ldots\}$ is bounded above but does not have a convergent subsequence.

8. Certainly Proposition 2.13 shows that the b_j converge to some limit β (see also Exercise 1 above). But the limit of the b_j is also the lim inf of the original sequence a_j. It follows then that there is a subsequence of the a_j that converges to β.

9. The sequence

$$3\,,\,3\,,\,3.1\,,\,3\,,\,3.1\,,\,3.14\,,\,3\,,\,3.1\,,\,3.14\,,\,3.141\,,\,3\,,\,3.1\,,\,3.14\,,\,3.141\,,\,3.1315\,,\ldots$$

has infinitely many different subsequences that converge to π.

11. If it is not true that every subsequence has a convergent subsequence, then some subsequence lacks a convergent subsequence. But then the full sequence cannot converge.

The converse direction is similar.

2.3 Limsup and Liminf

1. Consider the sequence

$$0, 1, 2, 3, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots .$$

Then the supremum of this set of numbers is 3, while the limsup is 0. A similar example applies to the inf and liminf.

3. Let $\alpha \equiv \limsup a_j$ and $\beta \equiv \liminf a_j$. Let $A_j = \sup\{a_j, a_{j+1}, a_{j+2}, \ldots\}$ and $B_j = \inf\{a_j, a_{j+1}, a_{j+2}, \ldots\}$. Then

$$\begin{aligned}\sup\{1/a_j, 1/a_{j+1}, 1/a_{j+2}, \cdots\} &= 1/\inf\{a_j, a_{j+1}, a_{j+2}, \cdots\} \\ &= 1/B_j.\end{aligned}$$

Thus $\limsup 1/a_j = 1/\beta$.

Analogously one shows that $\liminf_{j\to\infty} 1/a_j = 1/\alpha$.

5. Let $\alpha = \liminf a_j = \limsup a_j$. Seeking a contradiction suppose that $\{a_j\}$ does not converge. Then there exist $\epsilon > 0$ and a subsequence $\{a_{j_k}\}$ such that for all k

$$|a_{j_k} - \alpha| > \epsilon.$$

Let $\beta = \limsup a_{j_k} (\neq \alpha)$ and $a_{j_{k_\ell}}$ be a subsequence such that $\lim_{\ell\to\infty} a_{j_{k_\ell}} = \beta$. But $\{a_{j_{k_\ell}}\}$ is a subsequence of the original sequence. By Corollary 2.33,

$$\liminf a_j \le \lim_{\ell\to\infty} a_{j_{k_\ell}} \le \limsup a_j$$

and by the Pinching Principle

$$\lim_{\ell\to\infty} a_{j_{k_l}} = \alpha.$$

This contradiction shows that $\{a_j\}$ converges to α.

9. When dealing with $\limsup(a_j \cdot b_j)$ we have to be careful of the signs. If a_j and b_j are all non-negative numbers, then

$$\begin{aligned}\limsup(a_j \cdot b_j) &= \lim_{k\to\infty}(a_{j_k} \cdot b_{j_k}) \\ &= \lim_{k\to\infty} a_{j_k} \cdot \lim_{k\to\infty} b_{j_k} \\ &\le \alpha \cdot \beta.\end{aligned}$$

Notice that in the inequality we have used that fact that all the quantities involved are non-negative ($x_1 < y_1$ and $x_2 < y_2$ implies $x_1 \cdot x_2 \le y_1 \cdot y_2$ only if x_1, x_2, y_1, y_2 are non-negative). Using this comment, it is easy to construct sequences $\{a_j\}$ and $\{b_j\}$ of negative numbers for which

$$\limsup(a_j \cdot b_j) > \limsup a_j \cdot \limsup b_j.$$

11. Since the values $\cos j$ are dense in the interval $[-1, 1]$, it follows that the limsup of $\cos j$ is $+1$ and the liminf of the sequence is -1. A similar assertion holds for the limsup and liminf of $\sin j$.

2.4 Some Special Sequences

1. Let $r = p/q = m/n$ be two representations of the rational number r. Recall that for any real α, the number α^r is defined as the real number β for which

$$\alpha^m = \beta^n.$$

Let β' satisfy

$$\alpha^p = \beta'^q .$$

We want to show that $\beta = \beta'$. we have

$$\begin{aligned} \beta^{n \cdot q} &= \alpha^{m \cdot q} \\ &= \alpha^{p \cdot n} \\ &= \beta'^{q \cdot n}. \end{aligned}$$

By the uniqueness of the $(n \cdot q)^{th}$ root of a real number it follows that

$$\beta = \beta',$$

proving the desired equality. The second equality follows in the same way. Let

$$\alpha = \gamma^n.$$

Then

$$\alpha^m = \gamma^{n \cdot m}.$$

Therefore, if we take the n^{th} root on both sides of the above inequality, we obtain

$$\gamma^m = (\alpha^m)^{1/n}.$$

Recall that γ is the n^{th} root of α. Then we find that

$$(\alpha^{1/n})^m = (\alpha^m)^{1/n}.$$

Using similar arguments, one can show that for all real numbers α and β and $q \in \mathbb{Q}$

$$(\alpha \cdot \beta)^q = \alpha^q \cdot \beta^q.$$

Finally, let α, β, and γ be positive real numbers. Then

$$\begin{aligned} (\alpha \cdot \beta)^\gamma &= \sup\{(\alpha \cdot \beta)^q : q \in \mathbb{Q}, q \leq \gamma\} \\ &= \sup\{\alpha^q \beta^q : q \in \mathbb{Q}, q \leq \gamma\} \\ &= \sup\{\alpha^q : q \in \mathbb{Q}, q \leq \gamma\} \cdot \sup\{\beta^q : q \in \mathbb{Q},\ q \leq \gamma\} \\ &= \alpha^\gamma \cdot \beta^\gamma. \end{aligned}$$

3. It suffices to notice that, for any fixed x,

$$\begin{aligned}\lim_{j\to\infty}\left(1+\frac{x}{j}\right)^j &= \lim_{j\to\infty}\left\{\left(1+\frac{x}{j}\right)^{j/x}\right\}^x \\ &= \left\{\lim_{j/x\to\infty}\left\{1+\frac{x}{j}\right\}^{j/x}\right\}^x \\ &= e^x.\end{aligned}$$

5. Write

$$\begin{aligned}\frac{j^j}{(2j)!} &= \frac{j\cdots j}{1\cdots j\cdot j+1\cdots 2j} \\ &\leq \frac{1}{1\cdots j} \\ &= \frac{1}{j!}.\end{aligned}$$

Then

$$\begin{aligned}\lim_{j\to\infty}\frac{j^j}{(2j)!} &\leq \lim_{j\to\infty}\frac{1}{j!} \\ &= 0.\end{aligned}$$

7. We write $F(x) = a_0 + a_1x + a_2x^2 + \cdots$. Here the a_j's are the terms of the Fibonacci sequence and the letter x denotes an unspecified variable. What is curious here is that we do not care about what x is. We intend to manipulate the function F in such a fashion that we will be able to solve for the coefficients a_j. Just think of $F(x)$ as a polynomial with a *lot* of coefficients.

Notice that

$$xF(x) = a_0x + a_1x^2 + a_2x^3 + a_3x^4 + \cdots$$

and

$$x^2F(x) = a_0x^2 + a_1x^3 + a_2x^4 + a_3x^5 + \cdots .$$

Thus, grouping like powers of x, we see that

$$\begin{aligned}&F(x) - xF(x) - x^2F(x) \\ &= a_0 + (a_1 - a_0)x + (a_2 - a_1 - a_0)x^2 \\ &\qquad +(a_3 - a_2 - a_1)x^3 + (a_4 - a_3 - a_2)x^4 + \cdots .\end{aligned}$$

But the basic property that defines the Fibonacci sequence is that $a_2 - a_1 - a_0 = 0$, $a_3 - a_2 - a_1 = 0$, etc. Thus our equation simplifies drastically to

$$F(x) - xF(x) - x^2F(x) = a_0 + (a_1 - a_0)x.$$

We also know that $a_0 = a_1 = 1$. Thus the equation becomes

$$(1 - x - x^2)F(x) = 1$$

or

$$F(x) = \frac{1}{1 - x - x^2}. \tag{$*$}$$

It is convenient to factor the denominator as follows:

$$F(x) = \frac{1}{\left[1 - \frac{-2}{1-\sqrt{5}}x\right] \cdot \left[1 - \frac{-2}{1+\sqrt{5}}x\right]}$$

(just simplify the right hand side to see that it equals ($*$).

A little more algebraic manipulation yields that

$$F(x) = \frac{5+\sqrt{5}}{10}\left[\frac{1}{1 + \frac{2}{1-\sqrt{5}}x}\right] + \frac{5-\sqrt{5}}{10}\left[\frac{1}{1 + \frac{2}{1+\sqrt{5}}x}\right].$$

Now we want to apply the formula for the sum of a geometric series to each of the fractions in brackets ([ ]). For the first fraction, we think of $-\frac{2}{1-\sqrt{5}}x$ as λ. Thus the first expression in brackets equals

$$\sum_{j=0}^{\infty}\left(-\frac{2}{1-\sqrt{5}}x\right)^j.$$

Likewise the second sum equals

$$\sum_{j=0}^{\infty}\left(-\frac{2}{1+\sqrt{5}}x\right)^j.$$

All told, we find that

$$F(x) = \frac{5+\sqrt{5}}{10}\sum_{j=0}^{\infty}\left(-\frac{2}{1-\sqrt{5}}x\right)^j + \frac{5-\sqrt{5}}{10}\sum_{j=0}^{\infty}\left(-\frac{2}{1+\sqrt{5}}x\right)^j.$$

Grouping terms with like powers of x, we finally conclude that

$$F(x) = \sum_{j=0}^{\infty} \left[\frac{5+\sqrt{5}}{10} \left(-\frac{2}{1-\sqrt{5}} x \right)^j + \frac{5-\sqrt{5}}{10} \left(-\frac{2}{1+\sqrt{5}} x \right)^j \right] x^j.$$

But we began our solution of this problem with the formula

$$F(x) = a_0 + a_1 x + a_2 x^2 + \cdots .$$

The two different formulas for $F(x)$ must agree. In particular, the coefficients of the different powers of x must match up. We conclude that

$$a_j = \frac{5+\sqrt{5}}{10} \left(-\frac{2}{1-\sqrt{5}} \right)^j + \frac{5-\sqrt{5}}{10} \left(-\frac{2}{1+\sqrt{5}} \right)^j .$$

We rewrite

$$\frac{5+\sqrt{5}}{10} = \frac{1}{\sqrt{5}} \cdot \frac{1+\sqrt{5}}{2} \qquad \frac{5-\sqrt{5}}{10} = -\frac{1}{\sqrt{5}} \cdot \frac{1-\sqrt{5}}{2}$$

and

$$-\frac{2}{1-\sqrt{5}} = \frac{1+\sqrt{5}}{2} \qquad -\frac{2}{1+\sqrt{5}} = \frac{1-\sqrt{5}}{2}.$$

Making these four substitutions into our formula for a_j, and doing a few algebraic simplifications, yields

$$a_j = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^j - \left(\frac{1-\sqrt{5}}{2}\right)^j}{\sqrt{5}}$$

as desired.

9. We can write this sequence as

$$\left[\left(1 + \frac{1}{j^2} \right)^{j^2} \right]^{1/j} .$$

Of course the expression inside the brackets tends to e. So the entire sequence tends to the same limit as $e^{1/j}$, which is 1.

Chapter 3

Series of Numbers

3.1 Convergence of Series

1. (a) Notice that

$$\frac{2^{2j+2}/(j+1)!}{2^{2j}/j!} = \frac{4}{j}\,.$$

Then, by the Ratio Test for Convergence, the series converges.

(b) We use the Ratio Test again:

$$\begin{aligned}
\frac{\bigl(2(j+1)!\bigr)/\bigl(3(j+1)\bigr)!}{(2j)!/(3j)!} &= \frac{(2j+2)!}{(3j+3)!}\cdot\frac{(3j)!}{(2j)!} \\
&= \frac{(2j+2)(2j+1)}{(3j+3)(3j+2)(3j+1)} \\
&\le \frac{(2j+2)^2}{(3j)^3} \\
&\le \frac{1}{27}
\end{aligned}$$

for j large enough. Thus the series converges.

(c) Write $j^j = e^{j\ln j}$. Then

$$\frac{\frac{(j+1)!}{(j+1)^{(j+1)}}}{\frac{j!}{j^j}}\,.$$

Then $\lim_{j\to\infty}\frac{a_{j+1}}{a_j} = 1/e$ and the series converges.

(d) We use Abel's criterion. Let $a_j = (-1)^j$ and $b_j = \frac{1}{3j^2-5j+6}$ for all j. Then $b_j \geq b_{j+1}$ and $\lim_{j\to\infty} b_j = 0$. Also the partial sums of the a_j's remain bounded by 1. Thus the series converges.

(e) Notice that
$$\frac{2j-1}{3j^2-2} \geq \frac{2j-1}{3j^2} \geq \frac{j}{3j^2} = \frac{1}{3j}.$$
Since the series $\sum_{j=1}^\infty \frac{1}{3j}$ diverges, so also does $\sum_{j=1}^\infty \frac{2j-1}{3j^2-2}$.

(f) We use the Comparison Test again. Now
$$\frac{2j-1}{3j^3-2} \leq \frac{2j}{3j^3-2} \leq \frac{2j}{j^3} = \frac{2}{j^2}.$$
Since $\sum_{j=1}^\infty \frac{2}{j^2}$ converges, so also does $\sum_{j=1}^\infty \frac{2j-1}{3j^3-2}$.

(g) This is comparable to $\sum 1/|1+\log j|^{j-1}$. That in turn converges by the root test.

(h) This converges by the integral test.

3. If $\sum b_j$ converges then $b_j \to 0$. Therefore $1/[1+b_j] \to 1$. Thus $\sum 1/[1+b_j]$ cannot converge.

5. FALSE. Let $a_j = j^2$.

7. Let $b_j = 2^j$. Then $\sum_j b_j$ diverges and also $\sum_j 2^{-j} b_j$ diverges.

9. Refer to Exercise 2. It follows that $\sum_j a_j^4$ will converge. For a similar reason, $\sum_j a_j^3$ will converge.

11. Let $a_j = (-1)^j/j$ and $b_j = (-1)^j/j$. Then $\sum_j a_j$ converges and $\sum_j b_j$ converges, but $\sum_j a_j b_j$ diverges.

3.2 Elementary Convergence Tests

1. If j is large then b_j is very small. So $p(b_j)$ is comparable to $a_m b_j^m$, where the mth term is the lowest order term of the polynomial. But clearly $\sum b_j^m$ converges by the Comparison Test.

3. With the Root Test both series give limit 1. With the Ratio Test both series give limit 1.

9. For a_j positive this is true just because $a_j/j \le a_j$ so the Comparison Test applies.

11. The Integral Test, together with integration by parts, can often handle such series. Also you can use the Comparison Test.

3.3 Advanced Convergence Tests

1. We see that $b_j/(1-b_j) \le 2b_j$. So the convergence follows from the Comparison Test.

3. By the Cauchy-Schwarz inequality,

$$\sum_{j=1}^{N} (b_j)^{1/2} \cdot \frac{1}{j^\alpha} \le \sum_{j=1}^{N} b_j \cdot \sum_{j=1}^{N} \frac{1}{j^{2\alpha}} \,.$$

Both series on the right converge.

When $\alpha = 1/2$, consider the example given by $b_j = 1/(j \cdot [\log(j+1)]^{3/2}$. The product series will then diverge.

5. Clearly $(-1)^{3j} = [(-1)^3]^j = (-1)^j$. Hence the partial sums of $(-1)^{3j}$ are bounded, and the $a_j \to 0$. So Abel's test applies and the series converges.

7. Clearly s_j converges to some positive number σ. Thus $s_j \cdot b_j$ is comparable to $\sigma \cdot b_j$. So $\sum s_j \cdot b_j$ converges. Also $1 + s_j$ converges to $1 + \sigma$. So $b_j/(1+s_j)$ is comparable to $b_j/(1+\sigma)$. Thus $\sum b_j/(1+s_j)$ converges.

9. It must be that the positive summands from among the b_j form a divergent series and the negative summands from among the b_j form a divergent series. Fix a real number α. Add up enough (but finitely many) of the positive b_j so that the sum just exceeds α. Then add on enough of the negative terms (but finitely many) so that the sum is just less than α. Keep alternating in this fashion. The resulting rearrangement will converge to α.

11. We use the Mean Value Theorem to estimate the numerator by

$$3 \cdot \frac{1}{2} \cdot (2j+3)^{-1/2} \cdot 2 \,.$$

So the fraction being summed has size

$$C \cdot \frac{1}{(j^{3/4} \cdot \sqrt{2j+3}} .$$

this converges by the Comparison Test.

3.4 Some Special Series

1. Consider the formula

$$(j+1)^3 - j^3 = 3j^2 + 3j + 1 .$$

Sum both sides from $j = 1$ to N. The lefthand side telescopes and we get

$$(N+1)^3 - 1 = 3\sum_{j=1}^{N} j^2 + 3\sum_{j=1}^{N} j + \sum_{j=1}^{N} 1 .$$

Using Gauss's formula for the second sum on the right, and evaluating the third sum explicitly, we get

$$(N+1)^3 - 1 = 3\sum_{j=1}^{N} j^2 = \frac{3N(N+1)}{2} + N .$$

Now we may solve for the sum that we seek and we find that

$$\sum_{j=1}^{N} j^2 = \frac{2N^3 + 3N^2 + N}{6} .$$

The formulas for the sum of the first N cubes or the first N quartics are derived similarly.

3. This series majorizes $\sum_j 1/j$. So it diverges.

5. If the polynomial is constant then of course the series diverges. If the polynomial has degree at least 1, then $|p(j)|$ is comparable to the absolute value of its lead term for j large. Thus, if the degree is 1, then the series diverges. If the degree is greater than 1, then the series converges. term

9. Of course the integers are a countable set. It is easy to see then that, for any k, the polynomials of degree k with integer coefficients form a countable set. Therefore the set of *all* polynomials with integer coefficients forms a countable set. Each such polynomial has finitely many roots. So the set of all roots of all polynomials with integer coefficients (that is, the algebraic numbers) forms a countable set. Since the reals are uncountable, we conclude that there are uncountably many transcendental numbers.

11. The Cauchy-Schwarz inequality shows that, if $\sum_j b_j^2$ converges then the given series converges. Alternatively, by the Comparison Test, if $b_j < j^{-\alpha}$ for some $\alpha > 0$ then the series converges.

3.5 Operations on Series

1. Since $b_j \to 0$, it follows that $|a_j b_j| \le |a_j$ for j large. So $\sum a_j b_j$ will converge absolutely.

3. We see that

$$c_n = \sum_{j=0}^{n} a_j b_{n-j} = \sum_{j=0}^{n} \frac{1}{2^j} \cdot \frac{1}{4^{n-j}} .$$

It would be difficult to calculate this sum explicitly.

5. If $p(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_k x^k$ and $q(x) = q_0 + q_1 x + q_2 x^2 + \cdots + q_k x^k$, then

$$p(x) \cdot q(x) = p_0 q_0 + (p_0 q_1 + p_1 q_0)x + (p_0 q_2 + p_1 q_1 + p_2 q_0)x^2 + \cdots \text{etc.}$$

We see that the coefficients of the product polynomial are the same as those in the Cauchy product.

7. Since $\sum_j 1/j^\alpha$ diverges for all $0 < \alpha < 1$, we see that there are uncountably many different divergent series. Since $\sum_j 1/j^\beta$ converges for all $0 < \beta < 1$, we see that there are uncountably many different convergent series.

9. If $\sum_j a_j$ converges then $a_j \to 0$. Hence $e^{a_j} \to 1$. So the series $\sum_j e^{a_j}$ cannot converge.

11. If $\alpha = \sum a_j x^j$, then we may think of

$$\exp(\alpha)$$

as either

$$e^{\sum a_j x^j}$$

or as

$$\prod_{j=0}^{\infty} \exp(a_j x^j)\,.$$

Infinite products of this type are studied in a complex analysis course.

Chapter 4

Basic Topology

4.1 Open and Closed Sets

1. Let $t \in T$. Choose s as in the definition of T. Let $\epsilon' = \epsilon - |s-t|$. Then it follows from the triangle inequality that the interval $(t-\epsilon', t+\epsilon') \subseteq T$. So T is open.

3. Let $X_j = [1/j, 1-1/j]$. Then $\cup_j X_j = (0,1)$.

5. Let

$$\mathcal{I} = \{I = (a,b) : a, b \in \mathbb{Q}\}.$$

We first prove that each interval can be written as the union of elements in $\mathcal{I}$. Let (α, β) be a bounded interval, $\{a_n\}$ and $\{b_n\}$ sequences of rational numbers such that

$$a_n \geq \alpha, \quad \text{and} \quad b_n \leq \beta \quad \text{for all} \quad n$$

and

$$a_n \to \alpha, \quad \text{and} \quad b_n \to \beta \quad \text{as} \quad n \longrightarrow \infty.$$

Then

$$(\alpha, \beta) = \bigcup_{n=1}^{\infty} (a_n, b_n).$$

Notice that the same proof applies to the cases $\beta = -\infty$ and $\alpha = \infty$. Then any open interval can be writen as countable union of elements of $\mathcal{I}$.

Finally, let E be any open set in $\mathbb{R}$. By Proposition 3 in the text we know that

$$E = \bigcup_{n=1}^{\infty} J_n,$$

where the J_n's are intervals. By the above argument, for each n we have

$$J_n = \bigcup_{j=1}^{\infty} I_j^n, \quad \text{where} \quad I_j^n \in \mathcal{I}.$$

This concludes the proof.

7. The set $[0, 1)$ is not open, but it is also not closed. The set is not closed, but it is also not open.

9. Let $E = [0, 1] \cup [2, 3] \cup [4, 5] \cup \cdots$ and $F = [1.1, 1.9] \cup [3.01, 3.99] \cup [5.001, 5.999] \cup \cdots$. Then each of E and F is closed, but the distance between the two sets is 0.

4.2 Further Properties of Open and Closed Sets

1. By definition, $\overline{S}$ is the intersection of all closed sets that contain S. So certainly $\overline{S}$ contains S. Since the intersection of closed sets is closed, so certainly is $\overline{S}$ closed. Now let $x \in \overline{S} \setminus \overset{\circ}{S}$. Let U be any neighborhood of x. If U were disjoint from the interior then x could not be in the boundary. So U intersects the interior. If U were disjoint from the complement of S, then U would lie in S. But then x would be an interior point of S. That is impossible. So U intersects the complement. Thus x is in the boundary.

3. If x is in the boundary of S then any neighborhood U of x contains points of S and points of ${}^cS$. Thus any neighborhood U of x contains points of ${}^c({}^cS)$ and points of ${}^cS$. So x is in the boundary of ${}^cS$. The argument in the other direction is the same.

5. Let $U_j = (-1/j, 1 + 1/j)$. Then each U_j is open, but their intersection is $[0, 1]$, which is closed. Let $V_j = (0, 1)$ for each j. Then each V_j is open but their intersection is $(0, 1)$, which is also open.

7. If x is a non-isolated boundary point of S, then every neighborhood of x must contain points of S other than x itself. Thus x is an accumulation point off S.

9. Let $S = \{(x, y) \in \mathbb{R}^2 : y = 1/x, x > 0\}$. Then this set is closed. But the projection of S to the x axis is $\{x \in \mathbb{R} : 0 < x < \infty\}$, which is open.

11. The distance is $1/6$.

4.3 Compact Sets

1. The point here is that ϵ is the same for all $x \in K$. For each x, let ϵ_x be such that $I_x = (x - \epsilon_x, x + \epsilon_x) \subseteq U$. The collection $\{I_x\}_{x \in K}$ is a covering of K. Select a finite subcovering $I_{x_1}, \ldots, I_{x_n}$. Let

$$\epsilon = \min\{\epsilon_{x_1}, \ldots, \epsilon_{x_n}\}.$$

Then, for all $x \in K$,

$$\begin{aligned} (x - \epsilon, x + \epsilon) &\subseteq (x - \epsilon_{x_j}, x + \epsilon_{x_j}) \qquad \text{(for some } j) \\ &\subseteq I_{x_j} \\ &\subseteq U. \end{aligned}$$

3. Let K be compact and E be closed. Then $K \cap E$ is closed. Also $K \cap E$ is a subset of K so it is bounded. Therefore $K \cap E$ is compact.

5. If $K \subseteq \mathbb{R}$ is compact then it is closed. So $\mathbb{R} \setminus K$ is open. Also K is bounded. So $\mathbb{R} \setminus K$ is unbounded. Thus $\mathbb{R} \setminus K$ is definitely not compact.

7. Fix an open set U. Let

$$K_j = \{x \in U : \operatorname{dist}(x, {}^cU) \geq 1/j\} \cap \{x \in \mathbb{R} : |x| \leq j\}.$$

Then K_j is closed because it is is defined by non-strict inequalities. And it is bounded because it is contained in an interval of radius $2j$. So it is compact. Also plainly $\cup K_j = U$.

9. The union of finitely many closed sets is closed. The union of finitely many bounded sets is bounded. That does the job.

11. For each $x \in K$, let B_x be the open interval with center x and radius δ. Then $\{B_x\}_{x \in K}$ is an open covering of K. Since K is compact, there is a finite subcovering.

4.4 The Cantor Set

1. If x is a point of the Cantor set and U any neighborhood of x then U will intersect the complement. So the Cantor set has no interior.

3. The lengths of the intervals removed in this fashion are

$$\frac{1}{5} + \frac{2}{25} + \frac{4}{125} + \cdots = \frac{1}{5} \sum_{j=0}^{\infty} \frac{2^j}{5^j} = \frac{1}{5} \sum_{j=0}^{\infty} \left(\frac{2}{5}\right)^j = \frac{1}{5} \cdot \frac{1}{1 - 2/5} = \frac{1}{3}\,.$$

So the total length of the intervals removed is 1/3. So this is clearly a different set from the classical Cantor ternary set. This new set has length 2/3. The same proof shows that it is uncountable. The same proof shows that it is perfect.

5. Start out with the closed unit square $U = [0,1] \times [0,1]$. At the first step, remove an open square, centered at $(1/2, 1/2)$, of side 1/3. At the next step, remove four squares of side 1/9. And so forth.

7. Just by counting, the set of points with finite ternary expansions is countable. The set of points with infinite ternary expansions is uncountable.

9. The sequence should be lacunary. That is, $a_{j+1} = \lambda a_j$ for some $0 < \lambda < 1$.

4.5 Connected and Disconnected Sets

3. Let S be the rational numbers in $[0,1]$. Note that if x, y are distinct points of S then there is an irrational number β between them. Let $U = \{x \in S : x < \beta\}$ and let $V = \{x \in S : x > \beta\}$. Then U and V disconnect S. But clearly the closure of S is $[0,1]$.

5. If A and B are disjoint then the answer is clearly "no." If A and B are not disjoint then we are taking the union of two intervals with nontrivial intersection. This union will be another interval, hence connected.

7. The sets $B(q_j, r_j)$, where q_j are points with rational coordinates and r_j are rational numbers (and B stands for the ball with center the first entry and radius the second entry), then each $B(q_j, r_j)$ is connected. But the topology for which this is a basis is the usual topology on $\mathbb{R}^n$. That includes both connected and disconnected open sets.

9. Refer to Exercise 5.

11. Write $\mathbb{R} = \{\text{rational numbers}\} \cup \{\text{irrational numbers}\}$. Then, just as in Exercise 3, each of these sets is totally disconnected.

4.6 Perfect Sets

1. Let $S_j = \mathbb{R} \setminus [0, 1/j]$. Then each S_j is open. Also $S_1 \subseteq S_2 \subseteq S_3 \subseteq \cdots$. But $\cup_j S_j = \mathbb{R} \setminus \{0\}$. In fact the same argument shows that this assertion is never true.

3. Yes, and this is a direct verification.

5. Let S be the set of rational numbers in $[0.1]$ and let T be the set of irrational numbers in $[0, 1]$. Then $S \cup T$ is perfect. But neither S nor T is perfect.

7. That this new set is topologically like the Cantor set is obvious. A simple calculation with geometric series shows that this new set has length $6/7$.

9. Let A be the set of condensation points for S. Let $x \in \overline{A}$. Let U be any neighbourhood of x. Let $a \in A \cup U$. Then U is a neighbourhood of a. Thus U contains uncountably many points of S. We have proved that in any neighbourhood of x there are uncountably many points of A, i.e. $x \in A$. We conclude that A is closed. The set A can be empty, for instance, when A is only countable. Actually, A is non-empty if and only if S is uncountable. In order to prove that S uncountable implies A non-empty, just repeat the proof of the Bolzano-Weierstrass theorem. In this way we obtain a point in any neighborhood of which there are uncountably many points of S.

Now we want to show that for S uncountable, the set A of its condensation points is perfect. Indeed, we know that A is closed and non-empty, so we need only show that every point of A is an acculation point for A. We make the following claim: An uncountable subset of $\mathbb{R}$ has more than one accumulation points (indeed it has infinitely many). Assume the claim for moment. Let $x \in A$. Let U be a neighbourhood of x. Then there are uncountably many points of S in $U \cap S$. Let y be an accumulation point of $U \cap S$ that is different from x. Then $y \in A$. Since in each neighborhood of x we can find an element of A, it follows that x is an accumulation point for A. Then A is perfect, if we prove the claim.

To prove the claim, argue by contradiction. Suppose that $\{s_\lambda\}$ is an uncountable set contained in $[0, 1]$ and that the only accumulation point is 1 (the general case can be easily reduced to this one). Then, for each n, the set $[0, 1 - \frac{1}{n}]$ contains at most finitely many points s_λ. But $[0, 1) = \cup_{n=1}^{\infty}[0, 1 - \frac{1}{n}]$, so there cannot be uncountably many points of $\{s_\lambda\}$ in $[0, 1]$.

Chapter 5

Limits and Continuity of Functions

5.1 Basic Properties of the Limit of a Function

1. Let $\ell = \lim_{x \to c} f(x)$ and let $m = \lim_{x \to c} g(x)$. Let $\epsilon > 0$. Choose $\delta_1 > 0$ so that $0 < |x - c| < \delta_1$ implies that $|f(x) - \ell| < \epsilon$. Also choose $\delta_2 > 0$ so that $0 < |x - c| < \delta_2$ implies that $|g(x) - m| < \epsilon$. Let $\delta = \min(\delta_1, \delta_2)$. If $|x - c| < \delta$ then

$$\ell \leq f(x) + \epsilon \leq g(x) + \epsilon \leq m + \epsilon + \epsilon .$$

So

$$\ell \leq m + 2\epsilon .$$

Since this is true for any $\epsilon > 0$, we conclude that

$$\ell \leq m .$$

There is no improvement of we assume that $f(x) < g(x)$ for $x \in A$. To see this, let $A = (-1, 0) \cup (0, 1)$, $c = 0$, $f(x) = -x^2$, $g(x) = x^2$. Then $f(x) < g(x)$ for all $x \in A$ yet $\lim_{x \to c} f(x) = 0 = \lim_{x \to c} g(x)$.

3. Suppose that $\lim_{x \to c} f(x) = \ell$ and $\lim_{x \to c} f(x) = m$. Let $\epsilon > 0$. Choose δ_1 such that $0 < |x - c| < \delta_1$ implies that $|f(x) - \ell| < \epsilon$. Also choose δ_2 such that $0 < |x - c| < \delta_2$ implies that $|f(x) - m| < \epsilon$. Then

$$|\ell - m| \leq |\ell - f(x)| + |f(x) - m| < \epsilon + \epsilon .$$

Since this is true for any $\epsilon > 0$, we conclude that $\ell = m$.

7. The Dirichlet function is

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q}\,, \\ 0 & \text{if} \quad x \notin \mathbb{Q}\,. \end{cases}$$

Then f is discontinuous at every real number.

9. If c is any real then $\lim_{x\to c^-} f(x) \le \lim_{x\to c^+} f(x)$. If in fact the left one-sided limit is strictly less than the right one-sided limit, then there is a rational number between the two limits. So there are at most countably many such points. At any point c where the two one-sided limits agree, the function is continous. So the function is continuous at uncountably many points and discontinuous at at most countably many points.

5.2 Continuous Functions

1. Let $\epsilon > 0$. Choose $\delta = [\epsilon/C]^{1/\alpha}$. If $|s - t| < \delta$, then we see that

$$|f(s) - f(t)| \le C \cdot |s - t|^\alpha < C \cdot \delta^\alpha = C \cdot ([\epsilon/C]^{1/\alpha})^\alpha = \epsilon\,.$$

So f is uniformly continuous.

3. Because if c is isolated and $\delta > 0$ is small then the set of x with $0 < |x - c| < \delta$ would be empty.

5. If f were not bounded then there would be a sequence a_j in the compact set so that $f(a_j) \to \pm\infty$. But the domain is bounded, so some subsequence a_{j_k} converges to a limit a_0. Then $f(a_{j_k}) \to f(a_0)$, and that is a contradiction.

7. Let the closed set be $\mathbb{R}$ and let the function be $\mathrm{Tan}^{-1} x$. Then the image is the interval $(-\pi/2, \pi/2)$.

9. The continuous image of a connected set is connected. So $f((a, b))$ will be an interval. It can be an open interval, a closed interval, or a half-open interval.

11. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if the inverse image $f^{-1}(E)$ of any closed set E is closed.

5.3 Topological Properties and Continuity

1. No, but it follows that f is nonnegative at all x. For example, let $f(x) = |x - \pi|$. This function is positive at all rational x. But $f(\pi) = 0$.

3. Let $x, y \in \mathbb{R}$. Then, for any $s \in S$,

$$|x - s| \leq |x - t| + |t - s|.$$

This, taking the infimum over s on the left,

$$f(x) \leq |x - t| + |t - s|.$$

Now taking the infimum over t on the right gives

$$f(x) \leq f(t) + |t - s|.$$

A similar argument shows that

$$f(t) \leq f(x) + |t - s|.$$

Putting these two inequalities together yields that

$$|f(t) - f(x)| \leq |t - s|.$$

This shows that f satisfies a Lipschitz condition, so is uniformly continuous.

7. Set $g(x) = f(x) - x$. If f does not have a fixed point then we see that $f(0) > 0$ and $f(1) < 1$. As a result, $g(0) > 0$ and $g(1) < 0$. By the Intermediate Value Property, there is a point ξ between 0 and 1 such that $g(\xi) = 0$. But then $f(\xi) = \xi$.

11. The image $f(A)$ can be any number of intervals between 0 and k. Just for instance, if $f(x) = \sin x$ and $A = [0, \pi/6] \cup [\pi/4, \pi/2] \cup [9\pi/4, 5\pi/2]$, then $f(A) = [0, 1/2] \cup [\sqrt{2}/2, 1]$. Small variations of this example can vary

13. Let $U = (-1, 1)$ and $f(x) = x^2$.

5.4 Classifying Discontinuities and Monotonicity

1. Define

$$f(x) = \begin{cases} -1 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \,. \end{cases}$$

and

$$g(x) = \begin{cases} 1 & \text{if} \quad x < 0 \\ -1 & \text{if} \quad x \geq 0 \,. \end{cases}$$

Then each of f and g is discontinuous at the origin but $f + g \equiv 0$ so is continuous at all points.

Notice that $F(x) = e^{f(x)}$ and $G(x) = e^{g(x)}$ are both discontinuous at the origin. But $F(x) \cdot G(x) \equiv 1$ so is continuous at all points.

If you replace G in the last paragraph by $1/G$, then you can replace the product by the quotient.

7. If f has a discontinuity of the first kind at c, then the left limit and the right limit at c disagree. So one can slip a rational number between these two limits. This assigns a rational number to each discontinuity of the first kind. So the set of these discontinuities is countable.

9. Let $A = \{a_1, a_2, \ldots\}$. Define

$$f(x) = \max\{a_j : a_j \leq x\} \,.$$

Then f will be step function with discontinuities of the first kind at each of the a_j.

10. Define

$$f(x) = \begin{cases} x - 1 & \text{if} \quad -\infty < x < 0 \\ x & \text{if} \quad 0 \leq x < 1 \\ x - 2 & \text{if} \quad 1 \leq x < 2 \\ x - 1 & \text{if} \quad 2 \leq x < \infty \,. \end{cases}$$

Then f is one-to-one and onto from $\mathbb{R}$ to $\mathbb{R}$ but f is *not* monotone increasing. Of course this particular f is not continuous, much less continuously differentiable.

In case f is continuously differentiable and maps $\mathbb{R}$ to $\mathbb{R}$ one-to-one and onto, suppose there is an interval on which f is both monotone increasing and monotone decreasing. Then f would have to have a local extremum in the interval, so it cannot be one-to-one. Thus f must be monotone on every interval. By connectivity, f is globally monotone.

11. The continuity is straightforward. It is also easy to check that $f''(x) \geq 0$ implies convexity as defined here.

Chapter 6

Differentiation of Functions

6.1 The Concept of Derivative

1. If

$$f(x) = \begin{cases} -1 & \text{if} \quad x \leq 0 \\ 1 & \text{if} \quad 0 < x \, , \end{cases}$$

then $f^2 \equiv 1$. So f^2 is differentiable at 0 but f is not. So the implication fails when $k = 2$.

For $k = 3$, if $f(x) \neq 0$ and f^3 is differentiable near x then f is continuous near x. So f is nonvanishing near x. Thus we may write

$$f(x) = [f^3(x)]^{1/3}$$

to see that f must be differentiable at x.

Similar negative arguments hold for k even and positive arguments hold for k odd.

3. The function f is left differentiable at c if

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}$$

exists and is finite. Right differentiability is defined similarly.

The function is left continuous at c if

$$\lim_{x \to c^-} f(x) = f(c) \, .$$

Right continuity is defined similarly.

If f is left differentiable at c and the left derivative is α then

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \alpha$$

so

$$\lim_{x \to c^-} (f(x) - f(c)) = \lim_{x \to c^-} (x - c) \cdot \alpha = 0 \,.$$

Hence f is left continuous at c.

The proof for right differentiability implying right continuity is similar.

5. Let

$$f(x) = \begin{cases} x^{4/3} \sin(1/x) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \,. \end{cases}$$

Then it is easy to see that f is continuous at the origin, but its derivative has a discontinuity of the second kind at the origin.

7. Let f be the function from Exercise 5 and define

$$g(x) = x + f(x) \,.$$

This g is a counterexample.

9. Since differentiable functions only have discontinuities of the second kind, the answer is “yes.”

6.2 The Mean Value Theorem and Applications

1. Suppose that f takes a local maximum at some $x > 0$. We may even suppose that this maximum is one-sided on the left. Scaling the coordinates, we may assume that $0 < x < 1$. Then

$$\begin{aligned} |f(x)| &= \left| \int_0^x f'(t)\, dt \right| \\ &\leq \int_0^x |f'(t)|\, dt \\ &\leq \int_0^x |f(t)|\, dt \\ &\leq x \cdot f(x) \,. \end{aligned}$$

If $f(x) \neq 0$, then we may conclude that $1 \leq x$, and that is a contradiction. So $f(x) = 0$ and f is the identically zero function.

3. If f is not monotone, then there are points $a < b < c$ in I such that $f(b)$ does not lie between $f(a)$ and $f(b)$ inclusive. But then either $f(b) < f(a)$ (in which case f has a local minimum between a and b) or $f(b) > f(c)$ (in which case f has a local maximum between a and b). Contradiction.

4. Let $f(x) = \ln x$ and $g(x) = x^{-\alpha}$. Notice that $f(1) \leq g(1)$ and

$$f'(x) = \frac{1}{x} \leq \frac{1}{\alpha} \cdot \left(\alpha x^{\alpha - 1}\right) = \frac{1}{\alpha} g'(x)$$

for $x \geq 1$. Now, by the fundamental theorem of calculus, it follows that $f(x) \leq g(x)$ for $x \geq 1$.

We see that the constant is $1/\alpha$. If the constant were independent of α then we could let $\alpha \to 0$ and conclude that $\ln x$ is bounded, which is false.

5. Write $f''(t) \geq c$ and integrate both sides from 1 to x. We obtain

$$f'(x) - f'(1) \geq c(x - 1)\,.$$

Now replace x by t and integrate both sides again from 2 to x. The result follows.

7. Let $f(x) = \sqrt{x}$. By the Mean Value Theorem,

$$f(x + 1) - f(x) = 1 \cdot f'(\xi)$$

for some $x < \xi < x + 1$. Thus, for x large and positive,

$$\sqrt{x + 1} - \sqrt{x} = \frac{1}{\sqrt{\xi}} \to 0$$

as $x \to +\infty$.

9. Similar to Exercises 7 and 8.

6.3 More on the Theory of Differentiation

1. We prove the inequality with the constant C replaced by $2C$. Then, using the results in Chapter 8, we show that we can actually obtain the constant to be exactly C. We apply the Mean Value Theorem twice, to f and to f'. Write

$$\begin{aligned}\left|\frac{f(x+h)+f(x-h)-2f(x)}{h^2}\right| &= \left|\frac{(f(x+h)-f(x))-(f(x)-f(x-h))}{h^2}\right| \\ &= \left|\frac{h\cdot f'(t)-h\cdot f'(s)}{h^2}\right|\end{aligned}$$

for some t between x and $x+h$ and some s between x and $x-h$. This last equals

$$\begin{aligned}&= \left|\frac{f'(t)-f'(s)}{h}\right| \\ &= \left|f''(y)\frac{t-s}{h}\right| \\ &\leq 2|f''(y)| \\ &\leq 2C.\end{aligned}$$

This gives the result with constant $2C$.

Now we apply the Fundamental Theorem of Calculus (Theorem 7.19 in the text). For all x and h it is easy to see that

$$\begin{aligned}f(x+h)+f(x-h)-2f(x) &= f(x+h)-f(x)-(f(x)-f(x-h)) \\ &= \int_{x-\frac{h}{2}}^{x+\frac{h}{2}} f'(t+\frac{h}{2})-f'(t-\frac{h}{2})dt \\ &= \int_{x-\frac{h}{2}}^{x+\frac{h}{2}}\int_{t-\frac{h}{2}}^{t+\frac{h}{2}} f''(u)dudt.\end{aligned}$$

Then, let $\sup|f''(x)|\leq C$. We have

$$\begin{aligned}|f(x+h)+f(x+h)-2f(x)| &\leq \int_{x-\frac{h}{2}}^{x+\frac{h}{2}}\int_{t-\frac{h}{2}}^{t+\frac{h}{2}} |f''(u)|dudt \\ &\leq C\cdot h^2.\end{aligned}$$

Thus
$$\left|\frac{f(x+h)+f(x-h)-2f(x)}{h^2}\right| \le C.$$

3. By differentiating directly, we can see that $|x|^\ell$ is in $C^{\ell-1,1}$.

5. The usual proof of the Mean Value Theorem shows that the Mean Value Theorem is true for left derivatives. Thus if a function has left derivative zero at every point then it is constant.

7. Let $f(x) = x \cdot \ln|x|$. By the Mean Value Theorem,
$$|f(h) - f(\epsilon)| = |h-\epsilon||f'(\xi)|$$
for some $\epsilon < \xi < h$. Thus
$$|f(h) - f(\epsilon)| \le |h-\epsilon|(1+|\ln\xi|)\,.$$
Now, as in the solution of Exercise 4 in the last section, the righthand side is $\le C_\alpha|h-\epsilon|^\alpha$ for any $0 < \alpha < 1$. Of course the product of the two functions is infinitely differentiable.

A similar argument shows that $g(x) = x/\ln|x|$ is in Lipschitz 1.

9. The Weierstrass nowhere differentiable function has this property.

11. Let $k > \ell$. Define $f(x) = |x|^{\ell+\beta}$, $0 < \beta < 1$. Then f is in $C^{\ell,\beta}$ but not in $C^{k,\alpha}$.

If instead $k = \ell$ and $\alpha > \beta$ then let $g(x) = |x|^{\ell+\beta}$, $0 < \beta < 1$. The g is in $C^{\ell,\beta}$ but not in $C^{k,\alpha}$.

Chapter 7

The Integral

7.1 Partitions and the Concept of Integral

1. First suppose that f is non-negative. Fix $K > 0$. Let $\delta > 0$. If f is unbounded then there exists an $x_0 \in [a, b]$ such that $f(x_0) > 2K/\delta$. Let $[\alpha, \beta]$ be a closed interval of length $\delta/2$ such that $[\alpha, \beta] \subseteq [a, b]$ and $x_0 \in [\alpha, \beta]$. Let $\mathcal{P} = \{p_0, p_1, \ldots, p_k\}$ be a partition of $[a, b]$ of mesh less than δ such that two successive p_j's, say $p_{\ell-1}, p_\ell$, are equal to α and β. Let $s_1, \ldots, s_k$ be selected so that $s_\ell = x_0$. Then

$$\mathcal{R}(f, \mathcal{P}) = \sum_j f(s_j)\Delta_j \geq f(x_0) \cdot (\beta - \alpha) > K.$$

We see that, no matter how fine the partition, the Riemann sums for f are unbounded. Hence f is not Riemann integrable.

The argument when f takes both positive and negative values is similar but technically more complicated. It parallels the proof that if a series has terms not tending to zero then it must diverge.

3. The function g is continuous on the interval $[-1, 1]$. So it is certainly Riemann integrable.

5. Since f is bounded from 0, and is continuous except on a set of measure 0, the function $1/f$ will also be continuous except on a set of measure 0. So it will be Riemann integrable.

7. Let

$$f(x) = \begin{cases} 1 & \text{if} \quad x \text{ is rational} \\ -1 & \text{if} \quad x \text{ is irrational.} \end{cases}$$

Then $f^2(x) \equiv 1$, so f^2 is integrable. But f is discontinuous at every point, so f is not integrable.

9. Let f be bounded on $[a, b]$ and let $\{t_1, t_2, \ldots\}$ be its points of discontinuity. Let $T = \{t, t_1, t_2, \ldots\}$. We will show that, given any $\epsilon > 0$, there exists a partition $\mathcal{P}$ such that

$$\sum_j (\sup_{I_j} f - \inf_{I_j} f)\Delta_j < \epsilon\,.$$

The plan is the following. We construct a partition that splits into a small part about the set T and on the remaining big part we use the continuity of f. For each $t_j \in T, j = 0, 1, 2, \ldots,$ define the interval

$$E_j = \left(t_j - \epsilon 2^{-j-1}, t_j + \epsilon 2^{-j-1}\right).$$

Finitely many E_j's cover T. Let $F_1, \ldots, F_n$ be these sets. Now we write

$$\bigcup_{j=1}^{n} F_j = I_1 \cup \cdots \cup I_N,$$

where the I_j are open intervals. Clearly $N \leq n$. The key fact is that

$$\begin{aligned} \sum_{j=1}^{N} \Delta_j &= \sum_{j=1}^{N} \text{length}(I_j) \\ &\leq \sum_{j=1}^{n} \text{length}(F_j) \\ &\leq \sum_{j=1}^{\infty} \text{length}(E_j) \\ &= \sum_{j=1}^{\infty} \epsilon 2^j \\ &= \epsilon. \end{aligned}$$

Next, the set $A \equiv [a,b] \setminus \bigcup_{j=1}^{\infty} E_j$ is compact. By hypothesis, f is continuous on A. Let $\delta > 0$ be such that so that for $x, y \in A$, $|x-y| < \delta$, we have $|f(x) - f(y)| < \epsilon$. Take any partition of A into intervals of length less than $\delta/2$. Call them $I_1, \ldots, I_M$. Now the endpoints of $I_1, \ldots, I_N$ and $J_1, \ldots, J_M$ form a partition of $[a,b] : \mathcal{P} = \{p_1, \ldots, p_L$. Let $M = \sup\{|f(x)| : x \in [a,b]\}$. Notice that $|\sup f - \inf f| \leq 2M$. Then, writing I_j' for the intervals induced by $\mathcal{P}$ and denoting their lengths by $|I_j'|$, we have

$$\begin{aligned}
\left|\sum_j (\sup_{I_j'} f - \inf_{I_j'} f)|I_j'|\right| &\leq \left|\sum_{I_j} (\sup_{I_j} f - \inf_{I_j} f)|I_j|\right| \\
&\quad + \left|\sum_{J_k} (\sup_{J_k} f - \inf_{J_k} f)|J_k|\right| \\
&\leq 2M\epsilon + \sum_{J_k} |J_k| \\
&\leq (2M + [a,b]) \cdot \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, it follows that f is integrable.

11. Let q_j be an enumeration of the rationals. Define

$$f_j(x) = \begin{cases} 1 & \text{if} \quad x = q_\ell \text{ and } \ell \in \{1, 2, \ldots, j\} \\ 0 & \text{if} \quad \text{otherwise.} \end{cases}$$

Let f be the Dirichlet function. Then $f_j(x) \to f(x)$ as $j \to \infty$. And each f_j is integrable.

7.2 Properties of the Riemann Integral

1. For $t > a$ we have

$$\begin{aligned}\frac{F(t)-F(a)}{t-a} - f(a) &= \frac{\int_a^t f(s)\,ds - \int_a^a f(s)\,ds}{t-a} - f(a) \\ &= \frac{\int_a^t f(s)\,ds}{t-a} - f(a) \\ &= \frac{\int_a^t [f(s)-f(a)]\,ds}{t-a}\,.\end{aligned}$$

If we choose $\epsilon > 0$ and then $\delta > 0$ so small that $|s-s'| < \delta$ implies that $|f(s)-f(s')| < \epsilon$ then, for $|t-a| < \delta$, we see that this last expression is majorized by

$$\frac{\int_a^t \epsilon\,ds}{t-a} = \epsilon\,.$$

This shows that the one-sided derivative exists and equals $f(a)$.

The result at the endpoint b is proved similarly.

3. Assume that f is bounded. Let $\mathcal{P}_\epsilon$ be a partition of the interval $[a+\epsilon, b]$. Then the Riemann sum $\mathcal{R}(f, \mathcal{P}_\epsilon$ can be augmented to a partition of $[a,b]$ by just adding the interval $[a, a+\epsilon]$. That adds a single summand to the Riemann sum, and the sums will still converge to $\int f$. So no new Riemann integrable functions arise.

The function $x^{-1/2}$ can now be integrated on $[0,1]$ by the method in the first paragraph.

For $j = 1, 2, \ldots,$ define

$$g(x) = (-1)^j \cdot 2^{j+1} \cdot \frac{1}{j} \qquad \text{if} \qquad x \in (1/2^{j+1}, 1/2^j]\,.$$

Then it is easy to see that $\int_{1/2^j}^1 g(x)\,dx$ gives rise to an alternating sum $\sum_j (-1)^j/j$. But $\int_{1/2^j}^1 |g(x)|\,dx$ gives rise to the sum $\sum_j 1/j$. The former converges while the latter does not.

5. Notice that, by integration by parts,

$$\widehat{f}(n) = \frac{1}{2\pi n} \int_0^{2\pi} f'(x) \cos nx \, dx \, .$$

Thus we see that $|\widehat{f}(n)| \leq C/n$, and the result follows.

7. This exercise is done just like Exercise 5.

9. Let f be Riemann integrable on $[a, b]$. Let $\mathcal{P}$ be a partition of $[a, b]$. On each interval $[x_{j-1}, x_j]$ in the partition construct a continuous function that lies between the min and the max of f on that interval. This gives a piecewise continous function g that approximates f in the integral norm. Now use piecewise linear splines to approximate g by a continuous function h. Finally, by the Weierstrass approximation theorem, approximate h by a polynomial in the uniform norm.

11. If f is a trigonometric polynomial (i.e., a finite linear combination of exponentials), then the assertion is obvious. For a a more general f, use the approximation proved in Exercise 9.

Chapter 8

Sequences and Series of Functions

8.1 Partial Sums and Pointwise Convergence

1. If f is bounded from 0, then it works. Say that $|f(x)| \geq c > 0$ for all $x \in S$. Let $\epsilon > 0$ be less than $\min\{c/2, c^4/4\}$.

Choose N so large that $j > N$ implies that $|f_j(x) - f(x)| < \epsilon$ for all $x \in S$. It follows that $|f_j(x)| > c/2$ for all $x \in S$. Then, for $x \in S$,

$$\left|\frac{1}{f_j(x)} - \frac{1}{f(x)}\right| = \left|\frac{f(x) - f_j(x)}{f_j(x)f(x)}\right| \leq \frac{\epsilon}{(c/2)(c)} = \frac{2\sqrt{\epsilon}\sqrt{\epsilon}}{c^2} \leq \sqrt{\epsilon}\,.$$

That shows that $1/f_j \to 1/f$ uniformly on S.

3. Of course we know that the Taylor series

$$\sum_{j=0}^{\infty} \frac{x^{2j+1}}{(2j+1)!}$$

converges uniformly to $\sin x$ on any compact interval. It follows then that

$$\sum_{j=0}^{\infty} \frac{x^{4j+2}}{(2j+1)!}$$

converges uniformly to $\sin x^2$ on any compact interval.

5. Let $f(x) = x^2$. For $k = 1, 2, \ldots$ connect the points $(0/2^k, (0/2^k)^2)$, $(1/2^k, (1/2^k)^2) \ldots, ((2^k - 1)/2^k, ((2^k - 1)/2^k)^2, ((2^k/2^k, (2^k/2^k)^2)$ in sequence with line segments. Call the resulting function whose graph is the union of these line segments h_k. Using the Mean Value Theorem together with the uniform continuity of f', we see that h_k approximates f uniformly with a degree of accuracy not exceeding $1/2^k$. So $h_k \to f$ uniformly on $[0, 1]$.

7. We see that

$$\left|\sum_{j=M}^{N} f_j(x)\right| = \left|\sum_{j=M}^{N} \int_0^x f_j'(t)\,dt\right| = \left|\int_0^x \sum_{j=M}^{N} f_j'(t)\,dt\right| \leq \int_0^x \left|\sum_{j=M}^{N} f_j'(t)\right| dt < \int_0^x \epsilon$$

if M and N are large enough. But that shows that the series for f_j is Cauchy, and hence it converges.

9. It is a standard result that if a power series

$$\sum_j a_j(x - b)^j$$

converges pointwise at a point $b^* \neq b$, and if $\delta = |b^* - b|$, then the series converges absolutely and uniformly on compact subintervals of $(b - \delta, b + \delta)$.

10. The series

$$\sum_{j=0}^{\infty} \frac{x^{2j+1}}{(2j+1)!}$$

converges uniformly on any compact interval to $\sin x$.

11. Refer to Exercise 9. If a power series converges at a point other than the center, then it converges on an interval.

8.2 More on Uniform Convergence

1. Clearly the partial sums S_N are continuous functions that converge uniformly as a sequence. So the limit function will be continuous.

3. The functions $f_j(x) = x + 1/j$ converge uniformly to the function $f(x) = x$ uniformly on $\mathbb{R}$. But the functions $f_j^2(x) = x^2 + 2x/j + 1/j^2$ do *not* converge uniformly to x^2 because of the presence of the terms $2x/j$.

We can solve this problem by mandating that the functions be defined on a compact set. Or we can assume that the functions are uniformly bounded.

5. This is false. Let $f_j(x) = -x^j$. Then these functions satisfy the hypotheses, but the limit function is

$$f(x) = \begin{cases} 0 & \text{if} \quad 0 \le x < 1 \\ -1 & \text{if} \qquad x = 1\,. \end{cases}$$

7. This is exactly like the proof that a sequence of real numbers is Cauchy if and only if it converges.

* **9.** Call the continuous function f. We may assume that $[a, b] = [0, 1]$. Let $\epsilon > 0$. Choose $\delta > 0$ such that $|s - t| < \delta$ implies that $|f(s) - f(t)| < \epsilon$. Choose an integer k so large that $1/k < \delta$. Consider the points $(0/k, f(0, k)), (1/k, f(1/k), \ldots, ((k-1)/k, f((k-1)/k)), (k/k, f(k/k))$. Connect these points in sequence with line segments. This gives the graph of a piecewise linear function h.

By the choice of δ, it is clear that $|f(x) - h(x)| < \epsilon$ for any $x \in [0, 1]$. So we have approximated f uniformly by a piecewise linear function h.

8.3 Series of Functions

1. Suppose that the series $\sum_j f_j(x)$ converges pointwise on an open interval I and that $\sum_j f_j'(x)$ converges uniformly to a limit function g on I. Then the sum function f is differentiable and $f' = g$. This is proved by applying Theorem 8.13 to the partial sums of this series.

3. A polynomial p of degree at least 1 cannot be bounded because, for x large, $p(x)$ behaves like the lead term of the polynomial. But $\sin x$ and $\cos x$ are both bounded by 1. So they cannot be polynomials of degree at least 1. They also cannot be constant polynomials. So they are not polynomials at all.

5. A polynomial of degree k vanishes identically after taking $k+1$ derivatives. But no derivative of $\tan x$ ever vanishes identically, and no derivative of $\ln x$ vanishes identically.

9. Let us consider the series $x^2 + \sum_{j=1}^{\infty} 2^{-j} e^{-x^2}$. It clearly converges absolutely and uniformly on compact sets. But there are no convergent M_j to bound the terms of the series.

11. Apply the theorems in the text to the partial sums of the series.

8.4 The Weierstrass Approximation Theorem

1. Let $\epsilon > 0$. Choose J so large that, when $j > J$, then $|f_j(t) - f(t)| < \epsilon/2$ for all real t. Fix $x \in \mathbb{R}$. Choose $\delta > 0$ such that if $|t - x| < \delta$ then $|f(t) - f(x)| < \epsilon/2$. Now if j is so large that $1/j < \delta$ and also $j > J$ then

$$\begin{aligned} |f_j(x+1/j) - f(x)| &\leq |f_j(x+1/j) - f(x+1/j)| + |f(x+1/j) - f(x)| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon\,. \end{aligned}$$

That gives the result.

3. Consider the approximation theorem on the interval $[-1, 1]$. An odd function on this interval cannot be approximated by even polynomials.

5. The functions $f_j(x) = 2^{-j} \sin(2^j x)$ converge uniformly to zero, but the derivatives do not. Now use the Weierstrass theorem to approximate these functions by polynomials.

7. We know that any continuous function is the limit of a sequence of step functions. This is easily converted to a statement about sums.

11. On the unit circle, a polynomial has the form $a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k = a_0 + a_1 e^{i\theta} + a_2 e^{2i\theta} + \cdots + a_k e^{ik\theta}$.

Chapter 9

Elementary Transcendental Functions

9.1 Power Series

1. Let f, g be real analytic functions such that the composition $f \circ g$ makes sense. In order to show that $f \circ g$ is real analytic we need to show that for each $x_0 \in \operatorname{dom} g$, there exist $\delta > 0$ and C and $R > 0$ such that for all $x \in [x_0 - \delta, x_0 + \delta]$,

$$\left| \frac{(f \circ g)^{(k)}(x)}{k!} \right| \leq C \frac{1}{R^k}.$$

(See the remark at the end of Section 10.2 in the text.) This will show (see Exercise 8) that the power series of $f \circ g$ at x_0 converges $f \circ g$.

We have that

$$\frac{d^k}{dx^k}(f \circ g) = \sum \frac{k!}{i!j! \cdots h!} \frac{d^m}{dx^m} f \cdot \left(\frac{g'}{1!} \right)^i \left(\frac{g''}{2!} \right)^j \cdots \left(\frac{g^{(\ell)}}{\ell!} \right)^h,$$

where $m = i + j + \cdots + h$ and the sum is taken over all integer solutions of the equation

$$i + 2j + \cdots + \ell h = k.$$

This formula is the formula for the k^{th} derivative of a composite function. Now using the estimate

$$|f^{(k)}(x)| \leq C \cdot \frac{k!}{R^k},$$

valid for all real analytic functions with suitable constants C and R, we have

$$\begin{aligned}|\frac{d^k}{dx^k}(f\circ g)(x)| &\leq C^{k+1}\cdot\sum\frac{k!}{i!j!\cdots h!}\cdot\frac{m!}{R^m}\cdot\frac{1}{R^i}\frac{1}{R^{2j}}\cdots\frac{1}{R^{\ell h}}\\ &\leq C^{k+1}\cdot\sum\frac{k!}{i!j!\cdots h!}\cdot\frac{m!}{R^{2m}}\\ &= C^{k+1}\frac{k!}{R^{2k}},\end{aligned}$$

which implies that $f\circ g$ is real analytic.

3. By replacing $f(x)$ by $f(x-a)$ we can assume that $a=0$. Then

$$f(x)=\sum_{j=0}^{\infty}a_jx^j$$

has infinitely many zeroes in $(-r,r)$ that accumulate at 0. Hence, $f(0)=0$ and $a_0=0$. Thus,

$$\begin{aligned}f(x) &= \sum_{j=0}^{\infty}a_jx^j=\sum_{j=1}^{\infty}a_jx^j=x\sum_{j=1}^{\infty}a_jx^{j-1}\\ &\equiv x\sum_{j=0}^{\infty}b_jx^j \qquad (b_j=a_{j+1})\\ &\equiv x\cdot g(x).\end{aligned}$$

Since f has infinitely many zeroes, so does g. Applying the above argument to g we find that $0=b_0=a_1$. Proceding in this fashion we obtain that $a_j=0$ for all j, i.e. f identically 0.

5. We write

$$\begin{aligned}\frac{\sin x}{e^x} &= \frac{x-x^3/3!+x^5/5!-+\cdots}{1+x+x^2/2!+\cdots}\\ &= (x-x^3/3!+x^5/5!-+\cdots)\cdot\frac{1}{1-(-x-x^2/2!-x^3/3!-\cdots)}\\ &= (x-x^3/3!+x^5/5!-+\cdots)\cdot(1+(-x-x^2/2!-x^3/3!-\cdots)+(-x-x^2/2!-\\ &= x+x^2+\frac{x^3}{3}-\frac{7x^4}{6}\end{aligned}$$

7. We write

$$\begin{aligned}
\frac{\ln x}{\sin \pi x} &= \frac{(x-1)-(x-1)^2/2!+2(x-1)^3/3!-3(x-1)^4/4!+\cdots}{1-\pi^2(x-1)^2/8+\pi^2(x-1)^4/384+\cdots} \\
&= \frac{(x-1)-(x-1)^2/2!+2(x-1)^3/3!-3(x-1)^4/4!+\cdots}{1-(\pi^2(x-1)^2/8-\pi^2(x-1)^4/384)+\cdots} \\
&= ((x-1)-(x-1)^2/2!+2(x-1)^3/3!-3(x-1)^4/4!+\cdots) \\
&\quad \times(1+(\pi^2(x-1)^2/8-\pi^4(x-1)^4/384) \\
&\quad\quad +(\pi^2(x-1)^2/8-\pi^4(x-1)^4/384)^2) \\
&= (x-1)-\frac{(x-1)^2}{2}+\left(\frac{1}{3}+\frac{\pi^2}{8}\right)(x-1)^3-\left(\frac{1}{8}+\frac{\pi^2}{16}\right)(x-1)^4
\end{aligned}$$

9. Guess a solution of the form $f(x)=\sum_{j=0}^{\infty} a_j x^j$. Substituting this guess into the differential equation gives

$$\left(\sum_{j=0}^{\infty} a_j x^j\right)' + \left(\sum_{j=0}^{\infty} a_j x^j\right) = x\,.$$

We rewrite the lefthand side as

$$\sum_{j=1}^{\infty} j a_j x^{j-1} + \sum_{j=0}^{\infty} a_j x^j = x\,.$$

We can shift the index in the first sum on the left and combine the two sums. The result is

$$\sum_{j=0}^{\infty}[(j+1)a_{j+1}+a_j]x^j = x\,.$$

We see then that

$$\begin{aligned}
1\cdot a_1 + a_0 &= 0 \\
2\cdot a_2 + a_1 &= 1 \\
3\cdot a_3 + a_2 &= 0
\end{aligned}$$

and so forth. We set $a_0 = c$ and then iteratively solve these equations to learn that

$$\begin{aligned} a_0 &= c \\ a_1 &= -c \\ a_2 &= (1+c)/2 \\ a_3 &= -(1+c)/6 \\ a_4 &= (1+c)/24 \end{aligned}$$

and so forth. In the end we find that the solution is

$$\begin{aligned} f(x) &= c - cx + (1+c)\sum_{j=2}^{\infty} \frac{(-x)^j}{j!} \\ &= c - cx + (c+1)e^{-x} - (c+1) - (c+1)(-x) \\ &= x - 1 + (c+1)e^{-x} \,. \end{aligned}$$

This solution f may be checked explicitly. It is plainly real analytic, as the power series expansion is written explicitly.

11. Use the formula for the derivative of an inverse function to estimate the size of the coefficients.

9.2 More on Power Series: Convergence Issues

1. The radius of convergence is R. Just use the Comparision Test.

3. The examples are

(a) $\sum_{j=0}^{\infty} x^j$

(b) $\sum_{j=0}^{\infty} x^j/j$

(c) $\sum_{j=0}^{\infty} (-x)^j$

(d) $\sum_{j=0}^{\infty} x^j/j^2$

5. Use the Root Test to compare the radii of convergence. The final statement of the proposition is obtained just by calculating the derivative.

7. The radius of convergence is still 1. The power series does not converge at either 1 or -1. The function does not extend to either 1 or -1.

In this case we do not need to pass to complex functions (although we could). Now it is clear that the function has a singularity at $x = 1$. So the radius of convergence of the power series about 0 cannot be any greater than 1.

9. As in Exercise 6, the function $f(x) = 1/(1 + x^2)$ gives an example.

11. Calculate the derivatives of f and evaluate them at $x = c$.

9.3 The Exponential and Trigonometric Functions

5. Certainly

$$\frac{\pi}{2} = \mathrm{Sin}^{-1} 1 = \int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx$$

which is approximated by

$$\sum_{j=1}^{k} \frac{1}{\sqrt{1-(j/k)^2}} \cdot \frac{1}{k} \, .$$

For k, large enough we would get the approximation.

7. We sketch the proof of (d) just to give the idea.

Now

$$\sin 2s = \sum_{j=0}^{\infty} \frac{(2s)^{2j+1}}{(2j+1)!} \, ,$$

$$\sin s = \sum_{j=0}^{\infty} \frac{s^{2j+1}}{(2j+1)!} \, ,$$

and

$$\cos s = \sum_{j=0}^{\infty} \frac{s^{2j}}{(2j)!} \, .$$

We can calculate $2 \sin s \cos s$ by hand and compare the coefficients with those of $\sin 2s$ to verify the identity/

11. We know that
$$\cos 2x = \cos^2 x - \sin^2 x\,.$$
Hence
$$\cos 2x = \cos^2 x - (1 - \cos^2 x) = 2\cos^2 x - 1$$
so
$$\cos^2 x = \frac{1 + \cos 2x}{2}\,. \qquad (*)$$
Also
$$\cos 2x = (1 - \sin^2 x) - \sin^2 x$$
so
$$\sin^2 x = \frac{1 - \cos 2x}{2}\,.$$
Thus
$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1 - \cos 2x}{1 + \cos 2x}$$
so that
$$\tan^4 x = \frac{1 - \cos 2x + \cos^2 2x}{1 + \cos 2x + \cos^2 2x}\,.$$
Now we apply formula $(*)$ with the role of x played by $2x$. We obtain
$$\tan^4 x = \frac{1 - \cos 2x + \frac{1+\cos 4x}{2}}{1 + \cos 2x + \frac{1+\cos 4x}{2}} = \frac{3 - 2\cos 2x + \cos 4x}{3 + 2\cos 2x + \cos 4x}\,.$$

9.4 Logarithms and Powers of Real Numbers

1. Part (a) follows from the usual formula for the derivative of an inverse.

Part (b) follows from part (a).

Part (c) is immediate by inspection.

Part (d) is immediate by inspection.

Part (e) is true because the graph of the exponential function is asymptotic to the negative x-axis.

Parts (f) and (g) correspond naturally to properties of the exponential that we have already established.

3. First,

$$\phi(1) = \phi(1 \cdot 1) = \phi(1) + \phi(1)$$

so that

$$\phi(1) = 0\,.$$

Next,

$$\begin{aligned}
\frac{\phi(s+h) - \phi(s)}{h} &= \frac{\phi(s(1+h/s)) - \phi(s)}{h} \\
&= \frac{\phi(s) + \phi(1+h/s) - \phi(s)}{h} \\
&= \frac{1}{s} \cdot \frac{\phi(1+h/s) - \phi(1)}{h/s}\,.
\end{aligned}$$

And the limit of this last expression is $(1/s)\phi'(1)$. That shows that ϕ is differentiable at any point s.

5. We write

$$\frac{j^{j/2}}{j!} \approx \frac{j^{j/2}}{\sqrt{2\pi j}(j/e)^j} = j^{-j/2}e^{-j}/\sqrt{2\pi j}\,.$$

Here we have used Stirling's formula. The limit of the last expression is 0.

7. Let p be a polynomial of degree at least 1, and let p' be its derivatve. Then p' is a polynomial of degree at least 0. We may as well assume that $p'(x)$ is positive for x large. So there is a constant $C > 0$ such that

$$\frac{1}{x} \le C \cdot p'(x)$$

for $x \ge 1$. Integrating, we find that

$$\ln x \le C \cdot |p(x)|\,.$$

9. For b and c rational this is obvious from the defintions. For the general case take suprema.

11. Apply the real analytic implicit function theorem to the equation that defines W.

13. The differential equation

$$y'' + y = 0$$

has as solutions

$$y = \cos x \,,$$

$$y = \sin x \,,$$

and

$$y = e^{ix} \,.$$

Since the solution space of this differential equation is two-dimensional, we conclude that e^{ix} must be a linear combination of $\sin x$ and $\cos x$:

$$e^{ix} = A \cos x + B \sin x \,. \qquad (\star)$$

Since $e^{i0} = 1$, $\cos 0 = 1$, and $\sin 0 = 1$, we find that

$$1 = A \,.$$

Differentiating $(\star)$ and again substituting $x = 0$ gives that $B = i$. So we find that

$$e^{ix} = \cos x + i \sin x \,.$$

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