DIFFERENT FORMULATIONS OF THE DISCONTINUOUS GALERKIN METHOD FOR THE VISCOUS TERMS*

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Abstract. Discontinuous Galerkin method is a finite element method using completely discontinuous piecewise polynomial space for the numerical solution and the test functions. Until recently it was mainly used for solving convection problems involving only first spatial derivatives. Recently the method has been extended successfully to solve convection diffusion problems involving second derivative viscous terms. In this paper we will use simple examples to illustrate the basic ideas and "pitfalls" for using the discontinuous Galerkin method on the viscous terms.

Key words. discontinuous Galerkin method, viscous terms, convection diffusion problem.

AMS subject classifications. 65M60

1. Introduction.

The discontinuous Galerkin method is a class of finite element methods using completely discontinuous piecewise polynomial space for the numerical solution and the test functions. One certainly needs to use more degrees of freedom because of the discontinuities at the element boundaries, however this also gives one a room to design suitable inner boundary treatments (the so-called fluxes) to obtain highly accurate and stable methods in many difficult situations.

Until recently, the discontinuous Galerkin method was mainly used to solve first order hyperbolic problems. An example in the two space dimensional time dependent setting is

$$u_t + f(u)_x + g(u)_y = 0. \tag{1.1}$$

The first discontinuous Galerkin method was introduced in 1973 by Reed and Hill [13], in the framework of neutron transport, i.e. equation (1.1) without the time dependent term u_t and with linear $f(u) = au$ and $g(u) = bu$ where a and b do not depend on u. A major development of the discontinuous Galerkin method is carried out by Cockburn, Shu and their collaborators in a series of papers [4, 5, 6, 7], in which they established a framework to easily solve *nonlinear* time dependent problems (1.1) using explicit, nonlinearly stable high order Runge-Kutta time discretizations [14] and discontinuous Galerkin discretization in space with exact or approximate Riemann solvers as interface fluxes and TVB (total variation bounded) nonlinear limiters to achieve non-oscillatory properties for strong shocks.

The discontinuous Galerkin method for (1.1) has found rapid applications in such diverse areas as aeroacoustics, electro-magnetism, gas dynamics, granular flows, magneto-hydrodynamics, meteorology, modeling of shallow water, oceanography, oil recovery simulation, semiconductor device simulation, transport of contaminant in porous media, turbomachinery, turbulent flows, viscoelastic flows and weather forecasting, among many others.

The discontinuous Galerkin method has the following attractive properties:

- It can be easily designed for any order of accuracy in space and time. In fact, p-version or spectral element type version can be designed [11];

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- It can easily handle adaptivity strategies since refinement or unrefinement of the mesh can be achieved without taking into account of the continuity restrictions typical of conforming finite element methods. Moreover, the degree of the approximating polynomial can be easily changed from one element to the other.
- It is an explicit method, thus efficient for solving the hyperbolic problem (1.1). No global linear or nonlinear systems need be solved;
- It combines the flexibility of finite element methods in the easy handling of complicated geometry, with the high resolution property for discontinuous solutions of finite difference and finite volume methods through monotone fluxes or approximate Riemann solvers applied at the element interfaces and limiters;
- It has nice stability properties: a local cell entropy inequality for the square entropy can be proven [10] for general triangulation for any scalar nonlinear conservation laws (1.1) in any spatial dimensions and for any order of accuracy, even without the need of nonlinear limiters. So far this is the only class of high order methods having provable cell entropy inequalities in such a general setting. This implies nonlinear L^2 stability and entropy consistency even for discontinuous solutions;
- The method is highly compact: the evolution of information in any element depends only on the information of itself and its immediate neighbors, regardless of the order of accuracy. This is in contrast with high order finite volume schemes which must use wide stencils for high order reconstruction. This compactness is responsible for the efficient parallel implementation of the method, see, e.g. [3].

For more details of the discontinuous Galerkin method and its recent development and applications, we refer the readers to the survey article by Cockburn, Karniadakis and Shu [9], the references listed therein, and other papers in that special Springer volume dedicated exclusively to the discontinuous Galerkin method.

Recently, motivated by the successful numerical experiments of Bassi and Rebay [1], Cockburn and Shu developed the so-called local discontinuous Galerkin method in treating the second order viscous terms and proved the stability and convergence with optimal error estimates [8]. At about the same time, Baumann and Oden [2] introduced a new discontinuous Galerkin method for the discretization of the second order viscous terms, see also the paper by Oden, Babuška and Baumann [12]. In this paper we will use simple examples to illustrate the basic ideas of both approaches and compare their performances. We will also emphasize the "pitfalls" for using the discontinuous Galerkin method on the viscous terms.

2. Discontinuous Galerkin method for first order convection problems. We shall first describe the discontinuous Galerkin method for the first order convection problem (1.1). To simplify the presentation and without loss of generalities we shall use the one dimensional linear version of (1.1) as an example:

$$u_t - u_x = 0. \tag{2.1}$$

We shall solve (2.1) for $x \in [0,2\pi]$ with periodic boundary conditions and with an initial condition $u(x,0) = \sin(x)$.

Let's denote $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, for $j = 1, ..., N$, as a mesh for $[0,2\pi]$, where $x_{\frac{1}{2}} = 0$ and $x_{N+\frac{1}{2}} = 2\pi$. We denote the center of each cell by $x_j = \frac{1}{2}\left(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}}\right)$ and the

size of each cell by $\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$. The cells do not need to be uniform but for simplicity we will perform calculations in this paper only with uniform meshes and will denote the uniform mesh size by Δx.

If we multiply (2.1) by an arbitrary test function $v(x)$, integrate over the interval I_j, and integrate by parts, we get

$$\int_{I_j} u_t v dx + \int_{I_j} u v_x dx - u(x_{j+\frac{1}{2}}, t)v(x_{j+\frac{1}{2}}) + u(x_{j-\frac{1}{2}}, t)v(x_{j-\frac{1}{2}}) = 0. \tag{2.2}$$

This is the starting point for designing the discontinuous Galerkin method. We replace both the solution u and the test function v by piecewise polynomials of degree at most k. That is, $u, v \in V_{\Delta x}$ where

$$V_{\Delta x} = \{v : \ v \text{ is a polynomial of degree at most } k \text{ for } x \in I_j, \ \ j = 1, ..., N\}. \tag{2.3}$$

With this choice, there is an ambiguity in (2.2) in the last two terms involving the boundary values at $x_{j\pm\frac{1}{2}}$, as both the solution u and the test function v are *discontinuous* exactly at these boundary points. This is a nuisance but also an opportunity: one could cleverly design these terms so that the resulting numerical method is stable and accurate. To motivate the ideas, let's look at the simplest case $k = 0$. That is, the solution as well as the test functions are piecewise constants. If we denote by u_j the value of u (which is constant in each cell) in the cell I_j, (2.2) would become the familiar first order upwind finite volume scheme

$$\frac{d}{dt}u_j - \frac{1}{\Delta x_j}(u_{j+1} - u_j) = 0$$

if we perform the following in (2.2):

1. Replace the boundary terms $u(x_{j\pm\frac{1}{2}}, t)$ by single valued numerical fluxes $\hat{u}_{j\pm\frac{1}{2}} = \hat{u}(u^-_{j\pm\frac{1}{2}}, u^+_{j\pm\frac{1}{2}})$. These fluxes in general depend both on the left limit (e.g. $u^-_{j+\frac{1}{2}} = \lim_{x\to x^-_{j+\frac{1}{2}}} u(x,t)$) and on the right limit (e.g. $u^+_{j+\frac{1}{2}} = \lim_{x\to x^+_{j+\frac{1}{2}}} u(x,t)$). For the equation (2.1), the flux $\hat{u}_{j+\frac{1}{2}}$ is taken as $u^+_{j+\frac{1}{2}}$ according to upwinding.
2. Replace the test function v at the boundaries by the values taken from inside the cell I_j, namely $v^-_{j+\frac{1}{2}}$ and $v^+_{j-\frac{1}{2}}$.

The scheme now becomes: find $u \in V_{\Delta x}$ such that, for all test functions $v \in V_{\Delta x}$,

$$\int_{I_j} u_t v dx + \int_{I_j} u v_x dx - \hat{u}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + \hat{u}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0 \tag{2.4}$$

where the numerical flux $\hat{u}_{j+\frac{1}{2}} = u^+_{j+\frac{1}{2}}$.

After picking a local basis and inverting a local $(k+1)\times(k+1)$ mass matrix (by hand), the scheme (2.4) can be written as

$$\frac{d}{dt}u_j + \frac{1}{\Delta x_j}(Au_j + Bu_{j+1}) = 0 \tag{2.5}$$

where u_j is a small vector of length $k+1$ containing the coefficients of the solution u in the local basis inside cell I_j, and A and B are $(k+1)\times(k+1)$ constant matrices which can be computed once and for all and stored at the beginning of the code.

TABLE 2.1

L^2 and L^∞ errors and orders of accuracy for the discontinuous Galerkin method (2.4) applied to the linear equation (2.1) with an initial condition $u(x,0) = \sin(x)$, $t = 2\pi$. Third order Runge-Kutta in time.

	$k=1$				$k=2$			
Δx	L^2 error	order	L^∞ error	order	L^2 error	order	L^∞ error	order
$2\pi/20$	4.63E-03	—	1.34E-02	—	1.14E-04	—	5.10E-04	—
$2\pi/40$	1.09E-03	2.08	3.75E-03	1.84	1.42E-05	3.00	6.44E-05	2.98
$2\pi/80$	2.69E-04	2.02	9.84E-04	1.93	1.77E-06	3.00	8.08E-06	3.00
$2\pi/160$	6.69E-05	2.01	2.52E-04	1.97	2.21E-07	3.00	1.01E-06	3.00

Scheme (2.5) can then be easily discretized in time by the nonlinearly stable high order Runge-Kutta methods in [14]. We use the third order version in [14] and use sufficiently small time steps for the accuracy tests.

We remark that the method (2.5) is extremely simple to code and easy to parallelize.

For illustration purpose we show in Table 2.1 the L^2 and L^∞ errors and numerically observed orders of accuracy for the two cases $k = 1$ and 2 (piecewise linear and piecewise quadratic cases) for $t = 2\pi$ (after one time period). We can clearly see that an order of $k + 1$ is achieved.

To illustrate the power of the method for discontinuous solutions, even without using the nonlinear limiters, we solve equation (2.1) with a discontinuous initial condition $u(x,0) = 1$ for $x \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ and $u(x,0) = 0$ elsewhere inside $[0, 2\pi]$, extended periodically. We solve the problem for 50 time periods to $t = 100\pi$, using $N = 40$ cells, with $k = 1$ (piecewise linear) and $k = 6$ (piecewise sixth degree polynomial), and show the results in Fig. 2.1, where the solid line is the exact solution and the dashed line and square symbols (only the middle point value of each cell is plotted) are the numerical solutions. We can clearly observe that the second order method ($k = 1$) has a considerable smearing of the discontinuity at this long time but the seventh order method ($k = 6$) are still able to hold on to the structure of the solution without noticeable smearing. Notice that this is a particularly tough test case as discontinuities for linear equations like (2.1), which are called "contact discontinuities" in the literature, are subject to severe numerical dissipation (smearing) and are very difficult to resolve sharply by a numerical method.

3. Naive generalization of the discontinuous Galerkin method to the second order diffusion problem — a "pitfall". We now turn our attention to the convection diffusion problems containing second derivatives. The difficulty can be illustrated by the following simple heat equation:

$$u_t - u_{xx} = 0. \tag{3.1}$$

again for $x \in$ [0,2π] with periodic boundary conditions and with an initial condition $u(x,0) = \sin(x)$.

If we proceed as before we obtain the following equality similar to (2.2):

$$\int_{I_j} u_t v dx + \int_{I_j} u_x v_x dx - u_x(x_{j+\frac{1}{2}}, t) v(x_{j+\frac{1}{2}}) + u_x(x_{j-\frac{1}{2}}, t) v(x_{j-\frac{1}{2}}) = 0. \tag{3.2}$$

The only difference between (2.2) and (3.2) is that, in all the terms except the first one, u in (2.2) is replaced by u_x in (3.2). A very natural way to extend the scheme

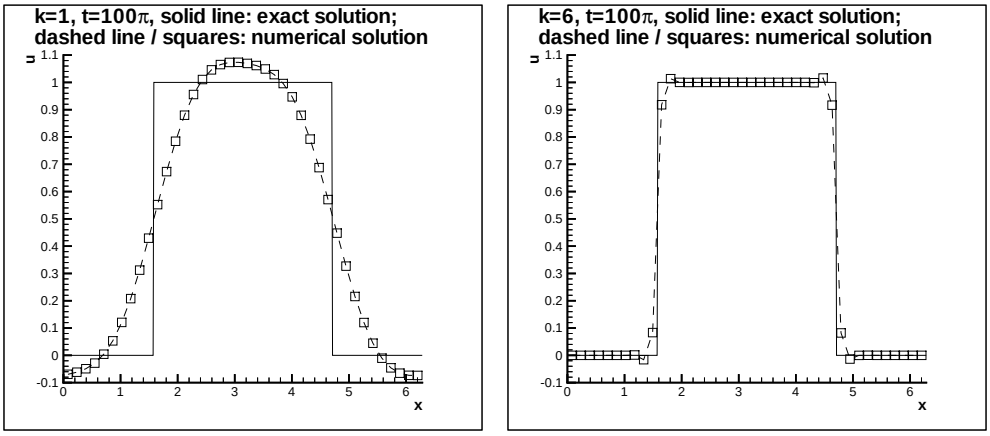

FIG. 2.1. *The discontinuous Galerkin method (2.4) applied to the linear equation (2.1) with a square initial condition, $t = 100\pi$. 40 cells. Third order Runge-Kutta in time. Solid line: the exact solution; Dashed line and squares symbols: the computed solution at the cell centers. Left: $k = 1$; Right: $k = 6$.*

(2.4) would be simply to replace u by u_x: find $u \in V_{\Delta x}$ such that, for all test functions $v \in V_{\Delta x}$,

$$\int_{I_j} u_t v dx + \int_{I_j} u_x v_x dx - \hat{u_x}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + \hat{u_x}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0 \tag{3.3}$$

where, for the lack of upwinding mechanism in a heat equation one naturally takes a central flux $\hat{u_x}_{j+\frac{1}{2}} = \frac{1}{2}\left((u_x)^-_{j+\frac{1}{2}} + (u_x)^+_{j+\frac{1}{2}}\right)$.

One might be more careful and notice that, for the piecewise constant case $k = 0$, (3.3) becomes the ridiculous

$$\frac{d}{dt} u_j = 0 \tag{3.4}$$

where u_j is the value of u (constant in each cell) in cell I_j, clearly inconsistent with the original PDE (3.1). However, one might be tempted to believe that this is just a special case and starting from $k = 1$ things will be OK, as then the first equality in the scheme (3.3) with the choice $v = 1$ in I_j becomes

$$\frac{d}{dt} u_j - \frac{1}{\Delta x_j}\left(\hat{u_x}_{j+\frac{1}{2}} - \hat{u_x}_{j-\frac{1}{2}}\right) = 0$$

which is quite reasonable in appearance.

We remark that, in the actual computation, the scheme is similar to (2.5) and takes the form

$$\frac{d}{dt} u_j + \frac{1}{\Delta x_j^2}\left(A u_{j-1} + B u_j + C u_{j+1}\right) = 0 \tag{3.5}$$

where u_j is a small vector of length $k+1$ containing the coefficients of the solution u in the local basis inside cell I_j, and A, B, C are $(k+1) \times (k+1)$ constant matrices which can be computed once and for all and stored at the beginning of the code. Again, the third order Runge-Kutta method [14] can be used.

TABLE 3.1

L^2 and L^∞ errors and orders of accuracy for the inconsistent discontinuous Galerkin method (3.3) applied to the heat equation (3.1) with an initial condition $u(x,0) = \sin(x)$, $t = 0.8$. Third order Runge-Kutta in time.

	$k=1$				$k=2$			
Δx	L^2 error	order	L^∞ error	order	L^2 error	order	L^∞ error	order
$2\pi/20$	1.78E-01	—	2.58E-01	—	1.85E-01	—	2.72E-01	—
$2\pi/40$	1.76E-01	0.016	2.50E-01	0.025	1.78E-01	0.049	2.55E-01	0.089
$2\pi/80$	1.75E-01	0.004	2.48E-01	0.012	1.77E-01	0.013	2.51E-01	0.025
$2\pi/160$	1.75E-01	0.001	2.48E-01	0.003	1.76E-01	0.003	2.50E-01	0.007

We compute with the scheme (3.3) and show in Table 3.1 the L^2 and L^∞ errors and numerically observed orders of accuracy for the two cases $k = 1$ and 2 (piecewise linear and piecewise quadratic cases) to $t = 0.8$. Clearly there is an order one error for both cases which does not decrease with a mesh refinement! We plot the solutions with 160 cells in Fig. 3.1 and can clearly see that the computed solutions have completely incorrect amplitudes. The scheme is not consistent!

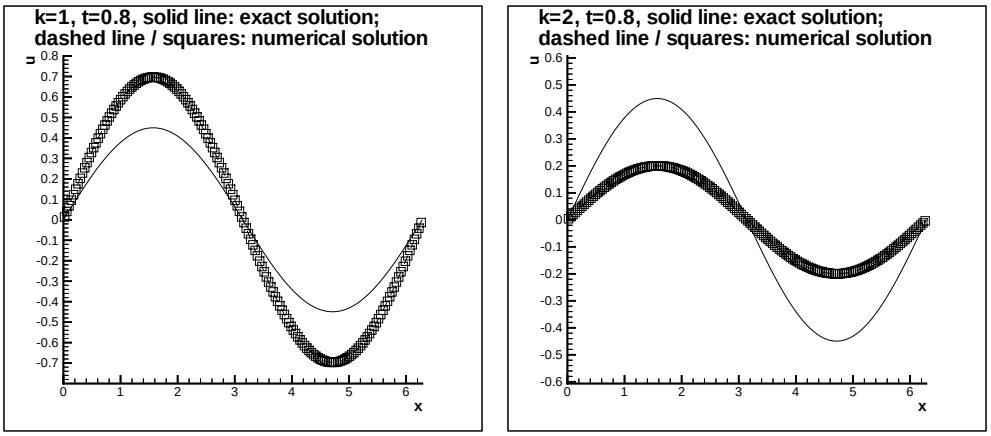

FIG. 3.1. *The inconsistent discontinuous Galerkin method (3.3) applied to the heat equation (3.1) with an initial condition $u(x,0) = \sin(x)$. $t = 0.8$. 160 cells. Third order Runge-Kutta in time. Solid line: the exact solution; Dashed line and squares symbols: the computed solution at the cell centers. Left: $k = 1$; Right: $k = 2$.*

This is a very subtle inconsistency: the exact solution of the PDE (3.1) does satisfy the scheme (3.3) *exactly*! Hence one might base the judgment on one's experience with finite difference and conclude that the method is consistent. Indeed, since the method is extremely non-conformal (i.e. the space $V_{\Delta x}$ in (2.3) is too irregular for the heat equation (3.1) which needs an H^1 space), the scheme (3.3) suffers from the "variational crimes" as described by Strang and Fix [15].

It is actually very dangerous that the scheme (3.3) produces stable but completely incorrect solution. If one was in a hurry and did not want to do the ground work of testing the method on the simple heat equation first which has a known exact solution, but rather went to solve the complicated Navier-Stokes equations and produced beautiful color pictures, one would not be able to tell that the result is actually wrong! In fact, the incorrect scheme (3.3) has been used in the literature for discretizing the viscous terms in the Navier-Stokes equations (I will spare the reference here).

In the next two sections we will describe "small" modifications to the incorrect

scheme (3.3) to obtain two classes of stable and accurate schemes for (3.1). We will also compare their numerical performances.

4. The local discontinuous Galerkin method for the second order diffusion problem. If we rewrite the heat equation (3.1) as a first order system

$$u_t - q_x = 0, \qquad q - u_x = 0, \tag{4.1}$$

we can then *formally* use the same discontinuous Galerkin method as in section 2 for the convection equation to solve (4.1), resulting in the following scheme: find $u, q \in V_{\Delta x}$ such that, for all test functions $v, w \in V_{\Delta x}$,

$$\begin{aligned} \int_{I_j} u_t v dx + \int_{I_j} q v_x dx - \hat{q}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + \hat{q}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0 \\ \int_{I_j} q w dx + \int_{I_j} u w_x dx - \hat{u}_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} + \hat{u}_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}} = 0, \end{aligned} \tag{4.2}$$

where, again for the lack of upwinding mechanism in a heat equation one naturally first tries the central fluxes:

$$\hat{u}_{j+\frac{1}{2}} = \frac{1}{2}\left(u^-_{j+\frac{1}{2}} + u^+_{j+\frac{1}{2}}\right), \qquad \hat{q}_{j+\frac{1}{2}} = \frac{1}{2}\left(q^-_{j+\frac{1}{2}} + q^+_{j+\frac{1}{2}}\right). \tag{4.3}$$

We emphasize that the above formulation of the discontinuous Galerkin scheme is only formally similar to that of the convection equation in section 2. In fact, there is no time derivative in the second equation in (4.1) and it is *not* a hyperbolic problem even though it is written into a system form with only first derivatives. If we view the scheme (4.2) as a mixed finite element method then it lacks the usual sophisticated matching of the two solution spaces for u and q (the same space is used for both of them). "Common sense" in traditional finite elements would hint that scheme (4.2) has no chance to work. However, Bassi and Rebay [1] were brave enough to try this method on the viscous terms in the Navier-Stokes equations and seemed to have obtained very good results. Motivated by their work, Cockburn and Shu [8] analyzed this method and obtained conditions on the choice of the fluxes $\hat{u}_{j+\frac{1}{2}}$ and $\hat{q}_{j+\frac{1}{2}}$ which guarantee stability, convergence and a sub-optimal error estimate of order k for piecewise polynomials of degree k. It turns out that the central fluxes (4.3) used by Bassi and Rebay [1] do satisfy these conditions. No wonder they converge in practice!

We remark that the appearance of the auxiliary variable q is superficial: when a local basis is chosen in cell I_j then q is eliminated and the actual scheme for u takes a form similar to (3.5). We will come back to this issue later.

There are two problems associated with the choice of the central fluxes in (4.3):

1. It spreads to five cells when a local basis is chosen for u in cell I_j. After q is eliminated the scheme becomes

$$\frac{d}{dt}u_j + \frac{1}{\Delta x_j^2}\left(Au_{j-2} + Bu_{j-1} + Cu_j + Du_{j+1} + Eu_{j+2}\right) = 0$$

where u_j is a small vector of length $k+1$ containing the coefficients of the solution u in the local basis inside cell I_j, and A, B, C D, E are $(k+1)\times(k+1)$ constant matrices which can be computed once and for all and stored at the beginning of the code. The stencil here is wider than that in (3.5).

TABLE 4.1

L^2 and L^∞ errors and orders of accuracy for the local discontinuous Galerkin method (4.2) with fluxes (4.4) applied to the heat equation (3.1) with an initial condition $u(x,0) = \sin(x)$, $t = 0.8$. Third order Runge-Kutta in time.

	$k=1$				$k=2$			
Δx	L^2 error	order	L^∞ error	order	L^2 error	order	L^∞ error	order
$2\pi/20$, u	1.92E-03	—	7.34E-03	—	4.87E-05	—	2.30E-04	—
$2\pi/20$, q	1.93E-03	—	7.33E-03	—	4.87E-05	—	2.30E-04	—
$2\pi/40$, u	4.81E-04	2.00	1.84E-03	1.99	6.08E-06	3.00	2.90E-05	2.99
$2\pi/40$, q	4.81E-04	2.00	1.84E-03	1.99	6.08E-06	3.00	2.90E-05	2.99
$2\pi/80$, u	1.20E-04	2.00	4.62E-04	2.00	7.60E-07	3.00	3.63E-06	3.00
$2\pi/80$, q	1.20E-04	2.00	4.62E-04	2.00	7.60E-07	3.00	3.63E-06	3.00
$2\pi/160$, u	3.00E-05	2.00	1.15E-04	2.00	9.50E-08	3.00	4.53E-07	3.00
$2\pi/160$, q	3.00E-05	2.00	1.15E-04	2.00	9.50E-08	3.00	4.53E-07	3.00

2. The order of accuracy is one order lower for odd k. That is, for odd k the proof of the sub-optimal error estimate of order k is actually sharp.

Both problems can be cured by a clever choice of fluxes, proposed in Cockburn and Shu [8]:

$$\hat{u}_{j+\frac{1}{2}} = u^-_{j+\frac{1}{2}}, \qquad \hat{q}_{j+\frac{1}{2}} = q^+_{j+\frac{1}{2}}. \tag{4.4}$$

i.e. we alternatively take the left and right limits for the fluxes in u and q (we could of course also take the pair $u^+_{j+\frac{1}{2}}$ and $q^-_{j+\frac{1}{2}}$ as the fluxes). Notice that the evaluation of (4.4) is simpler than that of the central fluxes in (4.3). We recover exactly the scheme in the form of (3.5) (of course with different constant matrices A, B and C) when a local basis is chosen. Hence the computational cost and storage requirement of scheme (4.2) with the fluxes (4.4) is the same as that of the inconsistent scheme (3.3), even though we now have nominally an additional auxiliary variable q! We can also prove that now the order of accuracy becomes $k+1$ for all k.

For illustration purpose we show in Table 4.1 the L^2 and L^∞ errors and numerically observed orders of accuracy, for both u and q, for the two cases $k = 1$ and 2 (piecewise linear and piecewise quadratic cases) to $t = 0.8$. Clearly $(k+1)$-th order of accuracy is achieved for both odd and even k and also the same order of accuracy is achieved for q which approximates u_x. We thus obtain the advantage of mixed finite element methods in approximating the derivatives of the exact solution to the same order of accuracy as the solution themselves, yet without additional storage or computational costs for the auxiliary variable q!

5. The Baumann-Oden discontinuous Galerkin method for the second order diffusion problem. Another possible modification to the inconsistent scheme (3.3) is given by Baumann and Oden [2], see also Oden, Babuška, and Baumann [12]. Basically, extra boundary terms were added to the element boundaries such that, when one takes $v = u$ and sum over all cells, the boundary contribution disappears and one gets a nice L^2 norm stability control. The scheme now becomes: find $u \in V_{\Delta x}$ such that, for all test functions $v \in V_{\Delta x}$,

$$\begin{aligned} &\int_{I_j} u_t v dx + \int_{I_j} u_x v_x dx - \hat{u_x}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + \hat{u_x}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} \\ &-\frac{1}{2}(v_x)^-_{j+\frac{1}{2}} \left(u^+_{j+\frac{1}{2}} - u^-_{j+\frac{1}{2}}\right) - \frac{1}{2}(v_x)^+_{j-\frac{1}{2}} \left(u^+_{j-\frac{1}{2}} - u^-_{j-\frac{1}{2}}\right) = 0 \end{aligned} \tag{5.1}$$

where, again for the lack of upwinding mechanism in a heat equation one naturally takes a central flux $\hat{u_x}_{j+\frac{1}{2}} = \frac{1}{2}\left((u_x)^-_{j+\frac{1}{2}} + (u_x)^+_{j+\frac{1}{2}}\right)$. Notice that the extra terms added do not make the system symmetric.

We remark that for $k = 0$ the scheme (5.1) again degenerates to the ridiculous (3.4) hence is inconsistent with the PDE (3.1). The scheme can only be used for $k \geq 1$.

For coding purpose (5.1) is the most convenient form, however it might be more illustrative if we rewrite (5.1) into a global form: find $u \in V_{\Delta x}$ such that, for all test functions $v \in V_{\Delta x}$,

$$\int_0^{2\pi} u_t v dx + \sum_{j=1}^{N} \left(\int_{I_j} u_x v_x dx + \hat{u_x}_{j+\frac{1}{2}}[v]_{j+\frac{1}{2}} - \hat{v_x}_{j+\frac{1}{2}}[u]_{j+\frac{1}{2}} \right) = 0 \tag{5.2}$$

where $[w] \equiv w^+ - w^-$ denotes the jump of the function w at the interface and the flux for v_x is also a central flux $\hat{v_x}_{j+\frac{1}{2}} = \frac{1}{2}\left((v_x)^-_{j+\frac{1}{2}} + (v_x)^+_{j+\frac{1}{2}}\right)$. The anti-symmetry nature of the boundary terms (which disappear when one takes $v = u$) is clearly seen in the global formulation (5.2).

We remark that once again we recover exactly the scheme in the form of (3.5) (of course with different constant matrices A, B and C) when a local basis is chosen. Hence the computational cost and storage requirement of scheme (5.1) is the same as that of the inconsistent scheme (3.3) or as that of the local discontinuous Galerkin method (4.2)-(4.4). There is no saving in the computational cost here over the method (4.2)-(4.4) even though the latter has nominally an additional auxiliary variable q. This statement is valid when a linear PDE is solved. For nonlinear problems the computational cost of the Baumann-Oden method (5.1) may be smaller than that of the local discontinuous Galerkin method (4.2)-(4.4).

The order of accuracy for the scheme (5.1) is k for even k (sub-optimal) and $k+1$ for odd k (optimal).

For illustration purpose we show in Table 5.1 the L^2 and L^∞ errors and numerically observed orders of accuracy, for the two cases $k = 1$ and 2 (piecewise linear and piecewise quadratic cases) to $t = 0.8$. Clearly $(k+1)$-th order of accuracy is achieved for the odd $k = 1$ and k-th order of accuracy is achieved for the even $k = 2$. Comparing with the results in Table 4.1 of the local discontinuous Galerkin method, we can see that, for the same mesh, the Baumann-Oden method (5.1) has larger errors than the local discontinuous Galerkin method (4.2)-(4.4) even for odd k where both are accurate of order $k + 1$. For even k the Baumann-Oden method (5.1) is much inferior to the local discontinuous Galerkin method (4.2)-(4.4) as the former is one order lower in accuracy.

6. Concluding remarks. We have discussed three different formulations of the discontinuous Galerkin methods for the heat equation (3.1). For practical implementations of linear problems all of them are of the simple form (3.5) and have the same computational cost and storage requirement. For nonlinear problems the computational cost of the Baumann-Oden method may be smaller than that of the local discontinuous Galerkin method. The first approach in section 3 produces a numerically stable but inconsistent method, giving nice looking but completely wrong solutions. This example serves as a warning to "pitfalls" in using discontinuous Galerkin methods for higher order derivative terms. The last two approaches are both stable and convergent and have comparable computational efficiency. The local discontinuous

TABLE 5.1
L^2 and L^∞ errors and orders of accuracy for the Baumann-Oden discontinuous Galerkin method (5.1) applied to the heat equation (3.1) with an initial condition $u(x,0) = \sin(x)$, $t = 0.8$. Third order Runge-Kutta in time.

	$k=1$				$k=2$			
Δx	L^2 error	order	L^∞ error	order	L^2 error	order	L^∞ error	order
$2\pi/20$	6.40E-03	—	1.25E-02	—	4.00E-03	—	5.64E-03	—
$2\pi/40$	1.60E-03	2.00	3.14E-03	2.00	1.03E-03	1.95	1.46E-03	1.95
$2\pi/80$	4.00E-04	2.00	7.85E-04	2.00	2.61E-04	1.99	3.68E-04	1.99
$2\pi/160$	9.99E-05	2.00	1.96E-04	2.00	6.53E-05	2.00	9.23E-05	2.00

Galerkin method is symmetric and more flexible in numerical fluxes and can achieve uniform $(k+1)$-th order accuracy for both u and u_x for all k, while the Baumann-Oden method approximates u to $(k+1)$-th order accuracy for odd k but only to k-th order accuracy for even k. For the same mesh and when k is odd hence both methods are of the same order $k+1$, the local discontinuous Galerkin method has smaller errors than the Baumann-Oden method for the heat equation we have tested. When k is odd the local discontinuous Galerkin method is much more accurate than the Baumann-Oden method as the latter is one order lower in accuracy. The conclusions drawn in this paper, although given only for the simple heat equation, are valid for more complex convection diffusion problems such as the Navier-Stokes equations.

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