Solutions Manual for *Bayesian Methods for the Physical Sciences*

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Contents

Chapter 2

Exercise 1

A single spin of a roulette wheel contains 38 possible outcomes (at least in the USA, as opposed to the French which have 37, we will work with the USA roulette wheel), the digits 1-36, and the additional events "0" and "00".

Exercise 1a

The probability of observing a "00" is 1/38 (i.e., the number of possible ways to observe this event divided by the total number of outcomes).

Exercise 1b

From Table 1, the probability is the same for all the requested outcomes, which is 9/38.

Exercise 1c

Using Table 1, the probability of observing a red is 18/38 or 9/19.

Exercise 1d

No they do not sum to one, i.e., $4 \times 9/38 = 36/38 < 1$. The slight difference is that the "0" and the "00" are not considered odd, even, red, or black according the rules of roulette.

	Red	Black	Total
Odd	9/38	9/38	18/38
Even	9/38	9/38	18/38
Total	18/38	18/38	

Table 1: Table of probabilities for Exercise 1 of Chapter 2.

Exercise 2

Using the results from the section on independence, i.e., if two random variables x and y are independent with densities $p(x)$ and $p(y)$, then their joint distribution is

$$p(x, y) = p(x)p(y).$$

The product of our two normal densities is,

$$p(\mathtt{x}[1], \mathtt{x}[2]) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(\mathtt{x}[1] - \mathtt{m}[1])^2\right] \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(\mathtt{x}[2] - \mathtt{m}[2])^2\right]$$
$$= \frac{1}{2\pi} \exp\left\{-\frac{1}{2}[(\mathtt{x}[1] - \mathtt{m}[1])^2 + (\mathtt{x}[2] - \mathtt{m}[2])^2]\right\}.$$

Exercise 3

Recall that the marginal distribution of a continuous random variable is

$$p(x) = \int p(x, y)dy.$$

For a bivariate normal, we will compute

$$p(\mathtt{x}[1]) = \int_{-\infty}^{\infty} p(\mathtt{x}[1], \mathtt{x}[2])d\mathtt{x}[2] = \int_{-\infty}^{\infty} \frac{1}{2\pi \mathtt{s}[1]\mathtt{s}[2]\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}(w^2 - 2\rho wz + z^2)\right] \mathtt{s}[2]dz$$

where $w = \frac{\mathtt{x}[1]-\mathtt{m}[1]}{\mathtt{s}[1]}$, $z = \frac{\mathtt{x}[2]-\mathtt{m}[2]}{\mathtt{s}[2]}$, and $d\mathtt{x}[2] = \mathtt{s}[2]dz$.

Next remove the terms from the integrand which do not involve z and complete the square:

$$p(\mathtt{x}[1]) = \frac{\exp\left[\frac{-w^2}{2(1-\rho^2)}\right]}{2\pi \mathtt{s}[1]\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left[\frac{-1}{2(1-\rho^2)}(z^2 - 2\rho wz + \rho^2 w^2 - \rho^2 w^2)\right] dz$$
$$= \frac{\exp\left[\frac{-w^2}{2(1-\rho^2)}\right] \exp\left[\frac{\rho^2 w^2}{2(1-\rho^2)}\right]}{2\pi \mathtt{s}[1]\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left[\frac{-1}{2(1-\rho^2)}(z - \rho w)^2\right].$$

Notice that the integrand is the kernel for a normal distribution with variance $(1-\rho^2)$ and mean ρw. And so this integrates to $\sqrt{2\pi}\sqrt{1-\rho^2}$. With further simplification we get

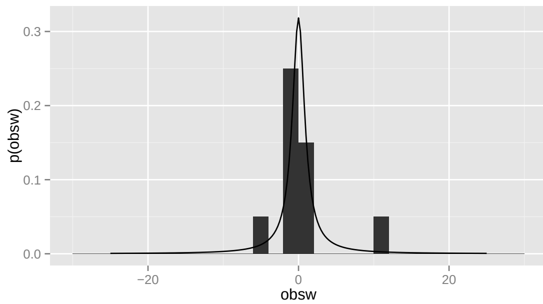
$$p(\mathtt{x}[1]) = \frac{\exp\left[\frac{-w^2}{2}\right]}{2\pi \mathtt{s}[1]\sqrt{1-\rho^2}} \sqrt{2\pi}\sqrt{1-\rho^2} = \frac{\exp\left[\frac{-w^2}{2}\right]}{\mathtt{s}[1]\sqrt{2\pi}}.$$

Exercise 4

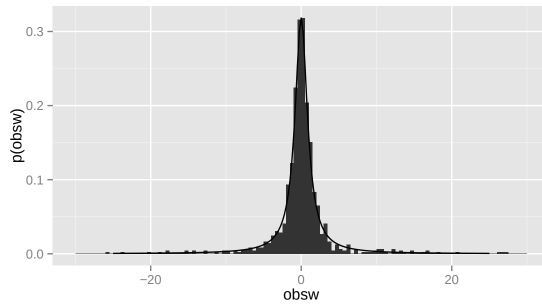
Figure 1 shows the results from this simple simulation. Notice that as the simulation size increases, the histogram better matches the actual density. Similar results occur with the alternate definition of a Cauchy distribution.

Exercise 5

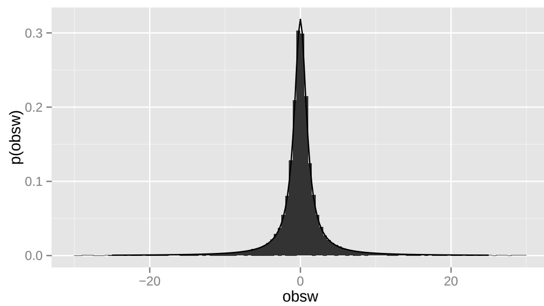
We use a gamma(1,1) distribution for these exercises.



(a)



(b)



(c)

Figure 1: Simulated Cauchy random variables for Exercise 4 of Chapter 2 for $n = 10$ (Figure 1a), $n = 1000$ (Figure 1b), and $n = 10,000$ (Figure 1c).

Exercise 5a

Using our simulations we calculate $[E(\texttt{obsx})]^2 = 1.005$ and $E(\texttt{obsy}) = 2.005$. Notice that $[E(\texttt{obsx})]^2 < E(\texttt{obsy})$.

Exercise 5b

Next we calculate $\sqrt{\texttt{obsx}[i]}$ and $1 + 2\texttt{obsx}[i]$. These results are $E(\sqrt{\texttt{obsx}}) = 0.8875$ and $\sqrt{E(\texttt{obsx})} = 1.001$ yielding $E(\sqrt{\texttt{obsx}}) < \sqrt{E(\texttt{obsx})}$. Lastly, $E(1 + 2\texttt{obsx}) = 3.005$ and $1 + 2E(\texttt{obsx}) = 3.005$ giving $E(1 + 2\texttt{obsx}) = 1 + 2E(\texttt{obsx})$.

In summary, we observe $[E(x)]^2 < E(x^2)$ (a convex function), $\sqrt{E(x)} > E(\sqrt{x})$ (a concave function), and $1 + 2E(x) = E(1 + 2x)$ (a linear function). The interested reader should consult Jensen's inequality to see this result in its full generality.

Exercise 6

We leave this solution to the discretion of the instructor.

Chapter 4

Exercise 1

From the problem's setup we have

$$p(\texttt{obsy}|\texttt{s}) = \texttt{s}\exp(-\texttt{s} * \texttt{obsy}),$$

and

$$p(\texttt{s}|\texttt{a},\texttt{b}) = \frac{\texttt{b}^{\texttt{a}}}{\Gamma(\texttt{a})}\texttt{s}^{\texttt{a}-1}\exp\left(-\texttt{s} * \texttt{b}\right).$$

We begin with Bayes' theorem:

$$p(x|y) \propto p(y|x)p(x).$$

Using the densities for a gamma and exponential distribution, we have

$$\begin{aligned} p(\texttt{s}|\texttt{obsy}) &\propto [\texttt{s}\exp(-\texttt{s} * \texttt{obsy})]\frac{\texttt{b}^{\texttt{a}}}{\Gamma(\texttt{a})}\texttt{s}^{\texttt{a}-1}\exp\left(-\texttt{s} * \texttt{b}\right), \\ &\propto [\texttt{s}\exp(-\texttt{s} * \texttt{obsy})]\texttt{s}^{\texttt{a}-1}\exp\left(-\texttt{s} * \texttt{b}\right) \\ &= \texttt{s}^{\texttt{a}}\exp[-\texttt{s} * (\texttt{obsy} + \texttt{b})] \end{aligned}$$

which is proportional to a gamma distribution with parameters $\texttt{a} + 1$ and $\texttt{obsy} + \texttt{b}$, i.e.,

$$p(\texttt{s}|\texttt{obsy},\texttt{a},\texttt{b}) = \frac{(\texttt{obsy} + \texttt{b})^{\texttt{a}+1}}{\Gamma(\texttt{a}+1)}\texttt{s}^{\texttt{a}}\exp[-\texttt{s} * (\texttt{obsy} + \texttt{b})].$$

We verify this result using JAGS with $\texttt{obsy} = 0.819$ and the following code:

```
model{
obsy ~ dexp(s)# likelihood
s~dgamma(1,1)# prior
}
```

Figure 2 shows the analytically-derived posterior (solid line) superimposed on the numerically-sampled posterior draws (histogram) obtained from JAGS.

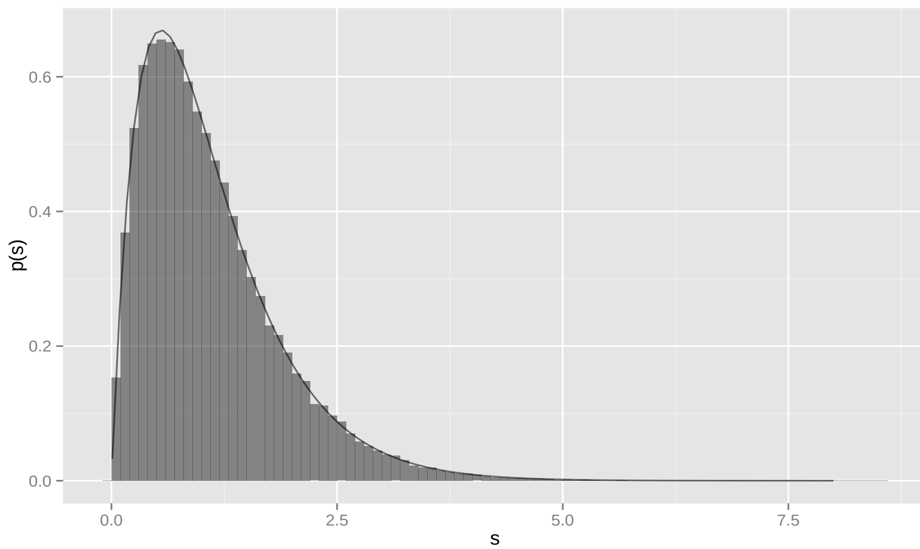


Figure 2: Numerically-sampled posterior distribution (histogram) and analytically derived posterior distribution (solid line) for Exercise 1 of Chapter 4.

Exercise 2

We are given that `obsm` $\sim$ dlnorm(`m`,`prec`) where `prec` is known and `m` $\sim$ dnorm($\mathtt{a}, \mathtt{b}$). Using Bayes' theorem,

$$
\begin{aligned}
\mathtt{p}(\mathtt{m}|\mathtt{obsm}) &\propto \exp\left[-\frac{\mathtt{prec}}{2} * (\log(\mathtt{obsm}) - \mathtt{m})^2\right] \exp\left[-\frac{\mathtt{b}}{2} * (\mathtt{m} - \mathtt{a})^2\right] \\
&\propto \exp\left\{-\frac{1}{2} * \left[\mathtt{prec} * (\log^2(\mathtt{obsm}) - 2\log(\mathtt{obsm}) * \mathtt{m} + \mathtt{m}^2) + \mathtt{b} * (\mathtt{m}^2 - 2\mathtt{m} * \mathtt{a} + \mathtt{a}^2)\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2} * \left[\mathtt{prec} * (\mathtt{m}^2 - 2\log(\mathtt{obsm}) * \mathtt{m}) + \mathtt{b} * (\mathtt{m}^2 - 2\mathtt{m} * \mathtt{a})\right]\right\} \\
&= \exp\left\{-\frac{\mathtt{prec} + \mathtt{b}}{2} * \left[\mathtt{m}^2 - 2\mathtt{m}\frac{\log(\mathtt{obsm}) * \mathtt{prec} + \mathtt{a} * \mathtt{b}}{\mathtt{prec} + \mathtt{b}}\right]\right\} \\
&= \exp\left[-\frac{\mathtt{prec} + \mathtt{b}}{2} * \left(\mathtt{m} - \frac{\log(\mathtt{obsm}) * \mathtt{prec} + \mathtt{a} * \mathtt{b}}{\mathtt{prec} + \mathtt{b}}\right)^2\right],
\end{aligned}
$$

where the last step follows from completing the square. Notice that this is proportional to a normal probability density function with mean $(\log(\mathtt{obsm}) * \mathtt{prec} + \mathtt{a} * \mathtt{b})/(\mathtt{prec} + \mathtt{b})$ and precision $\mathtt{prec} + \mathtt{b}$.

We verify this result using the following code in JAGS:

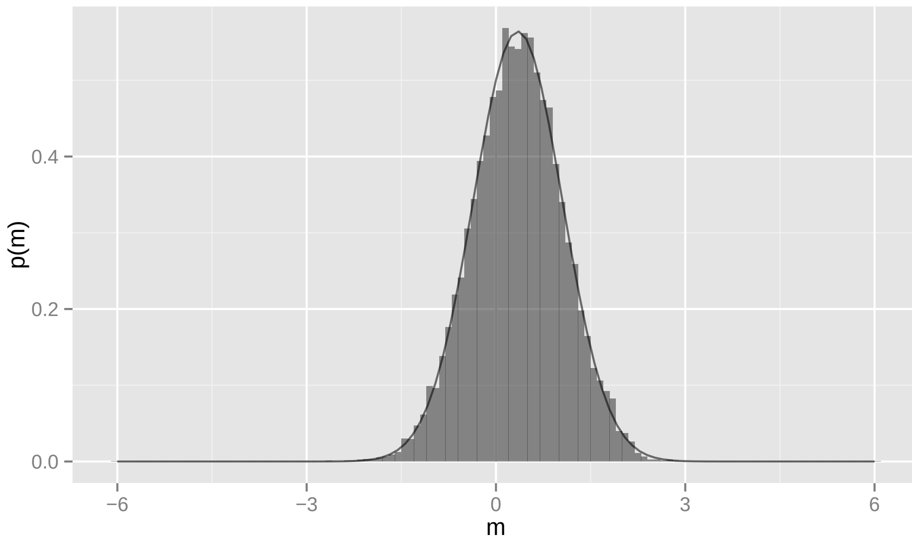


Figure 3: Numerically-sampled posterior distribution (histogram) and analytically derived posterior distribution (solid line) for Exercise 2 of Chapter 4.

```
model{
obsm ~ dlnorm(m,1)# likelihood
m~dnorm(0,1)# prior
}.
```

Figure 3 shows the analytically-derived posterior (solid line) superimposed on the numerically-sampled posterior draws (histogram) obtained from JAGS when we observe $\texttt{obsm} = 2$ from a lognormal distribution.

Exercise 3

From the problem's setup we have

$$p(\texttt{obsx}|\texttt{b}) = \frac{1}{\texttt{b}}$$

for $\texttt{obsx} \in (0, \texttt{b})$ and

$$p(\texttt{b}|\texttt{s}, \texttt{f}) = \texttt{s}\frac{\texttt{f}^{\texttt{s}}}{\texttt{b}^{\texttt{s}+1}}$$

for $\texttt{b} > \texttt{f}$. Begin with Bayes' theorem:

$$p(x|y) \propto p(y|x)p(x).$$

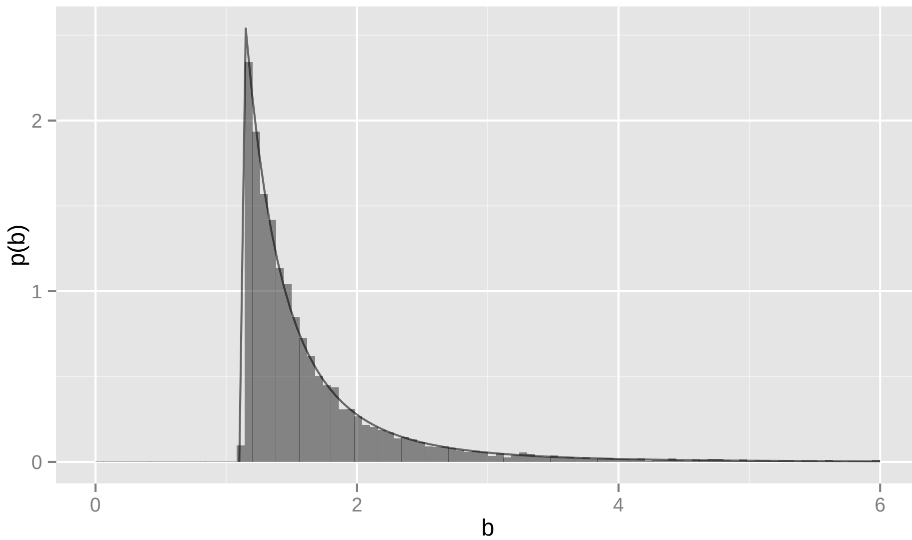


Figure 4: Numerically-sampled posterior distribution (histogram) and analytical posterior distribution (solid line) for Exercise 3 of Chapter 4.

For our exercise, we have

$$
\begin{aligned}
p(\mathtt{b}|\mathtt{obsx},\mathtt{s},\mathtt{f}) &\propto \frac{I(\mathtt{b} > \mathtt{obsx})}{\mathtt{b}}\mathtt{s}\frac{\mathtt{f}^{\mathtt{s}}}{\mathtt{b}^{\mathtt{s}+1}}I(\mathtt{b} > \mathtt{f}),\\
&= \mathtt{s}\frac{\mathtt{f}^{\mathtt{s}}}{\mathtt{b}^{\mathtt{s}+2}}I(\mathtt{b} > \max\{\mathtt{f},\mathtt{obsx}\})
\end{aligned}
$$

which is the density of a Pareto distribution with parameters $\max\{\mathtt{f},\mathtt{obsx}\}$ and $\mathtt{s}+1$ and $I(x \in A)$ is the indicator function where $I(x \in A) = 1$ if $x \in A$ and 0 otherwise.

We verify this result using the following code in JAGS:

```
model{
obsx ~ dunif(0,b)  # likelihood,
b~dpar(2,1) #prior
}.
```

Figure 4 shows the analytically posterior (solid line) superimposed on the numerically-sampled posterior (shaded density) obtained from JAGS when we observed $\mathtt{obsx} = 1.138$.

Exercise 4

We are assuming that $\texttt{n} \sim \texttt{dpois}(\texttt{s})$ and in general $\texttt{s} \sim \texttt{dunif}(\texttt{a}, \texttt{b})$. Then the posterior distribution is

$$
\begin{aligned}
p(\texttt{s}|\texttt{n}) \propto & \frac{\texttt{s}^{\texttt{obsn}} e^{-\texttt{s}}}{\texttt{obsn!}} \frac{1}{\texttt{b} - \texttt{a}} I[\texttt{s} \in (\texttt{a}, \texttt{b}))] \\
\propto & \texttt{s}^{\texttt{obsn}} e^{-\texttt{s}} I[\texttt{s} \in (\texttt{a}, \texttt{b}))] \\
\propto & \texttt{s}^{\texttt{obsn}+1-1} e^{-\texttt{s}/1} I[\texttt{s} \in (\texttt{a}, \texttt{b}))]
\end{aligned} \tag{1}
$$

Notice that $\texttt{s}^{\texttt{obsn}+1-1} e^{-\texttt{s}/1}$ in Equation (1) is proportional to a Gamma density with parameters $\texttt{obsn} + 1$ and 1. Hence,

$$
p(\texttt{s}|\texttt{obsn}) = \frac{1}{\Gamma(\texttt{obsn} + 1)} \texttt{s}^{\texttt{obsn}+1-1} e^{-\texttt{s}/1} I[\texttt{s} \in (\texttt{a}, \texttt{b}))] \tag{2}
$$

Finally, since $\texttt{s} \in (\texttt{a}, \texttt{b})$, we must normalize the distribution in (2). So,

$$
p(\texttt{s}|\texttt{obsn}) = \frac{c}{\Gamma(\texttt{obsn} + 1)} \texttt{s}^{\texttt{obsn}+1-1} e^{-\texttt{s}/1} I[\texttt{s} \in (\texttt{a}, \texttt{b}))] \tag{3}
$$

where

$$
c = \frac{1}{\int_{\texttt{a}}^{\texttt{b}} \frac{1}{\Gamma(\texttt{obsn}+1)} \texttt{s}^{\texttt{obsn}+1-1} e^{-\texttt{s}/1} d\texttt{s}}.
$$

To finish the problem, simply set $\texttt{a} = 0$ and $\texttt{b} = \texttt{smax}$.

We verify this result using the following code in JAGS:

```
model{
obsn ~ dpois(s)  #likelihood
s ~ dunif(0,5) #prior
}.
```

Figure 5 shows the analytically-derived posterior (solid line) superimposed on the numerically-sampled posterior (shaded density) obtained from JAGS with $\texttt{obsn} = 3$.

Exercise 5

In general, for $\texttt{obsn} \sim \texttt{dpois}(s)$, the range of $\texttt{s}$ being unrestricted, and with $\texttt{s} \sim \texttt{dgamma}(k, \theta)$, the posterior distribution is $\texttt{s} \sim \text{Gamma}[k + \texttt{obsn}, \theta/(\theta + 1)]$. So, for $\texttt{s} \in (\texttt{a}, \texttt{b})$,

$$
p(\texttt{s}|\texttt{obsn}) = \frac{c(\theta + 1)^k}{\theta^k} \frac{1}{\Gamma(k + \texttt{obsn})} \texttt{s}^{k+\texttt{obsn}-1} e^{-\texttt{s}*(\theta+1)/\theta} I[\texttt{s} \in (\texttt{a}, \texttt{b}))] \tag{4}
$$

where

$$
c = \frac{1}{\int_{\texttt{a}}^{\texttt{b}} \frac{(\theta+1)^k}{\theta^k} \frac{1}{\Gamma(k+y)} \texttt{s}^{k+\texttt{obsn}-1} e^{-\texttt{s}*(\theta+1)/\theta} d\texttt{s}}.
$$

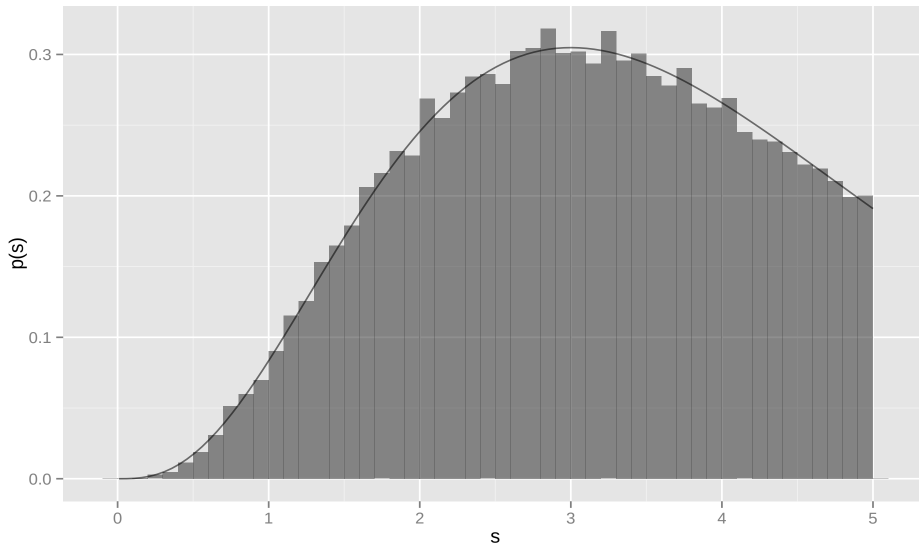


Figure 5: Numerically-sampled posterior distribution (histogram) and analytically derived posterior distribution (solid line) for Exercise 4 of Chapter 4.

To finish the derivation, simply set $\mathtt{a} = 0$ and $\mathtt{b} = \mathtt{smax}$.

We verify this result using the following code in JAGS:

```
model{
obsn ~ dpois(s)  #likelihood
s ~ dgamma(5,1)T(0,8)  #prior with truncation
}.
```

Figure 6 shows the analytically-derived posterior (solid line) superimposed on the numerically-sampled posterior draws (shaded histogram) obtained from JAGS with truncation point $\mathtt{smax} = 8$, and $\mathtt{obsn} = 3$.

Exercise 6

Our model is $\mathtt{obsn} \sim \mathtt{dbin}(\mathtt{f}, \mathtt{n})$ where $\mathtt{f} \sim \mathtt{dunif}(0, 1)$. And so, using a Bayes' theorem

$$p(f|\mathtt{obsn}, \mathtt{n}) \propto \mathtt{f}^{\mathtt{obsn}}(1-\mathtt{f})^{\mathtt{n}-\mathtt{obsn}}.$$

Integrating the above expression yields

$$\int_0^1 \mathtt{f}^{\mathtt{obsn}}(1-\mathtt{f})^{\mathtt{n}-\mathtt{obsn}} d\mathtt{f} = B(\mathtt{obsn}+1, \mathtt{n}-\mathtt{obsn}+1)$$

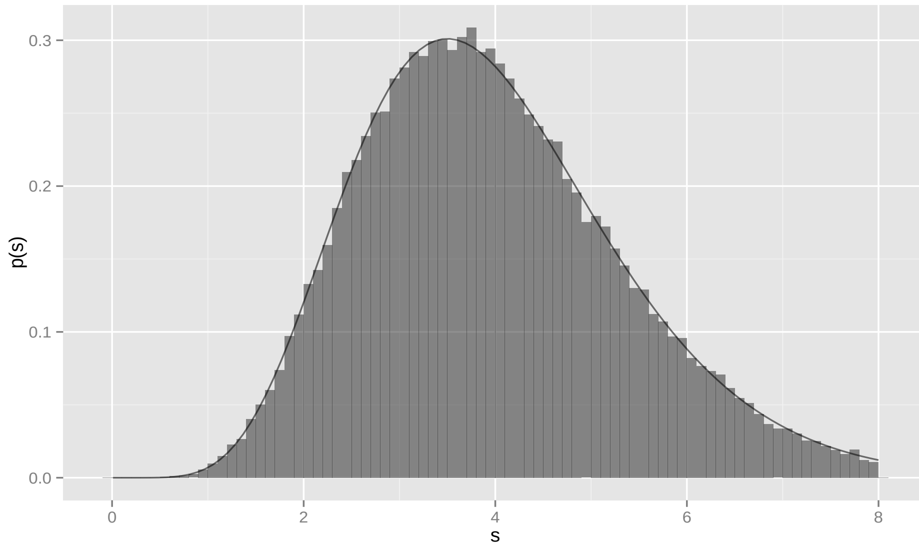


Figure 6: Numerically-sampled posterior distribution (histogram) and analytically derived posterior distribution (solid line) for Exercise 5 of Chapter 4.

where $B(\texttt{obsn}+1, \texttt{n}-\texttt{obsn}+1)$ is the beta function. Hence, the posterior distribution is

$$p(f|\texttt{obsn},\texttt{n}) = \frac{1}{B(\texttt{obsn}+1,\texttt{n}-\texttt{obsn}+1)}\texttt{f}^{\texttt{obsn}}(1-\texttt{f})^{\texttt{n}-\texttt{obsn}}$$

where by using the alternate definition of the beta function gets the expression in the book:

$$B(\texttt{obsn}+1,\texttt{n}-\texttt{obsn}+1) = \frac{(\texttt{obsn})!(\texttt{n}-\texttt{obsn})}{(\texttt{n}+1)!}.$$

We verify this result using the following code in JAGS:

```
model{
obsn ~ dbin(f,n)  #likelihood
f ~ dbeta(1,1)    #prior
}.
```

Figure 7 shows the analytically-derived posterior (solid line) superimposed on the numerically-sampled posterior draws (shaded density) obtained from JAGS with $\texttt{obsn} = 2$ and $\texttt{n} = 10$.

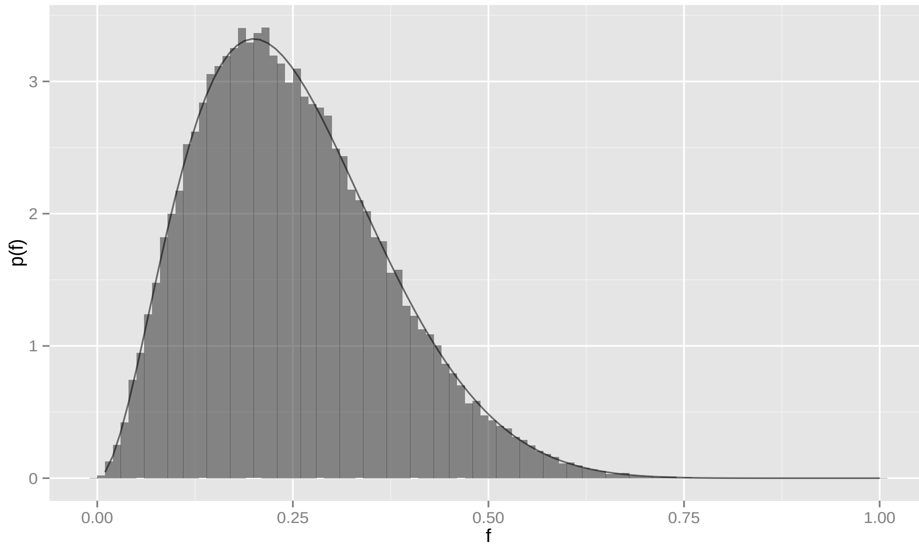


Figure 7: Numerically-sampled posterior distribution (histogram) and analytically derived posterior distribution (solid line) for Exercise 6 of Chapter 4.

Exercise 7

We are assuming $\texttt{obsm} \sim \text{dnorm}(\texttt{m}, \sigma^2)$ where σ^2 is known. Additionally, our prior is $\texttt{m} \sim \text{dunif}(\texttt{a}, \texttt{b})$. The posterior is

$$\texttt{p}(\texttt{m}|\texttt{obsm}) \propto \exp\left[\frac{-(\texttt{obsm} - \texttt{m})^2}{2\sigma^2}\right] I(\texttt{a} < \texttt{m} < \texttt{b})$$

which is proportional to a normal distribution with mean $\texttt{obsm}$ and standard deviation σ. Now we are restricting $\texttt{m}$ to be in $(\texttt{a}, \texttt{b})$, so we need to renormalize our normal distribution to account for this.

Define

$$c = \frac{1}{\sqrt{2\pi}\sigma} \int_{\texttt{a}}^{\texttt{b}} \exp\left[\frac{-(\texttt{m} - \texttt{obsm})^2}{2\sigma^2}\right] d\texttt{m}.$$

Then we can renormalize our posterior by dividing by c, i.e.,

$$\texttt{p}(\texttt{m}|\texttt{obsm}) = \frac{1}{c\sqrt{2\pi}\sigma} \exp\left[\frac{-(\texttt{obsm} - \texttt{m})^2}{2\sigma^2}\right] I(\texttt{a} < \texttt{m} < \texttt{b}).$$

For our specific problem, $1/c \approx 19.8$ (this was calculated using software using $\sigma = 3.3$, $\texttt{obsm} = -5.4$, and $(\texttt{a}, \texttt{b}) = (0, 30)$).

We verify this result using the following code in JAGS:

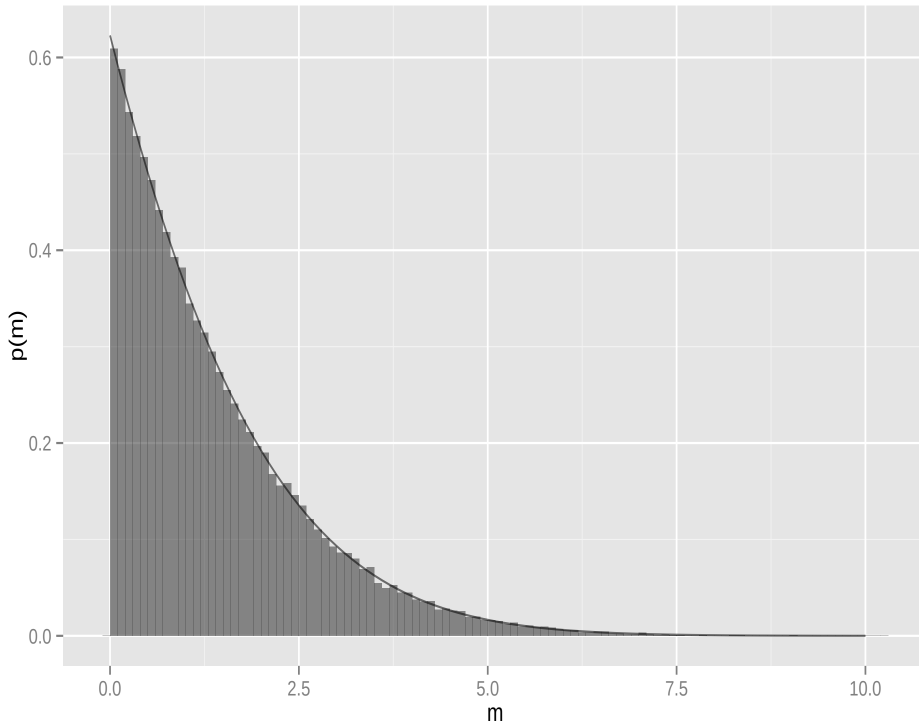


Figure 8: Numerically-sampled posterior distribution (histogram) and analytically derived posterior distribution (solid line) for Exercise 7 of Chapter 4.

```
model{
obsm ~ dnorm(m,pow(sig,-2)   #likelihood
m ~ dunif(0,30)       #prior
}.
```

Figure 8 shows the analytically-derived posterior (solid line) superimposed on the Numerically-sampled posterior draws (shaded density) obtained from JAGS with `sig` $= 3.3$ and `obsm` $= -5.4$.

Chapter 5

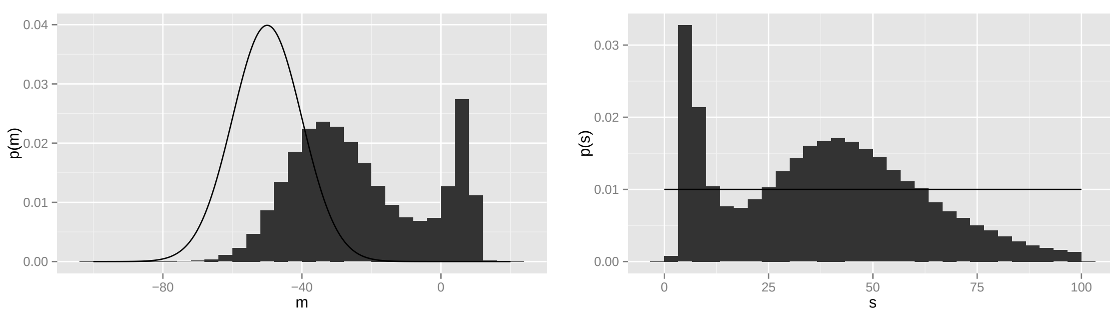
Exercise 1

To analyze the data we use the following JAGS code:

```
model{
##Likelihood
for(i in 1:length(y)){
y[i]~dnorm(m,pow(s,-2))
}
##Prior
s ~dunif(0,100)
m ~ dnorm(-50, .01)
}.
```

Exercise 1a

Figure 9 gives the prior and posteriors for `m` and `s`. These distributions are bimodial, indicating that the posterior is being influenced in different ways by the prior (potentially at odds with the data) and the data.



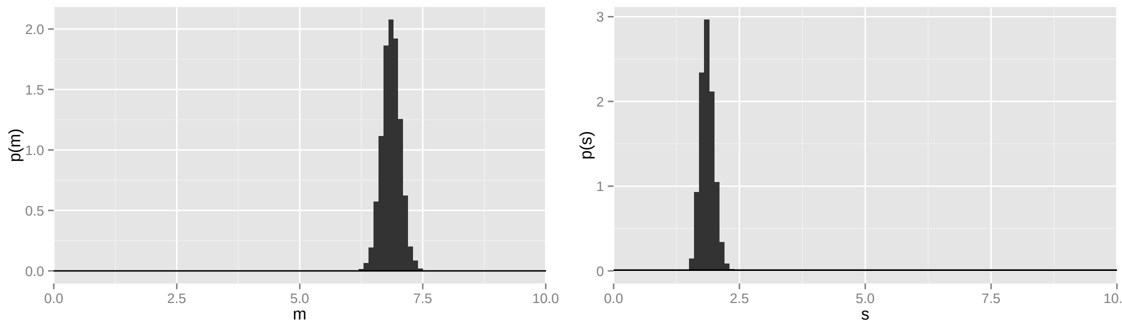
(a) (b)

Figure 9: Prior (solid line) and posterior distribution (histogram) for `m` (Figure 9a) and `s` (Figure 9b) for Exercise 1a of Chapter 5.

Exercise 1b

To simulate the 100 new data points, use the following JAGS code:

```
model{
y~dnorm(7,0.25)
```



(a) (b)

Figure 10: Prior (solid line) and posterior distribution (histogram) for `m` (Figure 10a) and `s` (Figure 10b) for Exercise 1b of Chapter 5.

```
}.
```

Using the newly simulated data and the JAGS code at the beginning of this exercise, Figure 10 gives the priors and posteriors for `m` and `s`. The posterior distribution is now primarily influenced by the data through the likelihood.

Exercise 1c

We omit part c because it is similar to Exercise 1b.

Exercise 1d

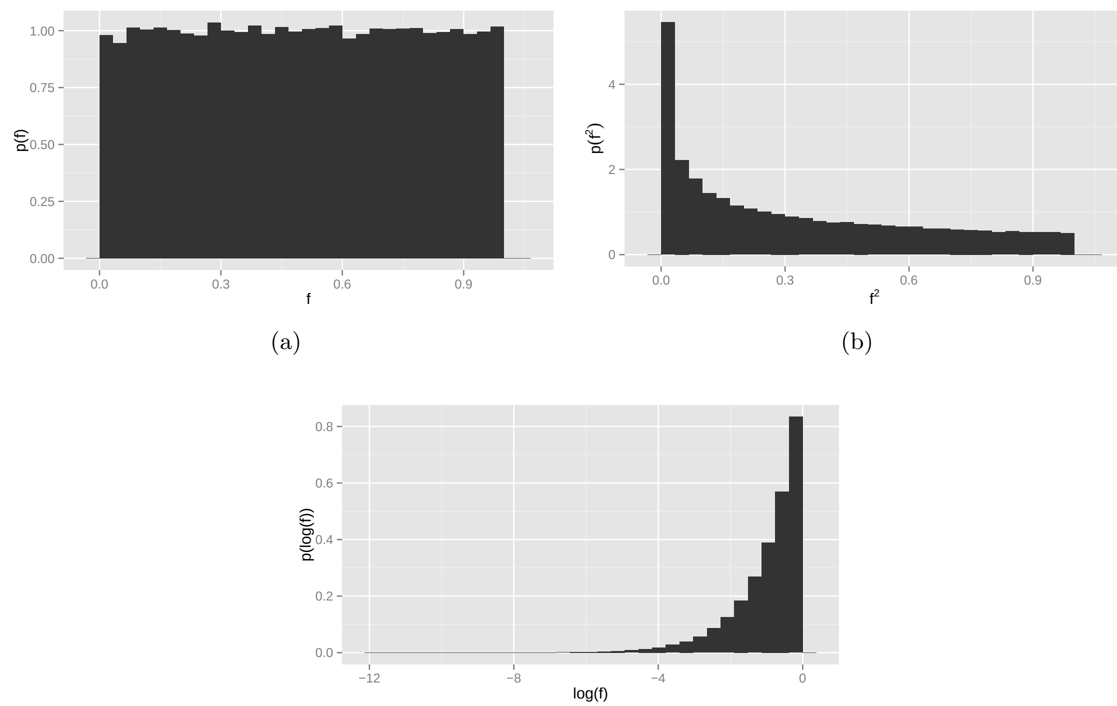
As we collect more data, even if our prior is in contention with the data, the posterior will reflect the information within the data. However, Exercise 1a should be warning, great care must be made when selecting priors.

Exercise 2

The following code can be used for the simulations:

```
model{
f~dunif(0,1)
f2 <-pow(f,2)
logf<-log(f)/log(10)
}.
```

Figure 11 shows the results of the simulations. Figure 11a shows our assumed flat prior. Figure 11b and 11c show the induced prior for $\mathtt{f}^2$ and log(`f`). For the parameters $\mathtt{f}^2$ and



(a) (b) (c)

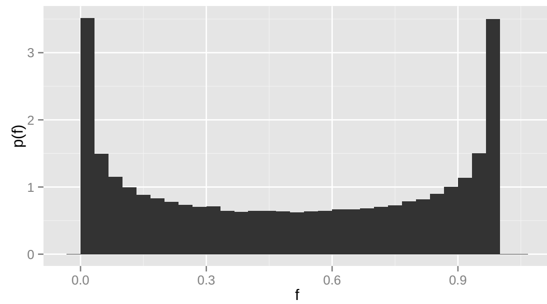
Figure 11: Simulated prior distributions for `f` (Figure 11a), $\texttt{f}^2$ (Figure 11b), and log(`f`) (Figure 11c) for Exercise 2 of Chapter 5.

log(`f`), the priors are no longer non-informative but now favor parameter values on the left portion of the plot (for $\texttt{f}^2$) or the right portion (for log(`f`)). This should teach us that as data analysts, we must be careful in choosing which parameter we are interested in estimating and the prior we place on that parameter.

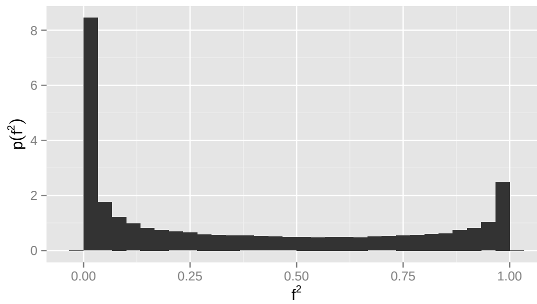
Exercise 3

The following code can be used for the simulations:

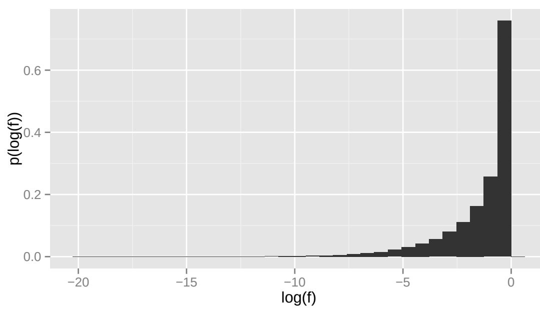
```
model{
f~dbeta(0.5,0.5)
f2 <-pow(f,2)
logf<-log(f)/log(10)
}.
```



(a)



(b)



(c)

Figure 12: Simulated prior distributions for `f` (Figure 12a), $\texttt{f}^2$ (Figure 12b), and log(`f`) (Figure 12c) for Exercise 3 of Chapter 5.

Figure 12 shows the results of the simulations. Figure 12a shows our Jeffrey's prior. Figure 12b and 12c show the induced prior for $\texttt{f}^2$ and log(`f`). Notice that for the parameter $\texttt{f}^2$, the distribution is similar in shape to the distribution of `f` which might be appealing to the researcher. However, for log(`f`), this is the not the case. Again this reemphasizes the fact that as data analysts, we must be careful in choosing which parameter we are interesting in estimating and the prior we place on that parameter.

Exercise 4

We begin using Bayes' theorem

$$\begin{aligned}
p(\mathtt{m}|\mathtt{obsy},\mathtt{prec}) &\propto \exp\left[-\frac{\mathtt{prec}}{2}*(\mathtt{obsy}-\mathtt{m})^2\right]\exp\left[-\frac{\mathtt{b}}{2}*(\mathtt{m}-\mathtt{a})^2\right] \\
&= \exp\left[-\frac{\mathtt{prec}}{2}*(\mathtt{obsy}-\mathtt{m})^2-\frac{\mathtt{b}}{2}*(\mathtt{m}-\mathtt{a})^2\right] \\
&= \exp\left[-\frac{\mathtt{prec}}{2}(\mathtt{obsy}^2-2\mathtt{obsy}*\mathtt{m}+\mathtt{m}^2)-\frac{\mathtt{b}}{2}(\mathtt{m}^2-2\mathtt{m}*\mathtt{a}+\mathtt{a}^2)\right] \\
&\propto \exp\left[-\frac{\mathtt{prec}}{2}*(\mathtt{m}^2-2\mathtt{obsy}*\mathtt{m})-\frac{\mathtt{b}}{2}*(\mathtt{m}^2-2\mathtt{m}*\mathtt{a})\right] \\
&= \exp\left[-\frac{(\mathtt{prec}+\mathtt{b})}{2}*\mathtt{m}^2-(\mathtt{obsy}*\mathtt{prec}+\mathtt{b}*\mathtt{a})*\mathtt{m}\right] \\
&= \exp\left[-\frac{(\mathtt{prec}+\mathtt{b})}{2}*\left(\mathtt{m}-\frac{\mathtt{obsy}*\mathtt{prec}+\mathtt{b}*\mathtt{a}}{\mathtt{prec}+\mathtt{b}}\right)^2\right]
\end{aligned}$$

where the last equality follows from completing the square. Notice that this is proportional to a normal distribution with mean $(\mathtt{obsy}*\mathtt{prec}+\mathtt{b}*\mathtt{a})/(\mathtt{prec}+\mathtt{b})$ and precision parameter $\mathtt{prec}+\mathtt{b}$. Hence the posterior distribution is

$$p(\mathtt{m}|\mathtt{obsy},\mathtt{prec}) = \frac{\mathtt{prec}+\mathtt{b}}{\sqrt{2\pi}}*\exp\left[-\frac{(\mathtt{prec}+\mathtt{b})}{2}*\left(\mathtt{m}-\frac{\mathtt{obsy}*\mathtt{prec}+\mathtt{b}*\mathtt{a}}{\mathtt{prec}+\mathtt{b}}\right)^2\right].$$

Notice now that this is the same family of distributions that the prior is within! This is the essential idea to conjugacy, the posterior is within the same distributional family as the prior.

Exercise 5

For this exercise, we have $\mathtt{obsn} \sim \mathtt{dnorm}(\mathtt{s},\mathtt{pow}(2,-2))$, $\mathtt{s} = \mathtt{tmps}^{-1/1.5}$, and $\mathtt{tmps} \sim \mathtt{dunif}(0,100)$. Recall we observe $\mathtt{obsn} = 4$. Therefore the JAGS code reads:

```
model{
#Likelihood
obsn ~ dnorm(s,pow(2, -2))
#Prior
s <- pow(tmps, -0.666666666)
tmps~ dunif(0,100).
}
```

Figure 13 displays the numerically-sampled posterior distribution (dark histogram) and the prior distribution (red histogram) for s. Notice that even though we assumed a symmetric distribution for s, the posterior's mean or median is still smaller than the observed value.

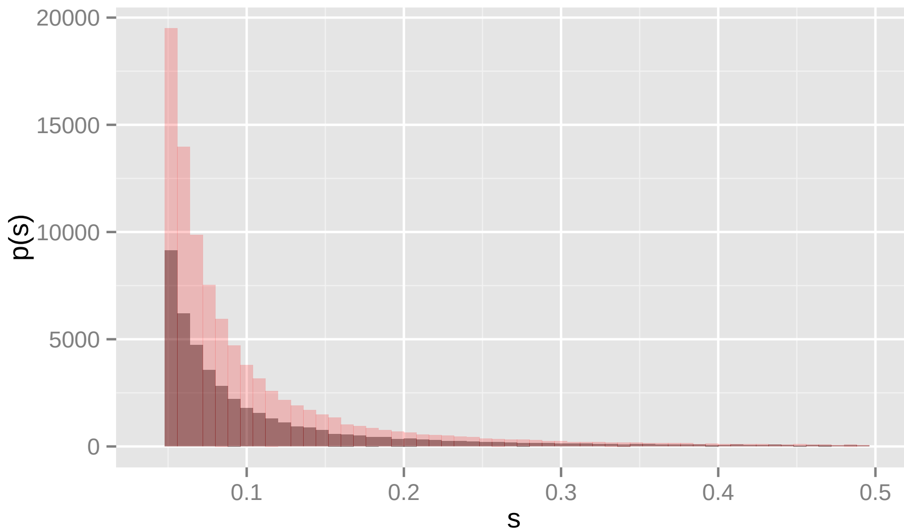


Figure 13: Numerically-sampled posterior distribution (dark histogram) and the prior distribution (red histogram) for Exercise 5 of Chapter 5.

Exercise 6

Exercise 6a

This solution is left to the student.

Exercise 6b

This solution is left to the student.

Chapter 6

Exercise 1

The following JAGS code is an adaption of the JAGS code in Section 6.1.2:

```
model{
##Likelihood
obstot~dpois(s + bkg)
##Prior
##Remember to supply bkg
##Make a large
s ~dunif(0,a)
}.
```

Then, for example, if we set $\mathtt{a} = 1 \times 10^8$, $\mathtt{obstot} = 7$, and $\mathtt{bkg} = 1$ (i.e., the fourth line and eight column), the table yields (3.17, 12.66) and the interval we get from JAGS is (3.717, 10.73). Similarly, for $\mathtt{a} = 1 \times 10^8$, $\mathtt{obstot} = 2$, and $\mathtt{bkg} = 3.5$, the table yields (0, 3.39) and the interval we get from JAGS is (0.182,3.42). We get reasonable agreement though not perfect, perhaps due to not using the same priors as the original authors.

Exercise 2

Note that when $\nu = 0$, we have a flat (and improper) prior for `s` and `bkg` which we approximate with wide uniform distributions. The following JAGS code is an adaption of the JAGS code in Section 6.1.2:

```
model{
##Likelihood
obstot~dpois(s+bkg)
obsbkg~dpois(bkg/C)
##Prior
##Make a and b large
s ~dunif(0,a)
bkg ~ dunif(0,b)
}.
```

Figure 14 gives the posterior for the given the data values in the exercise. There is good agreement between the numerical and analytical results.

Exercise 3

We leave this solution for the student.

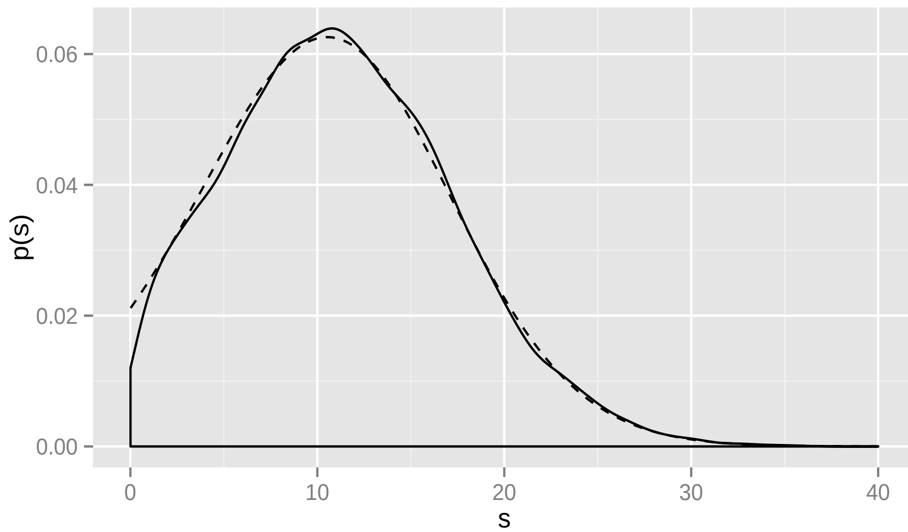


Figure 14: Numerically-sampled posterior distribution (solid line) and analytically derived posterior distribution (dashed line) for Exercise 2 of Chapter 6.

Exercise 4

We leave this solution for the student.

Exercise 5

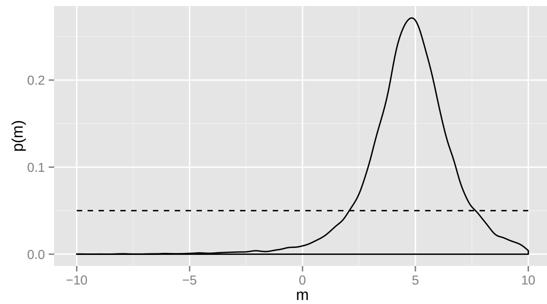
We leave this solution for the student.

Exercise 6

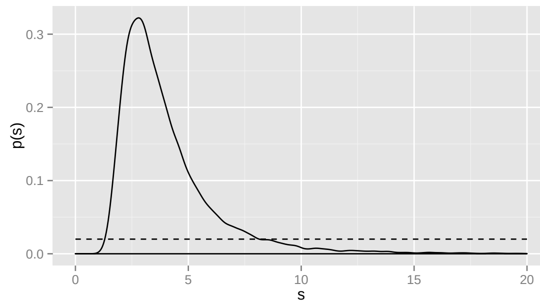
We use the following JAGS code:

```
model{
##Likelihood
for(i in 1:length(y)){
y[i]~dnorm(m,pow(s,-2))
}
##Prior
s ~dunif(0,50)
m ~ dunif(-10, 10)
}.
```

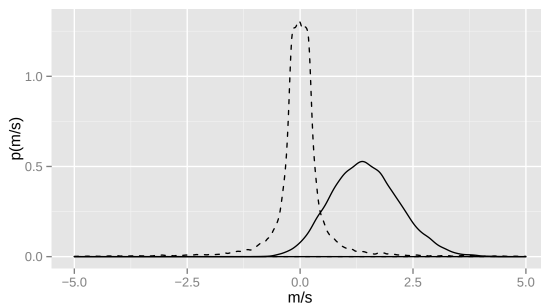
This code gives the three plots in Figure 15.



(a)



(b)



(c)

Figure 15: Prior (dashed line) and posterior (solid line) distributions for `m` (Figure 15a), `s` (Figure 15a), and `m/s` (Figure 15c) for Exercise 6 of Chapter 6.

Exercise 7

We use the following JAGS code:

```
model{
##Likelihood
for(i in 1:length(dens)){
dens[i]~dweib(a,b)
}
##Prior
a ~ dunif(0,50)
b ~ dunif(0,50)
}.
```

This code gives the two plots in Figure 16

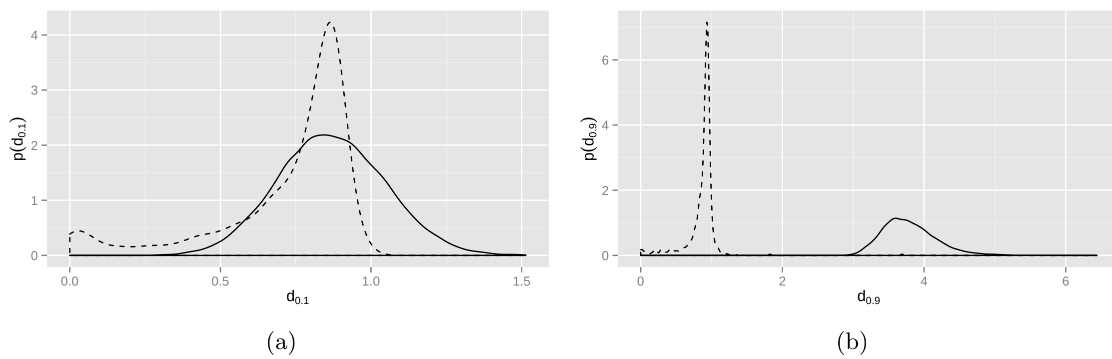


Figure 16: Prior (dashed line) and posterior (solid line) distributions for $d_{0.10}$ (Figure 16a), and $d_{0.90}$ (Figure 16b) for Exercise 7 of Chapter 6.

Exercise 8

We use the following JAGS code:

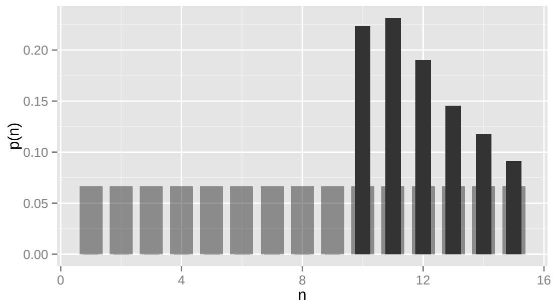
```
model{
##Likelihood
for(i in 1:length(obsx)){
obsx[i]~dbin(f,n)
}
##Prior
f ~ dunif(0,1)
n ~ dcat(pi)
}.
```

This code gives the two plots in Figure 17.

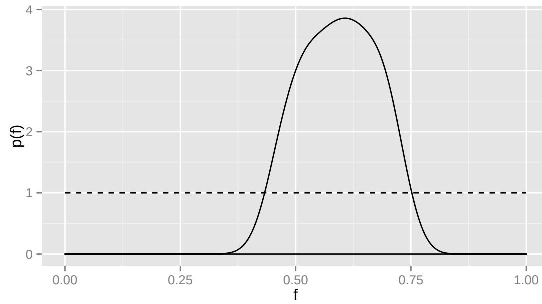
Exercise 9

We use the following JAGS code:

```
model{
##Likelihood
for(i in 1:length(speed)){
speed[i]~dgamma(a,b)
}
##Prior
a ~ dunif(0,1000)
b ~ dunif(0,1000)
```



(a)



(b)

Figure 17: Figure 17a gives the prior (gray bars) and the posterior (black bars) for `n` in Exercise 3. Figure 17b gives the prior (dashed line) and posterior (solid line) distributions for `f` for Exercise 8 of Chapter 6.

```
}.
```

This code gives Figure 18 . The median of the posterior distribution for the mode is estimated to be 897.24 with a 95 % posterior interval (851.42, 941.79). These results are quite different than the accepted value. This could be due to unknown biases in the original experiment (such as issues with calibration of instrumentation).

Exercise 10

We begin with the following code:

```
model{
##Likelihood
for(i in 1:length(obsheat)){
obsheat[i]~dnorm(heat,prec)
}
##Prior
heat ~ dnorm(54.59,1000)
rho ~ dunif(0,1)
prec <- pow(rho*heat,-2)
}.
```

Figure 19 displays the distributions for both `heat` and `rho`. Notice that for `rho`, the posterior is centered on such a small region for `rho` that the prior-density values do not appear. Additionally, the prior density for `rho` has been scaled so that it can be viewed in the figure.

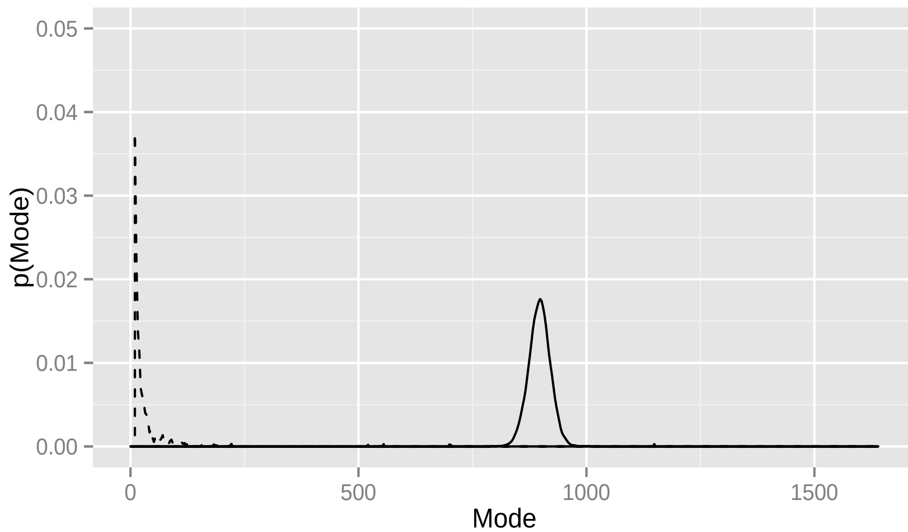
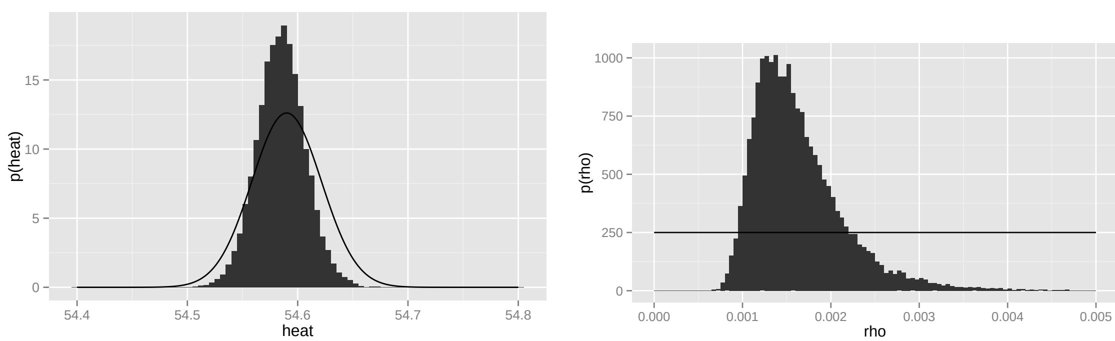


Figure 18: Posterior distribution (solid line) and prior distribution (dashed line) for the mode of the gamma distribution for Exercise 9 of Chapter 6.



(a) (b)

Figure 19: Posterior (histogram) and prior (solid line) distribution for the `heat` (Figure 19a) and for `rho` (Figure 19b) for Exercise 10 of Chapter 6. Note the prior density for `rho` has been scaled so that it can be viewed in the figure.

Exercise 11

We leave this solution to discretion of the instructor.

Chapter 8

Exercise 1

We leave this solution for the student.

Exercise 2

We leave this solution for the student.

Exercise 3

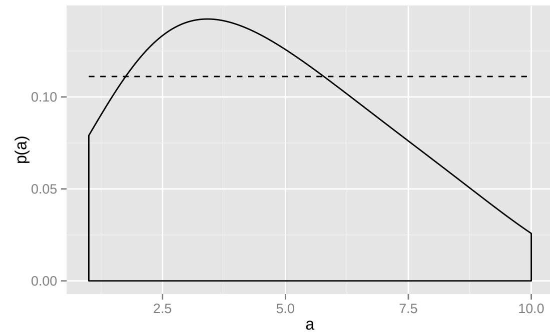
We leave this solution for the student.

Exercise 4

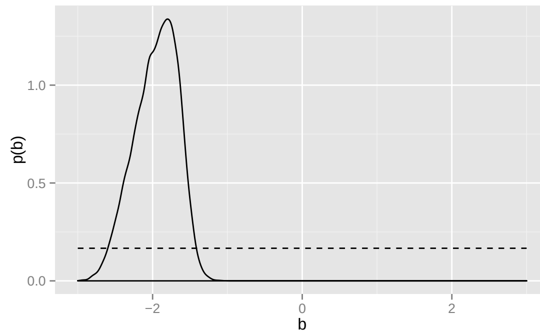
The following code illustrates how we obtained the posterior distribution, along with the priors we selected based on the information given in the problem:

```
model{
for(i in 1:length(obsY)){
obsY[i] ~ dnorm(Z[i], prec)
Z[i] <- a + b*obsX[i]
}
a ~ dunif(1,10)
b ~ dunif(-3,3)
prec ~ dunif(0.03,4)
}.
```

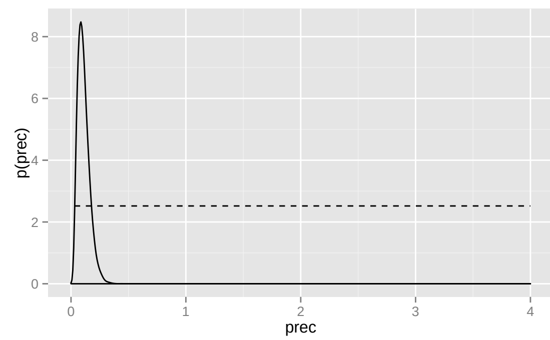
Figure 20 gives the prior and posterior for `a`, `b`, and `prec`.



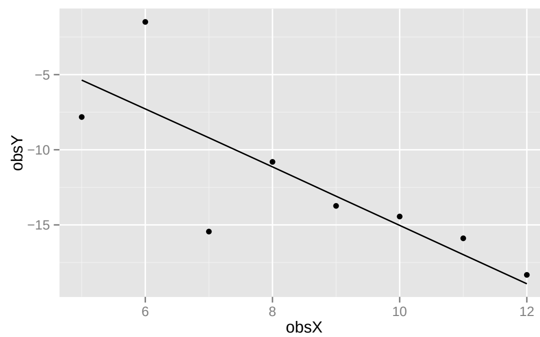
(a)



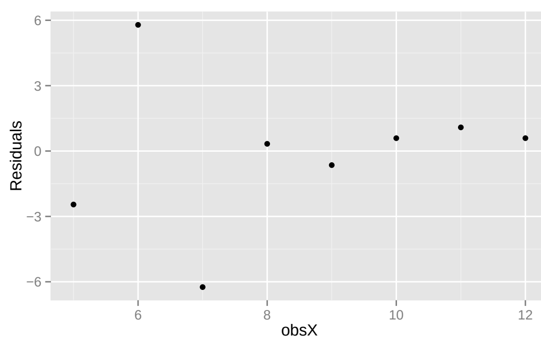
(b)



(c)

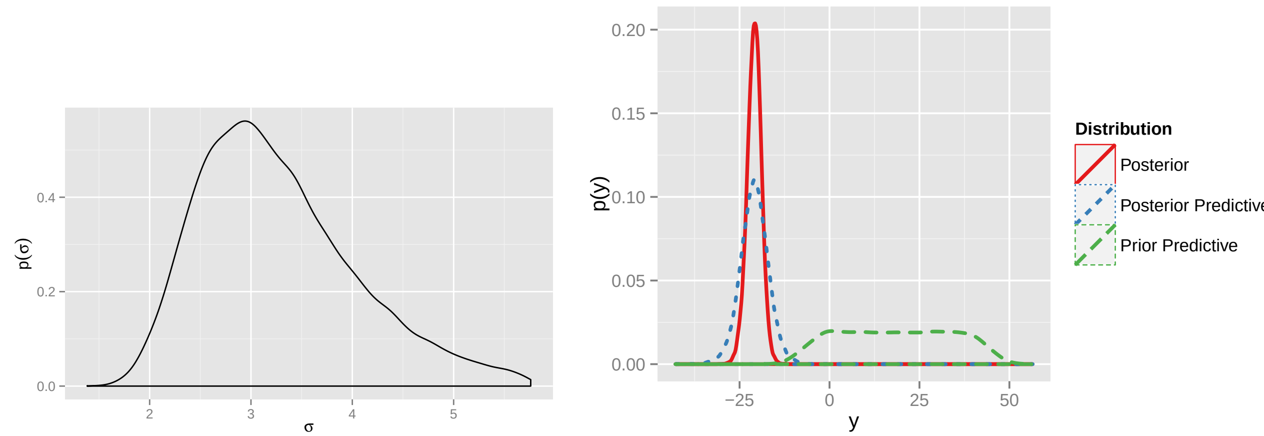


(d)



(e)

Figure 20: Posterior (solid line) and prior (dashed line) distributions for `a` (Figure 20a), `b` (Figure 20b), and `prec` (Figure 20c) for Exercise 4 of Chapter 8, the model fit plotted with the data (Figure 20d), and the residuals (Figure 20e). The prior in Figure 20c was scaled for readability.



(a) (b)

Figure 21: Posterior distributions for σ (Figure 21a). Figure 21b gives the posterior distribution for $\mathtt{a}+\mathtt{b}*13$ (solid red line), prior predictive distribution for $\mathtt{a}+\mathtt{b}*13$ (short-dashed green line), and posterior predictive distribution for $\mathtt{a}+\mathtt{b}*13$ (long-dashed blue line) for Exercises 5a-5d of Chapter 8.

Exercise 5

Exercise 5a

Figure 21a shows the posterior distributions for σ.

Exercise 5b - Exercise 5d

Figure 21b gives the posterior distribution for $\mathtt{a}+\mathtt{b}*13$ (solid line), prior predictive distribution for $\mathtt{a}+\mathtt{b}*13$ (long-dashed line), and posterior predictive distribution for $\mathtt{a}+\mathtt{b}*13$ (short-dashed line). Notice that the posterior distribution is narrower than both predictive distributions. This is because this is the distribution for the mean of `obsY` at `obsX` $=13$ and the only source of uncertainty comes from its posterior. The predictive distributions require additional uncertainty for the sampling of a new data point from the distribution of `obsY`. In other words, the posterior predictive distribution not only accounts for uncertainty from the model parameters, but also from observing future values of `obsY`.

Exercise 6

For the Hubble data set we adopted as priors `b`$\sim$ `dunif`(-10000, 10000) and `prec`$\sim$ `dunif`(0.005, 100) so that the JAGS model reads:

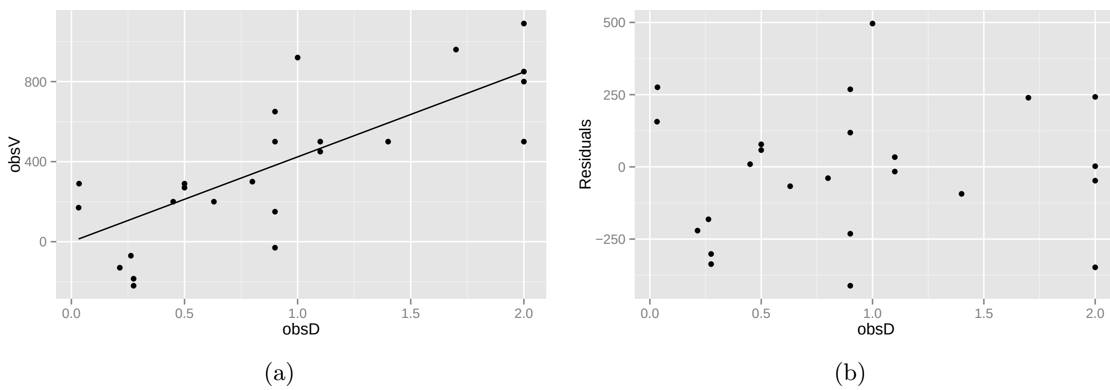


Figure 22: Hubble data plotted with the model fit (Figure 22a) and the residuals (Figure 22b) for Exercise 6 of Chapter 8.

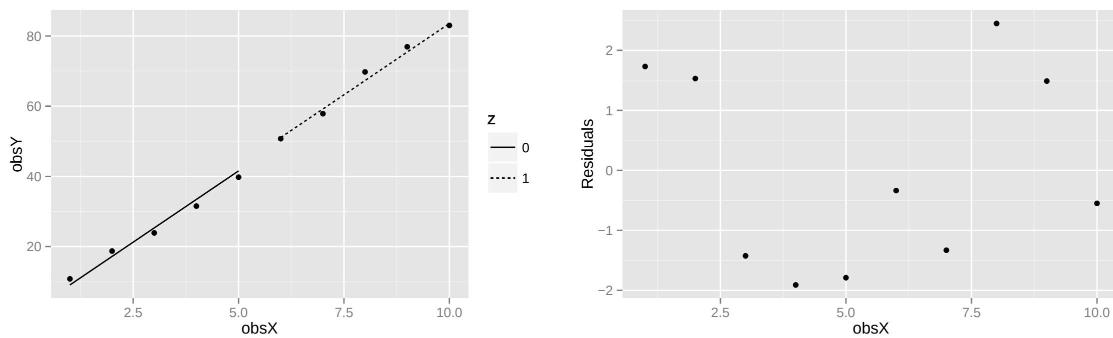
```
model{
##Likelihood
for(i in 1:length(obsV)){
obsV[i] ~ dnorm(b*obsD[i], prec)
}
b ~ dunif(-10000, 10000)
prec ~ dunif(0.005, 100)
} .
```

The posterior mean for `b` is $\texttt{b} = 423.95$ with a 95% credible interval $(418.87, 429.21)$. Figure shows the Hubble data with the model fit and the residuals.

Exercise 7

For this exercise we use the following code:

```
model{
##Likelihood
for(i in 1:length(obsY)){
obsY[i] ~ dnorm( a + b*obsX[i] + r*Z[i], prec)
}
a ~ dunif(0,2)
b ~ dunif(0,16)
r ~ dunif(0,2)
prec ~ dunif(0.577,1)
}.
```



(a)

(b)

Figure 23: The data plotted with the model fit (Figure 23a) and the residuals (Figure 23b) for Exercise 7 of Chapter 8.

The posterior mean for `a` is 0.922 with a 95% credible interval $(0.067, 1.912)$, the posterior mean for `b` is 8.12 with a 95% credible interval $(7.91, 8.35)$, and the posterior mean for `r` is 1.47 with a 95% credible interval $(0.22, 1.98)$, and the posterior mean for `prec` is 0.61 with a 95% credible interval $(0.5, 0.93)$. Figure shows the data with the model fit and the residuals.