Chapter 2 Probability

2.1 Introduction

Probability is a number between 0 and 1 which measures an unpredictable future event. We will spend this chapter looking at the concept of probability and its properties.

2.2 Probability and Inference

Understanding probability is very useful in statistical inference analyzes.

Exercise 2.2 (Probability and Inference: age distribution)

Is age distribution of a *random* sample of 463 people living in Uppsala, a city in Sweden, the same or different from the age distribution of *all* of Sweden?

age	Uppsala (out of $n = 463$)	Sweden (p_i)
under 5	47	0.067
5 to 16	75	0.141
16 to 65	296	0.695
over 65	45	0.097

1. *Comparing age distributions*. To make the two age distributions comparable, it makes sense to multiply the probabilities of the various age categories in Sweden by 463, the number of people in Uppsala.

age	Uppsala	(comparable) Sweden, $n \times p_i$
under 5	47	$463 \times 0.067 \approx 31$
5 to 16	75	$463 \times 0.141 \approx 65$
16 to 65	296	$463 \times 0.695 \approx 322$
over 65	45	$463 \times 0.097 \approx 45$

(i) True (ii) False

- 2. Same if "close"; different if not. Since the Uppsala age distribution appears to "close"¹ to the (comparable) Sweden age distribution, we could *infer* the age distribution in Uppsala (circle one) is the same as / is different from the age distribution for all of Sweden.
- 3. Probability was used in at least two places when performing statistical inference in this question. A *random* sample was chosen from Uppsala which requires knowledge about probability. Also, the age categories for all of Sweden were specified by probabilities.
 - (i) True (ii) False

2.3 A Review of Set Notation

A capital letter A, for example, denotes a set of elements (points). The *n* elements in set A are denoted $A = \{a_1, a_2, ..., a_n\}$. The set that contains *no* elements is called *null* or *empty set* and denoted by \emptyset . The set that contains *all* elements under consideration is the *universal* set, S. Set A is a subset of B, $B \subset A$, if every element in A is also in B. Sets can be manipulated in three basic ways:

- *intersection*, described using "and" or mathematical symbol, \cap ,
- union, described using "or" or mathematical symbol, \cup , and
- complementation, using mathematical symbol, \overline{A} .

Exercise 6.3 (A Review of Set Notation)

1. Venn diagrams. Consider the Venn diagrams in Figure 2.1. In diagram (a), the gray shaded area is (circle one) (i) \overline{G} (ii) $E \cup G$ (iii) $E \cap G$ In diagram (b), the gray shaded area is (circle one) (i) \overline{G} (ii) $E \cup G$ (iii) $E \cap G$ In diagram (c), the gray shaded area is (circle one) (i) \overline{G} (ii) $E \cup G$ (iii) $E \cap G$

¹"Close" is usually measured by a chi-square test *statistic*. This is not discussed in detail in this *probability* course, but would be discussed in a *statistics* course. Just accept that the two columns of numbers do, indeed, appear to be "close" to one another.



Figure 2.1: Venn diagrams

2. More Venn diagrams. Consider the Venn diagrams in Figure 2.2.



Figure 2.2: More Venn diagrams

In diagram (a), the gray shaded area is (circle one)	
(i) $(E \cap \overline{G}) \cup (F \cap \overline{G}) \cup (E \cap F)$ (ii) $(E \cap F) \cup (E \cap G)$	(iii) $\overline{(E \cup F)}$
In diagram (b), the gray shaded area is (circle one)	
(i) $(E \cap \overline{G}) \cup (F \cap \overline{G}) \cup (E \cap F)$ (ii) $(E \cap F) \cup (E \cap G)$	(iii) $\overline{(E \cup F)}$
In diagram (c), the gray shaded area is (circle one)	
(i) $(E \cap \overline{G}) \cup (F \cap \overline{G}) \cup (E \cap F)$ (ii) $(E \cap F) \cup (E \cap G)$	(iii) $\overline{(E \cup F)}$

3. Set Operations. Let $E = \{a, b, c, d, e, f\},\$ $F = \{e, f, g, h\},\$ $G = \{i\}$ and the universal set is $S = \{a, b, c, d, e, f, q, h, i, j\}$. (a) Since, for example, $\overline{E} = \{g, h, i, j\},\$ (iii) {*a*, *b*, *c*, *d*, *i*, *j*} $\overline{F} = (\text{circle one}) (i) \{g, h, i, j\}$ (ii) {*a*, *b*, *c*, *d*} (b) Since, for example, $E \cup F = \{a, b, c, d, e, f, g, h\}$ and $F \cup E = (\text{circle one})$ (ii) {*a*, *b*, *c*, *d*} (iii) {*a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*} (i) {**g**, **h**, **i**, **j**} then $E \cup F = F \cup E$ (commutative law) (c) Since, for example, $E \cap F = \{e, f\}$ and $F \cap E = (\text{circle one})$ (ii) {*a*, *b*, *c*, *d*} (i) {*e*, *f*} (iii) $\{a, b, c, d, e, f, g, h\}$ $E \cap F = F \cap E$ (commutative law) (d) The intersection of the set E and the complement of this set, \overline{E} , is the empty set, $E \cap \overline{E} = \emptyset$. (Can anything be both in a set and not in a set at the same time?) (i) **True** (ii) **False** The sets E and \overline{E} are said to be *mutually exclusive*². (e) Since $E \cap F = (\text{circle one})$ (ii) {*a*, *b*, *c*, *d*} (iii) {*a*, *b*, *c*, *d*, *i*, *j*} (i) {*e*, *f*} then $(E \cap F) \cap G = (\text{circle one})$ (iii) **Ø** (i) {*e*, *f*} (ii) {*a*, *b*, *c*, *d*} On the other hand, since $F \cap G = (\text{circle one})$ (i) {*e*, *f*} (ii) {*a*, *b*, *c*, *d*} (iii) Ø then $E \cap (F \cap G) = (\text{circle one})$ (ii) {*a*, *b*, *c*, *d*} (iii) Ø (i) {*e*, *f*} So $(E \cap F) \cap G = E \cap (F \cap G)$ (associative law) (f) $E \cup (F \cup G) = (E \cup F) \cup G$ (associative law) (i) **True** (ii) **False** (g) Since $E \cup F = (\text{circle one})$ (iii) {*a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*} (i) {*e*, *f*} (ii) {*a*, *b*, *c*, *d*} then $(E \cup F) \cap G = (\text{circle one})$ (iii) Ø (i) {*e*, *f*} (ii) {*a*, *b*, *c*, *d*} on the other hand, since

 $E \cap G = (\text{circle one})$

²Any two sets A and B where $A \cap \overline{B} = \emptyset$ are said to be mutually exclusive (or disjoint).

(i)
$$\{e, f\}$$
 (ii) $\{a, b, c, d\}$ (iii) \emptyset
and $F \cap G = (circle one)$
(i) $\{e, f\}$ (ii) $\{a, b, c, d\}$ (iii) \emptyset
then $(E \cap G) \cup (F \cap G) = (circle one)$
(i) $\{e, f\}$ (ii) $\{a, b, c, d\}$ (iii) \emptyset
and so $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$ (distributive law)
(h) Another Example of the Distribution Law
 $(E \cup G) \cap (F \cup G) = (E \cap F) \cup (E \cap G) \cup (G \cap F) \cup (G \cap G)$
(i) **True** (ii) **False**
(i) Since $E \cup F = (circle one)$
(i) $\{e, f\}$ (ii) $\{a, b, c, d\}$ (iii) $\{a, b, c, d, e, f, g, h\}$ then $(E \cup F) = (circle one)$
(i) $\{e, f\}$ (ii) $\{a, b, c, d\}$ (iii) $\{i, j\}$
on the other hand, since
 $\overline{E} = (circle one)$
(i) $\{g, h, i, j\}$ (ii) $\{a, b, c, d\}$ (iii) $\{i, j\}$
on the other hand, since
 $\overline{E} = (circle one)$
(i) $\{e, f\}$ (ii) $\{a, b, c, d\}$ (iii) $\{a, b, c, d, i, j\}$ then $\overline{E} \cap \overline{F} = (circle one)$
(i) $\{e, f\}$ (ii) $\{a, b, c, d\}$ (iii) $\{a, b, c, d, i, j\}$ then $\overline{E} \cap \overline{F} = (circle one)$
(i) $\{e, f\}$ (ii) $\{a, b, c, d\}$ (iii) $\{i, j\}$
and so $(E \cup F) = \overline{E} \cap \overline{F}$ (simple case of DeMorgan's Laws).
(j) $\overline{(E \cap F)} = \overline{E} \cup \overline{F}$ (another simple case of DeMorgan's Laws)
(i) **True** (ii) **False**

4. Two Dice. Two dice are tossed. Let A be the event that the first die is odd and let B be the event the sum of the die is seven. Consider the Venn diagram corresponding to this experiment in Figure 2.3.



Figure 2.3: Venn diagram for tossing two dice

- (a) $A \cap B = (\text{circle one})$ (i) $\{(1,6), (3,4), (5,2)\}$ (ii) $\{(1,6), (3,4), (5,2), (6,1), (4,3), (2,5)\}$ (iii) $\{(1,6), (3,4), (5,2), (6,1), (4,3), (2,5)\}$
- (b) number of elements in $A \cap B$ is (circle one) (i) **3** (ii) **4** (iii) **5**
- (c) number of elements in A is (circle one)
 (i) 18 (ii) 19 (iii) 20
- (d) number of elements in B is (circle one) (i) **5** (ii) **6** (iii) **7**
- (e) number of elements in A ∪ B is (circle one)
 (i) 19 (ii) 20 (iii) 21
 The number of elements in A ∪ B is calculated here by drawing a Venn diagram and counting the number of elements in A ∪ B³.
- (f) number of outcomes in A ∪ B is (circle one)
 (i) number in A number in B number in A ∩ B
 (ii) number in A number in B + number in A ∩ B
 - (ii) number in A number in D + number in A + D
 - (iii) number in A + number in B number in $A \cap B$

The number of elements in $A \cup B$ is calculated here by decomposing $A \cup B$ into smaller subsets A, B and $A \cap B$, counting elements in the smaller subsets and then using a formula (the "addition" formula) on the number of elements in these subsets to determine $A \cup B^4$.

2.4 A Probabilistic Model for an Experiment: The Discrete Case

Probability is a number assigned to any event in a sample space, S. The assignment of probability P(A) to event A; in other words, the function P that maps event A to probability P(A), obeys the following three axioms.

Axiom 1 $0 \le P(A) \le 1$

Axiom 2 P(S) = 1

³We shall shortly see the method used to calculate the number of elements in this example is closely related to the *sample-point* method used in calculating the *probability* of an *event*.

⁴We shall shortly see the method used to calculate the number of elements in this example is closely related to the *event-decomposition* method used in calculating the *probability* of an *event*.

Axiom 3 For any sequence⁵ of mutually exclusive⁶ events A_1, A_2, A_3, \ldots

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots) = P(A_1) + P(A_2) + P(A_3) + \cdots$$

Exercise 2.4(A Probabilistic Model for an Experiment: The Discrete Case)

1. *Terminology: Three Coins.* Three coins are flipped. Consider the Venn diagram corresponding to this situation in Figure 2.4.



Figure 2.4: Two Venn diagrams for flipping three coins

- (a) A (chance) *experiment* is a process which results in a set of observations where each observation has a chance of occurrence. Flipping a fair coin three times is an example of a chance experiment.
 (i) **True** (ii) **False**
- (b) The sample space, S, is a list of all the possible outcomes (or sample points) of a chance experiment. The sample space for flipping a fair coin three times is (choose one)
 - (i) **{HHH, HHT, HTH, HTT}**
 - (ii) {THH, THT, TTH, TTT}
 - (iii) {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

Sample space really just means "set".

- (c) A discrete sample space has a finite or countable number of sample points. The sample space for flipping a fair coin three times is an example of a discrete sample space.
 - (i) **True** (ii) **False**

 $P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + P(A_3) + \dots + P(A_n)$

⁶Recall, mutually exclusive means: $A_i \cap A_j = \emptyset$, $i \neq j$.

⁵If the sequence is *finite*, $A_1, A_2, A_3, \ldots, A_n$, then

- (d) An event is a subset of a sample space. The event "flipping exactly three heads" is A = {E₁} = {HHH}. The event "flipping exactly one head" is C = (choose one or more)
 (i) {HTT, THT, TTH}
 - (ii) $\{E_4, E_6, E_7\}$
 - (iii) $E_4 \cup E_6 \cup E_7$
 - (iv) $E_4 \cap E_6 \cap E_7$
- (e) A *simple event* consists of one sample point. Choose the one correct statement.
 - (i) Events A, B and C are all examples of simple events.
 - (ii) Only events A and B are examples of simple events.
 - (iii) Only event A is an example of a simple event.

Simple events are denoted by the letter E with a subscript.

- 2. Definition of Probability: Three Coins. Consider the Venn diagram in Figure 2.4. Assume the coin is fair; that is, the chance of any one of eight mutually exclusive simple events occurring is $\frac{1}{8}$: $P(E_1) = \frac{1}{8}$, $P(E_2) = \frac{1}{8}$, ..., $P(E_8) = \frac{1}{8}$.
 - (a) Axiom 1 implies any event has a probability between (and including) zero (0) and one (1); for example, 0 ≤ P(A) = P(E₁) = ¹/₈ ≤ 1.
 (i) **True** (ii) **False**
 - (b) Since all simple events are mutually exclusive, axiom 3 implies $P(B) = P(E_2 \cup E_3 \cup E_5 \cup E_8) = P(E_2) + P(E_3) + P(E_5) + P(E_8) = (i) \frac{1}{8} \quad (ii) \frac{3}{8} \quad (iii) \frac{4}{8} \quad (iv) \frac{8}{8}.$
 - (c) Axiom 3 implies the chance of flipping exactly one head, $P(C) = P(E_4 \cup E_6 \cup E_7) = P(E_4) + P(E_6) + P(E_7) =$ (i) $\frac{1}{8}$ (ii) $\frac{3}{8}$ (iii) $\frac{4}{8}$ (iv) $\frac{8}{8}$.
 - (d) Axiom 3 implies, since events A, B and C are mutually exclusive, $P(A \cup B \cup C) = P(A) + P(B) + P(C) = \frac{1}{8} + \frac{4}{8} + \frac{3}{8}$. (i) **True** (ii) **False**
 - (e) Axiom 2 implies $P(S) = P(A \cup B \cup C) = 1$. (i) **True** (ii) **False**

2.5 Calculating the Probability of an Event: The Sample–Point Method

Two possible ways of finding probability on a sample space include the sample-point and event-decomposition methods⁷. The sample-point method is discussed in this

⁷The two methods are not mutually exclusive. They are often both used together to determine the probability of an event in a sample space.

section. This method is closely associated with the three axioms of probability. It involves listing all of the simple events in an experiment, assigning probabilities to these simple events and then determining the probability of an event which consists of a subset of simple events.

Exercise 2.5 (Calculating the Probability of an Event: The Sample–Point Method)

1. Choosing two candidates for one job. Seven candidates, three of which are females, are being interviewed for a job. The three females are Kathy, Susan and Jamie; the four males are Tom, Tim, Tyler and Toothy. Two candidates are chosen for a job interview. Assume each candidate is chosen at random, that candidates are equally qualified. All possible 21 choices (simple events) are listed in Figure 2.5. Notice, for example, simple event E_1 , "Kathy and Susan", but not "Susan and Kathy", has been listed. Also notice "Susan and Susan" has not been listed⁸.



Figure 2.5: Venn diagram for picking two candidates for one job

- (a) Since all choices are equally probable, $P(E_1) = P(E_2) = \cdots = P(E_{21}) = (\text{circle one})$ (i) $\frac{1}{7}$ (ii) $\frac{1}{21}$ (iii) $\frac{1}{49}$.
- (b) The probability both candidates chosen are female is $P(FF) = P(E_1 \cup E_2 \cup E_7) = P(E_1) + P(E_2) + P(E_7) = \text{(circle one)}$ (i) $\frac{3}{21}$ (ii) $\frac{19}{21}$ (iii) $\frac{21}{21}$.
- (c) The probability both candidates chosen are male is P(MM) = (circle one) (i) $\frac{3}{21}$ (ii) $\frac{6}{21}$ (iii) $\frac{19}{21}$.
- (d) The probability exactly one of the two candidates is a female is P(FM) = (circle one) (i) $\frac{4}{21}$ (ii) $\frac{12}{21}$ (iii) $\frac{18}{21}$ (iv) $\frac{20}{21}$.

⁸The *order* of the couples does *not* matter to us and each member of the couple has been sampled with *out* replacement from the seven candidates: *combinations* of couples have been listed.

2. Choosing two candidates for two different jobs. Assume, now, two candidates are chosen for two different jobs, where one job has a \$20,000 salary and the other job has a \$40,000 salary. All possible 42 choices (simple events) are listed in Figure 2.6. Notice, for example, both simple event E_1 , "Kathy (\$40K) and Susan (\$20K)", and also simple event E_7 "Susan (\$40K) and Kathy (\$20K)", have both been listed. Once again, also notice "Susan and Susan" has not been listed⁹.



Figure 2.6: Venn diagram for picking two candidates for two jobs

- (a) Since all choices are equally probable, $P(E_1) = P(E_2) = \cdots = P(E_{42}) = (\text{circle one})$ (i) $\frac{1}{21}$ (ii) $\frac{2}{42}$ (iii) $\frac{1}{42}$.
- (b) The probability both candidates chosen are female¹⁰ is $P(FF) = P(E_1 \cup E_2 \cup E_7 \cup E_8 \cup E_{13} \cup E_{14}) = (\text{circle one})$ (i) $\frac{3}{42}$ (ii) $\frac{6}{42}$ (iii) $\frac{21}{42}$.
- (c) The probability both candidates chosen are male is P(MM) = (circle one) (i) $\frac{12}{21}$ (ii) $\frac{12}{42}$ (iii) $\frac{21}{42}$.
- (d) The probability exactly one of the two candidates is a female who receives a \$40,000 salary is (circle one) (i) $\frac{12}{21}$ (ii) $\frac{12}{42}$ (iii) $\frac{24}{42}$ (iv) $\frac{30}{42}$.

2.6 Tools for Counting Sample Points

Counting can be an important part of determining various probabilities. To help in understanding the various counting rules, we find out about an important method

 $^{^{9}}$ The *order* of the couples *does* matter to us and, as before, each member of the couple has been sampled with *out* replacement from the seven candidates: *permutations* of couples have been listed.

¹⁰Since this question does not specify order, it is not surprising that the probability calculated here in the permutation (order matters) case is the same as the answer determined previously in the combination (order does not matter) case.

of visualizing counting techniques, "marbles in boxes", and discuss the notions of whether order matters or not, sampling with or without replacement and whether objects are distinguishable or not. We look at a number of counting rules, including, most importantly, the *mn* rule of counting, permutations and combinations. Finally, all of the counting rules are used to calculate various probabilities.

Exercise 2.6 (Tools for Counting Sample Points)

1. Marbles in boxes analogy: 3-digit street numbers. It is often useful to think of the analogy of placing marbles in a number of side-by-side boxes when counting sample points. For example, as indicated in Figure 2.7, there are a total of $10 \times 10 \times 10 = 1000$ three-digit street numbers.



Figure 2.7: Marbles In Boxes: 3–Digit Street Numbers

- (a) Number of 3-digit numbers if first number cannot be zero is (circle one or more) (i) $9 \times 10 \times 10$ (ii) 900 (iii) 999.
- (b) Number of 3-digit numbers if first number must be the number 3, is (circle one or more) (i) $1 \times 10 \times 10$ (ii) 100 (iii) 900.
- (c) Number of 3-digit numbers if first number must be 3 and second number cannot 9, is (circle one)
 (i) 9 × 9 × 10
 (ii) 9 × 10 × 10
 (iii) 1 × 9 × 10.
- (d) Number of 3-digit numbers with exactly two 3s, is (circle one)
 - (i) $1 \times 1 \times 9 = 9$
 - (ii) $1 \times 9 \times 1 = 9$
 - (iii) $9 \times 1 \times 1 = 9$
 - (iv) 9 + 9 + 9 = 27.

(Hint: The correct answer is the fourth one, 27, and is obtained by adding

the first three possible answers together. Each of the first three answers represent a different way to have exactly two 3s in three digits.)

- 2. Things to Look For When Counting: Order Matters or Not, Sampling With or Without Replacement and Distinguishable or Not. The marbles-in-boxes analogy for counting could be thought of as a function with a domain (distinguishable or not), rule (sampling with or without replacement) and a range (order matters or not).
 - (a) Order Matters or Not. Let's say we wanted to count the total number of ways of picking a committee of 2 people from 3 people, Jim, Sue and Ali. The three possible committees include (Jim, Sue), (Jim, Ali) and (Sue, Ali). Since we have the committee (Jim, Sue), we would not include the committee (Sue, Jim), because they are the same committee. In other words, we do not include both (Jim, Sue) and (Sue, Jim) in our count of all possible committees because the order in which people are chosen for the committee does not matter.
 - i. When counting all the possible ways of choosing numbers for a 3-digit street number, do we assume the 3-digit numbers (9,3,4) and (3,4,9) count as two different 3-digit numbers (order matters) or, rather, as the same 3-digit number (order does not matter)? We assume
 (i) order matters (ii) order does not matter.
 - ii. When counting all the possible ways of parking three of five cars in three side-by-side parking spots, do we assume the (Jim's car, Sue's car, Ali's car) and (Sue's car, Jim's car, Ali's car) parking arrangements count as two different parking arrangements (order matters) or, rather, as the same parking arrangement (order does not matter)? We assume (circle one)

(i) order matters (ii) order does not matter.

- iii. When counting all the possible ways of dealing out five cards for a poker hand from a deck of 52 playing cards, do we assume the hands (10c,Jd,Qh,2h,3s) and (2h,3s,Jd,Qh,10c) (where, for example, "10c" represents the 10 of clubs and "Qh" represents the queen of hearts) count as two different hands each of which would be played in a different way (order matters) or, rather, as the same hand which is played in the same way (order does not matter)? We assume (circle one)
 (i) order matters (ii) order does not matter.
- iv. When counting the number of ways ten marbles, numbered 0, 1, ..., 9, can be placed in three side-by-side boxes, do we assume the marble arrangements (1,2,3) and (3,2,1) count as two different arrangements (order matters) or not (order does not matter). We assume

 (i) order matters
 (ii) order does not matter.

- v. When counting the number of ways four letters, L, I, F, E, can be formed into three–letter words, do we assume the letter arrangements (L,I,F) and (F,I,L) count as two different arrangements (order matters) or not (order does not matter). We assume (circle one)
 (i) order matters (ii) order does not matter.
- vi. If we assume order matters, we count (circle one) (i) **more** (ii) **less**.
- (b) Sampling With or Without Replacement. The three possible committees of 2 people picked from 3 people, Jim, Sue and Ali, are: (Jim, Sue), (Jim, Ali) and (Sue, Ali). It is not possible to have a committee of (Jim, Jim) since there is only one Jim. We cannot "sample" Jim from the available three people and then "replace" him back into the group to be possibly sampled again. Jim is sampled without replacement from the group of three people.
 - i. When counting all the possible ways of choosing numbers for a 3-digit street number, are we able to repeatedly sample and replace "1" from the ten possible digits, 0, 1, ..., 9, to form the 3-digit number (1,1,1) (sample with replacement) or not (sample without replacement)? We (i) sample with replacement (ii) sample without replacement.
 - ii. When counting all the possible ways of parking three of five cars in three side–by–side parking spots, are we able to repeatedly sample and replace "Sue's car" from the five possible cars to form the parking arrangement (Sue's car, Sue's car, Sue's car) (sample with replacement) or not (sample without replacement)? We (choose one)

(i) sample with replacement (ii) sample without replacement.

iii. When counting all the possible ways of dealing out five cards for a poker hand from a deck of 52 playing cards, are we able to repeatedly sample and replace the 10 of clubs from the deck to form the hand (10c,10c,10c,10c,10c) (sample with replacement) or not (sample with-out replacement)? We (choose one)

(i) sample with replacement (ii) sample without replacement.

- iv. When counting the number of three–letter words that can be formed from four letters, L, I, F, E, we might sample with replacement or not. *If* we sample with*out* replacement, then the word (F,F,F) (circle one) (i) is possible (ii) is not possible.
- v. When counting the number of ways ten marbles, numbered 0, 1, ..., 9, can be placed in three side-by-side boxes, we might sample with replacement or not. *If* we sample *with* replacement, then the marble arrangement (1,1,1) (circle one)
 - (i) is possible (ii) is not possible.
- vi. If we sample with replacement, we count (circle one)

(i) more (ii) less.

- (c) Distinguishable or Not. Possible two-letter words that can be formed from the three letters U, M, M, are (U, M) and (M, M). Notice that the letters "U" and "M" are distinguishable from one another, whereas the two "M"s are not. Indistinguishability refers to the "marbles" before they are placed in the side-by-side boxes.
 - i. When counting all the possible ways of choosing numbers for a 3-digit street number, are all of the ten possible digits, 0, 1, ..., 9, we can choose from distinguishable from one another or are some of them indistinguishable from one another? We assume (circle one)
 - (i) all marbles are distinguishable from one another.
 - (ii) some marbles are indistinguishable from one another.
 - ii. When counting all the possible ways of parking three of five cars (Sue's car, Bob's car, Mike's car, Judy's car, Darlene's car) in three different parking spots, we assume (circle one)
 - (i) all cars are distinguishable from one another.
 - (ii) some cars are indistinguishable from one another.
 - iii. When counting all the possible ways of parking three of five cars (two GMs, three Chryslers) in three different parking spots, we assume(i) all cars are distinguishable from one another.
 - (ii) some cars are indistinguishable from one another.
 - iv. When counting all the possible ways of dealing out five cards for a poker hand from a deck of 52 playing cards, we assume (circle one)
 - (i) all cards are distinguishable from one another.
 - (ii) some cards are indistinguishable from one another.
 - v. When counting all the possible ways of dealing out five cards when we are interested in the suits only (for example, the hand (10c,Jd,Qh,2h,3s) would be considered as one club, one diamond, two hearts and a spade) from a deck of 52 playing cards, we assume
 - (i) all cards are distinguishable from one another.
 - (ii) some cards are indistinguishable from one another.
 - vi. When counting the number of three–letter words that can be formed from four letters, L, I, F, E, we must assume (circle one)
 - (i) all letters are distinguishable from one another.
 - (ii) some letters are indistinguishable from one another.
 - vii. When counting the number of ways ten marbles (four blue, two red and four green), can be placed in three side–by–side boxes, we must assume (circle one)
 - (i) all marbles are distinguishable from one another.
 - (ii) some marbles are indistinguishable from one another.

- viii. If some items are indistinguishable from one another, then the order of these items does not matter in the sense it is not possible to count different arrangements of these indistinguishable items. In other words, if we some items are indistinguishable with one another, we count
 (i) more (ii) less.
- ix. If all items are distinguishable from one another, then their order may matter (think parking three cars in parking spots) or order may not matter (think number of five card poker hands).
 - (i) **True** (ii) **False**
- x. With distinguishable "marbles", it is still possible to create arrangements in the side-by-side boxes in which groups of *in*distinguishable objects occur because a sampling *with* replacement procedure has been undertaken. For example, sample three digits from the ten distinguishable digits, 0, 1, ..., 9, with replacement to arrive at number 300, for example, with two (indistinguishable) zeros.
 - (i) **True** (ii) **False**
- 3. The mn rule for counting sample points. If (chance) experiment 1 results in m possible outcomes, a_1, a_2, \ldots, a_m , and for each outcome from experiment 1 there results n possible outcomes from (chance) experiment 2, b_1, b_2, \ldots, b_n , then, together, there are mn possible paired outcomes, $(a_1, b_1), (a_1, b_2), \ldots, (a_m, b_n)$, from the two experiments.
 - (a) If there are three eligible treasurer candidates (T₁, T₂ and T₃) and two eligible secretary candidates (S₁ and S₂), there are 3×2 = 6 possible (treasurer, secretary) pairings: {(T₁, S₁), (T₁, S₂), (T₂, S₁), (T₂, S₂), (T₃, S₁), (T₃, S₂)}.
 (i) **True** (ii) **False**
 - (b) The number of ways of choosing a treasurer and secretary pair, when there are *four* eligible treasurer candidates and *eight* eligible secretary candidates, is (circle one)
 - (i) 4 + 8 = 12 (ii) $4 \times 8 = 32$.
 - (c) The number of ways of rolling a pair of dice, when there are six possible values for die 1 and six possible values for die 2, is (circle one)
 (i) 6 + 6 = 12 (ii) 6 × 6 = 36.
 - (d) The number of ways of choosing a treasurer, secretary and president, when there are *three* eligible treasurer candidates, *five* eligible secretary candidates and *two* eligible president candidates, is (circle one)
 (i) 3 + 5 + 2 = 10 (ii) 3 × 5 × 2 = 30.
 - (e) The number of ways of flipping a coin four times, when there are two possible values for each flip of the coin (heads or tails), is (circle one)
 (i) 2 + 2 + 2 + 2 = 8 (ii) 2 × 2 × 2 × 2 = 2⁴ = 16.

(f) The $n_1 \cdot n_2 \cdots n_r$ rule for counting sample points. If (chance) experiment 1 results in n_1 possible outcomes and for each outcome from experiment 1 there results n_2 possible outcomes from experiment 2, and for each outcome from the first two experiments there results n_3 possible outcomes from experiment 3, ..., then, together, there are $n_1 \cdot n_2 \cdots n_r$ possible r-tuple outcomes from the r experiments.

(i) **True** (ii) **False**

4. The permutation rule for counting sample points. The number of ways of ordering r objects (where the r objects have been taken from $n, n \ge r$, distinct objects) is

$$P_r^n = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

where $n! = n(n-1)\cdots(2)(1)$ and 0! = 1! = 1. A permutation is an arrangement of objects where the order of these objects *does* matter and the objects are sampled with *out* replacement.

(a) Parking r = 3 of n = 5 Cars. Five different cars could occupy the first parking spot, only four could occupy the second parking spot (since one car is in the first parking spot) and three could occupy the final parking spot in Figure 2.8. Consequently, there are

$$P_3^5 = 5 \times 4 \times 3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5 \times 4 \times 3}{2 \times 1} = 60$$

ways of doing this.

(i) **True** (ii) **False**

(TI-84+: type five (5), then MATH PRB nPr ENTER 3 ENTER.)



Figure 2.8: Marbles in boxes: 3 parking spots

(b) Parking r = 4 of n = 7 Cars. In this case, $P_4^7 =$ (choose one or more) i. $\frac{n!}{(n-r)!}$, where n = 7 and r = 4

(only groups of different cars, not arrangements in groups, counted)

- 5. Permutations, $n_1 \cdot n_2 \cdots n_r$ rule for counting and indistinguishable objects.
 - (a) Permutations and the mn rule for counting. Consider Figure 2.9. Nine different cars are to be driven into nine side—by—side parking spots. Three of the cars are driven by statistics students and there are three parking spots reserved for statistics students only. Six of the cars are driven by mathematics students and there are 6 parking spots reserved for these students only. Using the mn rule for counting, the number of ways the cars can be parked is (choose one or more)

(i)
$$P_3^3 \times P_6^6$$
 (ii) $3!6! = (6)(720) = 4320$ (iii) $9!$

(b) *Permutations*. Nine different cars are to be driven into nine side–by–side parking spots. Three of the cars are driven by statistics students and six of the cars are driven by mathematics students. The number of ways the cars can be parked is (choose one)

(i) 3! + 6! = 6 + 720 = 726 (ii) 3!6! = (6)(720) = 4320 (iii) 9!.



Figure 2.9: Marbles in boxes: statistics and mathematics parking spots

- (c) Permutations. How many different ways can a line of 5 females and 4 males waiting to buy books at the bookstore be arranged? Choose one.
 (i) 5!4! = (5)(4) = 20 (ii) 5!4! = (120)(24) = 2880 (iii) 9!.
- (d) Permutations and the mn rule for counting. How many different ways can a line of 5 females and 4 males waiting to buy books at the bookstore be arranged where the females are grouped together at the head of the line and the males are grouped together at the back of the line?
 (i) P₄⁵ = 5 (ii) 5!4! = (120)(24) = 2880 (iii) 9!.
- (e) Permutations and the mn rule for counting. How many different ways can a line of 5 females and 4 males waiting to buy books at the bookstore be arranged where the females are grouped together and the males are grouped together? Choose one.
 (i) 5!4! = (120)(24) = 2880 (ii) 2! × 5!4! = 2(120)(24) = 5760.
- (f) Permutations and the $n_1 \cdot n_2 \cdots n_r$ rule for counting. How many different ways can a line of 5 statistics students, 4 business students and 3 technology students, waiting to buy books at the bookstore, be arranged where the statistics students are grouped together at the head of the line, the business students are grouped together in the middle of the line and the technology students are grouped together at the end of the line? Choose one or more. (i) 5!4!3! (ii) 17280 (iii) $P_5^5 P_4^4 P_3^3$.
- (g) Permutations and the n₁ · n₂ · · · n_r rule for counting. How many different ways can a line of 5 statistics students, 4 business students and 3 technology students, waiting to buy books at the bookstore, be arranged where the students in each subject are grouped together in the line? Since each of the three groups can be arranged in 3! = 6 ways, the number of ways is

 (i) 5!4!3!
 (ii) 3! × 5!4!3!
 (iii) 12!.
- (h) *Permutations and indistinguishable objects.* How many different letter permutations can be formed from C, O, U, N, T, E, R, S? Since all eight letters

are distinguishable from one another, the number of letter permutations is (i) 8! (ii) $\frac{8!}{3!2!}$ (iii) $\frac{8!}{5!}$.

- (i) Permutations and indistinguishable objects. How many different letter permutations can be formed from A, A, R, D, V, A, R, K? Since there are eight letters in total and there are three indistinguishable A's and two indistinguishable R's, the number of letter permutations is (circle one)
 (i) 8! (ii) 8!/(312) (iii) 8!/(5!).
- 6. Partitioning n distinct objects into k distinct groups. If $n = n_1 + n_2 + \cdots + n_k$, the number of possible divisions of n distinct objects into k distinct nonoverlapping groups of sizes n_1, n_2, \ldots, n_k (where order does not matter in each nonoverlapping group) is given by

$$\begin{pmatrix} n\\ n_1 n_2 \cdots n_k \end{pmatrix} = \begin{pmatrix} n\\ n_1 \end{pmatrix} \begin{pmatrix} n-n_1\\ n_2 \end{pmatrix} \cdots \begin{pmatrix} n-n_1-\cdots-n_{k-1}\\ n_k \end{pmatrix} = \frac{n!}{n_1!n_2!\cdots n_k!}$$

(a) There are 15 different marbles. If 4 marbles are placed in jar 1, 6 are placed in jar 2 and 5 are placed in jar 3, how many different ways can the 15 marbles be divided up into the three jars? Circle one or more.

(i)
$$\begin{pmatrix} 15 \\ 465 \end{pmatrix}$$
 (ii) $\frac{15!}{4!6!5!}$ (iii) $630, 630$

(b) There are 9 faculty members in the Mathematical Sciences department who can teach either mathematics, statistics or physics. If, as shown in Figure 2.10, 4 of the faculty teach mathematics, 3 teach statistics and 2 teach physics, how many different ways can the nine faculty members be divided up to teach the three subjects? Circle one or more.

(i)
$$\begin{pmatrix} 9\\432 \end{pmatrix}$$
 (ii) $\begin{pmatrix} 9\\4 \end{pmatrix} \begin{pmatrix} 5\\3 \end{pmatrix} \begin{pmatrix} 2\\2 \end{pmatrix}$ (iii) $\frac{9!}{4!3!2!}$ (iv) 1260.



Figure 2.10: Marbles in boxes: faculty teaching three subjects

(c) Multinomial theorem

$$(x_1 + x_2 + \dots + x_k)^n = \sum_{(n_1,\dots,n_k): n_1 + \dots + n_k = n} \binom{n}{n_1 n_2 \cdots n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$$

where $\binom{n}{n_1 n_2 \cdots n_k}$ are called *multinomial coefficients*. For example, expand $(x_1 + x_2 + x_3)^2$. Circle one or more.

$$\begin{split} \text{i. } & \sum_{(n_1,\dots,n_r):\ n_1+n_2+n_3=2} \begin{pmatrix} 2\\n_1 \ n_2 \ n_3 \end{pmatrix} x_1^{n_1} x_2^{n_2} x_3^{n_3} \\ \text{ii. } & \begin{pmatrix} 2\\2 \ 0 \ 0 \end{pmatrix} x_1^2 x_2^0 x_3^0 \ + \ \begin{pmatrix} 2\\0 \ 2 \ 0 \end{pmatrix} x_1^0 x_2^2 x_3^0 \ + \ \begin{pmatrix} 2\\0 \ 0 \ 2 \end{pmatrix} x_1^0 x_2^0 x_3^2 \ + \\ & \begin{pmatrix} 2\\1 \ 1 \ 0 \end{pmatrix} x_1^1 x_2^1 x_3^0 \ + \ \begin{pmatrix} 2\\1 \ 0 \ 1 \end{pmatrix} x_1^1 x_2^0 x_3^1 \ + \ \begin{pmatrix} 2\\0 \ 1 \ 1 \end{pmatrix} x_1^0 x_2^1 x_3^1 \\ \text{iii. } & \frac{2!}{2!0!0!} x_1^2 x_2^0 x_3^0 \ + \frac{2!}{0!2!0!} x_1^0 x_2^2 x_3^0 \ + \frac{2!}{0!0!2!} x_1^0 x_2^0 x_3^2 \\ & + \frac{2!}{1!1!0!} x_1^1 x_2^1 x_3^0 \ + \frac{2!}{1!0!1!} x_1^1 x_2^0 x_3^1 \ + \frac{2!}{2!1!1!} x_1^0 x_2^1 x_3^1 \\ \text{iv. } & x_1^2 \ + x_2^2 \ + x_3^2 \ + 2x_1 x_2 \ + 2x_1 x_3 \ + 2x_2 x_3 \end{split}$$

7. The combination rule for counting sample points. The number of unordered groups of r objects chosen without replacement from $n, n \ge r$, distinct objects is

$$C_r^n = \begin{pmatrix} n \\ r \end{pmatrix} = \begin{pmatrix} n \\ r & n-r \end{pmatrix} = \begin{pmatrix} n \\ n-r \end{pmatrix} = \frac{n!}{r!(n-r)!} = \frac{P_r^n}{r!}.$$

A combination is an arrangement of objects where the order of these objects does *not* matter and the objects are sampled with*out* replacement. The combination rule is a special case of the partition rule above.

(a) The number of ways of dealing three cards from five cards (10,J,Q,K,A) is calculated by first assuming order matters (5 marbles, three side-by-side boxes),

$$P_3^5 = \frac{5!}{(5-3)!} = 5 \times 4 \times 3 = 60$$

and then "dividing out the order" (3! = 6 permutations of three cards), to arrive at

$$C_3^5 = \begin{pmatrix} 5\\3 \end{pmatrix} = \begin{pmatrix} 5\\3 & 5-3 \end{pmatrix} = \begin{pmatrix} 5\\5-3 \end{pmatrix} = \frac{5!}{3!(5-3)!} = \frac{P_3^5}{3!} = \frac{60}{6} =$$

(i) **9** (ii) **10** (iii) **11**.

(Use your calculator: type 5, then MATH PRB nCr ENTER 3 ENTER.)

(b) Number of ways of dealing r = 2 of n = 5 cards is (circle one or more) i. $\frac{5 \times 4}{2}$

ii.
$$\frac{5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1)2}$$
iii.
$$\frac{5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1)(2 \times 1)}$$
iv.
$$C_2^5 = \begin{pmatrix} 5\\2 \end{pmatrix} = \begin{pmatrix} 5\\3 \end{pmatrix} = \begin{pmatrix} 5\\2 3 \end{pmatrix} = \frac{5!}{3!2!} = 10$$
v.
$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$
, where $n = 5$ and $r = 2$

(c) How many different committees of 4 can be formed from a group of 40 people? Choose one or more.

(i)
$$\begin{pmatrix} 40 \\ 4 \end{pmatrix}$$
 (ii) $\begin{pmatrix} 40 \\ 40-4 \end{pmatrix}$ (iii) $\begin{pmatrix} 40 \\ 36 \end{pmatrix}$ (iv) $\begin{pmatrix} 40 \\ 4 & 36 \end{pmatrix}$

(d) Binomial theorem

$$(x+y)^n = \sum_{k=0}^n \left(\begin{array}{c}n\\k\end{array}\right) x^k y^{n-k}$$

where $\binom{n}{k}$ are called *binomial coefficients*. For example, expand $(x+y)^3$. Circle one or more.

i.
$$\sum_{k=0}^{3} \binom{3}{k} x^{k} y^{3-k}$$

ii.
$$\binom{3}{0} x^{0} y^{3} + \binom{3}{1} x^{1} y^{2} + \binom{3}{2} x^{2} y^{1} + \binom{3}{3} x^{3} y^{0}$$

iii.
$$\frac{3!}{0!3!} x^{0} y^{3} + \frac{3!}{1!2!} x^{1} y^{2} + \frac{3!}{2!1!} x^{2} y^{1} + \frac{3!}{3!0!} x^{3} y^{0}$$

iv.
$$y^{3} + 3x y^{2} + 3x^{2} y + x^{3}$$

(e) Combinations and the mn rule for counting. From a group of 6 women and 9 men, how many different committees consisting of 4 women and 3 men can be formed? Choose one or more.

(i)
$$\begin{pmatrix} 6\\4 \end{pmatrix} \begin{pmatrix} 9\\3 \end{pmatrix}$$
 (ii) $\begin{pmatrix} \frac{6\cdot5}{2\cdot1} \end{pmatrix} \begin{pmatrix} \frac{9\cdot8\cdot7}{3\cdot2\cdot1} \end{pmatrix}$ (iii) 1260.

(f) Combinations and the mn rule for counting. From a group of 6 women and 9 men, how many different committees consisting of (4 women and 3 men) or (3 women and 4 men) can be formed? Choose one or more.

(i)
$$\begin{pmatrix} 6\\4 \end{pmatrix} \begin{pmatrix} 9\\3 \end{pmatrix} + \begin{pmatrix} 6\\3 \end{pmatrix} \begin{pmatrix} 9\\4 \end{pmatrix}$$
 (ii) $\begin{pmatrix} \frac{6\cdot5}{4\cdot3\cdot2\cdot1} \end{pmatrix} \begin{pmatrix} \frac{9\cdot8\cdot7}{3\cdot2\cdot1} \end{pmatrix}$.

(g) Combinations and the mn rule for counting. From a group of 6 women and 9 men, how many different committees consisting of six (6) people can be formed? Choose one or more.

(i)
$$\begin{pmatrix} 6\\0 \end{pmatrix} \begin{pmatrix} 9\\6 \end{pmatrix} + \begin{pmatrix} 6\\1 \end{pmatrix} \begin{pmatrix} 9\\5 \end{pmatrix} + \dots + \begin{pmatrix} 6\\6 \end{pmatrix} \begin{pmatrix} 9\\0 \end{pmatrix}$$

(ii) $\begin{pmatrix} 6+9\\6 \end{pmatrix}$
(iii) $\begin{pmatrix} \frac{6\cdot5}{4\cdot3\cdot2\cdot1} \end{pmatrix} \begin{pmatrix} \frac{9\cdot8\cdot7}{3\cdot2\cdot1} \end{pmatrix}$

(h) A formula

$$\left(\begin{array}{c}n+m\\r\end{array}\right) = \left(\begin{array}{c}n\\0\end{array}\right) \left(\begin{array}{c}m\\r\end{array}\right) + \left(\begin{array}{c}n\\1\end{array}\right) \left(\begin{array}{c}m\\r-1\end{array}\right) + \dots + \left(\begin{array}{c}n\\r\end{array}\right) \left(\begin{array}{c}m\\0\end{array}\right)$$

(i) **True** (ii) **False**

(i) Combinations, the mn rule for counting. From a group of 6 women and 9 men, how many different committees consisting of 4 women and 3 men can be formed if two (2) of the women refuse to work with one another? Circle one.

(i)
$$\begin{pmatrix} 6\\4 \end{pmatrix} \times \begin{bmatrix} \begin{pmatrix} 2\\0 \end{pmatrix} \begin{pmatrix} 7\\3 \end{pmatrix} + \begin{pmatrix} 2\\1 \end{pmatrix} \begin{pmatrix} 7\\2 \end{bmatrix}$$

(ii) $\begin{pmatrix} 9\\3 \end{pmatrix} \times \begin{bmatrix} \begin{pmatrix} 2\\0 \end{pmatrix} \begin{pmatrix} 4\\4 \end{pmatrix} + \begin{pmatrix} 2\\1 \end{pmatrix} \begin{pmatrix} 4\\3 \end{bmatrix}$
(iii) $\begin{pmatrix} 9\\3 \end{pmatrix} \times \begin{bmatrix} \begin{pmatrix} 2\\2 \end{pmatrix} \begin{pmatrix} 4\\4 \end{pmatrix} + \begin{pmatrix} 2\\0 \end{pmatrix} \begin{pmatrix} 4\\3 \end{bmatrix}$
Hint: men combinations × [both feuding women missing + one feuding woman missing]

(j) Combinations and $n_1 \cdot n_2 \cdots n_r$ rule for counting. From a deck of 10 clubs, 9 spades, 11 diamonds and 12 hearts, how many different hands consisting of 3 clubs, 3 spades, 4 diamonds and 10 hearts can be formed? Choose one or more.

(i)
$$\begin{pmatrix} 10\\3 \end{pmatrix} \begin{pmatrix} 9\\3 \end{pmatrix} \begin{pmatrix} 11\\4 \end{pmatrix} \begin{pmatrix} 12\\10 \end{pmatrix}$$

(ii) $\begin{pmatrix} \frac{10!}{3!7!} \end{pmatrix} \begin{pmatrix} \frac{9!}{3!6!} \end{pmatrix} \begin{pmatrix} \frac{11!}{4!7!} \end{pmatrix} \begin{pmatrix} \frac{12!}{10!2!} \end{pmatrix}$
(iii) 219,542,400

(k) Multinomial, combinations and $n_1 \cdot n_2 \cdots n_r$ rule for counting. Eleven students are ranked 1 (first) to 11 (last) in a statistics contest. As shown in Figure 2.11, four of the students are from the United States, three are from Russia, three are from China and one is from the United Kingdom.



Figure 2.11: Marbles in boxes: ranks in statistics contest

- i. How many different ways can the (student) ranks be divided among the four different countries? Choose one.
 - (ii) $\begin{pmatrix} 11 \\ 3 3 3 1 \end{pmatrix}$ (iii) $\begin{pmatrix} 11 \\ 4 3 3 1 \end{pmatrix}$ $\begin{array}{c}11\\2&3&3&1\end{array}$
- ii. How many different ways can the (student) ranks be divided among the four different countries if the United States has two students in the top four and one in the bottom three? Choose one.

(i)
$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \begin{pmatrix} 7 \\ 4 & 2 & 1 \end{pmatrix}$$

(ii) $\begin{pmatrix} 4 \\ 2 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \begin{pmatrix} 7 \\ 2 & 3 & 2 \end{pmatrix}$
(iii) $\begin{pmatrix} 3 \\ 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \begin{pmatrix} 7 \\ 4 & 2 & 1 \end{pmatrix}$
Hint: US US O O | US O O | US O O
US on markin 2 of the function of better the condition of the second secon

US can appear in 2 of top four, 1 of middle four, 1 of bottom three and any other student, "O", of the seven remaining, are then placed accordingly.

- 8. Counting and probability.
 - (a) Permutations in Probability. Since there are $5 \times 4 \times 3 = 60$ permutations for 3 of five cars, A, B, C, D and E, to park in 3 parking spots, and, furthermore, $1 \times 4 \times 3 = 12$ ways for car A to park in the first spot, the chance car A parks in the first spot is $\frac{12}{60} = 0.2$ or 20%.

With this in mind, the chance that car A parks in the first spot, if

- i. 4 of 5 cars park in 4 parking spots, is (circle one) (i) $\frac{1 \times 4 \times 3 \times 2}{5 \times 4 \times 3 \times 2}$ (ii) $\frac{5 \times 4 \times 3 \times 2}{5 \times 4 \times 3 \times 2}$ (iii) $\frac{1 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2}$ ii. 4 of 7 cars park in 4 parking spots, is (circle one)
- (i) $\frac{1 \times 7 \times 6 \times 5}{5 \times 4 \times 3 \times 2}$ (ii) $\frac{1 \times 6 \times 5 \times 4}{7 \times 6 \times 5 \times 4}$ (iii) $\frac{1 \times 4 \times 3 \times 2}{7 \times 6 \times 5 \times 4}$
- iii. 4 of 6 cars park in 4 parking spots (remember: car A parks in the first spot) car B must appear in the second spot, is (circle one) (ii) $\frac{1 \times 1 \times 4 \times 3}{6 \times 5 \times 4 \times 3}$ (i) $\frac{1 \times 1 \times 6 \times 5}{6 \times 5 \times 4 \times 3}$ (iii) $\frac{1 \times 5 \times 4 \times 3}{6 \times 5 \times 4 \times 3}$
- (b) Permutations in Probability: Sitting Around A Table. Three girls and four boys play a game of "musical chairs" in which seven chairs are placed around a table. What is the probability the three females sit next to one another (and so this necessarily means the four boys must sit next to one another)?
 - i. How many ways can seven children in a line can be arranged? Choose one. (i) 5! (ii) **6!** (iii) **7**!
 - ii. Shifting six children of seven in a line to the right and moving the last child to the head of the line *does* change the arrangement of these children. In fact, there are seven shifts to the right (where the last

child is placed at the head of the line in each case) which cause a change in the arrangement of the children *in a line*.

(i) **True** (ii) **False**

iii. Seven children *around a table*, on the other hand, can all be shifted a total of seven locations to the right (or left) and their order does *not* change. Consequently, the total number of ways seven children be arranged around a table is equal to the number ways children can be arranged in a line divided by seven or (choose one)

(i) **5!** (ii)
$$\frac{7}{7}$$
 (iii) **6!**

- iv. Three girls grouped together and four boys grouped together around a table can be arranged in how many ways? Choose one.
 (i) 6! (ii) 3!3! (iii) 3!4!
- v. The probability the three females sit next to one another is (choose one) (i) $\frac{6!}{6!}$ (ii) $\frac{3!3!}{6!}$ (iii) $\frac{3!4!}{6!}$
- (c) Cards. Four cards are dealt at random from a deck of 52 playing cards.
 - i. The total number of ways of dealing four cards is (circle one)

(i)
$$\begin{pmatrix} 49\\ 3 \end{pmatrix}$$
 (ii) $\begin{pmatrix} 52\\ 4 \end{pmatrix}$ (iii) $\begin{pmatrix} 52\\ 2 \end{pmatrix}$ (iv) $\begin{pmatrix} 52\\ 1 \end{pmatrix}$

ii. The total number of ways of dealing four cards with three jacks is (circle one)

(i)
$$\begin{pmatrix} 4\\3 \end{pmatrix} \times \begin{pmatrix} 48\\1 \end{pmatrix}$$
 (ii) $\begin{pmatrix} 4\\2 \end{pmatrix} \times \begin{pmatrix} 48\\4 \end{pmatrix}$ (iii) $\begin{pmatrix} 4\\0 \end{pmatrix} \times \begin{pmatrix} 48\\1 \end{pmatrix}$
(The scalar bility of balls of a scalar bility of a scalar bili

iii. The probability of dealing four cards with three jacks is (circle one) $\begin{pmatrix} 4 \\ 4 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

(i)
$$\frac{\begin{pmatrix} 4\\3 \end{pmatrix} \begin{pmatrix} 48\\1 \end{pmatrix}}{\begin{pmatrix} 52\\2 \end{pmatrix}}$$
 (ii)
$$\frac{\begin{pmatrix} 4\\3 \end{pmatrix} \begin{pmatrix} 48\\1 \end{pmatrix}}{\begin{pmatrix} 52\\4 \end{pmatrix}}$$
 (iii)
$$\frac{\begin{pmatrix} 4\\0 \end{pmatrix} \begin{pmatrix} 48\\1 \end{pmatrix}}{\begin{pmatrix} 52\\4 \end{pmatrix}}$$

- (d) *Two Pair in Poker Hand.* What is the probability of being dealt two pair in a 5–card poker hand?
 - i. The total number of ways of dealing 5 cards is (circle one)

(i)
$$\begin{pmatrix} 52\\ 2 \end{pmatrix}$$
 (ii) $\begin{pmatrix} 52\\ 3 \end{pmatrix}$ (iii) $\begin{pmatrix} 52\\ 4 \end{pmatrix}$ (iv) $\begin{pmatrix} 52\\ 5 \end{pmatrix}$.

ii. The total number of ways of dealing a pair of *twos* and then a pair of *threes* is (circle one)

(i)
$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \begin{pmatrix} 44 \\ 1 \end{pmatrix}$$

(ii) $\begin{pmatrix} 4 \\ 3 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \begin{pmatrix} 44 \\ 1 \end{pmatrix}$
(iii) $\begin{pmatrix} 4 \\ 4 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix} \begin{pmatrix} 44 \\ 1 \end{pmatrix}$

iii. Since there are $\begin{pmatrix} 13\\2 \end{pmatrix}$ choices for picking any two pairs (without regard to their order), the number of ways of dealing two pair is (circle one)

(i)
$$\begin{pmatrix} 13\\0 \end{pmatrix} \times \begin{pmatrix} 4\\2 \end{pmatrix} \begin{pmatrix} 4\\2 \end{pmatrix} \begin{pmatrix} 44\\1 \end{pmatrix}$$

(ii) $\begin{pmatrix} 13\\1 \end{pmatrix} \times \begin{pmatrix} 4\\2 \end{pmatrix} \begin{pmatrix} 4\\2 \end{pmatrix} \begin{pmatrix} 44\\1 \end{pmatrix}$
(iii) $\begin{pmatrix} 13\\2 \end{pmatrix} \times \begin{pmatrix} 4\\2 \end{pmatrix} \begin{pmatrix} 4\\2 \end{pmatrix} \begin{pmatrix} 44\\1 \end{pmatrix}$

iv. The probability that two pair is dealt is (circle one)

(i)
$$\frac{\begin{pmatrix} 13\\ 0 \end{pmatrix} \times \begin{pmatrix} 4\\ 2 \end{pmatrix} \begin{pmatrix} 4\\ 2 \end{pmatrix} \begin{pmatrix} 44\\ 1 \end{pmatrix}}{\begin{pmatrix} 52\\ 5 \end{pmatrix}}$$

(ii)
$$\frac{\begin{pmatrix} 13\\ 1 \end{pmatrix} \times \begin{pmatrix} 4\\ 2 \end{pmatrix} \begin{pmatrix} 4\\ 2 \end{pmatrix} \begin{pmatrix} 44\\ 1 \end{pmatrix}}{\begin{pmatrix} 52\\ 5 \end{pmatrix}}$$

(iii)
$$\frac{\begin{pmatrix} 13\\ 2 \end{pmatrix} \times \begin{pmatrix} 4\\ 2 \end{pmatrix} \begin{pmatrix} 4\\ 2 \end{pmatrix} \begin{pmatrix} 44\\ 1 \end{pmatrix}}{\begin{pmatrix} 52\\ 5 \end{pmatrix}}$$

(iii)
$$\frac{\begin{pmatrix} 13\\ 2 \end{pmatrix} \times \begin{pmatrix} 4\\ 2 \end{pmatrix} \begin{pmatrix} 4\\ 2 \end{pmatrix} \begin{pmatrix} 4\\ 2 \end{pmatrix} \begin{pmatrix} 44\\ 1 \end{pmatrix}}{\begin{pmatrix} 52\\ 5 \end{pmatrix}}$$

- (e) *Bridge Hands.* All 52 cards are dealt out to four players. What is the probability one of the players receives 10 of 13 spades?
 - i. The total number of ways of dealing 52 cards to four people is (52)

(i)
$$\begin{pmatrix} 52 \\ 13 \end{pmatrix}$$
 (ii) $\begin{pmatrix} 52 \\ 13 13 \end{pmatrix}$
(iii) $\begin{pmatrix} 52 \\ 13 13 \end{pmatrix}$ (iv) $\begin{pmatrix} 52 \\ 13 13 13 \end{pmatrix}$ (iv) $\begin{pmatrix} 52 \\ 13 13 13 13 \end{pmatrix}$

.

ii. The total number of ways of dealing one *particular* player 10 of 13 spades is (circle one)

(i)
$$\begin{pmatrix} 13\\10 \end{pmatrix} \times \begin{pmatrix} 39\\3 \end{pmatrix}$$

(ii) $\begin{pmatrix} 13\\11 \end{pmatrix} \times \begin{pmatrix} 39\\2 \end{pmatrix}$
(iii) $\begin{pmatrix} 13\\12 \end{pmatrix} \times \begin{pmatrix} 39\\2 \end{pmatrix}$

iii. The total number of ways of dealing 39 cards to the remaining three people, (after dealing thirteen cards to a *particular* player) is (circle one)

(i)
$$\begin{pmatrix} 39\\ 13 \end{pmatrix}$$
 (ii) $\begin{pmatrix} 39\\ 13 \ 13 \end{pmatrix}$
(iii) $\begin{pmatrix} 39\\ 13 \ 13 \ 13 \end{pmatrix}$ (iv) $\begin{pmatrix} 39\\ 13 \ 13 \ 13 \ 13 \end{pmatrix}$

iv. Since there are four choices for choosing a particular player, the total number of ways 52 cards are dealt out to four players such that one of the players receives 10 of 13 spades is (circle one)

(i)
$$2 \cdot \begin{pmatrix} 39\\13 & 13 & 13 \end{pmatrix} \begin{pmatrix} 13\\10 \end{pmatrix} \times \begin{pmatrix} 39\\3 \end{pmatrix}$$

(ii) $3 \cdot \begin{pmatrix} 39\\13 & 13 & 13 \end{pmatrix} \begin{pmatrix} 13\\10 \end{pmatrix} \times \begin{pmatrix} 39\\3 \end{pmatrix}$
(iii) $4 \cdot \begin{pmatrix} 39\\13 & 13 & 13 \end{pmatrix} \begin{pmatrix} 13\\10 \end{pmatrix} \times \begin{pmatrix} 39\\3 \end{pmatrix}$.

v. The probability one of the players receives 10 of 13 spades is (circle one)

(i)
$$2 \cdot \frac{\begin{pmatrix} 39\\13 \ 13 \ 13 \end{pmatrix} \begin{pmatrix} 13\\10 \end{pmatrix} \times \begin{pmatrix} 42\\3 \end{pmatrix}}{\begin{pmatrix} 52\\13 \ 13 \ 13 \ 13 \end{pmatrix}}$$

(ii) $3 \cdot \frac{\begin{pmatrix} 39\\13 \ 13 \ 13 \end{pmatrix} \begin{pmatrix} 13\\10 \end{pmatrix} \times \begin{pmatrix} 42\\3 \end{pmatrix}}{\begin{pmatrix} 52\\13 \ 13 \ 13 \ 13 \end{pmatrix}}$
(iii) $4 \cdot \frac{\begin{pmatrix} 39\\13 \ 13 \ 13 \ 13 \end{pmatrix} \begin{pmatrix} 13\\10 \end{pmatrix} \times \begin{pmatrix} 39\\3 \end{pmatrix}}{\begin{pmatrix} 52\\13 \ 13 \ 13 \ 13 \end{pmatrix}}$.

9. Twelve-fold Way. Some counting methods, of placing r marbles into n boxes, can be categorized into following twelve-fold way table (order of marbles in each box is not counted):

	marbles, r	boxes, n	unrestricted: 0, 1 or more	at most 1 marble per box, ≤ 1	at least 1 marble per box, ≥ 1
l			marbles per box		
	D	D	n^r	P_r^n	n!S(r,n)
	Ι	D	C_r^{r+n-1}	C_r^n	C_{n-1}^{r-1}
	D	Ι	$\sum_{k=1}^{n} S(r,k)$	$\begin{array}{l} 1 \text{ if } r \leq n \\ 0 \text{ if } r > n \end{array}$	S(r,n)
	Ι	Ι	$\sum_{k=1}^{n} p_k(r)$	$\begin{array}{l} 1 \text{ if } r \leq n \\ 0 \text{ if } r > n \end{array}$	$p_n(r)$

D: distinguishable (labelled), I: indistinguishable (identical, unlabelled), combination $C_r^n = \frac{n!}{r!(n-r)!} = \binom{n}{r}$, permutation $P_r^n = \frac{n!}{(n-r)!}$ Stirling numbers of second kind $S(r,n) = \frac{1}{n!} \sum_{k=0}^n (-1)^k \binom{n}{k} (n-k)^r$ notice distinguishable box (D,D, ≥ 1) case with count n!S(r,n) is ordered version of indistinguishable box (D,I, ≥ 1) case with count S(r,n)

- (a) Count number of ways of putting r = 4 marbles and n = 6 boxes under following assumptions.
 - i. (D,D) unrestricted (none, 1 or more marbles can appear in each box) marble 1 placed in one of 6 boxes, same for marbles 2, 3, 4, $n^r = 6^4 = (i)$ **1296** (ii) **1297** (iii) **1298**.
 - ii. (D,D) at most 1 marble per box (permutation) marble 1 has 6 choices, marble 2 had 5 choices, ... $P_r^n = \frac{n!}{(n-r)!} = P_4^6 = 6 \times 5 \times 4 \times 3 = (i)$ **359** (ii) **360** (iii) **361**.
 - iii. (D,D) at least 1 marble per box only 4 marbles, so not possible to put at least 1 marble in 6 boxes,
 (i) 0 (ii) 1 (iii) 2.
 - iv. (I,D) unrestricted

o||oo||o| means boxes 1, 2, 3, 4, 5, 6 gets 1, 0, 2, 0, 1, 0 marbles, since arranging 4 marbles in 4 marbles + 6 boxes - 1 = 9 objects;

$$C_r^{r+n-1} = \begin{pmatrix} 4+6-1\\4 \end{pmatrix} = \begin{pmatrix} 9\\4 \end{pmatrix} = (i)$$
 126 (ii) **127** (iii) **128**.

v. (I,D) at most 1 marble per box (combination) order of 4 identical marbles in 4 of 6 distinguishable boxes does not matter,

$$C_r^n = \frac{n!}{r!(n-r)!} = \begin{pmatrix} n \\ r \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix} = (i) \mathbf{14} \quad (ii) \mathbf{15} \quad (iii) \mathbf{16}.$$



Figure 2.12: Twelve-Fold Way: 4 marbles, 6 boxes or 6 marbles; 4 boxes

vi. (I,D) at least 1 marble per box only 4 marbles, so not possible to put at least 1 marble in 6 boxes, (i) **0** (ii) **1** (iii) **2**. vii. (D,I) unrestricted $\sum_{k=1}^{n} S(r,n) = \sum_{k=1}^{6} S(4,k)$ $\overline{=S(4,1)} + S(4,2) + S(4,3) + S(4,4) + S(4,5) + S(4,6)$ = 1 + 7 + 6 + 1 + 0 + 0 = (i) 15 (ii) **16** (iii) **17**. viii. (D,I) at most 1 marble per box 4 marbles in 4 of 6 identical boxes same no matter which boxes (i) **0** (ii) **1** (iii) **2**. ix. (D,I) at least 1 marble per box only 4 marbles, so not possible to put at least 1 marble in 6 boxes, (i) **0** (ii) **1** (iii) **2**. x. (I,I) unrestricted 4 identical marbles placed in 1, 2, 3, 4, 5 and 6 identical boxes is (4); AND (0,4), (1,3) or (2,2); AND (0,0,4), (0,1,3), (0,2,2) or (1,1,2);

AND (0,0,0,4), (0,0,1,3), (0,0,2,2), (0,1,1,2), (1,1,1,1), AND (0,0,0,0,4), (0,0,0,1,3), (0,0,0,2,2), (0,0,1,1,2), (0,1,1,1,1), (0,0,0,0,0,4), (0,0,0,0,1,3), (0,0,0,0,2,2), (0,0,0,1,1,2), (0,0,1,1,1,1), $\sum_{k=1}^{n} p_k(r) = \sum_{k=1}^{6} p_k(4) =$ $= p_1(4) + p_2(4) + p_3(4) + p_4(4) + p_5(4) + p_6(4) = 1 + 3 + 4 + 5 + 5 + 5 =$ (i) **24** (ii) **25** (iii) **26**.

- xi. (I,I) at most 1 marble per box
 4 identical marbles in 4 of 6 identical boxes same so only (1,1,1,1,0,0) arrangement (i) 0 (ii) 1 (iii) 2.
- xii. (I,I) at least 1 marble per box only 4 marbles, so not possible to put at least 1 marble in 6 boxes, $p_n(r) = p_6(4) = (i) \mathbf{0}$ (ii) **1** (iii) **2**.
- (b) Count number of ways of putting r = 6 marbles and n = 4 boxes under following assumptions.
 - i. (D,D) unrestricted (none, 1 or more marbles can appear in each box) marble 1 placed in one of 4 boxes, same for marbles 2, ..., 6, n^r = 4⁶ = (i) 4096 (ii) 4097 (iii) 4098.
 - ii. (D,D) at most 1 marble per box (permutation)
 Since 6 marbles, not possible to put only at most 1 marble in 4 boxes,
 (i) 0 (ii) 1 (iii) 2.
 - iii. (D,D) at least 1 marble per box At least 1 marble can be placed in each of 4 boxes, for example, (3,1,1,1) and (1,3,1,1) different arrangements, $n!S(r,n) = n! \times \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^r$

$$= \sum_{k=0}^{4} (-1)^k \begin{pmatrix} 4\\k \end{pmatrix} (4-k)^6 = (i) \ \mathbf{1540} \qquad (ii) \ \mathbf{1550} \qquad (iii) \ \mathbf{1560}.$$

iv. (I,D) unrestricted

since arranging 6 marbles in 6 marbles + 4 boxes - 1 = 9 objects; for example o|o|oo|oo means boxes 1, 2, 3, 4 gets 1, 1, 2, 2 marbles,

$$C_r^{r+n-1} = \begin{pmatrix} 6+4-1\\4 \end{pmatrix} = \begin{pmatrix} 9\\6 \end{pmatrix} = (i) \ 82$$
 (ii) 83 (iii) 84 .

v. (I,D) at most 1 marble per box (combination)
Since 6 marbles, not possible to put only at most 1 marble in 4 boxes,
(i) 0 (ii) 1 (iii) 2.

vi. (I,D) at least 1 marble per box arranging 6 - 4 = 2 marbles in 6 - 4 marbles + 4 boxes - 1 = 5 objects; for example o|o|oo|oo means boxes 1, 2, 3, 4 gets 1, 1, 2, 2 marbles,

$$C_{r-n}^{r-n+n-1} = C_{r-n}^{r-1} = \begin{pmatrix} 5\\2 \end{pmatrix} = (i) \ \mathbf{10} \quad (ii) \ \mathbf{11} \quad (iii) \ \mathbf{12}.$$

- vii. (D,I) unrestricted $\sum_{k=1}^{n} S(r,n) = \sum_{k=1}^{4} S(6,k)$ = S(6,1) + S(6,2) + S(6,3) + S(6,4) $= 1 + 31 + 90 + 65 = (i) \mathbf{186} \quad (ii) \mathbf{187} \quad (iii) \mathbf{188}.$
- viii. (D,I) at most 1 marble per box
 Since 6 marbles, not possible to put only at most 1 marble in 4 boxes,
 (i) 0 (ii) 1 (iii) 2.
 - ix. (D,I) at least 1 marble per box at least 1 marble can be placed in each of 4 identical boxes, for example, (3,1,1,1) and (1,3,1,1) same arrangements,

$$S(r,n) = \frac{1}{n!} \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} (n-k)^{r}$$

= $\frac{1}{4!} \sum_{k=0}^{4} (-1)^{k} \binom{4}{k} (4-k)^{6} = (i) 64$ (ii) 65 (iii) 66.

x. (I,I) unrestricted

6 identical marbles placed in 1, 2, 3, and 4 identical boxes is (6); AND (0,6), (1,5), (2,4) or (3,3); AND (0,0,6), (0,1,5), (1,1,4), (1,2,3) or (2,2,2); AND (0,0,0,6), (0,0,1,5), (0,0,2,4), (0,0,3,3), or (0,1,1,4), (0,1,2,3), (0,2,2,2), (1,1,1,3), (1,1,2,2), so $\sum_{k=1}^{n} p_k(r) = \sum_{k=1}^{4} p_k(6) = p_1(6) + p_2(6) + p_3(6) + p_4(6) = 1 + 4 + 5 + 9 =$ (i) **10** (ii) **15** (iii) **19**.

- xi. (I,I) at most 1 marble per box
 Since 6 marbles, not possible to put only at most 1 marble in 4 boxes,
 (i) 0 (ii) 1 (iii) 2.
- xii. (I,I) at least 1 marble per box arrangements of 6 identical marbles placed in 4 identical boxes is either (3,1,1,1) or (2,2,1,1), so $p_n(r) = p_4(6) = (i)$ **0** (ii) **1** (iii) **2**.