Chapter 4

The System: Ship Dynamics

Learning Objectives:

1. **For Dynamics Review:**
    (a) Determine the equation of motion for a spring-mass-damper system with sinusoidal excitation and solve for motion amplitude, velocity, acceleration, and phase.
    (b) Explain the significance of under-damping, over-damping, or critically damping in a spring-mass-damper system.
    (c) Identify which of the seakeeping DOFs are under-damped or over-damped.
    (d) Explain what resonance is and why it is relevant to seakeeping
    (e) In your own words, describe the concepts of added mass and hydrodynamic damping
2. **For Model Testing in Regular Waves:**
    (a) Calculate the encounter frequency given the wavelength, heading, and ship speed.
    (b) Explain how encounter frequency depends on heading, ship speed, and wavelength.
    (c) Describe what a transfer function is.
    (d) Explain how the transfer function can help determine resulting ship motion given wave characteristics and why the limitation of linearity is important.
    (e) Describe the shape of a transfer function and what this means with respect to the magnitude of the ship response as the encounter frequency goes from very small though the natural frequency of the ship to very large.
3. **For Strip Theory:**
    (a) State the assumptions/limitations associated with strip theory and explain what they mean in your own words.
    (b) Identify Maxsurf Motions seakeeping output and explain what the results mean.

4. **For Roll Mitigation:**
   (a) Solve a roll seakeeping problem given a transfer function, vessel speed and wave heading, wave frequency to determine best action (how to change speed or heading) to reduce roll motions.
   (b) Describe a typical roll transfer function.
   (c) Explain the effect of damping on the roll response.
   (d) Calculate the roll natural frequency for a given ship.
   (e) Identify and explain different devices for reducing roll motion.
   (f) Calculate the damping factor given the roll decay coefficient.
5. **Laboratory Objectives:**
   (a) Describe proper ballasting techniques for seakeeping experiments.
   (b) Set proper pitch and yaw mass moments of inertia for models.
   (c) Calculate the pitch gyradius using the knife edge method.
   (d) Calculate the yaw gyradius using the bifilar suspension method.
   (e) Measure the heave and pitch amplitude responses to given wave excitation.
   (f) Measure the excitation (encounter) frequency and compare with the predicted values.
   (g) Develop a pitch and heave transfer function plot from experimental measurements.
   (h) Describe the expected heave and pitch motion responses to "short" and "long" wavelengths.
   (i) Explain resonance and the effects of damping on resonant response.
   (j) Describe the effects of a passive tank roll stabilization device on the roll motion for a model in beam seas.
   (k) Describe the effect of roll resonance on the roll amplitude magnification.
   (l) Explain the relationship between the roll motion and the wave frequencies for a model experiencing regular waves coming from the starboard beam.
   (m) Develop a realistic test plan for a seakeeping experiment.

A ship can be considered a mass that has damping and stiffness and is experiencing an oscillating excitation force. Chapter 3 dealt with the specifics of the exciting force. This chapter will deal with how a mass with damping and stiffness responds to a sinusoidal excitation. Let's review the six degrees of freedom associated with ship motion:

1. Surge	4. Roll
2. Sway	5. Pitch
3. Heave	6. Yaw

Three of these motions experience a "restoring force" due to buoyancy: *heave, roll*, and *pitch*. But, what is a restoring force and why is it a problem for a ship in waves?

4.1 A Review of Dynamics

Let's start with Newton's Second Law:

$$\sum \vec{F} = m\vec{a}$$

The forces on the left-hand side include any forces acting on the body - most obviously any external force on the system, $\vec{F}_{\text{external}}$, but also forces due to springs, $\vec{F}_{\text{spring}}$, or damping, $\vec{F}_{\text{damping}}$. Consider a single point mass on a spring, as shown in Figure 4.1. There are two forces in the system: the force of gravity due to the mass and the spring force due to the spring compression or extension. Figure 4.1 also shows the Free Body Diagram (FBD) of the system.

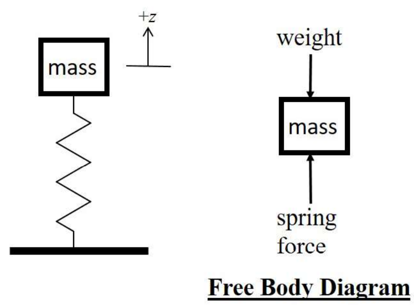


Figure 4.1: Point mass on a spring and the FBD (free body diagram)

The force due to gravity equals mg, where m is the mass and g is the acceleration due to gravity, and the force due to the spring equals $-kx$, where k is the spring stiffness and x is the distance the spring is compressed. Using Newton's Second Law and these forces, the Equation of Motion (EOM) for the system can be written:

$$\begin{aligned} -kx - mg &= ma \\ 0 &= ma + kx + mg \end{aligned}$$

Acceleration is the second time-derivative of position. If position is written as x, then acceleration can be written as $\ddot{x}$ and the EOM becomes:

$$0 = m\ddot{x} + kx + mg$$

For a ship, the stiffness is due to the buoyant force acting on the ship. Consider heave motion, for example. If you push the ship a foot down in the water, there is an extra buoyant force acting up on the ship in excess of the ship's displacement. If you then release the downward force on the ship, it will move up. Likewise, lifting a ship out of the water will result in *less* buoyant force than the ship's displacement, so when released the ship will move *down*. Thus, buoyancy is our restoring force, i.e. the spring in the system. Now, if we keep the gravity term, our x value equals the draft of a barge just for everything to be in equilibrium (in equilibrium there is no acceleration, so $\ddot{x} = 0$ and the position, x, equals the weight divided by the buoyant force). If we redefine the $x = 0$ to occur when the ship is in equilibrium (rather than when the ship is not in the water yet), the mg term goes away as we are only interested in *changes* to the equilibrium state. So, we can simplify to $0 = m\ddot{x} + kx$. Also, and just to mess with students who took classical dynamics, instead of using k to represent the stiffness coefficient, naval architects use c. So, the EOM now looks like,

$$0 = m\ddot{x} + cx$$

Ships don't just experience a buoyant force from the surrounding water, there is also hydrodynamic damping. Water is more viscous than air, so energy dissipates more quickly when moving in water. This damping force is proportional to the velocity, $-b\dot{x}$, where b is the damping coefficient and $\dot{x}$ is the velocity. This force is an additional force to the spring force.

$$0 = m\ddot{x} + b\dot{x} + cx$$

This equation can be considered *trivial* in the sense that if the ship is at its equilibrium draft and is not currently moving or accelerating then both sides of the equation are zero and $0 = 0$! However, if something gives the ship a bump, thus causing a positive change from equilibrium or an initial velocity, then we have a dynamic response that changes over time. For the case of spring-mass-damper with no external excitation force, the resulting motion is a decaying position that moves back towards the equilibrium position. If the system is **over-damped** the response will look like an exponential decay back to equilibrium. If the system is **under-damped** the position will be a decaying oscillation where the amplitudes of oscillation get smaller and smaller until equilibrium is finally reached. And if the system is **critically damped** the position returns to equilibrium in the shortest amount of time with one or no oscillations.

Ships, however, almost never operate in conditions where there is no heaving, rolling, or pitching because there is almost always an excitation force around - waves! We have to deal with the EOM that includes such external excitation forces. For now we will consider only regular waves - sinusoidal excitation with a single frequency and amplitude - moving on to more realistic waves in Chapter 5. Our wave excitation force can be written as:

$$F(t) = F_0 \sin \omega_e t$$

where F_0 is the forcing amplitude (related to the wave height) and ω_e is the frequency the wave moves past the ship. Figure 4.2 shows the updated free body diagram for our system including the mass, spring, damping, and excitation force.
Using the FBD and Newton's Second Law, our EOM for the full system becomes

$$F_0 \sin \omega_e t = m\ddot{x} + b\dot{x} + cx \tag{4.1}$$

The solution to this equation will be a system that has a transient oscillation until the damping has eliminated the ship's natural buoyant/damping response to the initial displacement and then an equilibrium solution that will have the same frequency as the excitation force. The solution will not necessarily have the same amplitude or phase as the excitation force, however. The solution can be written as

$$x(t) = X_0 \sin(\omega_e t - \phi)$$

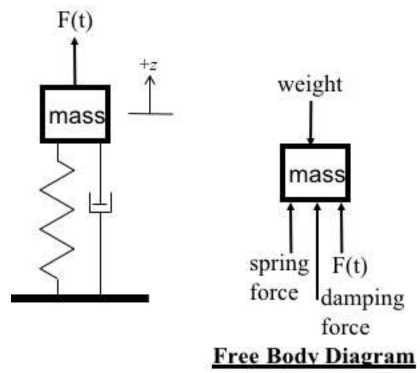


Figure 4.2: Point mass on a spring with damping and an excitation force and the FBD

where X_0 is the amplitude of the motion, ω_e is the frequency of the motion (and equal to the excitation frequency), and ϕ is the phase difference between the excitation sinusoidal motion and the resulting sinusoidal motion. The Appendix to this chapter goes through the derivation to solve for X_0 and ϕ. The solutions are:

$$X_0 = \frac{F_0}{\sqrt{(-\omega_e^2 m + c)^2 + (\omega_e b)^2}}$$
$$\tan\phi = \frac{\omega_e b}{-\omega_e^2 m + c}$$

The terms of the EOM can be described as:

$m\ddot{x}$:	can be considered an **inertial term**
$b\dot{x}$:	can be considered an **damping term**
cx:	can be considered an **stiffness term**
$F_0 \sin\omega_e t$:	can be considered the **excitation force term**

If the solution is $x(t) = X_0 \sin(\omega_e t + \phi)$, then the stiffness term has a sin phase ($cx = cX_0 \sin(\omega_e t + \phi)$), the damping term has a cos phase ($b\dot{x} = b\omega_e X_0 \sin(\omega_e t + \phi)$), and the inertial term has a sin phase ($m\ddot{x} = -m\omega_e^2 X_0 \sin(\omega_e t + \phi)$). Figure 4.3 shows how each of these terms can be treated as a separate effect on the final motion of the mass.
These terms can be related to some common concepts used in seakeeping (and vibration) to describe systems that experience *simple harmonic motion* (i.e. the system motion is sinusoidal due to the presence of a restoring force).

Natural Frequency - the frequency at which the system oscillates on its own when disturbed from equilibrium

$$\omega_n \equiv \sqrt{\frac{c}{m}}$$

Damping Factor - the amount of damping in the system (> 1 for over-damped, < 1 for under-damped, and $= 1$ for critically damped)

$$\eta \equiv \frac{b}{2\sqrt{mc}}$$

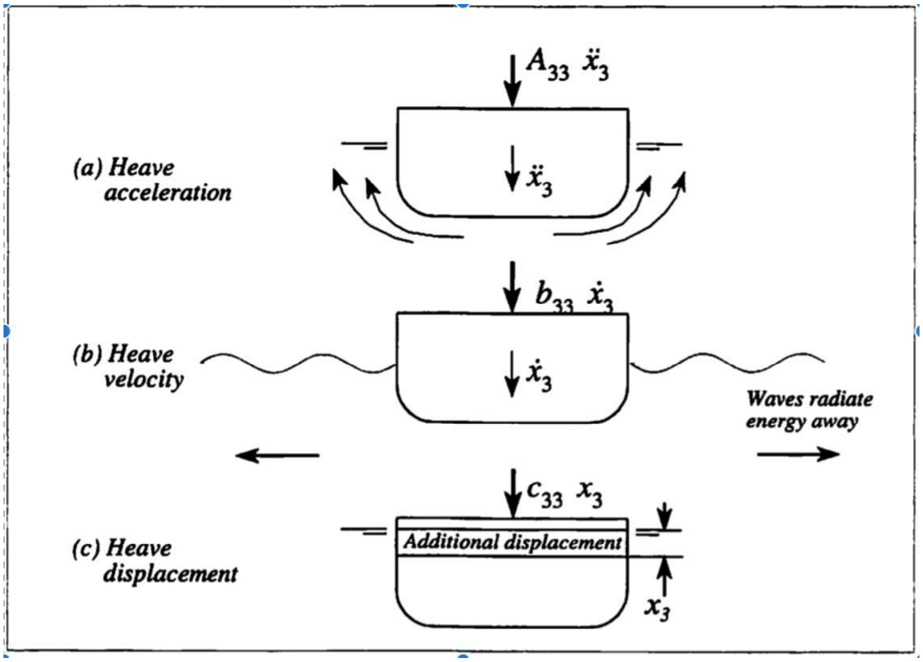


Figure 4.3: Effects of terms in the Equation of Motion (EOM) (Figure 3.4 in reference 2)

Tuning Factor - the ratio of the excitation frequency to the natural frequency

$$\Lambda \equiv \frac{\omega_e}{\omega_n}$$

These terms can be used in the EOM giving an EOM that equals:

$$\ddot{x} + 2\eta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m}\sin\omega_e t.$$

The solution for the response amplitude (X_0) and response phase (ϕ) can also be written using these concepts:

$$X_0 = \frac{F_0/c}{\sqrt{(1-\Lambda^2)^2 + (2\eta\Lambda)^2}} \tag{4.2}$$

$$\tan\phi = \frac{2\eta\Lambda}{1-\Lambda^2}. \tag{4.3}$$

To understand what all is going on with the response amplitude solution, consider spring-mass-damper system than is experiencing a constant force, F_0 (instead of a sinusoidally varying force). In this case, the equation of motion looks like

$$F_0 = m\ddot{x} + b\dot{x} + cx$$

However, if the force is a constant, the system will come to a state of equilibrium where the mass in not moving, i.e. $\ddot{x} = \dot{x} = 0$. Thus, the EOM simplifies to

$$F_0 = cX_0$$

and the solution is

$$X_0 = F_0/c.$$

Looking at the solution for X_0, this is the term in the numerator of equation 4.2. So, when a sinusoidal force is applied to a spring-mass-damper system, the resulting position amplitude is the "static" response (response if the force amplitude, F_0, were constantly applied) modified by the terms in the denominator. The modification of this static response is called the **Dynamic Magnification Factor** or MF:

$$\text{Dynamic Magnification Factor (MF)} = \frac{1}{\sqrt{(1-\Lambda^2)^2 + (2\eta\Lambda)^2}}. \tag{4.4}$$

This means that for a given forcing amplitude, F_0, the response amplitude changes depending on the damping factor (η) and the tuning factor (Λ). The damping factor relates to how much damping there is in the system. Looking at the MF, the larger η, the smaller the Magnification Factor. As one would expect, increasing damping reduces the magnitude of the response. The tuning factor relates to how close the excitation frequency (ω_e) is to the natural frequency (ω_n). Unlike for damping, it is not clear what happens as the tuning factor increases. Consider the case where the excitation frequency equals the natural frequency ($\omega_e = \omega_n$). In this case $\Lambda = 1$ and in the absence of damping ($\eta = 0$) the response amplitude would go to infinity. The presence of damping reduces the response amplitude, but the maximum response amplitude will still occur at the natural frequency ($\Lambda = 1$). The MF increases as Λ approaches 1 and then decreases as Λ becomes greater than 1. Figure 4.4 shows some example MF curves for different amounts of damping. As the damping increases the peak response amplitude decreases until the system becomes over-damped and there is no peak. This peak is called **resonance** and systems that are over-damped do not show any response amplitudes greater than the static response amplitude. For over-damped systems there is no magnification, no resonance.

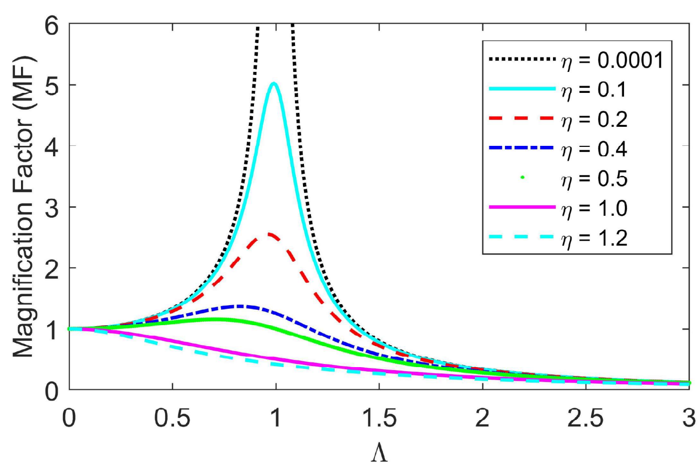


Figure 4.4: Example Dynamic Magnification Factor plots for different amounts of η (damping)

4.2 Added Mass and Hydrodynamic Damping

Moving through water is different than moving through air. When you push your flat palm through the air it does not feel like you are pushing on anything. The same motion feels very different when done under water. This is partly because water is more viscous than air (increased damping), but it is also because the water is denser than air and needs to be moved along with your palm. This motion requires the water surrounding your palm to be accelerated. The effect of this is to make your palm feel as if it had extra inertia (mass). This effect is known as **added mass**. This extra required force shows up in the equation of motion (EOM) as an addition to the mass of the object. The added mass represents the amount of fluid accelerated by the object. However, something to keep in mind is that the particles of fluid adjacent to the body will accelerate to varying degrees and the added mass value is a weighted integration of the entire fluid mass effected by the accelerating object. So, instead of $m\ddot{x} + b\dot{x} + cx = F_0 \sin \omega t$, the equation of motion becomes,

$$(a + m)\ddot{x} + b\dot{x} + cx = F_0 \sin \omega t.$$

The a stands for added mass (and now explains why naval architects have damping as b and stiffness as c, in contrast to the standard mechanical engineering expressions). Added mass depends primarily on the shape of the object, the type of motion (linear or rotation), and the direction of the motion. In this way, added mass differs from just *mass* since mass is a quantity independent of motion.

Hydrodynamic damping is related to the viscosity of the fluid (and hence the frictional drag), but when a free surface is involved the damping is dominated by the generation of waves. The larger the waves generated, the larger the hydrodynamic damping. This damping is proportional to the velocity of the direction as well. Both the added mass (a) and hydrodynamic damping coefficients (b) are a function of the frequency of oscillation.

Experimental Investigation in Sway

The concepts of damping and "added mass" are forces on an object moving in a fluid that are not explained by buoyancy or the mass of the object. To explore the concepts of added mass and hydrodynamic damping in a way that could be observed, we conducted an experiment on a U-shaped foam barge in the 120-ft towing tank in the USNA Hydromechanics Laboratory. Using the data collected, we were able to calculate the added mass and hydrodynamic damping coefficients and determined the dependence of the added mass and hydrodynamic damping on frequency. In this section we will go over the details, data, and analysis from this experiment to help explain the concepts of added mass and hydrodynamic damping.

The problem addressed in this experiment was a two-dimensional simplification of the problem of a ship moving in simple harmonic motion in a *calm* sea. The cross-section of the body had a semi-circular bottom with vertical sides, shown in the figure below. The instrumented model section had a length of 3 feet, a beam of 8 inches, and a draft of 5 inches. The displacement was about 15 pounds.

The model was attached to a scotch yoke that oscillated the model with pure sinusoidal sway motion. The only measurements taken were the sway **displacement** and the sway **force** over time.

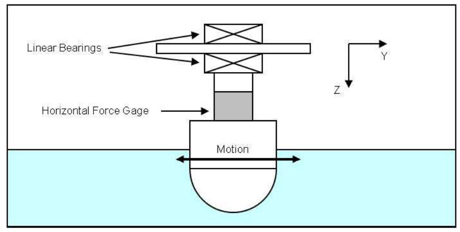


Figure 4.5: Cross-section of the "two-dimensional" ship model.

Experimental Procedure

To collect the data necessary to measure added mass and hydrodynamic damping, the following steps were taken:

1. We oscillated the model in sway in air and recorded the position and the force as a function of time
2. We oscillated the model in sway in water and recorded the position and the force as a function of time

Analysis Method

The damping force is the force component that is proportional to the velocity and the added mass force is the force component that is proportional to the acceleration. The equation of motion for the model is

$$(m + a_2)\ddot{x}_2 + b_2\dot{x}_2 + c_2 x_2 = F_0 \sin \omega_e t$$

where m is the mass of the model, a_2 is the added mass coefficient in sway, $\ddot{x}_2$ is the acceleration of the model in sway, b_2 is the damping coefficient in sway, $\dot{x}_2$ is the sway velocity, c_2 is the stiffness coefficient in sway, x_2 is the sway displacement, F_0 is the force amplitude in sway, and ω_e is the sway excitation frequency in radians/second. For sway motion there is no buoyancy (stiffness), so $c_2 = 0$. The solution to the equation of motion (i.e. the motion of the model in time) is of the form

$$x_2(t) = X_2 \sin(\omega_e t - \phi).$$

To solve for the motion and force amplitudes, the data collected was analyzed using a FFT to identify the actual oscillation frequency and amplitude at that frequency. For example, consider the sample in-air sway displacement plot below (Figure 4.6). The motion is fairly sinusoidal, although each oscillation varies slightly from the ones before it. However, performing an FFT on the data (see the Fourier Transform section in the **Irregular Waves Lab**), gives a pretty clear spike at a single amplitude (see Figure 4.7). We can then say that the sway amplitude of the data is 1.3 inches at an excitation frequency of 1.1 Hz. If we use that amplitude and frequency and create a sinusoidal function, we can compare that result with the actual data, see the plot below (Figure 4.8). In this case, the perfect sine wave matches the actual data fairly closely.

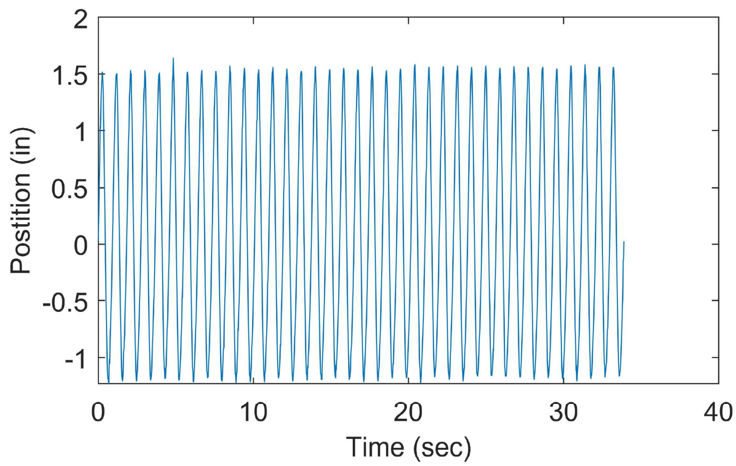


Figure 4.6: Sway Displacement in Air as a function of time

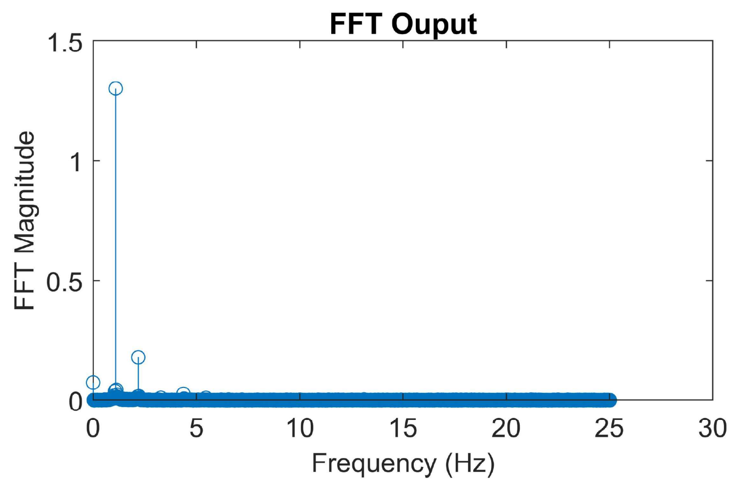


Figure 4.7: FFT of Sway Displacement in Air data

We can do the same for the force data - perform a FFT on the data and use the amplitude and frequency data to recreate the "pure" signal. The plots below (Figure 4.9) show the raw data signal and the FFT result. Compared to the sway displacement, the force FFT signal shows many more spikes. This means the raw data is not as good of a pure sinusoid and has higher frequency components. However, for this experiment, those higher frequency responses can be considered "noise" and we are only interested in the primary spike that occurs at the excitation frequency. Recreating the force as a function of time using the FFT result, we can compare that pure sine wave to the actual data, see Figure 4.10.

So, for the data we collected in this experiment, Figures 4.6 to 4.12, we found the values for X_0, F_0, and ω_e for the in-air and in-water experiments. Using these values, the known mass, the zero stiffness coefficient, and the equations for magnification (see the appendix in this chapter), we can solve for the added mass, a_2, and damping, b_2, in the air and again in the water. For the data shown, the in-air total mass is 1.74 slugs, the in-air damping is 0.06 lb-s/ft, the in-water total mass is 5.75 slugs, and the in-water hydrodynamic damping

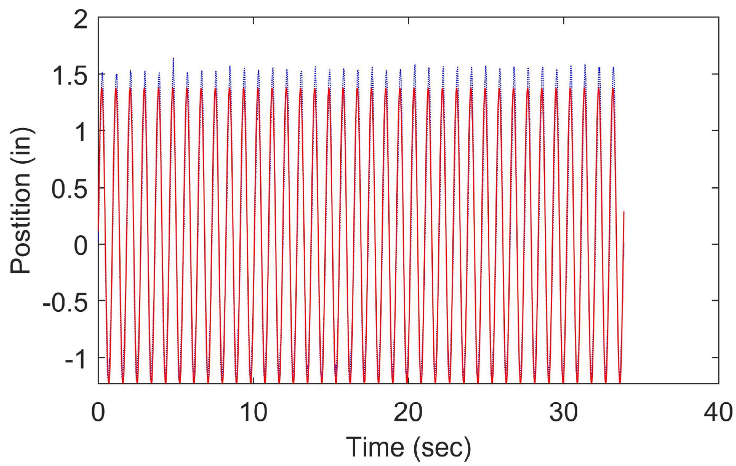


Figure 4.8: Data and Pure Sway Displacement in AIr

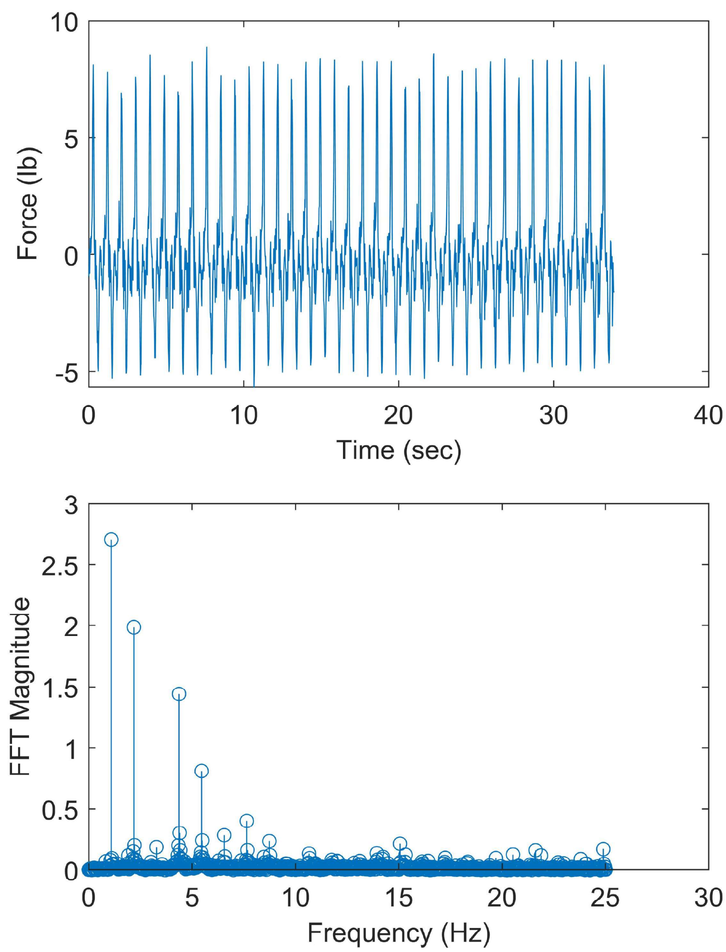


Figure 4.9: Sway Force as a function of time and FFT of Sway Force in Air data

is 6.68 lb-s/ft. The in-air added mass is almost negligible (air provides almost no added

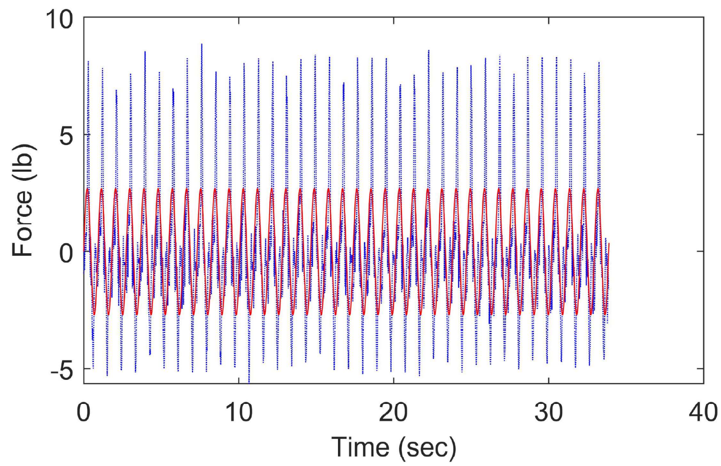


Figure 4.10: Data and Pure Sway Force in Air

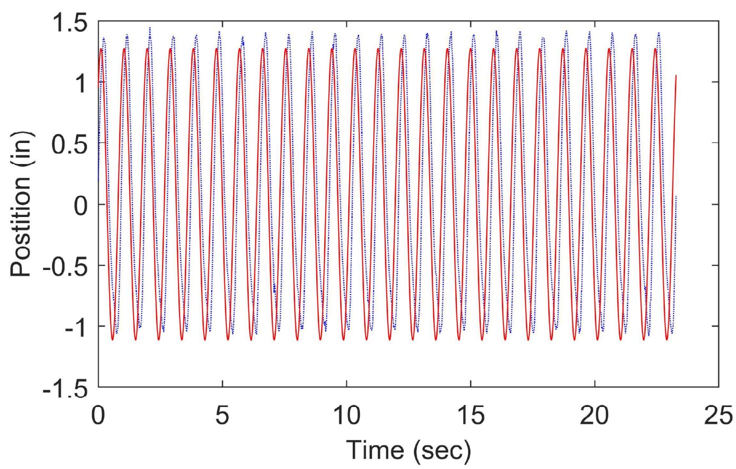


Figure 4.11: Sway Displacement Data for in Water Experiment

mass effect) meaning the mass in the equation of motion is really just equal to the mass of the model, while the added mass is very noticeable for the in-water experiment. There is a similar result for the damping coefficient, the hydrodynamic damping is significantly more than the aerodynamic damping.

By repeating this procedure at different excitation frequencies, we can see how added mass and damping vary with frequency of oscillation. Figure 4.13 shows the dependence of these coefficients on the frequency of oscillation. The added mass is close to zero in air and does not depend on the frequency. The same trends apply for the damping in air. In water, the added mass is frequency dependent and tends to decrease with increasing oscillation frequency. This means that at higher frequencies there is less of an effect from the added mass. The hydrodynamic damping in water also depends on frequency, but in this case the damping *increases* with increasing frequency. Therefore, the damping gets *more* significant at higher oscillation frequencies.

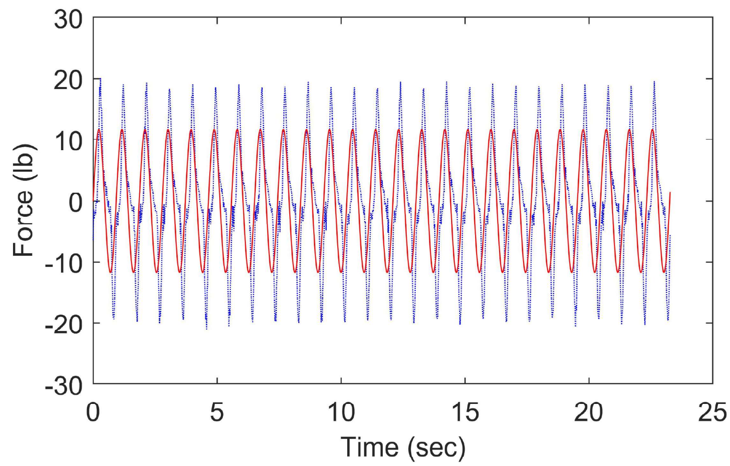


Figure 4.12: Sway Force Data for in Water Experiment

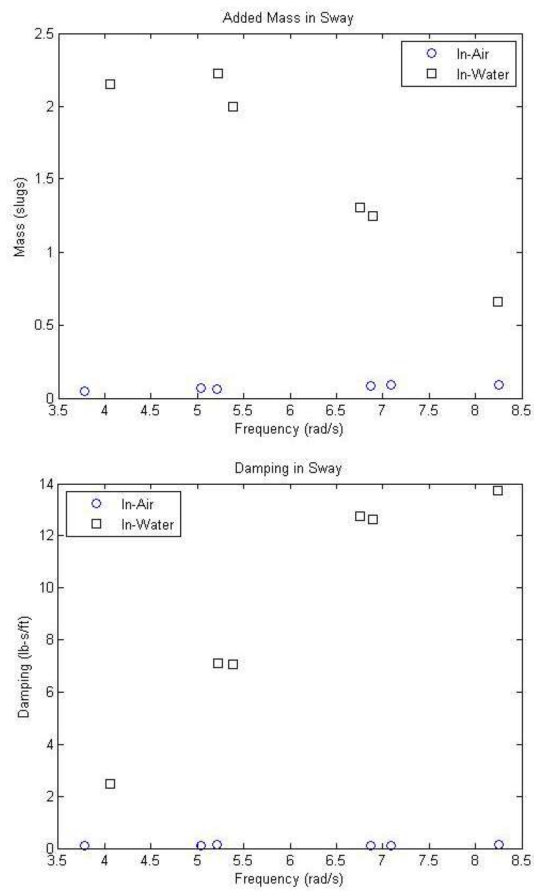


Figure 4.13: Added Mass and Damping of Sway Oscillation Model as a function of frequency

4.3 Ship Natural Frequencies

Each degree of freedom that has a restoring force has an associated natural frequency. So, for a ship, there is a natural frequency in heave, roll, and pitch. These natural frequencies depend on the mass and stiffness properties of the system.

To a first approximation, we will consider the heave and pitch motions to be uncoupled (i.e. *independent*). The natural frequency in heave is

$$\omega_{3n} = \sqrt{\frac{c_3}{m + a_3}} \tag{4.5}$$

and the natural frequency in pitch is

$$\omega_{5n} = \sqrt{\frac{c_5}{I_5 + a_5}} \tag{4.6}$$

We can actually reasonably estimate the added mass or inertia in heave and pitch, based on conventional ships, as equal to the actual mass or inertia. In other words,

$$a_3 \approx m$$

and

$$a_5 \approx I_5.$$

The natural frequency in roll can be similarly determined,

$$\omega_{4n} = \sqrt{\frac{c_4}{I_4 + a_4}}.$$

For roll, the estimated added inertia is about a quarter of the ship's roll moment of inertia, or

$$a_4 = 0.25 I_4.$$

However, the roll natural frequency of a ship is also strongly linked to the ship's metacentric height (as can be determined from the stiffness coefficient). So, the roll natural frequency can also be written as

$$\omega_{4n} = \sqrt{\frac{mg\bar{GM}}{1.25 I_4}}. \tag{4.7}$$

The roll damping increases with forward speed. The increase in damping results in a smaller maximum resonant peak, but also a slight reduction in the frequency at which the peak response will occur.

4.4 Ship Inertia - Pitch, Yaw, and Roll

The mass of a ship is determined by its total weight or displacement. The rotational inertia is determined by the distance of each weight from the combined center of gravity. The further the heaviest weights are from the CG, the larger the moment of inertia (the rotational inertia). If all the mass were located equidistant from the center of gravity the moment of inertia would be easy to calculate and would be equal to the total mass times the distance from the CG squared. Although the mass in a ship is never located equidistant from the center of gravity, we can find the representative distance the mass would need to be were the ship a sphere. This representative distance is the **radius of gyration**, k. If we have the radius of gyration, we can find the ship's moment of inertia,

$$I = mk^2.$$

There are three rotational degrees of freedom - roll, pitch, and yaw - and each have a subscript number associated with the direction. It turns out that for typical ship shapes, the radii of gyration have a relationship to the ship's geometry. So, in general,

$$\begin{aligned} k_4 &= 0.30\mathrm{B_{WL}} && \text{roll} \\ k_5 &= 0.25\mathrm{L_{PP}} && \text{pitch} \\ k_6 &= 0.25\mathrm{L_{PP}} && \text{yaw} \end{aligned}$$

4.5 Ship Transfer Function

To predict the ship motion in a set of regular waves, we need to have a way to predict the ship response as a function of the excitation amplitude and frequency. This is essentially the same as the dynamic magnification factor described in Section 4.1. Other names for this relationship include Frequency Response Function (FRF) and, for naval architects, Transfer Function. In all cases, the result can be represented as a plot with the ratio of ship response to excitation amplitude (X/ζ_0, where ζ_0 is the wave amplitude) on the vertical axis and the ratio excitation frequency to natural frequency ($\Lambda = \omega_e/\omega_n$) on the horizontal axis. Figure 4.14 shows a typical transfer function in roll.

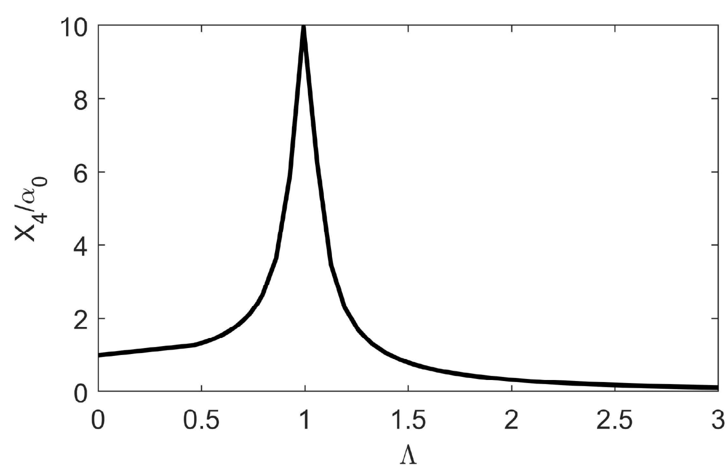


Figure 4.14: Typical Ship Transfer Function for Roll

The response depends on the ship mass, added mass, hydrodynamic damping, buoyancy, and excitation frequency in the direction of motion. If the ship were to not be moving (zero velocity), the excitation frequency would match the wave frequency. However, when the ship has velocity the excitation frequency depends on the ship speed, the wave frequency, and the relative direction of the ship and waves. This resulting excitation frequency is called the **encounter frequency** since it is the frequency at which the ship encounters the waves.

Encounter Frequency

Assuming the waves and ship are on a straight course, the frequency with which the ship will encounter a wave crest depends on the distance between the waves crests (λ - wavelength), the speed of the waves (c - which depends on the wavelength), the speed of the ship (U), and the relative angle between the ship heading and the wave heading (μ), see Figure 4.15. The encounter period is thus the distance traveled (λ) divided by the speed the ship encounters the waves ($c - U\cos\mu$). The encounter frequency is

$$\omega_e = \frac{2\pi}{T_e} = \frac{2\pi}{\lambda}(c - U\cos\mu).$$

Manipulating the equation a bit and substituting in the relationships between wavelength, wave speed, and wave frequency, the encounter frequency can be written as

$$\omega_e = \omega - \frac{\omega^2 U}{g}\cos\mu. \tag{4.8}$$

The heading period (the time between crests to pass) is given by

$$T_e = \frac{\lambda}{c - U\cos\mu}.$$

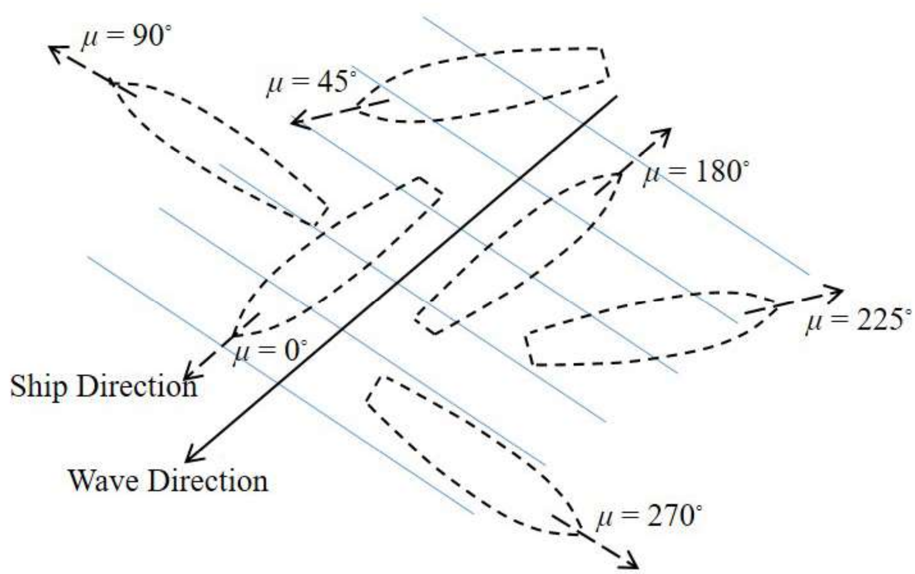


Figure 4.15: Relative ship and wave heading angle, μ

The heading angle determines the "type" of seas the ship experiences. For example, $\mu = 0^\circ$ means the ship and waves are heading in the same direction and the ship is experiencing **following seas**. When the ship is traveling directly at the oncoming waves ($\mu = 180°$), the ship is experiencing **head seas**. Quartering waves on the ship's starboard side are when μ is between 0° and 90° and quartering waves on the ship's port side are when μ is between 270° and 360°. Starboard beam seas are when $\mu = 90°$ and port beam seas are when $\mu = 270°$. When μ is between 90° and 180° the ship is experiencing bow waves on the starboard side and bow waves on the port side are when μ is between 180° and 270°.

To calculate the transfer function computationally (as opposed to experimentally), the hydrodynamic coefficients need to be known (added mass, mass, damping, and buoyancy).

Ship Transfer Function Example

Consider the following roll characteristics for a particular ship:

- mass = 4,898,300 kg
- $\mathrm{L_{PP}}$ = 86.5 m
- $k_4 = 0.25\mathrm{L_{PP}}$
- $a_4 = 0.25\mathrm{I}_4$
- c_4 = 715,000,000 N/rad
- b_4 = 143,000,000 N/(rad/s)

We would like to find the transfer function (or frequency response function) for this ship in roll. First we need to solve for all the terms in the EOM for roll: $(a_4 + I_4)$, ω_n, and η.

- To solve for the combined moment of inertia and added inertia in roll we need to find the ship's roll inertia, I_4.

$$I_4 = m\mathrm{k}_4^2 = (4898300) \cdot (0.25 \cdot 86.5) = 2290644073 \text{ kg} \cdot \text{m}^2$$

  We can then find the combined moment of inertia and added inertia using the relationships provided above:

$$a_4 + I_4 = 1.25 \cdot I_4 = 1.25(2290644073) = 2863305092 \text{ kg} \cdot \text{m}^2.$$

- To use the equation for the magnification factor (which gives us the ship transfer function), we also need to find the ship's natural frequency, ω_n

$$\omega_n = \sqrt{\frac{c_4}{a_4 + I_4}} = \sqrt{\frac{715000000}{2863305092}} = 0.50 \text{ rad/sec}$$

- and the damping factor, η

$$\eta = \frac{b_4}{2(a_4 + I_4)\omega_n} = \frac{143000000}{2 \cdot 2863305092 \cdot 0.50} = 0.05$$

- Now we can plug these values into the ship transfer function equation (i.e. the magnification equation) using $\Lambda = \omega_e/\omega_n$:

$$\text{ship transfer function} = \frac{1}{\sqrt{(1-\Lambda^2)^2 + (2\eta\Lambda)^2}} = \frac{1}{\sqrt{(1-\Lambda^2)^2 + (2 \cdot 0.05\Lambda)^2}}$$

- Let's consider some points of interest. What is the transfer function when the encounter frequency equals the natural frequency (i.e. $\omega_e = \omega_n$)? Plugging in $\Lambda = 1$, the transfer function is 10. What does this mean? It means that the roll response will be 10 times as large as the excitation magnitude. What about when the excitation frequency is close to zero? In this case, $\Lambda \approx 0$ and the ship transfer function equals 1. This means for very low excitation frequencies the roll response is the same as the excitation magnitude. Lastly, let's consider a very high excitation frequency, say $\Lambda = 10$. In this case the ship transfer function equals 0.01. This means the ship response is $1/100^{\text{th}}$ the magnitude of the excitation.

- To understand the ship's roll response over a range of frequencies, the best thing is to create a plot. Figure 4.16 shows the roll transfer function for this problem.

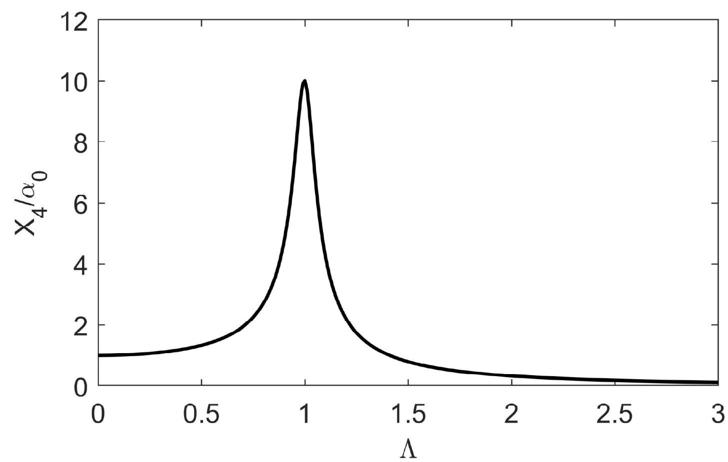


Figure 4.16: Example ship transfer function in roll

There are two ways to determine the transfer function for a ship - experimentally or theoretically. There are distinct limitations to both methods. We will be exploring both methods in this class.

Consider a wave elevation time history at point O that is of the form $\zeta = \zeta_0 \sin(\omega_e t)$ with the resulting ship motion described by $x_i = X_{i0} \sin(\omega_e t + \delta_i)$. The motion amplitude ($X_{i0}$) and phase ($\delta_i$) are functions of the ship speed (U), the ship heading relative to the waves (μ), and the encounter frequency (ω_e). The amplitudes are assumed to be proportional to the wave amplitude ($X_{i0} = \text{constant} \times \zeta_0$). Therefore, we typically express the motion amplitudes in non-dimensional form:

$\frac{X_{30}}{\zeta_0}$ heave transfer function

$\frac{X_{40}}{k\zeta_0}$ roll transfer function

$\frac{X_{50}}{k\zeta_0}$ pitch transfer function

where $k\zeta_0$ is the wave slope amplitude. Graphs of the resulting non-dimensional amplitudes are plotted as a function of the encounter frequency. In essence, a transfer function gives the proportion of wave amplitude or wave slope amplitude "transferred" by the ship "system" into ship motions.

The phase angle, δ_i gives the phase relationship between the motion and the wave, the maximum positive response occurs $+\delta_i/\omega_e$ seconds **before** the maximum wave depression. For $-\delta_i$ the motion lags the wave depression.

Let's start by examining the magnitude of a ship response operating in head seas ($\mu = 180°$). Consider very long waves (low frequency). When ω_e is very low, the dynamic effects associ-

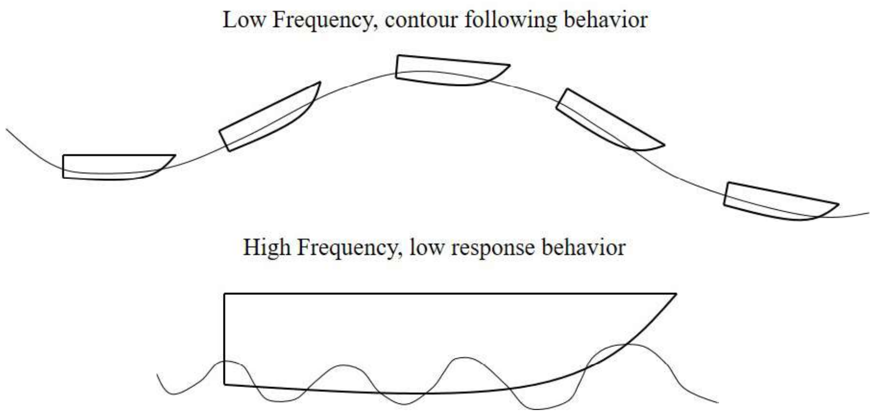


Figure 4.17: Relationship between wave amplitude and ship response

ated with added mass and damping are virtually negligible. Therefore, the excitations and motion responses experienced by the ship are almost entirely attributed to the buoyancy changes as the wave passes by - the maximum pitch occurs at the wave nodes (inflections) and the maximum heave response occurs at the crests and troughs. The result is motion amplitudes on the same order as the wave amplitudes. If you think of the wave length as much longer than the ship length, the ship will always be aligned with the wave surface (top of Figure 4.17). In this scenario, the transfer function is equal to one (for example, in heave $\frac{X_{30}}{\zeta_0} = 1$ at low encounter frequencies).

Now consider very short waves (high frequency). When ω_e is very high, the ship responses are reduced because the short waves do not excite the ship very much. In this scenario, there are many wavelengths along the length of the ship and the ship (bottom of Figure 4.17), in a way, can't decide if it wants to be up or down so it does neither. As the ship speed increases, the wavelengths which do not excite the ship are encountered over a wider range of frequencies. When operating in head seas, increasing the ship speed has the effect of increasing the encounter frequency for a given wave.

If the range of frequencies encountered includes the natural frequency of heave and/or pitch, the response may exhibit a resonant peak (see the Dynamics Review section). However, heave and pitch are both heavily damped so any peaks at resonance are never very pronounced.

What about the *phases* of the ship's response when operating in head seas? In the long waves (low ω_e) the heave motion is synchronized with the wave motions (maximum response at the crest and trough) while the pitch has a phase of 90°, so the maximum (positive) pitch occurs at the wave node. For short waves, there is little response so the phase is not very relevant. However, in theory the waves should be *out of phase* (that is, $\delta = 180°$) with the ship motions (for example, the maximum positive heave motion occurs at the wave trough and the maximum negative heave motion occurs at the wave crest).

Next, let's consider following waves. As in head seas, for long waves ($\omega_e \to 0$) the transfer function is 1 (for heave and pitch). The heave phase is close to zero over most of the encounter frequencies (i.e. the motion is nearly synchronized with the wave). The pitch phase is $-270°$ or $+90°$ over most of the range of encounter frequencies. In following waves the maximum bow up motion now *leads* the trough of the wave. In oblique waves the motion is no longer confined to the vertical plane, we can now have roll, sway, and yaw motions. For *long* oblique waves the ship appears to be crawling over a succession of long, shallow hills. From the ship's perspective, the wave length appears longer. The "effective wave length" depends on the heading angle and is, thus,

$$\frac{\lambda}{\cos\mu}.$$

The "effective wave slope" is $k\zeta_0 \cos\mu$. For headings forward of the beam ($90° < \mu < 180°$), the responses are broadly similar to the head seas responses.

4.6 Model Testing in Regular Waves

For an experimentally determined transfer function - whether in roll, pitch, or heave - the results are only relevant to the specific ship model tested at the tested speed. Therefore, each new geometry must have a new model built and tested at relevant speeds to find the transfer functions. We can create a set of regular waves to send the ship model through and measure the results. For a straight, long tow tank (like the ones at USNA) we can only test in head seas with the model in motion. For a stationary model, we can also test beam seas. The model tests in regular waves are concerned with the experimental determination of the motion transfer functions. We will determine the motion amplitudes experienced for a variety of different wavelengths or frequencies.

Model testing in regular head seas is an important part of determination of full-scale ship responses. The usual procedure is to test the model at a variety of speeds covering the operating speed range of the vessel. At each speed the model is tested in regular waves with a range of frequencies (wavelengths) such that the expected wave frequencies in a typical seaway are covered. For each test the heave and pitch response are recorded along with information about the waves generated for each run.

The usual form of output for motions in regular waves is a plot of the *average response amplitude normalized by the average wave amplitude* against the *encounter frequency*. When all of the test runs have been plotted on the same graph, a curve is faired through the data. This faired curve represents the Transfer Function of the response for the specified motion of the vessel at that specific speed. Squaring the transfer function gives what is known as the *Response Amplitude Operator (RAO)*. RAO's are used in the process of determining the ship's response in an irregular sea (as we will see later).

Typically we keep the *wave slope constant* while varying the wavelength. For these tests, the wave steepness must be kept small to ensure the responses are in the linear range (more on this requirement in the section on Strip Theory). To acquire ship phase information (with respect to the waves), it is necessary to measure the incident waves using a wave probe. The measurement must not be at a location where the model is influencing the waves and any wave generated by a probe must not influence the model.

Phase Shift for Wave Time History

In cases where we can't measure the waves at a location parallel to the model's CG, we need to introduce a time shift to the wave time history data. Consider the theoretical condition shown in Figure 4.18[1]. If the probe is located at position x_{1P} ahead of the CG and x_{2P} to starboard, the probe will record the waves x_P after (or before) they have passed the CG.

$$x_P = x_{1P}\cos\mu - x_{2P}\sin\mu.$$

If the waves are overtaking the model, the speed the waves encounter the ship is

$$c - U\cos\mu$$

a wave trough recorded at the probe would be at the CG at time

$$t_P = \frac{x_P}{c - U\cos\mu}.$$

So, the phase lead measured with reference to the waves recorded at the wave probe should

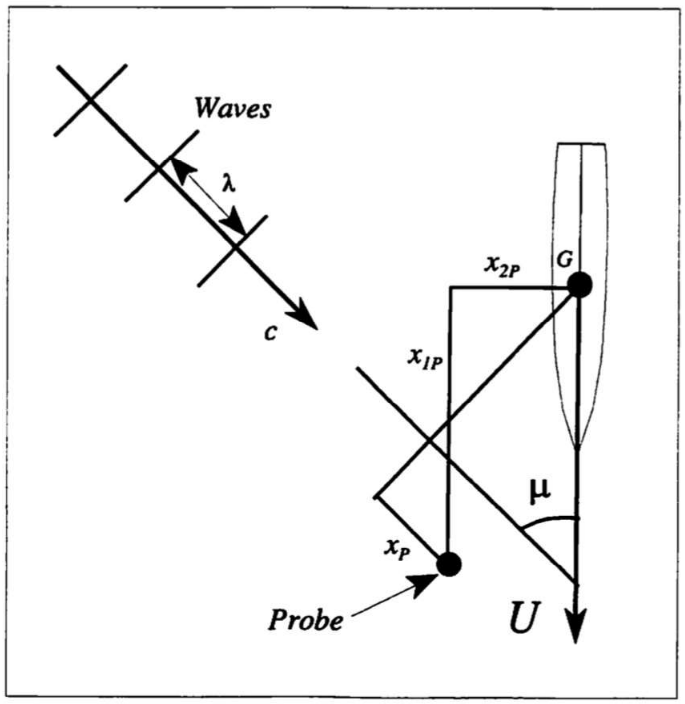


Figure 4.18: Time Shift for Wave Probe Data (Figure 10.15 from reference 2)

be *reduced* by the amount

$$\delta_P = \omega_e t_P = \frac{\omega_e(x_{1P}\cos\mu - x_{2P}\sin\mu)}{c - U\cos\mu}$$

For a standard straight tow tank, the only waves that can be tested are in head seas where $\mu = 180°$. In this case $\cos 180° = -1$ and $\sin 180° = 0$ so

$$\delta_P = \frac{\omega_e(-x_{1P})}{c + U}$$

and we will *increase* the phase by $(\omega_e x_{1P})/(c + U)$ for the waves in the experiment if the probe is located forward of the model.

[1]taken from reference 2

4.6.1 Experimental Model Ship Testing

Let's review the basics of resistance testing and then add the complexities introduced by the dynamic motions involved in testing in waves. Consider a full-scale boat with the following characteristics: You wish to build and test in waves a 5.35 scale model. We need to determine

Length Overall, ft	42.77
Max Beam, ft	13.1
Displacement, lbs	35,000
LCG (fwd of transom), ft	15.09
L_{PP}, ft	38.4
Pitch gyradius, ft	9.6
Speed, kts	40

Table 4.1: Model Testing Ship Characteristics

the **model length** (in feet), **model beam** (in feet), **displacement** (in pounds), **pitch gyradius** (k_5, in inches), and the **model speed** (in ft/s).

Remember from Resistance and Propulsion that there are three types of experimental "similarities" - *Geometric Similarity*, *Kinematic Similarity*, and *Dynamic Similarity*. To achieve **geometric similarity** we need to make sure all length measurements (for the model and the waves) have the same scale ratio between the full-scale and model. Since in this example the scale ratio is 5.35, the relationship between the model and full-scale geometric lengths is

$$R = \frac{L_S}{L_M} = 5.35$$
$$L_M = \frac{L_S}{R} = 8.0.$$

Therefore, the model must have the following geometric characteristics: The wave heights

	Ship	**Model**
Length Overall, ft	42.77	8.0
Max Beam, ft	13.1	2.45
LCG (fwd of transom), ft	15.09	2.82
L_{PP}, ft	38.4	7.18
Pitch gyradius, ft	9.6	1.79

Table 4.2: Model Testing Model Characteristics

and wavelengths scale in a similar way,

$$H_M = \frac{H_S}{R}$$
$$\lambda_M = \frac{\lambda_S}{R}.$$

Area measurements scale by R^2 and mass (or displacement) scales by R^3. The area is two lengths multiplied together, so if each length is scaled by R then the area is scaled by $R \cdot R = R^2$. The mass scales by R^3 because it is proportional to volume, which is *three* lengths multiplied together ($R \cdot R \cdot R = R^3$). The other concern between model and ship is that, generally, a full-scale ship is floating in salt water while a model is floating in fresh water. Therefore, we need to account for this difference in scaled displaced volume as well,

$$m_M = \frac{\rho_M m_S}{\rho_S R^3}$$

So, the moments of inertia scale by R^5! Remember, the moment of inertia can be expressed as mass times the gyradius squared ($m \cdot k^2$). So, if the mass is squared by R^3, when we multiply by k^2 we get another R^2 in the relationship. So,

$$I_M = \frac{\rho_M I_S}{\rho_S R^5}.$$

To achieve **kinematic similarity** we need to match the Reynold's numbers of the full-scale and model ships. To achieve **Dynamic Similarity** we need to match the Froude numbers of the full-scale and model ships. Due to the physical properties of our Earth, we can't simultaneously satisfy both the Reynold's and Froude scaling requirements. For tank testing we neglect the Reynold's number scaling (friction matching) and stimulate the flow as necessary for turbulent flow. We do match the Froude scaling. The Froude number is

$$\mathrm{Fr} = \frac{U}{\sqrt{gL}}$$

So, to satisfy dynamic similarity, the model speed must be equal to

$$U_M = U_S \sqrt{\frac{L_M}{L_S}}$$
$$U_M = \frac{U_S}{\sqrt{R}} = \frac{40 \cdot 1.688}{\sqrt{5.35}}$$

So, for our model the speed must be 29.2 ft/s. The wave frequency (or wave period) is related to the wave velocity. We can relate the frequencies of the full-scale and model-scale using the wavelengths and the relationships between frequency and wavelength shown in

Chapter 3. So we find the model wave frequency from

$$\omega = \sqrt{\frac{2\pi g}{\lambda}}$$
$$\omega_S = \sqrt{\frac{2\pi g}{\lambda_S}}$$
$$\omega_M = \sqrt{\frac{2\pi g}{\lambda_M}}$$
$$\omega_M = \sqrt{\frac{2\pi g}{(\lambda_S/R)}}$$
$$\omega_M = \sqrt{\frac{2\pi g}{\lambda_S}}\sqrt{R}$$
$$\omega_M = \omega_S\sqrt{R}$$

Using a similar process, the model wave period is

$$T_M = \frac{T_S}{\sqrt{R}}$$

Froude scaling gives us *velocity* scaling and geometric scaling gives us *position* scaling. What about *acceleration* scaling? Let's consider heave. Position scaling gives us

$$x_{3M} = \frac{x_{3S}}{R}$$

Velocity scaling gives us

$$\dot{x}_{3M} = \frac{\dot{x}_{3S}}{\sqrt{R}}$$

Why is there a square-root for scaling velocity when there isn't for position? This is because while position scales by R, *time* scales by $\sqrt{R}$ due to the requirements of Froude scaling. Since velocity has units of *length/time*, the scaling is $R/\sqrt{R}$ or $\sqrt{R}$. So, let's consider acceleration. The units of acceleration are *length/time*^{2}. Plugging in the length and time scaling relationships we get $R/(\sqrt{R})^2 = R/R = 1$. So, there is no scaling factor for acceleration! What you measure is what you get,

$$\ddot{x}_{3M} = \ddot{x}_{3S}$$

To prepare a model to be tested in waves, we need to have geometric scaling for the hull and waves, the model needs to ballasted to the scaled displacement and center of gravity location (correct calm water trim), and we need to have the correct scaled moments of inertia in pitch. This last point is different than your previous experience with resistance testing. To have the correct moment of inertia the *distribution* of the weight in the model must match the full-scale ship, not just the *average* of the weights. To get the moments of inertia correct in our model we need to **dynamically ballast the model**.

Dynamic Ballasting

When traditional model tests are conducted in still water for resistance and speed related sinkage and trim attitude changes, the geometrically similar model must be ballasted to the scaled waterline of the subject full-scale prototype. This means that both displacement, Δ, and longitudinal position of the center of gravity with respect to amidships, LCG, must be scaled geometrically; i.e.

$$\Delta_s = \Delta_m \times \frac{\rho_s}{\rho_m} \times \frac{R^3}{2240}$$

and

$$LCG_s = LCG_m \times R$$

where	Δ_s	= ship displacement in long tons
	Δ_m	= model displacement in pounds
	ρ_s	= density of water in which the ship floats
	ρ_m	= density of water in which the model floats
	R	= linear scale ratio

As long as the displacement and LCG scale as indicated, no other requirement must be met. The situation can be considered to be a steady state.

However, when models are to be tested in waves such that they oscillate in one or more of the six degrees of freedom possible for a rigid body, the added specification of weight distribution about the center of gravity becomes necessary. Specifically, when ship models are tested in long crested head (or following) sea conditions - as is typical in long, narrow towing tanks - the displacement, the LCG, and a quantitative measure of the distribution of weight (longitudinally) about the center must be modeled. The measure of longitudinal weight distribution about the center of gravity is the *longitudinal gyradius,* k_5 *(subscript* 5 *referring to pitch motion)*, and is defined and scaled as follows:

$$I_{55} = \int (x_{B1}^2 + x_{B3}^2)\, dm \qquad \text{mass moment of inertia about the } \mathrm{x}_{\mathrm{B2}} \text{ axis}$$

However, it is usually more practical to find the moment of inertia using:

$$I_{55} = mk_5^2$$
$$k_{5s} = k_{5m} \times R$$

Traditionally, a value of k_5 equal to about 25% of the length between perpendiculars, L_{PP}, is assumed for ships. Hence, models are ballasted accordingly. Ship models are ballasted to the correct displacement, LCG, and k_5, by the judicious placement of ballast weights within the test model.

When the weights are moved symmetrically away from the center of gravity of a ship, the gyradius of that ship will increase. No change in displacement or change in the position of the center of gravity will result from such a move. Increasing a ship's longitudinal gyradius should affect the natural pitching period and the magnitude of pitch response for a given wave excitation. Because of the strong coupling between pitch and heave motions for ships, it is likely that the heave motion will also be affected.

We will be learning two methods for measuring the pitch gyradius for ship models.

Knife Edge Method The model is hung as shown in Figure 4.19 so that it is supported by an installed transverse knife edge which is located at some known longitudinal position (e.g., Station 1) on the model. The pitch gyradius can then be computed from

$$k_5 = \sqrt{\frac{12g}{4\pi^2}T^2a - a^2}$$

where T = swing period in seconds
a = distance from the knife edge to the model center of gravity (inches)

Experience has shown that timing 50 cycles of model oscillations and then dividing the time by 50 provides adequate precision for the calculation of the swing period. It should be noted that the Knife Edge Method is really only practical for small models whose ballasted weights are securely fastened.

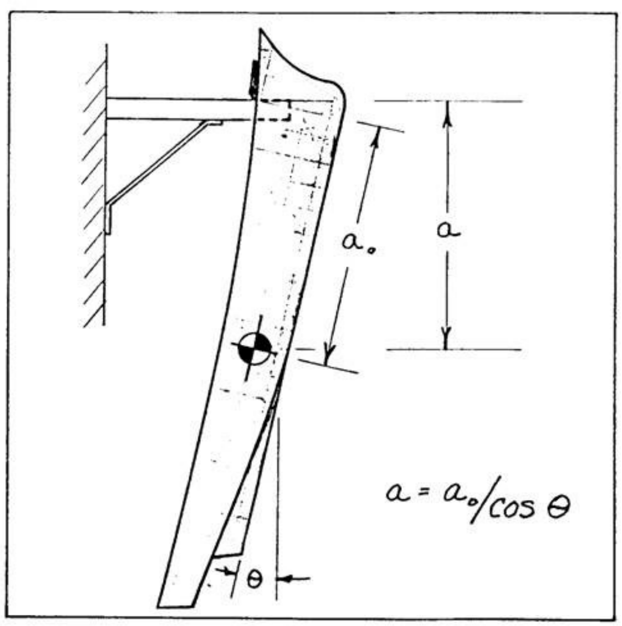


Figure 4.19: Knife Edge Method

Bifilar Suspension Method This method involves suspending the model horizontally from two eyes equidistant from the model center of gravity and at the same height above the model baseline as shown (Figure 4.20). The model is forced to oscillate horizontally about a vertical axis through the center of gravity. Thus the oscillation is in fact in YAW, not PITCH, and the resulting gyradius is k_6 vice k_5. For "normal ship forms" we tacitly *assume* that $k_6 = k_5$. This assumption loses validity as L/B decreases. Nonetheless, k_6 can be computed from

$$k_6 = \frac{T x_R}{2\pi}\sqrt{\frac{g}{h}} \qquad \text{ft, Eqn. 10.29, p.197 in Lloyd}$$

where $2x_R =$ distance between support wires (ft)
$T =$ oscillation period (sec)
$h =$ length of the support wires (ft)

This method is much more tractable for large models.

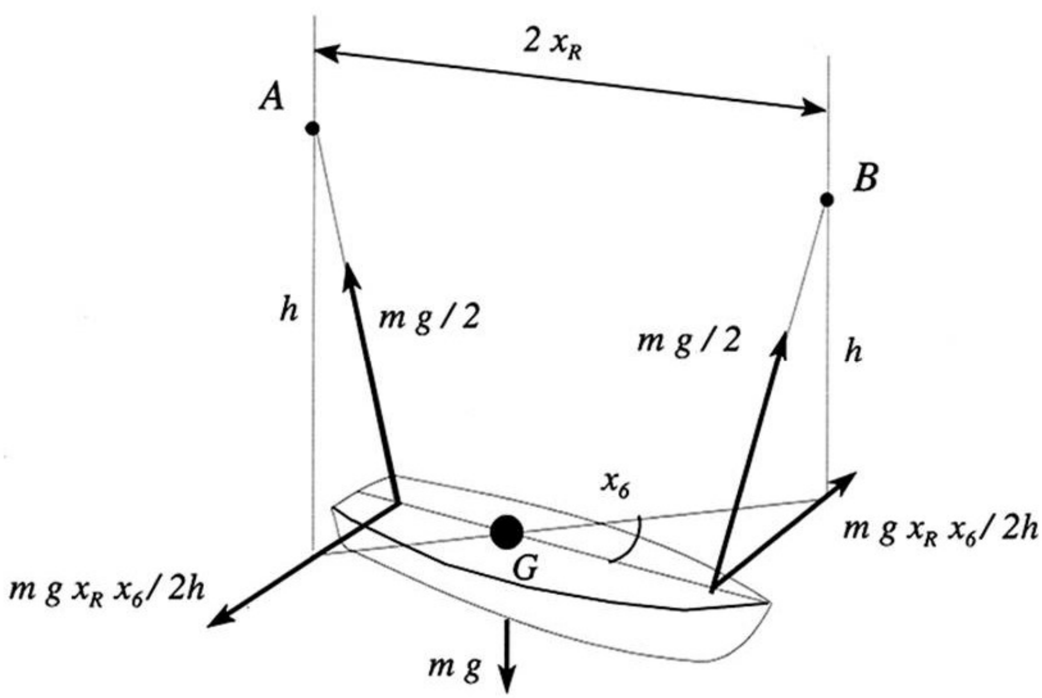


Figure 4.20: Bifilar Suspension Method (Lloyd, 1999)

Lamboley Method Developed by Gilbert Lamboley, this techniques involves swinging the model in pitch from pivots a know distance apart. The resultant period is a function of the distance from the pivots to the model center of gravity and the pitch gyradius. By employing two pivot heights and measuring periods for each, a simultaneous equation can be created to solve for both the distance to the CG and the pitch gyradius of the model. If the weight of the pivot gear is significant relative to the weight of the model, the pitch moment of inertia of the gear should be measured independently and accounted for in the final calculations.

$$T_1 = 2\pi\sqrt{\frac{d^2 + k_5^2}{gd}}$$

$$T_2 = 2\pi\sqrt{\frac{(d-x)^2 + k_5^2}{g(d-x)}}$$

where T is the swing period in seconds, d is the vertical distance from the pivot to the model CG, x is the vertical distance between the pivots, and k_5 is the pitch gyradius. Figure 4.21 shows how the rig is configured. If we solve for the intermediate quantity $c = g/(4\pi^2 x)$, then

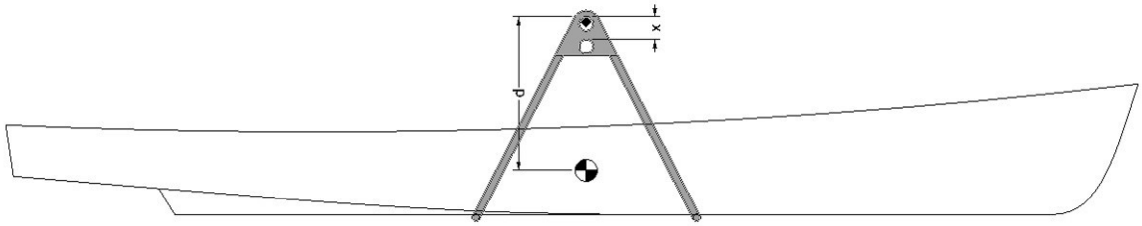


Figure 4.21: Set-up for Lamboley Method

$$d = \frac{x(cT_2^2 + 1)}{c(T_2^2 - T_1^2) + 2}$$

and

$$k_5 = \sqrt{(dxcT_1^2) - d^2}$$

In all of these methods, large amplitudes of motion are not necessary - there is very little damping due to air. The oscillation periods measured in either method bear no direct relationship to the model's natural period in pitch in water.

4.6.2 Summary of Planning Seakeeping Experiments

When traditional model tests are conducted in still water for resistance and speed-related sinkage and trim attitude changes, the geometrically similar model must be ballasted to the scaled waterline of the subject full-scale prototype. This means that both the displacement (Δ) and the longitudinal position of the center of gravity (LCG) must be scaled geometrically,

$$\Delta_S = \Delta_M \times \frac{\rho_S}{\rho_M} \times \frac{R^3}{2240}$$

$$LCG_S = LCG_M \times R$$

As long as the displacement and LCG are scaled as indicated, no other ballasting requirements must be met. The situation can be considered as a steady state test.

When models are tested in waves such that the model oscillates in one or more of the six degrees of freedom, the added specification of weight distribution about the center of gravity becomes necessary. Specifically, when ship models are tested in long crested head (or following) sea conditions - as is typical in long, narrow towing tanks - the displacement, LCG, and a quantitative measure of the distribution of weight (longitundinally) about the center must be modeled. The measure of longitudinal weight distribution about the CG is the *longitudinal gyradius*, k_5 (the subscript 5 referring to pitch), and is defined and scaled below. The mass moment of inertia about the x_B2 axis is

$$I_5 = \int (x_{B1}^2 + x_{B3}^2)\mathrm{d}m$$

However, it is usually more practical to find the moment of inertia using

$$\begin{aligned} I_5 &= mk_5^2 \\ k_{5S} &= k_{5M} \times R \end{aligned}$$

where k_5 is the pitch gyradius (the S subscript refers to the ship gyradius and the M subscript refers to the model gyradius). Traditionally, a value of k_5 equal to about 25% of the $\mathrm{L_{PP}}$ is assumed for ships. Hence, without specific mass moment of inertia information, models are ballasted to $k_5 = 25\%$ of $\mathrm{L_{PP}}$. Ship models are ballasted to the correct displacement, LCG, and k_5 by the judicious placement of ballast weights within the test model.

When the weights are moved symmetrically away from the center of gravity of a ship, the gyradius of that ship will increase. No change in displacement or change in the position of the center of gravity will result from such a move. Increasing a ship's longitudinal gyradius should affect the natural pitching period and the magnitude of the pitch response for a given wave excitation. Because of the strong coupling between pitch and heave motions for ships, it is likely that the heave motion will also be affected.

4.7 Strip Theory

The linearized equations of motion for the six degrees of freedom are as follows:

$$\begin{aligned}
(m + a_{11})\ddot{x}_1 + b_{11}\dot{x}_1 &= F_{w10}\sin(\omega_e t + \gamma_1) && \textbf{Surge} \\
(m + a_{22})\ddot{x}_2 + b_{22}\dot{x}_2 + a_{24}\ddot{x}_4 + a_{26}\ddot{x}_6 + b_{26}\dot{x}_6 + c_{26} &= F_{w20}\sin(\omega_e t + \gamma_2) && \textbf{Sway} \\
(m + a_{33})\ddot{x}_3 + b_{33}\dot{x}_3 + c_{33}x_3 + a_{35}\ddot{x}_5 + b_{35}\dot{x}_5 + c_{35}x_5 &= F_{w30}\sin(\omega_e t + \gamma_3) && \textbf{Heave} \\
a_{42}\ddot{x}_2 + b_{42}\dot{x}_2 + (I_4 + a_{44})\ddot{x}_4 + b_{44}\dot{x}_4 + c_{44}x_4 + a_{46}\ddot{x}_6 + b_{46}\dot{x}_6 + c_{46}x_6 &= F_{w40}\sin(\omega_e t + \gamma_4) && \textbf{Roll} \\
a_{53}\ddot{x}_3 + b_{53}\dot{x}_5 + c_{53}x_3 + (I_5 + a_{55})\ddot{x}_5 + b_{55}\dot{x}_5 + c_{55}x_5 &= F_{w50}\sin(\omega_e t + \gamma_5) && \textbf{Pitch} \\
a_{62}\ddot{x}_2 + b_{62}\dot{x}_2 + a_{64}\ddot{x}_4 + b_{64}\dot{x}_4 + (I_6 + a_{66})\ddot{x}_6 + b_{66}\dot{x}_6 + c_{66}x_6 &= F_{w60}\sin(\omega_e t + \gamma_6) && \textbf{Yaw}
\end{aligned}$$

In these equations, the coefficients have two subscripts - one refers to the direction of motion and the other refers to the direction of force. For example, c_{35} in the heave equation refers to the buoyancy force in the heave direction (3) due to a change in position in the pitch direction (5). In this example, if the ship pitches bow down, the ship will experience a pitching restoring force (trying to return the ship to bow up). The ship will *also* experience a general upwards force on the ship due to the bow pushing down into the water. Therefore, the ship will experience a heave force due to a pitch motion. For coefficients with both subscripts being the same number (for example, c_{55}) that is the force and motion in the same direction (so c_{55} refers to the restoring force in pitch due to pitch displacement).

The six degrees of freedom have the following sign conventions:

6DOF

x_1	surge	x_4	roll
x_2	sway	x_5	pitch
x_3	heave	x_6	yaw

Motions for the ship measured at the ship's center of gravity.

Solving the linearized equations of motion requires evaluation of the coefficients and the excitation amplitudes and phases. Considerable effort has therefore been devoted to developing theoretical methods of determining the coefficients and excitations to allow ship motions to be calculated without recourse to experiment. Strip theory is a method for determining the coefficients and excitations theoretically to allow ship motions to be calculated.

There are limitations concerning what assumptions <u>must</u> be made to use strip theory. The basic principle behind strip theory is that the hydrodynamic properties of a vessel (that is added mass, damping, and stiffness) may be predicted by dividing the vessel into a series for two-dimensional transverse strips, for which these properties may be computed. The global hydrodynamic values for the complete hull are then computed by integrating the two-dimensional values of the strips over the length of the ship. Linear strip theory assumes the vessel's motions are linear and harmonic, in which case the response of the vessel in both pitch and heave, for a given wave frequency and speed, will be proportional to the wave amplitude. and slope, respectively.

The basic assumptions (as stated in reference 2) required for linear strip theory are:

1. The fluid is inviscid, that is, viscous damping is ignored (although, the damping factor which the user enters in Maxsurf Motions for roll should include viscous roll damping, which is the primary source of damping for roll).

2. The ship is slender (i.e. the length is much greater than the beam or the draft, and the beam is much less than the wave length).

3. The hull is rigid so that no flexure of the structure occurs.

4. The speed is moderate so there is no appreciable planing lift.

5. The motions are small (or at least linear with wave amplitude).

6. The ship hull sections are wall-sided.

7. The water depth is much greater than the wave length so that deep water wave approximations may be applied.

8. The presence of the hull has no effect on the waves (Froude-Krilov hypothesis).

This theory is called **Strip Theory** because it represents the 3D underwater hull form by a series of 2D slices or strips. Each strip has associated local hydrodynamic properties (added mass, damping, and stiffness) which contribute to the coefficients for the complete hull in the equations of motion. Similarly the wave excitations experienced by the hull are composed of contributions from all of the strips.

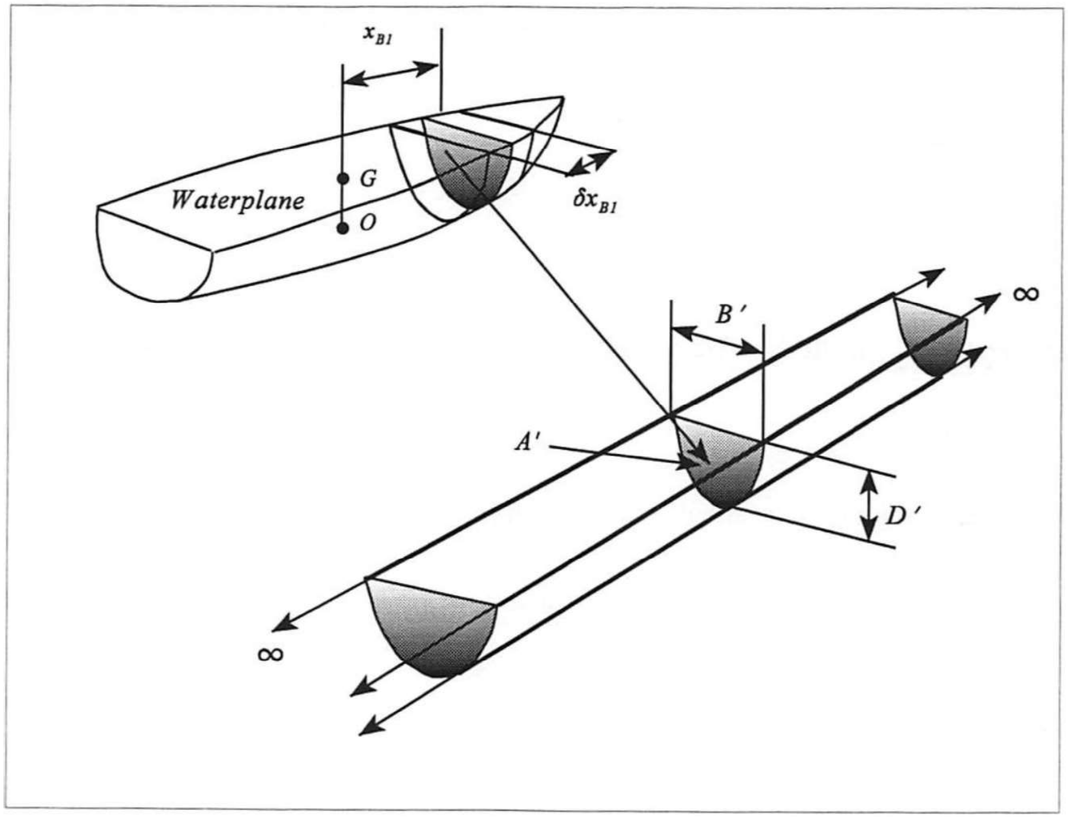


Figure 4.22: Concept of the ship "strips" used in Strip Theory (Figure 4.1 in reference 2)

Consider the hydrodynamic properties of each strip:

- added mass
- damping
- stiffness

These hydrodynamic properties for each strip contribute to the total hydrodynamic properties of the ship. How can we distinguish between a property for a *strip* and a property of the *ship*? By convention, properties of strips are written with a hashmark:

Ship	**Strip**
a_{33}	a'_{33}
b_{33}	b'_{33}
c_{33}	c'_{33}

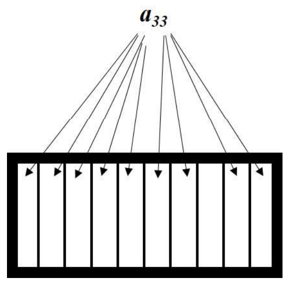


Figure 4.23: Strip Theory Barge Example

Consider a barge where each strip has an added mass of a'_{33}.

How do we find a_{33} of the total barge? How would we find the total **mass** if we had the mass of each strip? We could add each strip to find the total! Same principle for the added mass, except we will take the integral:

$$a_{33} = \int_0^{L_{PP}} a'_{33} \mathrm{d}x_{B1}.$$

Can a *strip* have an added mass in pitch? For strip theory to be successful, each strip must be extremely thin so that it can be considered a 2D strip. A 2D strip cannot have any pitch motions, so there can be no strip added mass in pitch! Which leaves us with the question of how to find the **total ship added mass in pitch**! Consider how we find the moment of inertia about G for a mass a'_{33} located x_{B1} away from the center of rotation: $I = mr^2$. We can use the same relationship for added mass in pitch of the total ship:

$$a_{55} = \int_0^{L_{PP}} (a'_{33} x_{B1}^2) \mathrm{d}x_{B1}.$$

Example: Consider a model barge with each section having a sectional added mass coefficient of 3.2 slugs/ft with 10 sections each of width 2 inches. Find the total ship added mass in heave (a_{33}) and pitch (a_{55}).

$$a_{33} = \int_0^{L_{PP}} a'_{33} \mathrm{d}x_{B1} = \int_0^{L_{PP}} 3.2 \mathrm{d}x_{B1} = 3.2 \int_0^{L_{PP}} \mathrm{d}x_{B1}$$

$$L = 10 * 2 = 20 \text{ inches} = 1.67 \text{ ft}$$

$$a_{33} = 3.2(1.67) = 5.34 \text{ slugs}$$

For added mass in pitch, I need to integrate each strip added mass over the distances from G. Given that each strip has the same added mass ($a'_{33} = 3.2$ slugs/ft), we can use Simpson's Rule to integrate this numerically,

x_{B1} (ft)	$a'_{33}x^2_{B1}$
−0.75	1.8
−0.58	1.09
−0.42	0.56
−0.25	0.2
−0.08	0.02
0.08	0.02
0.25	0.2
0.42	0.56
0.58	1.09
0.75	1.8

Using the symmetry in the problem, the Simpson's Rule integration looks like

$$a_{55} = 2\left[\frac{0.17}{3}(1.8 + 4(1.09) + 2(0.56) + 4(0.42) + 0.02)\right] = 0.918 \text{ slug} \cdot \text{ft}^2$$

4.7.1 Lewis Coefficients

Once the forces for inertia, damping, restoring, and exciting are known, the various expressions for the motion can be easily determined. In other words, before we can determine the heaving motion, we must evaluate the various strip coefficients! The strip inertial coefficient (added mass) can be expressed as a function of the added mass for a simple shape. In the case of Lewis coefficients, the shape is a circular segment of unit length and diameter B' (the beam of the "strip"). The added mass in heave for a circular segment of unit length and diameter B' is

$$\frac{1}{2}\rho\pi r^2 = \frac{\rho\pi(B')^2}{8}$$

where B' is twice the radius of the circular section. For shapes other than semi-circular, the added mass in heave is

$$a'_{33} = C\frac{\rho\pi(B')^2}{8}$$

where C is the ratio of the added mass of the section of unit length, section beam B', and section draft D' over half of the added mass for a circular segment of unit length and diameter B'. The C value is the coefficient for Lewis-form sections (derived through a method known as conformal mapping). There are plots that allow you to determine the appropriate Lewis-form coefficient given the section (or strip) beam at the waterline, draft, and cross-sectional area. **Lewis-forms have the same beam, draft and area as the actual ship section, but not the same shape.** There are similar charts for the damping Lewis-form coefficients.

4.8 Roll Mitigation

Because of the limited hydrodynamic damping available in roll (rolling causes only small waves, as opposed to the waves generated from ship heave or pitch), motions are generally large in roll and can be devastating (cause capsize, for example) if the encountered excitation wave frequency is too close to the ship's natural frequency in roll. Therefore, there is a great deal of effort put into minimizing roll motion in ships, or *roll mitigation*.

If the roll motions are too large crew comfort, economics, safety, and readiness can all be negatively affected. There are three main ways to reduce motions in waves:

Tuning stabilization Tuning stabilization changes the natural period (frequency) so that resonance is less likely to occur in the expected sea conditions. It is accomplished by hull form design and/or weight placement.

Damping stabilization Damping stabilization reduces the resonant peak. This is accomplished by increasing the damping of the system, sometimes by adding appendages to the hull.

Equilibrium stabilization Equilibrium stabilization creates an equal and opposite force approach. This is generally accomplished by applying a counter-acting force to maintain the ship in equilibrium, It relies on proper phasing of the forces and moments to reduce the motions.

In addition to the light amount of damping, roll is suitable for mitigation because it is a narrow-banded response. It means a stabilizer can be "tuned" to a single frequency. And being lightly damped means there can be large motions, but also that small increases in damping or counter-acting forces can make big differences. The total roll damping of a ship depends on four general categories: wave making (largest), viscous (eddies), skin friction, and appendage forces:

$$b_{44} = b_{\text{wave}} + b_{\text{visous}} + b_{\text{skin friction}} + b_{\text{appendage forces}}.$$

Methods of motion reduction are known as "stabilization". The term implies an increase in the stiffness coefficient (i.e., c_{44}), but it is more likely the "stabilization" method involves an increase in the motion damping (b_{44}). If a motion damper can double the decay coefficient,

$$\eta = \frac{b_{44}}{2\sqrt{c_{44}(I_4 + a_{44})}},$$

the roll amplitude at the natural frequency (ω_n) is halved, if the inherent damping is very small.

There are many different ways roll motion can be reduced, but in this chapter we are going to discuss three types of roll mitigation devices:

- Bilge Keels
- Active Fins

- Passive Tanks

For more roll mitigation devices, and further information on the three mentioned in this chapter, I refer you to the NSWC Carderock report "A Survey of Ship Motion Reduction Devices" by T.C. Smith and W.L. Thomas III.

Bilge Keels

Bilge keels are passive roll devices that increase the damping at all speeds and sea states. They consist of long narrow keels, mounted at the turn of the bilge. They work by generating drag forces which oppose the rolling motion of the ship. The advantages of bilge keels is that they are simple, inexpensive, and require no more maintenance than the hull. The disadvantage of bilge keels is that they increase the resistance of the ship (although the effects can be minimized by optimizing the design of the bilge keels for the design speed).

Active Fins

Active fins are active roll stabilizers that are mounted on rotatable stocks at the turn of the bilge near the middle of the ship. They work by using the angle of incidence between the fins and the flow of the water past the ship. The fins are continually adjusted by a control system that is sensitive to the rolling motion of the ship. The fins develop lift forces (due to the forward motion) that exert roll moments to oppose the moment applied by the waves. The advantages of active fins is that they are the most powerful and effective motion control device for high-speed applications. Their effectiveness increases with the square of the speed. Reductions of at least 50% in the average roll amplitudes are possible in moderate waves with a well-designed system. The disadvantages include that at low speeds the fins do not generate much lift (although they act somewhat like passive dampers). Also, their ability to reduce roll motion decreases in very severe sea states. They are a relatively sophisticated and expensive system and require considerable maintenance. Finally, at less than 10 knots they do not produce much lift, and at extreme speeds they can experience cavitation and separation.

Passive Tanks

Passive tanks are stabilizers that involve a sloshing liquid to produce damping and restoring forces. They work by shifting the weight of the liquid so that it exerts a roll moment on the ship and (by suitable design) this can be arranged to damp the roll motion. The natural frequency of the tank should be equal or near the ship's natural frequency. The tank is tuned by adjusting the amount of liquid in the tank or by the baffle design. A passive tank is a good choice if space and weight are not concerns. There are no moving parts and it requires very little maintenance. However, optimal tank placement is high in the ship and this makes access along the ship difficult. In addition, the free surface always reduces the metacentric height so roll stability is reduced. And all passive tanks *amplify* the roll motions at low encounter frequencies.

Motion Sickness Indices (MSI)

Ship roll motions can have very negative effects on passengers and crew. The people onboard can experience motion sickness and the roll motion can make it more difficult to move in a controlled and coherent manner so the performance of everyday tasks is impaired. The inner ear detects changes in magnitude and direction of apparent gravitational acceleration. Motion sickness is exacerbated if the person is

- confined below decks (can't see the horizon)
- facing diagonally across the ship
- anxious
- fatigued
- hungry
- smelling strong smells
- eating or smelling greasy foods
- reading
- drinking carbonated or alcoholic drinks

The symptoms of seasickness generally disappear after a few days at sea (the person becomes acclimated). Motions can impair the ability to work effectively even when there are no problems with seasickness. In these cases, the addage "one hand for the ship and one for yourself" is true.

The principle cause of motion sickness is believed to be the vertical acceleration experienced by the person (which varies with location on the ship). Other motions can cause motion sickness if sufficiently high, but are not common on conventional ships (the other motions aren't large enough). It is very difficult to predict the occurrence of seasickness. For one thing, individuals differ in their susceptibility to motions. Even a single individual's responses may vary from day to day, depending on the other factors mentioned above. Having a job to do versus thinking about how awful you feel can effect how much you suffer. Since a deterministic approach is not realistic (there exists no *if this, then that* relationship that is always true), a statistical approach is required.

The Motion Sickness Incidence is based on a 1974 experiment. The test measured the motion sickness response of over 300 American male college student (paid) volunteers who were not acclimated to motions. The students were tested in pairs in a ship motion simulator that had no windows and experienced sinusoidal vertical motion with amplitudes of about 3.5 meters ($\approx$ 23 ft overall height). The experimenters monitored the state of the participants nausea by having the students press buttons on a control panel. The experiments lasted up to 2 hours or until the subjects vomited. The results of this data allowed the Motion Sickness Incidence (defined as the percentage of subjects who vomited within two hours) equation:

$$\mathrm{MSI} = 100\big[0.5 + \mathrm{erf}\big(\frac{\log_{10}(\frac{|\ddot{s}_3|}{g}) - \mu_{\mathrm{MSI}}}{0.4}\big)\big] \tag{4.9}$$

where erf is the error function and equal to

$$\mathrm{erf}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \mathrm{e}^{\frac{-z^2}{2}} \mathrm{d}z$$

and the values are typically stored in tables for reference. The other variables in the equation are

$$\mu_{\mathrm{MSI}} = -0.819 + 2.32(\log_{10} \omega_e)^2$$

and $|\ddot{s}_3|$, which is the absolute value of the vertical (heave) acceleration averaged over a half cycle. The MSI increases as the magnitude of the vertical acceleration increases and it is most severe at a frequency of about 1.07 rad/sec. This frequency is very close to the average frequency for the vertical motions for many ships and explains why seasickness is such a common problem.

The applications of these results to the real life environment of a ship in rough weather requires us to make assumptions about the equivalence of the random motions of the ship and the sinusoidal motions of the simulator used in the experiment. We can assume the ship accelerations are distributed according to a normal (or Gaussian) probability distribution function (just like the waves the ship is encountering). Therefore, the average acceleration is

$$|\ddot{s}_3| = 0.798\sqrt{m_4}$$

where m_4 is the variance of the *vertical acceleration*. The average frequency of the motion *peaks* is

$$\bar{\omega}_e = \sqrt{\frac{m_4}{m_2}}$$

where m_2 is the variance of the *vertical displacement*. These approximations allow an estimate of the proportion of people who will suffer from seasickness in a given set of conditions at sea.

Motion Induced Interruptions (MII)

Another measure of importance for motions related to crew performance is the Motion Induced Interruptions value. This measures when a member of a crew would have to stop working at the current task and hold on to some convenient anchorage to prevent loss of balance. The MII is relevant to the effectiveness of the ship and the crew. Typically, MII is given as number of interruptions per minute.

4.9 Appendix: EOM Solution Derivation

The basic set-up for simple harmonic motion consists of a mass, a spring, a damper, and an external harmonic excitation. For the basic equation, the mass is variable m, the spring coefficient is c, the damping coefficient is b and the external excitation has magnitude F_0 and frequency ω_e. The equation of motion is a 2nd-order, linear, ordinary differential equation:

$$m\ddot{x} + b\dot{x} + cx = F_0 \sin \omega_e t$$

The basic solution to this equation is given as

$$x(t) = X_0 \sin(\omega_e t - \phi).$$

To solve for the unknown amplitude, X_0, and phase angle, ϕ, we need to plug the solution back into the governing equation. First we need to take the time-derivatives of the solution:

$$\begin{aligned} \dot{x}(t) &= \omega_e X_0 \cos(\omega_e t - \phi) \\ \ddot{x}(t) &= -\omega_e^2 X_0 \sin(\omega_e t - \phi) \end{aligned}$$

Now, plugging back into the governing equation:

$$-m\omega_e^2 X_0 \sin(\omega_e t - \phi) + b\omega_e X_0 \cos(\omega_e t - \phi) + cX_0 \sin(\omega_e t - \phi) = F_0 \sin \omega_e t.$$

We have two unknowns (X_0 and ϕ) and only one equation. We can separate the equation into two equations by choosing two times to evaluate at. To start, choose a time such that the $\sin \omega_e t$ term is equal to zero (i.e. $\omega_e t = 0$):

$$\begin{aligned} -m\omega_e^2 X_0 \sin(-\phi) + b\omega_e X_0 \cos(-\phi) + cX_0 \sin(-\phi) &= 0 \\ m\omega_e^2 X_0 \sin(\phi) + b\omega_e X_0 \cos(\phi) - cX_0 \sin(\phi) &= 0. \end{aligned}$$

This equation simplifies to

$$\boxed{\tan \phi = \frac{\omega_e b}{-\omega_e^2 m + c}.}$$

For the second equation, choose a time such that the $\sin \omega_e t$ term is equal to one (i.e. $\omega_e t = \pi/2$):

$$\begin{aligned} -m\omega_e^2 X_0 \sin(\pi/2 - \phi) + b\omega_e X_0 \cos(\pi/2 - \phi) + cX_0 \sin(\pi/2 - \phi) &= F_0 \\ -m\omega_e^2 X_0 \cos(\phi) + b\omega_e X_0 \sin(\phi) + cX_0 \cos(\phi) &= F_0. \end{aligned}$$

This equation simplifies to

$$Z = \frac{F_0}{(-\omega_e^2 m + c)\cos \phi + \omega_e b \sin \phi}.$$

Using the trigonometric identities for sin and cos in terms of tan:

$$\begin{aligned} \sin \phi &= \frac{\tan \phi}{\sqrt{1 + \tan^2 \phi}} \\ \cos \phi &= \frac{1}{\sqrt{1 + \tan^2 \phi}} \end{aligned}$$

we can plug the solution for $\tan\phi$ (from above) into the second equation:

$$X_0 = F_0 \frac{1}{(-\omega_e^2 m + c)\frac{1}{\sqrt{1+\tan^2\phi}} + \omega_e b \frac{\tan\phi}{\sqrt{1+\tan^2\phi}}}$$

$$X_0 = \frac{F_0}{\frac{1}{\sqrt{1+\tan^2\phi}}} \frac{1}{(-\omega_e^2 m + c) + \omega_e b \tan\phi}$$

$$X_0 = \frac{F_0}{\frac{1}{\sqrt{1+\tan^2\phi}}} \frac{1}{(-\omega_e^2 m + c) + \omega_e b \frac{\omega_e b}{-\omega_e^2 m + c}}$$

$$X_0 = \frac{F_0}{\frac{1}{\sqrt{1+\tan^2\phi}}} \frac{1}{(-\omega_e^2 m + c) + \frac{(\omega_e b)^2}{-\omega_e^2 m + c}}$$

Using the first equation, we can substitute

$$\sqrt{1+\tan^2\phi} = \sqrt{1 + (\frac{\omega_e b}{-\omega_e^2 m + +c})^2}$$

This leads to

$$X_0 = \frac{F_0 \sqrt{1 + (\frac{\omega_e b}{-\omega_e^2 (m+a) + c})^2}}{(-\omega_e^2 m + c) + \frac{(\omega_e b)^2}{-\omega_e^2 m + c}}$$

Simplifying this equation,

$$X_0 = \frac{F_0 \sqrt{(-\omega_e^2 m + c)^2 + (\omega_e b)^2}}{(-\omega_e^2 m + c)((-\omega_e^2 m + c) + \frac{(\omega_e b)^2}{-\omega_e^2 m + c})}$$

$$X_0 = F_0 \frac{\sqrt{(-\omega_e^2 m + c)^2 + (\omega_e b)^2}}{(-\omega_e^2 m + c)^2 + (\omega_e b)^2}$$

$$\boxed{X_0 = \frac{F_0}{\sqrt{(-\omega_e^2 m + c)^2 + (\omega_e b)^2}}}$$

The other form of the equation of motion for a *spring-mass-damper* system is written in terms of the natural frequency of the system (ω_n), the damping factor (η), and the tuning factor (Λ). These variables are related to the coefficients (mass, damping, and stiffness) in the following ways:

$$\omega_n^2 = \frac{c}{m}$$

$$\eta = \frac{b}{2m\omega_n}$$

$$\Lambda = \frac{\omega_e}{\omega_n}$$

Plugging these substitutions back into the equations for X_0 and ϕ requires reworking the equations so that the m, b, and c coefficients can be replaced by the ω_n, η, and Λ variables.

$$-\omega_e^2 m + c = \left(-\frac{\omega_e^2}{\omega_n^2} + 1\right)c$$

$$\omega_e b = \omega_e(2m\omega_n\eta) = 2\eta\frac{\omega_e}{\omega_n}\omega_n^2 m = 2\eta\frac{\omega_e}{\omega_n}\frac{c}{m}m = \left(2\eta\frac{\omega_e}{\omega_n}\right)c$$

Reworking the X_0 and ϕ equations results in

$$X_0 = \frac{F_0}{\sqrt{c^2(1-\Lambda^2)^2 + (2\eta\Lambda)^2 c^2}}$$

$$\boxed{X_0 = \frac{F_0/c}{\sqrt{(1-\Lambda^2)^2 + (2\eta\Lambda)^2}} \qquad \tan\phi = \frac{2\eta\Lambda}{1-\Lambda^2}}$$

Problems

Problems on Dynamics Review

1. The stiffness coefficient in sway, c_{22}, is equal to
   (i) zero
   (ii) greater than zero
   (iii) equal to the stiffness coefficient in heave

2. Consider a forcing function of the form $F_0 \sin(\omega_e t + \epsilon_e)$. The variable F_0 represents
   (i) the total forcing function
   (ii) the amplitude of the forcing function
   (iii) the height of the forcing function
   (iv) the frequency of the forcing function

3. If the equation for the amplitude response of a forced *spring-mass-damper* system is written as

$$X_0 = \frac{\frac{F_0}{c}}{\sqrt{(1-\Lambda^2)^2 + (2\eta\Lambda)^2}}$$

   The variable Λ refers to
   (i) the damping factor
   (ii) the magnification factor
   (iii) the tuning factor
   (iv) the static response

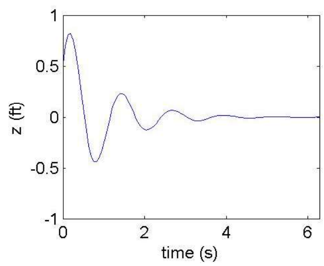
4. Consider motion described by $x(t) = x_0 \sin(\omega t + \epsilon)$.
   - What is the equation for the velocity, $\dot{x}(t)$?
   - What is the equation for the acceleration, $\ddot{x}(t)$?

5. A *spring-mass-damper* system has an effective mass of $m + a$, a damping coefficient of b, and a stiffness coefficient of c. The system forcing can be described by $F(t) = F_0 \sin(\omega_e t + \epsilon_e)$.
   - Write the equation of motion for this system.
   - What is the natural frequency of the system if the damping coefficient, b, was equal to zero?
   - What is the difference between the natural frequency of the system and the frequency, ω_e?

6. What is the equation for the tuning factor and what is the symbol representing it?

7. The damping factor is the ratio of the actual damping of the system to what quantity?

8. The sinusoidal curve shown below is experiencing
   (i) over-damping
   (ii) critical damping
   (iii) under-damping



9. Consider a *spring-mass-damper* system. The weight of the system is assumed to be 220 lbs and the restoring force provides a spring constant of 68.5 lb/ft. If the exciting force is assumed to have the form of a sine function with an amplitude of 224 lbs and a frequency of 3 rad/sec, what is the least amount of damping (b) needed to keep the amplitude of motion to less than (or equal to) 3.3 ft? What is the phase angle between the exciting force and the response?

10. An unknown mass m (kg) is attached to the end of a spring with an unknown stiffness, c (N/m). This system has a natural frequency of 1.5 Hz. When a second mass of 0.45 kg is added to the first mass, the system natural frequency becomes 1.25 Hz. Determine the unknown stiffness c and mass m. Write the equation of motion for the original unforced system.

11. A real world dynamic system can be modeled as a simple *spring-mass-damper* system. The mass of the system is assumed to be 0.1 tones, and the restoring force provides a spring constant of 2.5 kN/m. If the exciting force is assumed to have the form of a sine function with an amplitude of 1.6 kN and a frequency of 4.1 rad/sec, what is the least amount of damping (b) needed to keep the amplitude of motion to less than (or equal to) 1 m? What is the phase angle between the exciting force and the response? Using Matlab or Excel, plot the response with the found damping and show that the amplitude remains less than (or equal to) 1 m.

12. A new spar buoy, 0.5 m diameter, is being designed to carry oceanographic instrumentation in the Indian Ocean. At this stage in the design process, the heave motion of the buoy is being modeled as a simple *spring-mass-damper* system. The current design has an effective mass ($a + m$) of 650 kg. The design includes 200 kg of lead ballast in this mass. The restoring force is the change in buoyancy as the buoy heaves. The damping for a cylinder of this size is estimated to be 0.28 kN-sec/m. The wave-induced exciting force is assumed to have the form of a sine function with an amplitude of 1.65 kN and a frequency of 0.8 rad/sec. For this wave condition it is desired to keep the maximum amplitude of the motion to less than 1 m. The only flexibility in the design is the

amount of lead ballast. You can have either 0, 200, or 400 kg of ballast. What is your choice and why?

13. For a circular cylinder buoy with a diameter of 1 ft and a draft of 0.5 ft floating in salt water, what is the heave restoring coefficient (C_{33})?

14. The following equation describes a mass-spring system: $20\ddot{x} + 80x = 10 \sin 2t$. What is the natural frequency of the system?
    (i) 2 rad/sec
    (ii) 80 rad/sec
    (iii) 20 rad/sec
    (iv) 4 rad/sec
    (v) 0.5 rad/sec

Problems on Added Mass and Damping

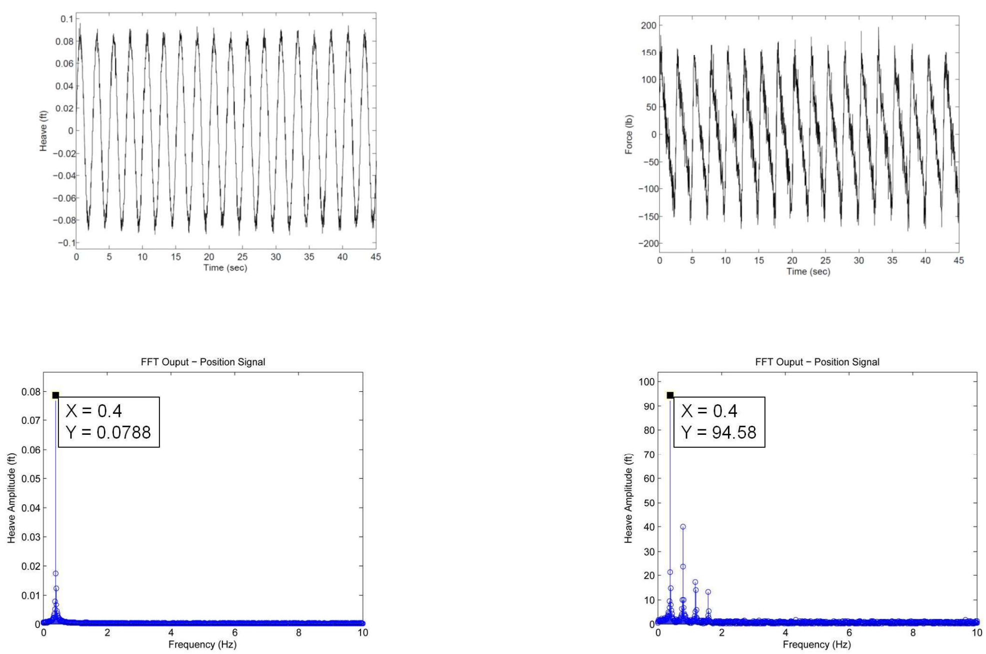
15. What is added mass?

16. A forced oscillation test was done and the required forces and motions were recorded. When determining the coefficients, the stiffness and the _____ are in-phase with each other.
    (i) added inertia or mass
    (ii) damping
    (iii) forcing

17. Consider an experiment where a model is forced to move in heave. You know the weight of the model ($W = mg$). You specify the amplitude of oscillation (Z) and the frequency of motion (ω), and you measure the magnitude of the force (F_z), and the phase angle of the response (ϕ).
    - What is the equation of motion in heave?
    - What is the form of the motion solution (in heave)?
    - What are the unknowns in your solution (*there should be two unknowns*).
    - Find an equation for each unknown variable.
    - What is the name of the digital signal processing technique that will be used on the collected data (position versus time and force versus time) that will extract the amplitude of motion (Z), the frequency of motion (ω), the magnitude of the force (F_z), and the phase angle (ϕ)?

18. You have recently gotten very confident with your Matlab skills and have decided to write a program to predict the motions in heave for the DDG-51 model operating in regular waves. The model has a displacement of 402.25 lbs and a L_{PP} of 12.945 ft. To solve for the response motion you need to have values for the heave stiffness coefficient (c_{33}), heave damping coefficient (b_{33}), and the heave added mass coefficient (a_{33}). You

know you can calculate the heave stiffness coefficient and you find it to be **1049.5 lb/s**. To find the damping and added mass coefficients you decide to do a forced oscillation test in heave. Below is the data and the FFT output for a single frequency. The FFT analysis gives you a phase angle of 2.88°.



- Write the equation of motion for the system (keep everything in variables).
- What is the frequency of ocsillation for this data (in rad/sec)?
- *optional*: Solve for the heave damping (b_{33}) and the heave added mass coefficient (a_{33}).

19. Explain the difference between added mass and hydrodynamic damping.

Problems on Model Testing in Regular Waves

20. A ship is traveling at 20 kts in head seas. The waves have a frequency of 0.5 rad/sec. What is the encounter frequency of the ship?

21. A ship is traveling at 12 kts in beam seas. The waves have a frequency of 0.76 rad/sec. What is the encounter frequency of the ship?

22. Consider a ship traveling in regular head seas at 10 knots (1.688 ft/s per 1 knot) with a wave frequency of 3.18 rad/sec. What is the ship's encounter frequency?

23. What effect does ship length have on seakeeping for a given wavelength?

24. Consider the following information:

U (kts)	μ (deg)	λ (m)	ζ_0 (m)	ω_{n5} (rad/sec)	ω_{n3} (rad/sec)
15	180	10	0.333	0.6	1.2

ω (rad/sec)	X_3 (m)	X_5 (deg)	$k\zeta_0$ (rad)	ω_e (rad/sec)
0.641	0.314	13.652	0.209	0.964

- What are the pitch and heave transfer functions for this condition?
- What do you expect the phase relationships to be for a point on the **pitch** transfer function curve at an encounter frequency of 0.3 rad/sec?

25. A box-shaped lighter ($C_B = 1.0$) has a L_{WL} of 60 m, a beam of 12 m, a full load draft of 3 m, and a freeboard in the full load condition of 1 m. The longitudinal radius of gyration, k_5, is $0.27L_{WL}$. Assume the weight of the cargo and the structure is evenly distributed along its length (LCG = 0 measured from midships) and that the added mass in heave (a_{33}) is 1.34 times the mass of the ship. You have been asked to evaluate the vessel's pitching characteristics for a planned 8 knot ocean transit. As a worst case scenario, you expect the lighter will encounter waves with a 100 m wavelength, a 4 m wave height, and a wave slope of 0.13. The waves are coming from 45° off the starboard bow ($\mu = 135°$). Use the information provided below to

- Find the natural frequency and period in pitch
- Find the steady state amplitude and phase angle
- Find the transfer function of the lighter in this sea condition.

$$
\begin{aligned}
(I_5 + a_{55}) &\approx (m + a_{33})k_5^2 \\
b_{55} &= 807,734 \text{ kN m/(rad/s)} \\
c_{55} &= mg(\frac{I_L}{\nabla}) \\
I_L &= \frac{BL^3}{12} \\
F_{W_{50}} &= 94,700 \text{ kNm} \\
\rho &= 1.025 \text{ tonnes/m}^3 \\
g &= 9.81 \text{ m/s}^2 \\
1 \text{ knot} &= 0.5144 \text{ m/s}
\end{aligned}
$$

26. Sketch the heave and pitch transfer functions for a conventional displacement ship. Identify any interesting aspects of the curve.

27. Assume a model test has been designed with a scale ratio of $R = L_s/L_M$. What is the proper scaling for the forward speed?

28. Consider a ship of length 42 ft traveling at 4.4 knots with a model length of 8 ft. The full-scale ship has a displacement of 17.86 LT and a pitch mass moment of inertia of 137,000 slugs $\cdot$ ft^2.

- What is the model scale or dimensional ratio?
- If the full-scale wave amplitude is 1.6 ft, what is the appropriate wave amplitude for the model (in inches)?
- What is the Froude number for this condition?
- What is the appropriate forward speed for the model (in ft/s)?
- If the full-scale wave period of 3.9 sec, what is the scaled wave period? What is the scaled wave frequency (in Hz)?
- What is the appropriate model displacement (in lbs)?
- What is the full-scale pitch gyradius as a percentage of length? *What is the equation for the mass moment of inertia as a function of gyradius?*
- What is the appropriate model-scale pitch gyradius (in inches)? What is the model scale pitch gyradius as a percentage of the length?
- What is the scaling factor for acceleration ($\ddot{x}$)?

29. Explain the purpose of the dynamic ballasting lab.

30. Consider the DDG-51 model used in the Dynamic Ballasting lab.

    - What is the correct model speed for a $1/36^{\text{th}}$ scale model if the full-scale speeds are 15 and 20 knots? *Give the answer in ft/s.*
    - What is the target k_5 for the DDG-51 model? *Give both the actual measurement and the value as a percent of L_{PP}.*

31. What measurements were taken during the labs measuring the model motions when operating in waves? What instrumentation was necessary? What preparation is needed for a seakeeping test that is not required for a resistance test and why?

32. Describe the heave motions when a ship is experiencing regular head seas whose predominant wavelength is much, much longer than the length of the ship. Be sure to mention both the magnitude and the phase of each motion. **Include a sketch!**

Problems on Strip Theory

33. List all of the strip theory assumptions and discuss the reasonableness with regard to the real world for each one.

34. Consider a box barge where the added mass in heave is known for each strip (a'_{33}). How is the total added mass for the barge, a_{33}, found?
    (i) sum the strip values
    (ii) sum the strip values times the distance of each strip from the LCG
    (iii) sum the strip values times the distance of each strip from the LCG squared
    (iv) sum the strip values times the absolute value of the distance from the LCG

35. **T** or **F**: The added inertia in pitch for a ship, a_{55}, can be found if the added inertia in heave is known for each strip, a'_{33}.

36. If the strip added mass is known (a'_{33}), the equation $\int a'_{33}x_{B1}^2 \mathrm{d}x_{B1}$ gives the
   (i) a_{33}
   (ii) a_{35}
   (iii) a_{55}
   (iv) a_{53}

Roll Motions

37. You are on a Navy destroyer experiencing significant roll amplitudes. The waves are coming from your starboard beam ($\mu = 90°$) at a modal period of about 6 to 7 seconds. The natural roll period of your ship is 7.5 seconds. You are experiencing engine problems that are preventing any change in speed.
   - Sketch a typical roll transfer function for a typical combatant craft.
   - What action do you recommend to the CO and why?

38. You are the OOD of a Navy destroyer underway in a moderate seaway at 15 knots. Looking at the current sea state you estimate the seaway to have a significant wave height of about 2.5 m and a modal period between 10 and 11 seconds. You have just steadied on the new base course that puts the predominant sea off your starboard quarter ($\mu = 45°$). The ship starts to experience very large rolls. You know that the natural period of your ship is about 9.5 seconds.
   - What is your encounter frequency?
   - What is your tuning factor (Λ)?
   - If you increase speed by 10 kts, how much do you change your encounter frequency ($\Delta\omega_e$)?
   - If you turned 15° to starboard, how much do you change your encounter frequency ($\Delta\omega_e$)?
   - What actions would you recommend to the CO and why (be specific!).

39. You are the OOD of a Navy destroyer underway in a moderate seaway at 18 knots. You estimate the seaway to have a significant wave height of about 3.0 m and a modal period between 9 and 10 seconds. You have just steadied on the new base course that puts the predominant sea on your starboard beam ($\mu = 90°$). The ship starts to experience very large rolls. You know the natural roll period of your ship is about 8.0 sec. You could alter course or speed, but you need to make an assigned rendezvous which is 150 nm away on the base course 10 hours from now. What actions would you recommend to the CO and why (be specific!).

40. A small cruise ship is headed to a destination that has no deep water port. The passengers will disembark via small boats. For disembarkation, the roll of the cruise ship must be kept small ($< 6°$). For the small roll angles being considered, damping can be neglected. The roll radius of gyration is $k_4 = 4.0$ m. What is the natural frequency of the ship if the metacentric height is 2.5 m? What is the natural frequency

if the metacentric height is 1 m? Does the relationship between metacentric height and roll natural frequency make sense?

41. A new cargo ship type features a relatively large beam and extreme sectional flare at the ship ends to allow many containers on deck. The ship has the following particulars: $L_{WL} = 94$ m, $B = 16$ m, $D = 5$ m, and $k_4 = 0.4L_{WL}$. Investigate whether there is a danger of capsizing due to resonance in wave induced rolling for possible metacentric heights of $0.30 < GM < 1.0$ m when the ship is in beam seas and the wavelengths are less than 100 m. *Consider the possible natural frequencies of the ship compared with the range of wave frequencies.*

42. Consider a surfaced submarine with speed $U = 6$ knots. Neglect the sail and model the submarine hull as a circular cross section. The submarine has the following particular: $L = 70.0$ m, diameter $= 8.0$ m, $KG = 3.60$ m,

    (a) What is the roll natural frequency?

    (b) If the waves are coming from the beam and have a length of 70 m, is the sub in danger of experiencing resonance?