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Fractional Hamilton's equations of motion in fractional time

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Abstract: The Hamiltonian formulation for mechanical systems containing Riemman–Liouville fractional derivatives are investigated in fractional time. The fractional Hamilton's equations are obtained and two examples are investigated in detail.

Keywords: Fractional calculus, fractional derivatives, fractional Lagrangian and Hamiltonian, fractional time.
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1 Introduction

Although the embedding space in our world is a three dimensional Euclidean space, the motion of material objects is not always in three dimensions. The dimensionality depends on constraint conditions [1].

Besides, in some applications, the fractional dimensions appear as an explicit param-

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eter when the physical problem is formulated in α dimensions in such a way that α may be extended to non-integer values, as occurs in Wilson's study of quantum field theory models in less than four-dimensions [2], or as in the approach to quantum mechanics proposed by Stillinger [3].

In [2, 3] it was pointed out that the fractional dimensional space represents an effective physical description of confinement in low-dimensional systems. In addition, the integer space is extended to the case of fractional space [2–6]. Some important applications of the fractional dimension of space can be found in [7] and some experimental results were reported in [8, 9]. Spacetime was modelled as a fractal subset of $\mathbb{R}^n$ in [10] and a framework of calculus on net fractals was obtained in [11].

Fractional calculus deals with the generalization of differentiation and integration to non-integer orders [13–18]. Various mathematicians have built up a large body of mathematical knowledge on fractional integrals and derivatives. Fractional calculus, as a natural generalization of classical calculus, has played a significant role in engineering, science, pure and applied mathematics in recent years [19–42].

Many applications of fractional calculus are based on replacing the time derivative in an evolution equation with a derivative of fractional order. A relation between stable distributions in probability theory and the fractional integral was obtained in [43]. The fractional integral and its physical interpretation was discussed in [44]. Under the condition that the electric and magnetic fields are defined on fractals and do not exist outside of fractals in Euclidean space, the fractional generalization of the integral Maxwell equations was considered in [45]. The results of many recent researchers illustrate that fractional derivatives seem to arise for deep mathematical reasons.The fractional derivatives arise as the infinitesimal generators of a class of translation invariant convolution semigroups. These semigroups appear universally as attractors. The fractional variational principles [23–27], [31–43] are under continuous development and some interesting applications were reported recently (see for example Refs. [40–42] and the references there in).

The above mentioned results suggest that interest in the fractional variational calculus is continuing but much remains to be investigated.

The main aim of this paper is to obtain the fractional Hamilton's equations for a discrete systems on a fractional space.

The paper is organized as follows:

In Section 2, some of the basic properties of the RL fractional derivatives are reviewed. Section 3 presents the fractional Hamilton's equations in fractional time. Two examples are described in Section 4. Conclusions are presented in section 5.

2 Mathematical tools

In this section, we formulate the problem in terms of the left and the right Riemann–Liouville (RL) fractional derivatives, which are defined as follows:

the left Riemann–Liouville fractional derivative [12–16]

$$ {}_a\mathbf{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int\limits_a^t (-\tau + t)^{n-\alpha-1} f(\tau) d\tau, \tag{1} $$

and the right Riemann–Liouville fractional derivative

$$ {}_t\mathbf{D}_b^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int\limits_t^b (\tau - t)^{n-\alpha-1} f(\tau) d\tau. \tag{2} $$

Here the order α fulfills $n-1 \leq \alpha < n$ and Γ denotes the Euler's Gamma function. It can be shown that if α becomes an integer, we recovered the usual definitions, namely,

$$ {}_a\mathbf{D}_t^{\alpha} f(t) = \left(\frac{d}{dt}\right)^{\alpha} f(t), \quad {}_t\mathbf{D}_b^{\alpha} f(t) = \left(-\frac{d}{dt}\right)^{\alpha} f(t), \;\; \alpha = 1, 2, ... \tag{3} $$

Fractional RL derivatives possess several interesting properties. The RL derivative of a constant is not zero, namely

$$ {}_a\mathbf{D}_t^{\alpha} C = C \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}. \tag{4} $$

The RL derivative of a power of t is given by

$$ {}_a\mathbf{D}_t^{\alpha} t^{\beta} = \frac{\Gamma(\alpha+1) t^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}, \tag{5} $$

such that $\alpha > -1, \beta \geq 0$. Finally, the fractional product rule becomes

$$ {}_a\mathbf{D}_t^{\alpha} (fg) = \sum_{j=0}^{\infty} \begin{pmatrix} \alpha \\ j \end{pmatrix} \left({}_a\mathbf{D}_t^{\alpha-j} f\right) \left(\frac{d^i g}{dt^j}\right). \tag{6} $$

By inspection we observe that the fractional product contains infinitely many terms and this product takes into account the memory effect.

The result given in (4) creates complications in solving the fractional differential equation by using RL fractional derivatives. Very recently, based on finite difference [46], an alternative definition was proposed for the Riemann–Liouville derivatives (for more details see Ref. [46] and the references there in). By using the approach presented in [46] the troublesome effects of the initial conditions in the RL fractional derivative are removed.

3 Fractional equations of motion

3.1 Fractional Euler–Lagrange equations

Let us consider the action function of the following form

$$ S = \frac{1}{\Gamma(\alpha)} \int_a^b L(\tau, {}_a\mathbf{D}_\tau^{\beta} q, {}_\tau\mathbf{D}_b^{\gamma} q)(t-\tau)^{\alpha-1} d\tau, \tag{7} $$

where $0 \leq \beta \leq 1, \quad 0 < \gamma < 1, \quad 0 \leq \alpha \leq 1$, t represents the observer time and τ denotes the intrinsic time.The appearance of the multi-time characteristic time is important in applications.

Let us the ϵ finite variations of function S then

$$\Delta_\epsilon S = \int_a^b L(q + \epsilon\ \delta q, {}_a\mathbf{D}_\tau^\beta q + \epsilon\ {}_a\mathbf{D}_\tau^\beta \delta q, {}_\tau\mathbf{D}_b^\gamma q + \epsilon\ {}_\tau\mathbf{D}_b^\gamma \delta q)(t-\tau)^{\alpha-1} d\tau. \tag{8}$$

This equation leads us to obtain the Euler–Lagrange equations of motion which reads as

$$\frac{\partial L}{\partial q} + \frac{1}{(t-\tau)^{\alpha-1}} \left[{}_\tau\mathbf{D}_b^\beta \left(\frac{\partial L}{\partial({}_a\mathbf{D}_\tau^\beta q)} (t-\tau)^{\alpha-1} \right) + {}_a\mathbf{D}_\tau^\gamma \left(\frac{\partial L}{\partial({}_\tau\mathbf{D}_b^\gamma q)} (t-\tau)^{\alpha-1} \right) \right] = 0. \tag{9}$$

For $\beta = \gamma = 1$ and assuming that the Lagrangian depends only on ${}_a\mathbf{D}_\tau^\beta q$ or ${}_\tau\mathbf{D}_b^\beta$ we obtain [31]

$$\frac{\partial L}{\partial q} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\alpha-1}{t-\tau}\frac{\partial L}{\partial \dot{q}} = 0. \tag{10}$$

From (10) we observe the presence of a fractional generalized external force acting on the system. The presence of this term (see Ref. [47] and the references therein) provides potential applications of the present approach in cosmology, finance and in all processes involving dissipative Lagrangians. By inspection, using (10) we obtain for $\alpha = 1$ the classical Euler–Lagrange equations.

3.2 The fractional Hamilton's equations of motion

Let us define the Lagrangian of the system in the form

$$L^* = L(\tau, {}_a\mathbf{D}_\tau^\beta q, {}_\tau\mathbf{D}_b^\gamma q)(t-\tau)^{\alpha-1}. \tag{11}$$

Following references [32, 40], we define the generalized momenta as

$$p_\mu^\alpha = \frac{\partial L^*}{\partial_a \mathbf{D}_\tau^\mu q}, \tag{12}$$

$$p_\nu^\alpha = \frac{\partial L^*}{\partial_\tau \mathbf{D}_b^\nu q}. \tag{13}$$

The canonical Hamiltonian is defined as

$$H = p_\mu^\alpha {}_a\mathbf{D}_\tau^\mu q + p_\nu^\alpha {}_\tau\mathbf{D}_b^\nu q - L^*. \tag{14}$$

Making use of the generalized momenta and using the equations of motion, we obtain the canonical Hamiltonian in terms of the canonical phase space $(q, p_\mu^\alpha, p_\nu^\alpha)$ as follows

$$H = H(q, p_\mu^\alpha, p_\nu^\alpha, \tau). \tag{15}$$

The equations of motion are given by

$$\frac{\partial H}{\partial p_\mu^\alpha} = {}_a\mathbf{D}_\tau^\mu q, \quad \frac{\partial H}{\partial p_\nu^\alpha} = {}_\tau\mathbf{D}_b^\nu q, \quad \frac{\partial H}{\partial q} = {}_\tau\mathbf{D}_b^\mu p_\mu^\alpha + {}_a\mathbf{D}_\tau^\nu p_\nu^\alpha, \quad \frac{\partial H}{\partial \tau} = -\frac{\partial L^*}{\partial \tau}. \tag{16}$$

4 Examples

4.1 Fractional free particle

As a first example let us consider the action function

$$S = \int_a^b \frac{1}{2}\dot{x}(\tau)^2 (t-\tau)^{\alpha-1} d\tau. \tag{17}$$

We propose the fractional Lagrangian corresponding to (17) as

$$S' = \int_a^b \frac{1}{2}\left({}_a\mathbf{D}_\tau^\beta x(\tau)\right)^2 (t-\tau)^{\alpha-1} d\tau. \tag{18}$$

The generalized momenta have the following

$$p_\beta^\alpha = {}_a\mathbf{D}_\tau^\beta x(\tau)(t-\tau)^{\alpha-1}. \tag{19}$$

The canonical Hamiltonian is calculated as

$$H = \frac{p_\beta^{\alpha 2}}{2(t-\tau)^{\alpha-1}}. \tag{20}$$

The Hamilton's equation of motion becomes

$${}_\tau\mathbf{D}_b^\beta p_\beta^\alpha = {}_\tau\mathbf{D}_b^\beta \left({}_a\mathbf{D}_\tau^\beta x(\tau)(t-\tau)^{\alpha-1}\right) = 0. \tag{21}$$

The solution of (21) is given by

$$x(\tau) = C\frac{(\tau-a)^{\beta-1}}{\Gamma(\beta)} + \frac{\int_a^\tau \frac{[(b-\sigma)(\tau-\sigma)]^{\beta-1}}{(t-\sigma)^{\alpha-1}} d\sigma}{\Gamma(\beta)}. \tag{22}$$

It was observed that for $\beta \to 1$, the equation (21) becomes

$$\ddot{x}(\tau)(t-\tau) - \dot{x}(\tau)(\alpha-1) = 0, \tag{23}$$

having a solution:

$$x(\tau) = C_1 + C_2(-t+\tau)^{-\alpha+2}. \tag{24}$$

For integer dimensional case, when $t=0$ and in the limit $\alpha \to 1$ the classical solution was recovered, namely

$$x(\tau) = C_1 + C_2\tau, \tag{25}$$

where C_1 and C_2 are constants.

4.2 Fractional simple pendulum

As a second example let us consider the fractional generalization of simple pendulum of length l attracted to the circumference of a body of negligible radius and mass m. The classical Lagrangian is

$$L = \frac{1}{2}\dot{\theta}^2 - \frac{1}{2}mgl\theta^2. \tag{26}$$

Here θ denotes the angular coordinate. The fractional Lagrangian for this systems has the form

$$L = \left(\frac{1}{2}\left({}_a\mathbf{D}_\tau^\beta\theta\right)^2 - \frac{1}{2}mgl\theta^2\right)(t-\tau)^{\alpha-1}. \tag{27}$$

The generalized momenta are calculated as

$$p_\beta^\alpha = {}_a\mathbf{D}_\tau^\beta\theta(\tau)(t-\tau)^{\alpha-1}. \tag{28}$$

The canonical Hamiltonian is calculated as

$$H = \frac{p_\beta^{\alpha 2}}{2(t-\tau)^{\alpha-1}} + \frac{1}{2}mgl\theta^2(t-\tau)^{\alpha-1}. \tag{29}$$

The Hamilton's equations of motion lead to

$$\frac{{}_\tau\mathbf{D}_b^\beta\left(({}_a\mathbf{D}_\tau^\beta\theta)(t-\tau)^{\alpha-1}\right)}{(t-\tau)^{\alpha-1}} - mgl\theta = 0 \tag{30}$$

For $\beta \to 1$, we have

$$\ddot{\theta}(\tau) + \frac{(\alpha-1)}{t-\tau}\dot{\theta}(\tau) + mgl\theta(\tau) = 0. \tag{31}$$

The solution of (31) is given by

$$\begin{aligned}\theta(\tau) &= C_1 e^{-i\sqrt{mgl}\tau} KummerM\left(-\frac{1}{2}+\frac{\alpha}{2}, \alpha-1, -2i\sqrt{mgl}(t-\tau)\right)\\ &+ C_2 e^{-i\sqrt{mgl}\tau} KummerU\left(-\frac{1}{2}+\frac{\alpha}{2}, \alpha-1, -2i\sqrt{mgl}(t-\tau)\right),\end{aligned} \tag{32}$$

where *KummerM* and *KummerU* are Kummer functions and C_1 and C_2 are constants. For $\alpha = 1$ and $t = 0$, the classical solution is reobtained.

5 Conclusions

In this study the fractional Hamilton's and Euler–Lagrange equations were investigated. The fractional canonical equations on a classical space are obtained. The classical Euler–Lagrange are obtained for $\beta \to 1$ and $\alpha \to 1$. The fractional free particle and the fractional pendulum on fractional space were analyzed and the analytical solutions of their corresponding fractional Hamiltonian equations were obtained. The results so obtained can be applied to describe the weak dissipative and nonconservative dynamical systems.

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