RELATIVISTIC MECHANICS AND MAXWELL"S EQUATIONS

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I. MOTION OF A POINT PARTICLE; LORENTZ FORCE

We will discuss relativistic mechanics from an unusual point of view using the principle of minimal action. It will serve as a warm up for doing the same thing with the electromagnetic field. I will assume familiarity with lagrangian mechanics in classical physics but this knowledge is not strictly necessary. Imagine a point particle moving in spacetime. Its trajectory describes a world-line and can be described by a function $x^\mu(s)$; for every value of s we have the position of the particle and the time when it passed through that point. Notice that different parametrizations of the world-line correspond to the same physical trajectory, all it matters is the graph of $x^\mu(s)$ in spacetime, not the value of s corresponding to the different points of the curve. The principle of minimal action says that the trajectory actually followed by the particle will be the one which minimizes (extremizes, actually) a functional of $x^\mu(s)$ called the action $S[x^\mu(s)]$. But that doesn't say much because we don't know what $S[x^\mu(s)]$ is. Let us then try to guess what $S[x^\mu(s)]$ could be. First, as mentioned above, the physical trajectory, that is, $\vec{x}$ as a function of $t = x^0/c$ can be parametrized in different ways. But since they describe the same physical situation, S should be the same, regardless of the parametrization chosen. Second, S should be a Lorentz scalar. Otherwise the path that minimizes S in one reference frame would be different from the path that minimizes the action in another reference frame. The resulting theory wouldn't be Lorentz invariant. One obvious quantity satisfying these properties is the length of the spacetime path:

$$S[x^\mu(s)] = mc \int ds \sqrt{\frac{dx^\mu}{ds}\frac{dx_\mu}{ds}}. \tag{1.1}$$

The mc constant is just conventional and is included to provide the correct dimensions, it won't alter the trajectory that minimizes S. The square root term is just the length of the "velocity" $\frac{dx^\mu}{ds}$. Notice that the "velocity" $\frac{dx^\mu}{ds}$ depends on the parametrization but S doesn't. If we use a different parametrization $x^\mu(s')$

$$\int ds' \sqrt{\frac{dx^\mu}{ds'}\frac{dx_\mu}{ds'}} = \int ds \left|\frac{ds'}{ds}\right| \sqrt{\frac{dx^\mu}{ds}\frac{ds}{ds'}\frac{dx_\mu}{ds}\frac{ds}{ds'}} = \int ds \sqrt{\frac{dx^\mu}{ds}\frac{dx_\mu}{ds}}. \tag{1.2}$$

Now, given an initial and final spacetime points for the trajectory, which path will minimize S, that is, which path will minimize the length? The straight line, of course! A straight line in space time corresponds to a particle moving at constant velocity in space, which is, of course, the correct result. But for more complicated actions it is not so easy to guess the minimizing path. Minimization problems of this kind are the object of the "Calculus of Variations". We will now take a little mathematical detour and explain how this works. If you know about Euler-Lagrange equation you can skip most of it.

Variational Calculus and functional derivatives in one page

How do you minimize (or maximize or extremize) a function $f(x)$ of one variable x? You find the point(s) x_0 where its derivative vanishes

$$\frac{df(x)}{dx}|_{x_0} = 0. \tag{1.3}$$

How do you minimize (or maximize or extremize) a function $f(x^i)$ of several variables $x^i, i = 1, \cdots, N$? ($f(x^i)$ is a short hand for $f(x^1, x^2, \cdots, x^N)$). We find a point with coordinates x_0^i where all the partial derivatives vanish

$$\frac{\partial f(x^i)}{\partial x^j}|_{x_0^i} = 0. \tag{1.4}$$

The partial derivative above is

$$\frac{\partial f}{\partial x^j} = \lim_{\Delta \to 0} \frac{f(x^i + \Delta\delta^{ij}) - f(x^i)}{\Delta}. \tag{1.5}$$

We can think of the functional $S[x(s)]$ as a function of an infinite number of variables $x(s)$, one for every value of s. The analogy is

$$\begin{aligned} i &\to s, \\ x^i &\to x(s), \\ f(x^i) &\to S[x(s)]. \end{aligned} \tag{1.6}$$

The role of the partial derivative is taken by the "functional derivative"

$$\frac{\delta F[x(s)]}{\delta x(s')} = \lim_{\Delta \to 0} \frac{F[x(s) + \Delta\delta(s - s')] - F[x(s)]}{\Delta}, \tag{1.7}$$

in other words, you increase the value of the function $x(s)$ at $s = s'$ and see how the functional $S[x(s)]$ changes. All you really need to know about functional derivatives are the properties (easily derived from the definition

$$\begin{aligned} \frac{\delta F[x(s)]}{\delta x(s')} &= \frac{dF}{dx}\frac{\delta x(s)}{\delta x(s')}, \\ \frac{\delta x(s)}{\delta x(s')} &= \delta(s - s'), \\ \frac{\delta}{\delta x(s')}\frac{dx(s)}{ds} &= \frac{d\delta(s - s')}{ds}. \end{aligned} \tag{1.8}$$

How do we minimize the functional

$$F[x(s)] = \int ds\ \mathcal{F}\left(x(s), \frac{dx(s)}{ds}\right) ? \tag{1.9}$$

The extremum $x_0(s)$ will be the one where the functional derivative vanishes

$$\begin{aligned} 0 = \left.\frac{\delta F[x(s)]}{\delta x(s')}\right|_{x(s)=x_0(s)} &= \int ds \left.\left[\frac{\partial \mathcal{F}}{\partial x}\frac{\delta x(s)}{\delta x(s')} + \frac{\partial \mathcal{F}}{\partial \frac{dx}{ds}}\frac{\delta \frac{dx(s)}{ds}}{\delta x(s')}\right]\right|_{x(s)=x_0(s)} = \int ds \left.\left[\frac{\partial \mathcal{F}}{\partial x}\delta(s - s') + \frac{\partial \mathcal{F}}{\partial \frac{dx}{ds}}\frac{d}{ds}\delta(s - s')\right]\right|_{x(s)=x_0(s)} \\ &= \int ds \left.\left[\frac{\partial \mathcal{F}}{\partial x}\delta(s - s') - \frac{d}{ds}\frac{\partial \mathcal{F}}{\partial \frac{dx}{ds}}\delta(s - s')\right]\right|_{x(s)=x_0(s)} = \left.\frac{\partial \mathcal{F}}{\partial x} - \frac{d}{ds}\frac{\partial \mathcal{F}}{\partial \frac{dx}{ds}}\right|_{x(s)=x_0(s)}. \end{aligned} \tag{1.10}$$

The equation above is known as the Euler-Lagrange equation, the central result of the calculus of variation.

We can now apply the Euler-Lagrange equations to the problem of minimizing the action for a free particle. The only adaptation is that the action depends on four functions $x^\mu(s)$, one for each value of μ. We take one particular frame of reference and use the time, as measured in that frame, as the parameter s: $x^\mu(t) = (ct, \vec{x}(t)$:

$$\begin{aligned} S &= mc\int \sqrt{\frac{dx^\mu}{dt}\frac{dx_\mu}{dt}} \\ &= mc\int dt\sqrt{c^2 - \left(\frac{d\vec{x}}{dt}\right)^2} \\ &= mc^2\int dt\sqrt{1-\frac{\dot{\vec{x}}^2}{c^2}}, \end{aligned} \tag{1.11}$$

where the dot represents a time derivative. The Euler-Lagrange equation leads to

$$\frac{\ddot{\vec{x}}}{\sqrt{1-\frac{\dot{\vec{x}}^2}{c^2}}} + \dot{\vec{x}}.\ddot{\vec{x}}\frac{\dot{\vec{x}}}{(1-\frac{\dot{\vec{x}}^2}{c^2})^{3/2}} = 0 \tag{1.12}$$

Multiplying this equation by $\dot{\vec{x}}$ gives $\dot{\vec{x}}.\ddot{\vec{x}} = 0$ so the component of the acceleration tangential to the velocity vanishes. Multiplying the same equation by any vector $\vec{n}$ normal to the velocity gives $\vec{n}.\ddot{\vec{x}} = 0$ so the component of the acceleration normal to the velocity also vanishes. Since the acceleration vanishes the trajectories minimizing the action are straight lines with constant speed, as expected.

Things are more interesting when there's an external force acting on the particle. How can we include them? We can change the action by the inclusion of "external fields" while keeping S i) a Lorentz scalar and ii) independent of how $x^\mu(s)$ is parametrized. One could, for instance, imagine that there is a scalar field $\Phi(x)$ permeating the spacetime ad modify the action as

$$S = \int ds(mc - \Phi(x(s)))\sqrt{\frac{dx^\mu}{ds}\frac{dx_\mu}{ds}}. \tag{1.13}$$

Or one could have a 4-vector field permeating spacetime changing the action to

$$S = \int ds\left[mc\sqrt{\frac{dx^\mu}{ds}\frac{dx_\mu}{ds}} + \frac{q}{c}A_\mu(x(s))\frac{dx^\mu}{ds}\right]. \tag{1.14}$$

Finally, one could have a rank-two symmetric tensor $g_{\mu\nu}$ and the action

$$S = mc\int ds\sqrt{g_{\mu\nu}(x(s))\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}}. \tag{1.15}$$

I can't think of any other possibility. It turns out that the field $g_{\mu\nu}$ exists inNature and describes gravity, $A_\mu(x)$ also exists in Nature and describes electromagnetic forces. Apparently Nature decided not to use the scalar field $\Phi(x)$.

In this class we are interested in the coupling with the vector field $A_\mu(x)$. But does the coupling with $A_\mu(x)$ really describes electromagnetic forces? Suppose we change $a_\mu(x)$ (from now on we'll call it the "vector potential") by

$$A_\mu \to A_\mu + \partial_\mu\chi, \tag{1.16}$$

where χ is some scalar function. What happens to the action under this transformation?

$$S \to S + \frac{q}{c}\int_{s_i}^{s_f} ds\partial_\mu\chi\frac{dx^\mu}{ds} = S + \frac{q}{c}\int_{x^\mu(s_i)}^{x^\mu(s_f)} dx^\mu\partial_\mu\chi = S + \frac{q}{c}\Big|_{\chi(x^\mu(s_f))}^{\chi(x^\mu(s_i))}. \tag{1.17}$$

The change in the action is determined by the value of χ at the initial and final points and it doesn't depend on the trajectory in between them. Thus the action with A_μ or $A_\mu + \partial_\mu\chi$ is minimized by the same trajectory. The invariance of the physical results under the change $A_\mu \to A_\mu + \partial_\mu\chi$ is called "gauge invariance". We can now find the equations of motion of a article moving under the influence of the vector potential as described by the action in eq. (1.14). Choosing the time in an specific frame as the parameter s we have

$$S = \int dt\sqrt{c^2 - \dot{\vec{x}}^2} + \frac{q}{c}\int (A_0 c - \dot{\vec{x}}.\vec{A}). \tag{1.18}$$

The derivatives we need to calculate are

$$\begin{aligned}\frac{\partial L}{\partial \dot{x}^i} &= -\frac{mc}{\sqrt{c^2 - \dot{\vec{x}}^2}}\dot{x}^i - \frac{q}{c}A^i, \\ \frac{\partial L}{\partial x^i} &= q\partial_i A_0 - \frac{q}{c}\dot{x}^j\partial_i A^j.\end{aligned} \tag{1.19}$$

Combining them together we have

$$\begin{aligned}0 &= \frac{d}{dt}\left[-\frac{mc}{\sqrt{c^2 - \dot{\vec{x}}^2}}\dot{x}^i - \frac{q}{c}A^i\right] - q\partial_i A_0 + \frac{q}{c}\dot{x}^j\partial_i A^j \\ &= \frac{d}{dt}\left[-\frac{mc}{\sqrt{c^2 - \dot{\vec{x}}^2}}\dot{x}^i\right] - \frac{q}{c}(\partial_j A^i \dot{x}^j + \frac{\partial A^i}{\partial t}) - q\partial_i A_0 + \frac{q}{c}\dot{x}^j\partial_i A^j \\ &= \frac{d}{dt}\left[-\frac{m\dot{x}^i}{\sqrt{1 - \frac{\dot{\vec{x}}^2}{c^2}}}\right] - q(\partial_i A_0 + \frac{1}{c}\frac{\partial A^i}{\partial t}) + \frac{q}{c}\dot{x}^j(\partial_i A^j - \partial_j A^i),\end{aligned} \tag{1.20}$$

or, writing in the more usual fashion

$$\frac{d\vec{p}}{dt} = q(\vec{E} + \frac{\dot{\vec{x}}}{c} \times \vec{B}), \tag{1.21}$$

with

$$\begin{aligned}\vec{p} &= m\gamma\dot{\vec{x}}, \\ (\vec{E})^i &= -\partial_i A_0 - \frac{1}{c}\frac{dA^i}{dt}, \\ (\vec{B})^k &= \epsilon^{ijk}\partial_j A^k.\end{aligned} \tag{1.22}$$

Of course, the right-hand side of the equation of motion is the Lorentz force and the $\vec{E}$ and $vecB$ fields are the usual electric and magnetic fields written in terms of the (3D) scalar potential A_0 and the (3D) vector potential $\vec{A}$. We see then that electromagnetism emerged from very mild assumptions (Lorentz invariance and the choice of a vector field instead of a scalar or a tensor). Even the precise form of the Lorentz force law is determined by these general principles.

Our derivation of the equations of motion above has one flaw: it is not *explicitly* Lorentz invariant. We can obtain a different form of the same equation of motion by making an invariant choice of the parameter s and repeating the derivation. Let us choose s to be the proper time τ, defined by $d\tau = dt/\gamma$. The physical interpretation of τ is that it is the time measured by a clock moving with the particle. We have

$$v^\mu = \frac{dx^\mu(\tau)}{d\tau} = \gamma\frac{dx^\mu(t)}{dt} = \gamma\frac{d}{dt}(ct, \vec{x}) = (c\gamma, \gamma\vec{v}) \tag{1.23}$$

The Euler-Lagrange equation is then

$$\begin{aligned}
0 &= \frac{d}{d\tau}\frac{\partial L}{d\frac{dx^\mu}{d\tau}} - \frac{\partial L}{dx^\mu} \\
&= \frac{d}{d\tau}\left(m\frac{dx^\mu}{d\tau}\right) + \frac{q}{c}\partial_\nu A_\mu \frac{dx^\nu}{d\tau} - \frac{q}{c}\partial_\mu A_\nu \frac{dx^\nu}{d\tau}
\end{aligned} \tag{1.24}$$

or

$$\boxed{\frac{dp_\mu}{d\tau} = \frac{q}{c}F_{\mu\nu}\frac{dx^\nu}{d\tau}}, \tag{1.25}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.26}$$

and $p^\mu = mdx^\mu/d\tau$. We will leave as an exercize to show that the two forms of the equation of motion for a point particle are indeed equivalent. You might wonder why eq. (1.25) contains four euqations while eq. (1.21) constains only three. The answer is the four equations contained in eq. (1.25) are not independent. At this point one should stop and solve eq. (1.25) for some simple external field configurations. But this is more of a classical (relativistic but not quantum mechanical) mechanics problem than an electrodynamics problem. We will review a few on the homework. You should know the solution for i) a constant, homogeneous electric field, ii) a constant, homogeneous magnetic field and, if you have any ambition iii) an *almost* homogeneous magnetic field. A few others are also soluble (crossed homogeneous electric and magnetic fields, a plane wave, ...).

Eq. (1.25) is obviously relativistic invariant as it is written in terms of tensors. Notice that the electric and magnetic fields form together a four-tensor. This has as a consequence that, under a boost, electric and magnetic fields are *not* invariant. What looks like a magnetic field for one observer is an electric field for another. Let us look at the $F_{\mu\nu}$ tensor in a particular frame and write it in terms of the electric and magnetic fields. Tod o this one needs to be careful with conventions, signs, ... We have

$$\begin{aligned}
F_{0i} &= \partial_0 A_i - \partial_i A_0 = -\partial_0 A^i - \partial_i A_0 = -\frac{dA^i}{cdt} - \partial_i A_0 = E^i, \\
F_{12} &= \partial_1 A_2 - \partial_2 A_1 = -\partial_1 A^2 + \partial_2 A^1 = -B^3, \\
&\vdots
\end{aligned} \tag{1.27}$$

Writing is as a matrix with the index μ indexing the rows and the index ν indexing the columns we have

$$F_{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{pmatrix} \tag{1.28}$$

The contravariant tensor $F^{\mu\nu}$ would be like above but with $\vec{E} \to -\vec{E}$ instead (only the entries with one time and one spatial index flip sign as going from lower to upper indices).

Since we know how tensor components change under a change of basis we can easily find how the electric and magnetic fields transform under a boost.

$$
\begin{aligned}
F'^{\mu\nu} &= \begin{pmatrix} 0 & -E'^1 & -E'^2 & -E'^3 \\ E'^1 & 0 & -B'^3 & B'^2 \\ E'^2 & B'^3 & 0 & -B'^1 \\ E'^3 & -B'2 & B'^1 & 0 \end{pmatrix} \\
&= \Lambda^\mu_\lambda \Lambda^\nu_\rho F^{\lambda\rho} \\
&= \begin{pmatrix} \gamma & 0 & 0 & -\gamma v/c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma v/c & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B2 & B^1 & 0 \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & -\gamma v/c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma v/c & 0 & 0 & \gamma \end{pmatrix} \\
&= \begin{pmatrix} 0 & \gamma(-E^1 + \frac{v}{c}B^2) & \gamma(-E^2 - \frac{v}{c}B^1) & -E^3 \\ \gamma(E^1 - \frac{v}{c}B^2) & 0 & -B^3 & \gamma(B^2 - \frac{v}{c}E^1) \\ \gamma(E^2 + \frac{v}{c}B^1) & B^3 & 0 & \gamma(-B^1 + \frac{v}{c}E^2) \\ E^3 & \gamma(-B^2 + \frac{v}{c}E^1) & \gamma(B^1 - \frac{v}{c}E^2) & 0 \end{pmatrix}.
\end{aligned}
\tag{1.29}
$$

A more civilized way of writing these relations is

$$
\begin{aligned}
E'_{||} &= E_{||}, \\
B'_{||} &= B_{||}, \\
E'_\perp &= \gamma(E_\perp + \frac{v}{c} \times B_\perp), \\
B'_\perp &= \gamma(B_\perp - \frac{v}{c} \times E_\perp),
\end{aligned}
\tag{1.30}
$$

where $||$ and $\perp$ denote the components parallel and perpendicular to the boost direction. Without the formalism of tensors and Lorentz transformations this is a relatively difficult result to find; yet, we did it in one line. You may find this demonstration almost too slick, too short, to satisfy you. A more physical approach may make this more concrete. The homework will fill this gap.

II. MAXWELL'S EQUATIONS

Up to now we consider the motion of a particle under the influence of a given external field. The particle itself didn't generate a field. This is a good approximation to some circumstances like, for instance, an electron inside a TV tube (if you don't know what a TV tube is, ask your parents). The few electron in the beam directed towards the screen create a field that is negligible compared to the field generated by the electronics of the tube. In general, however, the fields have their own dynamics. In fact, fields can exist on their own, with no charges around (think about the light of a distant star). To describe the dynamics of the electromagnetic fields we need a term in the action describing it. We can pretty much guess what it has to be. The same rules we used in the particle action are in effect here. In order to get a relativistically invariant dynamics the action has to be a Lorentz scalar. Also, it has to be gauge invariant as we know that fields related by a gauge transformation describe the same physics. Gauge invariance forbids terms like $A_\mu A^\mu$. $F^{\mu\nu}$ though is gauge invariant and we can use it to build the action. A few possibilities are allowed

$$
\begin{aligned}
S_1 &= \int d^4x F_{\mu\nu} F^{\mu\nu}, \\
S_2 &= \int d^4x F_{\mu\nu} F_{\alpha\beta} \epsilon^{\alpha\beta\mu\nu}, \\
S_3 &= \int d^4x (F_{\mu\nu} F^{\mu\nu})^2, \\
&\vdots
\end{aligned}
\tag{2.1}
$$

The inclusion of S_2 does not change the equations of motion as it can be written as a total divergence. If it did, it would lead to disaster because S_2 is a Lorentz scalar but it flips sign under a parity ($\vec{r} \to -\vec{r}$) transformation. As a consequence, including this term would lead to the laws of electromagnetism to break parity and this is not observed in Nature (although weak nuclear forces are know to break parity). So we forget about S_2. S_3, and similar terms involving more powers of $F_{\mu\nu}$ are negligible, compared to S_2 in the limit of weak fields. Thus, we will take the action for the electromagnetic field to be

$$S = \frac{1}{16\pi} \int d^4x F_{\mu\nu} F^{\mu\nu}. \tag{2.2}$$

The normalization is arbitrary and simply defines the units we are going to measure the electromagnetic field and charge. We can change the 16π factor by absorbing it into A_μ. But then, in order for the Lorentz force not to change we need to change the charge q. Thus the normalization of the action can be absorbed in to the units we measure charge.

The presence of terms in the action with more than two powers of F would lead to *non*-linear equations of motion for the electromagnetic field. The superposition of two of its solutions would *not* be another solution. In other words, the electromagnetic fields would interact with each other, just like the light sabers on Star Wars. How big do the fields have to be in order for us to notice the higher powers of F? Even if we initially drop S_3, quantum effects generate effects that mock terms in the action with higher powers of F (the so-called Euler-Heisenberg lagrangian, the Euler here is not the Euler you are thinking about). Quantum electrodynamics, in fact, predicts what the coefficient should be. It turns out that the fields in the most powerful lasers we can make are just a little too weak for the nonlinear effects to be observed. But maybe we will see then within our lifetimes. For the present course we will just disregard this possibility and stick to eq. (2.2). Adding to it the action for the charged particles we have

$$S = \frac{1}{16\pi} \int d^4x F_{\mu\nu} F^{\mu\nu} + \sum_a \int d\tau \;\; m^{(a)} c \sqrt{\frac{dx^{\mu(a)}(\tau)}{ds} \frac{dx^{(a)}_\mu(\tau)}{d\tau}} + \frac{q^{(a)}}{c} \frac{dx^{\mu(a)}(s)}{d\tau} A_\mu(x^{(a)}(\tau)), \tag{2.3}$$

where a indexes the different charged particles and their masses/charges. The term coupling the particles to the fields can be rewritten as

$$\sum_a q^{(a)} \frac{dx^\mu}{d\tau} = \sum_a q^{(a)} (c, \vec{v})^{(a)} = \sum_a (c\rho, \vec{j})^{(a)} = j^\mu, \tag{2.4}$$

where ρ and $\vec{j}$ are the charge density and current. In terms of the four-current j^μ the action is written as

$$S = \frac{1}{16\pi} \int d^4x F_{\mu\nu} F^{\mu\nu} + \sum_a \int d\tau \;\; m^{(a)} c \sqrt{\frac{dx^{\mu(a)}(\tau)}{ds} \frac{dx^{(a)}_\mu(\tau)}{d\tau}} + \frac{1}{c} \int d^4x \; j^\mu(x) A_\mu(x). \tag{2.5}$$

What equations of motion follow from this action? The variation in relation to the particle paths just give their equations of motion; the field action doesn't depend on $x^\mu(s)$ so it doesn't alter the equation of motion for $x^\mu(s)$ (eq.(1.25)). In order to minimize the action we need in relation to A_μ we need a slight generalization of the Euler-Lagrange equations we derived before. A_μ is function of the four variables x^μ, not one (s) like before. But that doesn't make much of a difference. Variation of $A_\mu(x)$ gives then We have then

$$\begin{aligned} 0 &= \frac{1}{16\pi} \partial_\mu \frac{\partial F_{\alpha\beta} F^{\alpha\beta}}{\partial \partial_\mu A_\nu} - \frac{1}{c} \frac{\partial j^\mu(x) A_\mu(x)}{\partial A_\nu} \\ &= \frac{1}{16\pi} \partial_\mu \frac{\partial \eta^{\alpha\gamma} \eta^{\beta\delta} (\partial_\alpha A_\beta - \partial_\alpha A_\beta)(\partial_\gamma A_\delta - \partial_\delta A_\gamma)}{\partial \partial_\mu A_\nu} - \frac{1}{c} j^\nu \\ &= \frac{1}{4\pi} \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{c} j^\nu, \end{aligned} \tag{2.6}$$

that is

$$\boxed{\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu}. \tag{2.7}$$

Let us see at how this equation looks like in 3D notation. Taking $\nu = 0$ in eq. (2.7) gives

$$\partial_0 \underbrace{F^{00}}_{=0} + \partial_1 \underbrace{F^{10}}_{E^1} + \partial_2 \underbrace{F^{20}}_{E^2} + \partial_3 \underbrace{F^{30}}_{E^3} = \frac{4\pi}{c} \underbrace{j^0}_{c\rho} \Rightarrow \nabla.\vec{E} = 4\pi\rho. \tag{2.8}$$

Taking $\nu = 3$

$$\underbrace{\frac{d}{cdt}}\partial_0 \underbrace{F^{03}}_{-E^3} + \partial_1 \underbrace{F^{13}}_{B^2} + \partial_2 \underbrace{F^{23}}_{-B^1} + \partial_3 \underbrace{F^{33}}_{0} = \frac{4\pi}{c} j^3 \Rightarrow (\nabla \times \vec{B})^3 - \frac{1}{c}\frac{dE^3}{dt} = \frac{4\pi}{c} j^3, \tag{2.9}$$

and similarly for $\nu = 1, 2$. So eq.(2.7) contains the inhomogeneous Maxwell's equations. What about the homogeneous ones? Remember that, in our formalism, $F_{\mu\nu}$ is just a short hand for $\partial^\mu A^\nu - \partial^\nu A^\mu$. Thus,

$$\partial_\mu F_{\nu\lambda} + \partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} = 0 \tag{2.10}$$

vanishes identically. Eq.(2.7) is equivalent to the homogeneous Mazwell's equations. In fact, take $\mu = 1, \nu = 2$ and $\lambda = 3$. We have then

$$\begin{aligned} 0 &= \partial^1 F^{23} + \partial^3 F^{12} + \partial^2 F^{31} \\ &= \partial_1 B^1 + \partial_2 B^2 + \partial_3 B^3 \\ &= \nabla.\vec{B}. \end{aligned} \tag{2.11}$$

Taking $\mu = 0, \nu = 1$ and $\lambda = 2$:

$$\begin{aligned} 0 &= \underbrace{\partial^0}_{\frac{\partial}{c\partial t}} F^{12} + \partial^2 F^{01} + \partial^1 F^{20} \\ &= -\frac{1}{c}\frac{\partial B^3}{\partial t} + \partial_2 E^1 - \partial_1 E^2 \\ &= -(\nabla \times \vec{E})^3 - \frac{1}{c}\frac{\partial B^3}{\partial t}, \end{aligned} \tag{2.12}$$

and similar for other components.

III. CONCLUSION

Why do we do all this?