A BRIEF HISTORY OF DETERMINACY

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§1. Introduction. Determinacy axioms are statements to the effect that certain games are *determined*, in that each player in the game has an optimal strategy. The commonly accepted axioms for mathematics, the Zermelo–Fraenkel axioms with the Axiom of Choice (ZFC; see [Jec03, Kun83]), imply the determinacy of many games that people actually play. This applies in particular to many **games of perfect information**, games in which the players alternate moves which are known to both players, and the outcome of the game depends only on this list of moves, and not on chance or other external factors. Games of perfect information which must end in finitely many moves are determined. This follows from the work of Ernst Zermelo [Zer13], Dénes Kőnig [Kőn27] and László Kálmar [Kal1928–29], and also from the independent work of John von Neumann and Oskar Morgenstern (in their 1944 book, reprinted as [vNM04]).

As pointed out by Stanisław Ulam [Ula60], determinacy for games of perfect information of a fixed finite length is essentially a theorem of logic. If we let x_1,y_1,x_2,y_2,...,x_n,y_n be variables standing for the moves made by players player I (who plays x_1,...,x_n) and player II (who plays y_1,...,y_n), and A (consisting of sequences of length $2n$) is the set of runs of the game for which player I wins, the statement

$$\exists x_1 \forall y_1 \ldots \exists x_n \forall y_n \langle x_1, y_1, \ldots, x_n, y_n \rangle \in A \tag{1}$$

essentially asserts that the first player has a winning strategy in the game, and its negation,

$$\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \langle x_1, y_1, \ldots, x_n, y_n \rangle \notin A \tag{2}$$

essentially asserts that the second player has a winning strategy.[1]

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[1] If there exists a way of choosing a member from each nonempty set of moves of the game, then these statements are actually equivalent to the assertions that the corresponding strategies exist. Otherwise, in the absence of the Axiom of Choice the statements above can hold without the corresponding strategy existing.

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We let ω denote the set of natural numbers $0, 1, 2, \ldots$; for brevity we will often refer to the members of this set as "integers". Given sets X and Y, ${}^{X}Y$ denotes the set of functions from X to Y. The **Baire space** is the space ${}^{\omega}\omega$, with the product topology. The Baire space is homeomorphic to the space of irrational real numbers (see [Mos09, p. 9], for instance), and we will often refer to its members as "reals" (though in various contexts the Cantor space ${}^{\omega}2$, the set of subsets of ω ($\wp(\omega)$) and the set of infinite subsets of ω ($[\omega]^{\omega}$) are all referred to as "the reals").

Given $A \subseteq {}^{\omega}\omega$, we let $G_{\omega}(A)$ denote the game of perfect information of length ω in which the two players collaborate to define an element f of ${}^{\omega}\omega$ (with player I choosing $f(0)$, player II choosing $f(1)$, player I choosing $f(2)$, and so on), with player I winning a run of the game if and only if f is an element of A. A game of this type is called an **integer game**, and the set A is called the **payoff set**. A **strategy** in such a game for player player I (player II) is a function Σ with domain the set of sequences of integers of even (odd) length such that for each $a \in \mathrm{dom}(\Sigma)$, $\Sigma(a)$ is in ω. A run of the game (partial or complete) is said to be **according to** a strategy Σ for player player I (player II) if every initial segment of the run of odd (nonzero even) length is of the form $a^\frown\langle\Sigma(a)\rangle$ for some sequence a. A strategy Σ for player player I (player II) is a **winning strategy** if every complete run of the game according to Σ is in (out of) A. We say that a set $A \subseteq {}^{\omega}\omega$ is **determined** (or the corresponding game $G_{\omega}(A)$ is determined) if there exists a winning strategy for one of the players. These notions generalize naturally for games in which players play objects other than integers (for instance, **real games**, in which they play elements of ${}^{\omega}\omega$) or games which run for more than ω many rounds (in which case player player I typically plays at limit stages).

The study of determinacy axioms concerns games whose determinacy is neither proved nor refuted by the Zermelo–Fraenkel axioms ZF (without the Axiom of Choice). Typically such games are infinite. Axioms stating that infinite games of various types are determined were studied by Stanisław Mazur, Stefan Banach and Ulam in the late 1920s and early 1930s; were reintroduced by David Gale and Frank Stewart [GS53] in the 1950s and again by Jan Mycielski and Hugo Steinhaus [MS62] in the early 1960s; gained interest with the work of David Blackwell [Bla67] and Robert Solovay in the late 1960s; and attained increasing importance in the 1970s and 1980s, finally coming to a central position in contemporary set theory.

Mycielski and Steinhaus introduced the Axiom of Determinacy (AD), which asserts the determinacy of $G_{\omega}(A)$ for all $A \subseteq {}^{\omega}\omega$. Work of Banach in the 1930s shows that AD implies that all sets of reals satisfy the property of Baire. In the 1960s, Mycielski and Stanisław Świerczkowski proved that AD implies that all sets of reals are Lebesgue measurable, and Mycielski showed that AD implies countable choice for reals. Together, these results show that determinacy provides a natural context for certain areas of mathematics, notably analysis, free of the paradoxes induced by the Axiom of Choice.

Unaware of the work of Banach, Gale and Stewart [GS53] had shown that AD contradicts ZFC. However, their proof used a wellordering of the reals given by the Axiom of Choice, and therefore did not give a nondetermined game of this type with definable payoff set. Starting with Banach's work, many simply definable payoff sets were shown to induce determined games, culminating in D. Anthony Martin's celebrated 1974 result [Mar75] that all games with Borel payoff set are determined. This result came after Martin had used measurable cardinals to prove the determinacy of games whose payoff set is an analytic sets of reals.

The study of determinacy gained interest from two theorems in 1967, the first due to Solovay and the second to Blackwell. Solovay proved that under AD, the first uncountable cardinal ω_1 is a measurable cardinal, setting off a study of strong Ramsey properties on the ordinals implied by determinacy axioms. Blackwell used open determinacy (proved by Gale and Stewart) to reprove a classical theorem of Kazimierz Kuratowski. This also led to the application, by John Addison, Martin, Yiannis Moschovakis and others, of stronger determinacy axioms to produce structural properties for definable sets of reals. These axioms included the determinacy of $\utilde{\boldsymbol{\Delta}}^1_n$ sets of reals, for $n \geq 2$, statements which would not be proved consistent relative to large cardinals until the 1980s.

The large cardinal hierarchy was developed over the same period, and came to be seen as a method for calibrating consistency strength. In the 1970s, various special cases of $\utilde{\boldsymbol{\Delta}}^1_2$ determinacy were located on this scale, in terms of the large cardinals needed to prove them. Determining the consistency (relative to large cardinals) of forms of determinacy at the level of $\utilde{\boldsymbol{\Delta}}^1_2$ and beyond would take the introduction of new large cardinal concepts. Martin (in 1978) and W. Hugh Woodin (in 1984) would prove $\utilde{\boldsymbol{\Pi}}^1_2$-determinacy and $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$ respectively, using hypotheses near the very top of the large cardinal hierarchy. In a dramatic development, the hypotheses for these results would be significantly reduced through work of Woodin, Martin and John Steel. The initial impetus for this development was a seminal result of Matthew Foreman, Menachem Magidor and Saharon Shelah which showed, assuming the existence of a supercompact cardinal, that there exists a generic elementary embedding with well-founded range and critical point ω_1. Combined with work of Woodin, this yielded the Lebesgue measurability of all sets in the inner model $\mathbf{L}(\mathbb{R})$ from this hypothesis. Shelah and Woodin would reduce the hypothesis for this result further, to the assumption that there exist infinitely many Woodin cardinals below a measurable cardinal.

Woodin cardinals would turn out to be the central large cardinal concept for the study of determinacy. Through the study of tree representations for sets of reals, Martin and Steel would show that $\underset{\sim}{\mathbf{\Pi}}^1_{n+1}$-determinacy follows from the existence of n Woodin cardinals below a measurable cardinal, and that this hypothesis was not sufficient to prove stronger determinacy results for the projective hierarchy. Woodin would then show that the existence of infinitely many Woodin cardinals below a measurable cardinal implies $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$, and he would locate the exact consistency strengths of $\underset{\sim}{\mathbf{\Delta}}^1_2$-determinacy and $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$ at one Woodin cardinal and ω Woodin cardinals respectively.

In the aftermath of these results, many new directions were developed, and we give only the briefest indication here. Using techniques from inner model theory, tight bounds were given for establishing the exact consistency strength of many determinacy hypotheses. Using similar techniques, it has been shown that almost every natural statement (*i.e.*, not invented specifically to be a counterexample) implies directly those determinacy hypotheses of lesser consistency strength. For instance, by Gödel's Second Incompleteness Theorem, ZFC cannot prove that the AD holds in $\mathbf{L}(\mathbb{R})$, as the latter implies the consistency of the former. Empirically, however, every natural extension T of ZFC of sufficient consistency strength (*i.e.*, such that Peano Arithmetic does not prove the consistency of of T from the consistency of ZF+ AD) does appear to imply that AD holds in $\mathbf{L}(\mathbb{R})$. This sort of phenomenon is taken by some as evidence that the statement that AD holds in $\mathbf{L}(\mathbb{R})$, and other determinacy axioms, should be counted among the true statements extending ZFC (see [KW], for instance).

The history presented here relies heavily on those given by Jackson [Jac10], Kanamori [Kan95, Kan03], Moschovakis [Mos09], Neeman [Nee04] and Steel [Ste08B]. As the title suggests, this is a selective and abbreviated account of the history of determinacy. We have omitted many interesting topics, including, for instance, Blackwell games [Bla69, Mar98, MNV03] and proving determinacy in second-order arithmetic [LSR87, LSR88B, KW10].

§2. Early developments. The first published paper in mathematical game theory appears to be Zermelo's paper [Zer13] on chess. Although he noted that his arguments apply to all games of reason not involving chance, Zermelo worked under two additional chess-specific assumptions. The first was that the game in question has only finitely many states, and the second was that an infinite run of the game was to be considered a draw. Zermelo specified a condition which is equivalent to having a winning strategy in such a game guaranteeing a win within a fixed number of moves, as well as another condition equivalent to having a strategy guaranteeing that one will not lose within a given fixed number of moves. His analysis implicitly introduced the notions of **game tree**, **subtree** of a game tree, and **quasi-strategy**.[2]

The paper states indirectly, but does not quite prove, or even define, the statement that in any game of perfect information with finitely many possible positions such that infinite runs of the game are draws, either one player has a strategy that guarantees a win, or both players have strategies that guarantee at least a draw. A special case of this fact is determinacy for games of perfect information of a fixed finite length, which is sometimes called Zermelo's Theorem.

Kőnig [Kőn27] applied the fundamental fact now known as **Kőnig's Lemma** to the study of games, among other topics. While Kőnig's formulation was somewhat different, his Lemma is equivalent to the assertion that every infinite finitely branching tree with a single root has an infinite path (a path can be found by iteratively choosing any successor node such that the tree above that node is infinite). Extending Zermelo's analysis to games in which infinitely many positions are possible while retaining the condition that each player has only finitely many options at each point, Kőnig used the statement above to prove that in such a game, if one player has a strategy (from a given point in the game) guaranteeing a win, then he can guarantee victory within a fixed number of moves. The application of Kőnig's Lemma to the study of games was suggested by von Neumann.

Kálmar [Kal1928–29] took the analysis a step further by proving Zermelo's Theorem for games with infinitely many possible moves in each round. His arguments proceeded by assigning transfinite ordinals to nodes in the game tree, a method which remains an important tool in modern set theory. Kálmar explicitly introduced the notion of a winning strategy for a game, though his strategies were also quasi-strategies as above. In his analysis, Kálmar introduced a number of other important technical notions, including the notion of a **subgame** (essentially a subtree of the original game tree), and classifying strategies into those which depend only on the current position in the game and those which use the history of the game so far.[3]

Games of perfect information for which the set of infinite runs is divided into winning sets for each player appear in a question by Mazur in the Scottish Book, answered by Banach in an entry dated August 4, 1935 (see [Mau81, p. 113]). Following up on Mazur's question (still in the Book), Ulam asked about games where two players collaborate to build an infinite sequence of 0's and 1's by alternately deciding each member of the sequence, with the winner determined by whether the infinite sequence constructed

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[2] As defined above, a strategy for a given player specifies a move in each relevant position; a quasi-strategy merely specifies a set of acceptable moves. The distinction is important when the Axiom of Choice fails, but is less important in the context of Zermelo's paper.

[3] See [SW01] for much more on these papers of Zermelo, Kőnig and Kálmar.

falls inside some predetermined set E. Essentially raising the issue of determinacy for arbitrary $G_\omega(E)$, Ulam asked: for which sets E does the first player (alternately, the second player) have a winning strategy? (Section 2.1 below has more on the Banach–Mazur game.)

Games of perfect information were formally defined in 1944 by von Neumann and Morgenstern [vNM04]. Their book also contains a proof that games of perfect information of a fixed finite length are determined (p. 123).

Infinite games of perfect information were reintroduced by Gale and Stewart [GS53], who were unaware of the work of Mazur, Banach and Ulam (Gale, personal communication). They showed that a nondetermined game can be constructed using the Axiom of Choice (more specifically, from a wellordering of the set of real numbers).[4] They also noted that the proof from the Axiom of Choice does not give a definable undetermined game, and raised the issue of whether determinacy might hold for all games with a suitably definable payoff set. Towards this end, they introduced a topological classification of infinite games of perfect information, defining a game (or the set of runs of the game which are winning for the first player) to be **open** if all winning runs for the first player are won at some finite stage (*i.e.*, if, whenever $\langle x_0, x_1, x_2, \ldots\rangle$ is a winning run of the game for the first player, there is some n such that the first player wins all runs of the game extending $\langle x_0, \ldots, x_n\rangle$). Using this framework, they proved a number of fundamental facts, including the determinacy of all games whose payoff set is a Boolean combination of open sets (*i.e.*, in the class generated from the open sets by the operations of finite union, finite intersection and complementation). The determinacy of open games would become the basis for proofs of many of the strongest determinacy hypotheses. Gale and Stewart also asked a number of important questions, including the question of whether all Borel games are determined (to be answered positively by Martin [Mar75] in 1974).[5] Classifying games by the definability of their payoff sets would be an essential tool in the study of determinacy.

2.1. Regularity properties. Early motivation for the study of determinacy was given by its implications for regularity properties for sets of

[4] Given a set Y, we let AC_Y denote the statement that whenever $\{X_a : a \in Y\}$ is a collection of nonempty sets, there is a function f with domain Y such that $f(a) \in X_a$ for all $a \in Y$. Zermelo's **Axiom of Choice** (AC) [Zer04] is equivalent to the statement that AC_Y holds for all sets Y. A linear ordering $\leq$ of a set X is a **wellordering** if every nonempty subset of X has a $\leq$-least element. The Axiom of Choice is equivalent to the statement that there exist wellorderings of every set.

Kőnig's Lemma is a weak form of the Axiom of Choice and cannot be proved in ZF (see [Lév79, Exercise IX.2.18]).

[5] The **Borel** sets are the members of the smallest class containing the open sets and closed under the operations of complementation and countable union. The collection of Borel sets is generated in ω_1 many stages from these two operations. A natural process assigns a measure to each Borel set (see, for instance, [Hal50]).

reals. In particular, determinacy of certain games of perfect information was shown to imply that every set of reals has the property of Baire and the perfect set property, and is Lebesgue measurable.[6] These three facts themselves each contradict the Axiom of Choice. We will refer to Lebesgue measurability, the property of Baire and the perfect set property as the **regularity properties**, the fact that there are other regularity properties notwithstanding.

Question 43 of the Scottish Book, posed by Mazur, asks about games where two players alternately select the members of a shrinking sequence of intervals of real numbers, with the first player the winner if the intersection of the sequence intersects a set given in advance. Banach posted an answer in 1935, showing that such games are determined if and only if the given set is either meager (in which case the second player wins) or comeager relative to some interval (in which case the first player wins). The determinacy of the restriction of this game to each interval implies then that the given set has the Baire property (see [Oxt80, pp. 27–30], [Kan03, pp. 373–374]). The game has come to be known as the Banach–Mazur game. Using an enumeration of the rationals, one can code intervals with rational endpoints with integers, getting a game on integers.

Morton Davis [Dav64] studied a game, suggested by Dubins, where the first player plays arbitrarily long finite strings of 0's and 1's and the second player plays individual 0's and 1's, with the payoff set a subset of the set of infinite binary sequences as before. Davis proved that the first player has a winning strategy in such a game if and only if the payoff set contains a perfect set, and the second player has a winning strategy if and only if the payoff set is finite or countably infinite. The determinacy of all such games then implies that every uncountable set of reals contains a perfect set (asymmetric games of this type can be coded by integer games of perfect information). It follows that under AD there is no set of reals whose cardinality falls strictly between $\aleph_0$ and $2^{\aleph_0}$.[7]

Mycielski and Świerczkowski [MŚ64] showed that the determinacy of certain integer games of perfect information implies that every subset of the real line is Lebesgue measurable. Simpler proofs of this fact were later given by Leo Harrington (see [Kan03, pp. 375–377]) and Martin [Mar03].

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[6] A set of reals X has the **property of Baire** if $X \triangle O$ is meager for some open set O, where the **symmetric difference** $A \triangle B$ of two sets A and B is the set $(A \setminus B) \cup (B \setminus A)$, where $A \setminus B = \{x \in A : x \notin B\}$. A set of reals X has the **perfect set property** if it is countable or contains a perfect set (an uncountable closed set without isolated points). A set of reals X is **Lebesgue measurable** if there is a Borel set B such that $X \triangle B$ is a subset of a Borel measure 0 set. See [Oxt80].

[7] *I.e.*, for every set X, if there exist injections $f\colon \omega \to X$ and $g\colon X \to 2^\omega$, then either X is countable or there exists a bijection between X and 2^ω.

By way of contrast, an argument of Vitali [Vit05] shows that under ZFC there are sets of reals which are not Lebesgue measurable. Banach and Tarski ([BT24], see also [Wag93]), building on work of Hausdorff [Hau14], showed that under ZFC the unit ball can be partitioned into five pieces which can be rearranged to make two copies of the same sphere, again violating Lebesgue measurability as well as physical intuition. As with the undetermined game given by Gale and Stewart, the constructions of Vitali and Banach–Tarski use the Axiom of Choice and do not give definable examples of nonmeasurable sets. Via the Mycielski–Świerczkowski theorem, determinacy results would rule out the existence of definable examples, for various notions of definability.

2.2. Definability. As discussed above, ZFC implies that open sets are determined, and implies also that there exists a nondetermined set. The study of determinacy was to merge naturally with the study of sets of reals in terms of their definability (*i.e.*, descriptive set theory), which can be taken as a measure of their complexity. In this section we briefly introduce some important definability classes for sets of reals. Standard references include [Mos80, Kec95]. While we do mention some important results in this section, much of the section can be skipped on a first reading and used for later reference.

A **Polish space** is a topological space which is separable and completely metrizable. Common examples include the integers ω, the reals $\mathbb{R}$, the open interval $(0, 1)$, the Baire space ${}^{\omega}\omega$, the Cantor space ${}^{\omega}2$ and their finite and countable products. Uncountable Polish spaces without isolated points are a natural setting for studying definable sets of reals. For the most part we will concentrate on the Baire space and its finite powers.

Following notation introduced by Addison [Add59B],[8] open subsets of a Polish space are called $\utilde{\mathbf{\Sigma}}^0_1$, complements of $\utilde{\mathbf{\Sigma}}^0_n$ sets are $\utilde{\mathbf{\Pi}}^0_n$, and countable unions of $\utilde{\mathbf{\Pi}}^0_n$ sets are $\utilde{\mathbf{\Sigma}}^0_{n+1}$. More generally, given a positive $\alpha < \omega_1$, $\utilde{\mathbf{\Sigma}}^0_\alpha$ consists of all countable unions of members of $\bigcup_{\beta<\alpha} \utilde{\mathbf{\Pi}}^0_\beta$, and $\utilde{\mathbf{\Pi}}^0_\alpha$ consists of all complements of members of $\utilde{\mathbf{\Sigma}}^0_\alpha$. The Borel sets are the members of $\bigcup_{\alpha<\omega_1} \utilde{\mathbf{\Sigma}}^0_\alpha$.

A **pointclass** is a collection of subsets of Polish spaces. Given a pointclass $\Gamma \subseteq \wp({}^{\omega}\omega)$, we let $\mathsf{Det}(\Gamma)$ and Γ-**determinacy** each denote the statement that $G_\omega(A)$ is determined for all $A \in \Gamma$. Philip Wolfe [Wol55] proved $\utilde{\mathbf{\Sigma}}^0_2$-determinacy in ZFC. Davis [Dav64] followed by proving $\utilde{\mathbf{\Pi}}^0_3$-determinacy. Jeffrey Paris [Par72] would prove $\utilde{\mathbf{\Sigma}}^0_4$-determinacy. However, this result was proved after Martin had used a measurable cardinal to prove analytic determinacy (see Section 5.2).

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[8] The papers [Add59B] and [Add59A] appear in the same volume of **Fundamenta Mathematicae**. The front page of the volume gives the date 1958–1959. The individual papers have the dates 1958 and 1959 on them, respectively.

Continuous images of $\mathbf{\Pi}^0_1$ sets are said to be $\mathbf{\Sigma}^1_1$, complements of $\mathbf{\Sigma}^1_n$ sets are $\mathbf{\Pi}^1_n$, and continuous images of $\mathbf{\Pi}^1_n$ sets are $\mathbf{\Sigma}^1_{n+1}$. For each $i \in \{0, 1\}$ and $n \in \omega$, the pointclass $\mathbf{\Delta}^i_n$ is the intersection of $\mathbf{\Sigma}^i_n$ and $\mathbf{\Pi}^i_n$. The **boldface projective pointclasses** are the sets $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, and $\mathbf{\Delta}^1_n$ for positive $n \in \omega$. These classes were implicit in work of Lebesgue as early as [Leb18]. They were made explicit in independent work by Nikolai Luzin [Luz25C, Luz25B, Luz25A] and Wacław Sierpiński [Sie25]. The notion of a boldface pointclass in general (*i.e.*, possibly non-projective) is used in various ways in the literature. We will say that a pointclass Γ is **boldface** (or **closed under continuous preimages** or **continuously closed**) if $f^{-1}[A] \in \Gamma$ for all $A \in \Gamma$ and all continuous functions f between Polish spaces (where A is a subset of the codomain). The classes $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, $\mathbf{\Delta}^0_\alpha$ are also boldface in this sense.

The pointclass $\mathbf{\Sigma}^1_1$ is also known as the class of **analytic sets**, and was given an independent characterization by Mikhail Suslin [Sus17]: A set of reals A is analytic if and only if there exists a family of closed sets D_s (for each finite sequence s consisting of integers) such that A is the set of reals x for which there is an ω-sequence S of integers such that $x \in \bigcap_{n\in\omega} D_{S\restriction n}$.[9] Suslin showed that there exist non-Borel analytic sets, and that the Borel sets are exactly the $\mathbf{\Delta}^1_1$ sets.

We let $\exists^0$ and $\exists^1$ denote existential quantification over the integers and reals, respectively, and $\forall^0$ and $\forall^1$ the analogous forms of universal quantification. Given a set $A \subseteq ({}^\omega\omega)^{k+1}$, for some positive integer k, $\exists^1 A$ is the set of $(x_1, \ldots, x_k) \in ({}^\omega\omega)^k$ such that for some $x \in {}^\omega\omega$, $(x, x_1, \ldots, x_k) \in A$, and $\forall^1 A$ is the set of $(x_1, \ldots, x_k) \in ({}^\omega\omega)^k$ such that for all $x \in {}^\omega\omega$, $(x, x_1, \ldots, x_k) \in A$. Given a pointclass Γ, $\exists^1\Gamma$ consists of $\exists^1 A$ for all $A \in \Gamma$, and $\forall^1\Gamma$ consists of $\forall^1 A$ for all $A \in \Gamma$. It follows easily that for each positive integer n, $\exists^1\mathbf{\Pi}^1_n = \mathbf{\Sigma}^1_{n+1}$ and $\forall^1\mathbf{\Sigma}^1_n = \mathbf{\Pi}^1_{n+1}$.

Given a pointclass Γ, $\breve{\Gamma}$ is the set of complements of members of Γ, and Δ_Γ is the pointclass $\Gamma \cap \breve{\Gamma}$; Γ is said to be **selfdual** if $\Delta_\Gamma = \Gamma$. A set $A \in \Gamma$ is said to be Γ-**complete** if every member of Γ is a continuous preimage of A. If Γ is closed under continuous preimages and Γ-determinacy holds, then $\breve{\Gamma}$-determinacy holds. Each of the regularity properties for a set of reals A are given by the determinacy of games with payoff set simply definable from A (indeed, continuous preimages of A), but not necessarily with payoff A itself. It follows that when Γ is a boldface pointclass, Γ-determinacy implies the regularity properties for sets of reals in Γ.

A simple application of Fubini's theorem shows that if Γ is a boldface pointclass and there exists in Γ a wellordering of a set of reals of positive Lebesgue measure, then there is a non-Lebesgue measurable set in Γ.

[9] For S a function with domain ω, and $n \in \omega$, $S\restriction n = \langle S(0), \ldots, S(n-1)\rangle$.

Skipping ahead for a moment, in the early 1970s Alexander Kechris and Martin, using a technique of Solovay called **unfolding**, proved that for each integer n, $\boldsymbol{\Pi}^1_n$-determinacy plus **countable choice for sets of reals**[10] implies that all $\boldsymbol{\Sigma}^1_{n+1}$ sets of reals are Lebesgue measurable, have the Baire property and have the perfect set property (see [Kan03, pp. 380–381]).

As developed by Stephen Kleene, the **effective** (or **lightface**) pointclasses Σ^0_n, Π^0_n, Δ^0_n [Kle43] and Σ^1_n, Π^1_n, Δ^1_n [Kle55C, Kle55B, Kle55A] are formed in the same way as their boldface counterparts, starting instead from Σ^0_1, the collection of open sets O such that the set of indices for basic open sets contained in O (under a certain natural enumeration of the basic open sets) is recursive (see [Mos09], for instance). Sets in Σ^0_1 are called **semirecursive**, and sets in Δ^0_1 are called **recursive**. Given $a \in {}^{\omega}\omega$, $\Sigma^0_1(a)$ is the collection of open sets O such that the set of indices for basic open sets contained in O is recursive in a, and the **relativized lightface projective pointclasses** $\Sigma^0_n(a)$, $\Pi^0_n(a)$, $\Delta^0_n(a)$, $\Sigma^1_n(a)$, $\Pi^1_n(a)$, $\Delta^1_n(a)$ are built from Σ^0_n in the manner above. It follows that each boldface pointclass is the union of the corresponding relativized lightface classes (relativizing over each member of ${}^{\omega}\omega$).

Following [Mos09], a pointclass is **adequate** if it contains all recursive sets and is closed under finite unions and intersections, bounded universal and existential integer quantification (see [Mos09, p. 119]) and preimages by recursive functions.[11] The relativized lightface projective pointclasses are adequate (see [Mos09, pp. 118–120]).

Given a Polish space $\mathfrak{X}$, an integer k, a set $A \subseteq \mathfrak{X}^{k+1}$ and $x \in \mathfrak{X}$, A_x is the set of $(x_1, \ldots, x_k)$ such that $(x, x_1, \ldots, x_k) \in A$. A set $A \subseteq \mathfrak{X}^{k+1}$ in a pointclass Γ is said to be **universal** for Γ if each subset of $\mathfrak{X}^k$ in Γ has the form A_x for some $x \in \mathfrak{X}$. Pointclasses of the form Σ^1_n, Π^1_n have universal members. Those of the form Δ^1_n do not. Each member of each boldface pointclass is of the form A_x for A a member of the corresponding effective class. Conversely, as each member of each lightface projective pointclass listed above is definable, each member of each corresponding boldface pointclass is definable from a real number as a parameter.

A set of reals is said to be Σ^2_1 (Π^2_1) if is definable by a formula of the form $\exists X \subseteq \mathbb{R}\, \varphi$ ($\forall X \subseteq \mathbb{R}\, \varphi$), where all quantifiers in φ range over the reals or the integers.

In the **Lévy hierarchy** [Lév65B], a formula φ in the language of set theory is Δ_0 (equivalently Σ_0, Π_0) if all quantifiers appearing in φ are

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[10] The statement that whenever X_n ($n \in \omega$) are nonempty sets of reals, there is a function $f \colon \omega \to \mathbb{R}$ such that $f(n) \in X_n$ for each n. Countable choice for sets of reals is a consequence of AD, as shown by Mycielski [Myc64] (see Section 2.3).

[11] A function f from a Polish space $\mathfrak{X}$ to a Polish space $\mathfrak{Y}$ is said to be **recursive** if the set of pairs $x \in \mathfrak{X}$, $n \in \omega$ such that $f(x)$ is in the nth basic open neighborhood of $\mathfrak{Y}$ is semi-recursive.

bounded (see [Jec03, Chapter 13]); Σ_{n+1} if it has the form $\exists x\psi$ for some Π_n formula ψ; and Π_{n+1} if it has the form $\forall x\psi$ for some Σ_n formula ψ. A set is Σ_n-*definable* if it can be defined by a Σ_n formula (and similarly for Π_n). We say that a model M is **Γ-correct**, for a class of formulas Γ, if for all φ in Γ and $x \in M$, $M \models \varphi(x)$ if and only if $\mathbf{V} \models \varphi(x)$. If M is a model of ZF, we say that a set in M is Σ_n^M if it is definable by a Σ_n formula relativized to M (and similarly for other classes of formulas).

Gödel's inner model $\mathbf{L}$ is the smallest transitive model of ZFC containing the ordinals. For any set A, Gödel's constructible universe $\mathbf{L}$ generalizes to two inner models $\mathbf{L}(A)$ and $\mathbf{L}[A]$, developed respectively by András Hajnal [Haj56, Haj61] and Azriel Lévy [Lév57, Lév60] (see [Jec03, Chapter 13] or [Kan03, p. 34]). Given a set A, $\mathbf{L}(A)$ is the smallest transitive model of ZF containing the transitive closure of $\{A\}$ and the ordinals,[12] and $\mathbf{L}[A]$ is the smallest transitive model of ZF containing the ordinals and closed under the function $X \mapsto A \cap X$. Alternately, $\mathbf{L}(A)$ is constructed in the same manner as $\mathbf{L}$, but introducing the members of the transitive closure of the set $\{A\}$ at the first level, and $\mathbf{L}[A]$ is constructed as $\mathbf{L}$, but by adding a predicate for membership in A to the language. When A is contained in $\mathbf{L}$, $\mathbf{L}(A)$ and $\mathbf{L}[A]$ are the same. While $\mathbf{L}[A]$ is always a model of AC, $\mathbf{L}(A)$ need not be. Indeed, $\mathbf{L}(\mathbb{R})$ is a model of AD in the presence of suitably large cardinals, and is thus a natural example of a "smaller universum" as described in the quote from [MS62] in Section 2.3.

Though it can be formulated in other ways, we will view the set $0^\#$ ("zero sharp") as the theory of a certain class of ordinals which are indiscernibles over the inner model $\mathbf{L}$. This notion was independently isolated by Solovay [Sol67A] and by Jack Silver in his 1966 Berkeley Ph.D. thesis (see [Sil71C]). The existence of $0^\#$ cannot be proved in ZFC, as it serves as a sort of transcendence principle over $\mathbf{L}$. For instance, if $0^\#$ exists then every uncountable cardinal of $\mathbf{V}$ is a strongly inaccessible cardinal in $\mathbf{L}$.[13] For any set X there is an analogous notion of $X^\#$ ("X sharp") serving as a transcendence principle over $\mathbf{L}(X)$ (see [Kan03]).

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[12] A set x is **transitive** if $z \in x$ whenever $y \in x$ and $z \in y$. The **transitive closure** of a set x is the smallest transitive set containing x.

[13] A cardinal κ is **strongly inaccessible** if it is uncountable, regular and a strong limit (*i.e.*, $2^\gamma < \kappa$ for all $\gamma < \kappa$). If κ is a strongly inaccessible cardinal, then $\mathbf{V}_\kappa$ is a model of ZFC. Hence, by Gödel's Second Incompleteness Theorem, the existence of strongly inaccessible cardinals cannot be proved in ZFC. See [Jec03] for the definition of $\mathbf{V}_\alpha$, for an ordinal α.

2.3. The Axiom of Determinacy. The Axiom of Determinacy, the statement that all length ω integer games of perfect information are determined, was proposed by Mycielski and Steinhaus [MS62].[14] In a passage that anticipated a commonly accepted view of determinacy, they wrote

> It is not the purpose of this paper to depreciate the classical mathematics with its fundamental "absolute" intuitions on the universum of sets (to which belongs the axiom of choice), but only to propose another theory which seems very interesting although its consistency is problematic. Our axiom can be considered as a restriction of the classical notion of a set leading to a smaller universum, say of determined sets, which reflect some physical intuitions which are not fulfilled by the classical sets ... Our axiom could be considered as an axiom added to the classical set theory claiming the existence of a class of sets satisfying (A) and the classical axioms (without the axiom of choice).

Mycielski and Steinhaus summarized the state of knowledge of determinacy at that time, including the fact that AD implies that all sets of reals are Lebesgue measurable and have the Baire property, and they noted that by results of Kurt Gödel and Addison [Add59B], there is in Gödel's constructible universe $\mathbf{L}$ (and thus consistently with ZFC) a Δ^1_2 wellordering of the reals, and thus a Δ^1_2 set which is not determined.

In his [Myc64], Mycielski proved several fundamental facts about determinacy, including the fact that AD implies countable choice for set of reals (he credits this result to Świerczkowski, Dana Scott and himself, independently). Thus, while AD contradicts the Axiom of Choice, it implies a form of Choice which suffices for many of its most important applications, including the countable additivity of Lebesgue measure. Via countable choice for sets of reals, AD implies that ω_1 is regular.[15] Mycielski also showed that AD implies that there is no uncountable wellordered sequence of reals. In conjunction with the perfect set property, this implies that under determinacy, $\omega_1^{\mathbf{V}}$ is a strongly inaccessible cardinal in the inner model $\mathbf{L}$ (and even in $\mathbf{L}[a]$ for any real number a), a fact which was to be greatly extended by Solovay, Martin and Woodin. Harrington [Har78] would show that Π^1_1-determinacy implies that $0^{\#}$ exists, and thus that Π^1_1-determinacy is not provable in ZFC.

[14] We continue to use the now-standard abbreviation AD for the Axiom of Determinacy; it was called (A) in [MS62].

[15] The ordinal ω_1 is the first uncountable ordinal. A cardinal κ is **regular** if, for every $\gamma < \kappa$, every function $f\colon \gamma \to \kappa$ has range bounded in κ. Under ZFC, every successor cardinal is regular. Solomon Feferman and Azriel Lévy [FL63] (see also [HR98, pp. 153–154]) showed that the singularity of ω_1 is consistent with ZF. Moti Gitik [Git80] showed that it is consistent with ZF (relative to large cardinals) that ω is the largest regular cardinal.

In the same paper, Mycielski showed that ZF implies the existence of an undetermined game of perfect information of length ω_1 where the players play countable ordinals instead of integers. An interesting aspect of the proof is that it does not give a specific undetermined game. As a slight variant on Mycielski's argument, consider the game in which the first player plays a countable ordinal α (and then makes no other moves for the rest of the game) and the second player plays a sequence of integers coding α, under some fixed coding of hereditarily countable sets by reals.[16] Since the first player cannot have a winning strategy in this game, determinacy for the game implies the existence of an injection from ω_1 into $\mathbb{R}$, which contradicts AD but is certainly by itself consistent with ZF, as it follows from ZFC. Later results of Woodin would show that, assuming the consistency of certain large cardinal hypotheses, ZFC is consistent with the statement that every integer game of length ω_1 with payoff set definable from real and ordinal parameters is determined (see Section 6.3, and [Nee04, p. 298]). Mycielski noted that under AD there are no nonprincipal ultrafilters[17] on ω (this follows from Lebesgue measurability for all sets of reals plus a result of Sierpiński [Sie38] showing that nonprincipal ultrafilters on ω give rise to nonmeasurable sets of reals), which implies that every ultrafilter (on any set) is countably complete (*i.e.*, closed under countable intersections). Finally, in a footnote on the first page of the paper, Mycielski reiterated a point made in the passage quoted above from his paper with Steinhaus, suggesting that an inner model containing the reals could satisfy AD. In a followup paper, Mycielski [Myc66] presented a number of additional results, including the fact that there is a game in which the players play real numbers whose determinacy implies *uniformization* (see Section 3.2) for subsets of the plane, another weak form of the Axiom of Choice.

In 1964, a year after Paul Cohen's invention of forcing, Solovay [Sol70] proved that if there exists a strongly inaccessible cardinal, then in a forcing extension there exists an inner model containing the reals in which every set of reals satisfies the regularity properties from Section 2.1. Shelah [She84] later showed that a strongly inaccessible cardinal is necessary, in the sense that the Lebesgue measurability of all sets of reals (and even the perfect set property for $\undertilde{\mathbf{\Pi}}^1_1$ sets) implies that ω_1 is strongly inaccessible in all models of the form $\mathbf{L}[a]$, for $a \subseteq \omega$. In the introduction to his paper, Solovay

[16] The **hereditarily countable** sets are those sets whose transitive closures are countable. Such sets are naturally coded by sets of integers.

[17] An **ultrafilter** on a nonempty set X is a collection U of nonempty subsets of X which is closed under supersets and finite intersections, and which has the property that for every $A \subseteq X$, exactly one of A and $X \setminus A$ is in U. An ultrafilter is **nonprincipal** if it contains no finite sets. The existence of nonprincipal ultrafilters on ω follows from ZFC, but (as this result shows) requires the Axiom of Choice.

conjectured (correctly, as it turned out) that large cardinals would imply that AD holds in $\mathbf{L}(\mathbb{R})$.

The year 1967 saw two major results in the study of determinacy, one by Blackwell [Bla67] and the other by Solovay. Reversing chronological order by a few months, we discuss Blackwell's result and its consequences in the next section, and Solovay's in Section 4.

§3. Reduction and scales. Blackwell [Bla67] used open determinacy to reprove a theorem of Kuratowski [Kur36] stating that the intersection of any two analytic sets A, B in a Polish space $\mathfrak{Y}$ is also the intersection of two analytic sets A' and B' such that $A \subseteq A'$, $B \subseteq B'$, and $A' \cup B' = \mathfrak{Y}$.[18] Briefly, the argument is as follows. Since A and B are analytic, there exist continuous surjections $f\colon {}^{\omega}\omega \to A$ and $g\colon {}^{\omega}\omega \to B$. For each finite sequence $\langle n_0, \dots, n_k\rangle$, let $\Omega(\langle n_0, \dots, n_k\rangle)$ be the set of $x \in {}^{\omega}\omega$ with $\langle n_0, \dots, n_k\rangle$ as an initial segment; let $R(\langle n_0, \dots, n_k\rangle)$ be the closure (in $\mathfrak{Y}$) of the f-image of $\Omega(\langle n_0, \dots, n_k\rangle)$; and let $S(\langle n_0, \dots, n_k\rangle)$ be the closure of the g-image of $\Omega(\langle n_0, \dots, n_k\rangle)$. Then for each $z \in \mathfrak{Y}$, let $G(z)$ be the game where players player I and player II build x and y in ${}^{\omega}\omega$, with player I winning if for some integer k, $z \in R(x{\restriction}k) \setminus S(y{\restriction}k)$, player II winning if for some integer k, $z \in S(y{\restriction}k) \setminus R(x{\restriction}(k+1))$, and the run of the game being a draw if neither of these happens. Roughly, each player is creating a real (x or y) to feed into his function, and trying to maintain for as long as possible that the corresponding output can be made arbitrarily close to the target real z; the loser is the first player to fail to maintain this condition. Let A' be the set of z for which player I has a strategy guaranteeing at least a draw, and let B' be the set of z for which player player II has such a strategy. Then the determinacy of open games implies that ${}^{\omega}\omega = A' \cup B'$, and $A \subseteq A'$, $B \subseteq B'$ and $A' \cap B' = A \cap B$ follow from the fact that A is the range of f and B is the range of g. The sets A' and B' are analytic, as A' is a projection of the set of pairs (φ, z) such that φ is (a code for) a strategy for player I in $G(z)$ guaranteeing at least a draw, which is Borel, and similarly for B'.[19]

3.1. Reduction, separation, norms and prewellorderings. In his [Kur36], Kuratowski defined the **reduction theorem** (now called the **reduction property**) for a pointclass Γ to be the statement that for any A, B in Γ there exist disjoint A', B' in Γ with $A' \subseteq A$, $B' \subseteq B$ and $A' \cup B' = A \cup B$. He showed in this paper that $\boldsymbol{\Pi}^1_1$ and $\boldsymbol{\Sigma}^1_2$ have the reduction property; Addison [Add59B] showed this for $\Pi^1_1(a)$ and $\Sigma^1_2(a)$, for each real number a. Blackwell's argument proves the reduction property for $\boldsymbol{\Pi}^1_1$, working with the corresponding $\boldsymbol{\Sigma}^1_1$ complements.

[18] Blackwell describes the discovery of his proof in [AA85, p. 26].

[19] A **projection** of a set $A \subseteq ({}^{\omega}\omega)^k$ (for some integer $k \geq 2$) is a set of the form $\{(x_0, \dots, x_{i-i}, x_{i+1}, \dots, x_{k-1}) \mid \exists x_i (x_0, \dots, x_{k-1}) \in A\}$, for some $i < k$.

Kuratowski also defined the **first separation theorem** (now called the **separation property**) for a pointclass Γ to be the statement that for any disjoint A, B in Γ there exists C in Δ_Γ with $A \subseteq C$ and $B \cap C = \varnothing$. This property had been studied by Sierpiński [Sie24] and Luzin [Luz30A] for initial segments of the Borel hierarchy. Kuratowski also noted that the reduction property for a pointclass Γ implies the separation property for $\breve{\Gamma}$. Luzin [Luz27, pp. 51–55] proved that the pointclass $\utilde{\mathbf{\Sigma}}^1_1$ satisfies the separation property, by showing that disjoint $\utilde{\mathbf{\Sigma}}^1_1$ sets are contained in disjoint Borel sets. Petr Novikov [Nov35] showed that $\utilde{\mathbf{\Pi}}^1_2$ satisfies the separation property and $\utilde{\mathbf{\Sigma}}^1_2$ does not. Novikov [Nov35] (in the case of $\utilde{\mathbf{\Sigma}}^1_2$ sets) and Addison [Add59B] showed that if Γ satisfies the reduction property and has a so-called **doubly universal** member, and Δ_Γ has no universal member, then Γ does not have the separation property, so $\breve{\Gamma}$ does not have the reduction property.[20] Addison [Add59A, Add59B] showed that if all real numbers are constructible, then the reduction property holds for $\utilde{\mathbf{\Sigma}}^1_k$, for all $k \geq 2$.

Inspired by Blackwell's argument, Addison and Martin independently proved that $\utilde{\mathbf{\Delta}}^1_2$-determinacy implies that $\utilde{\mathbf{\Pi}}^1_3$ has the reduction property. Since the pointclass $\utilde{\mathbf{\Sigma}}^1_3$ has a doubly universal member, this shows that $\utilde{\mathbf{\Delta}}^1_2$-determinacy implies the existence of a nonconstructible real. This fact also follows from Gödel's result (discussed in [Add59A]) that the Lebesgue measurability of all Δ^1_2 sets implies the existence of a nonconstructible real. Determinacy would soon be shown to imply stronger structural properties for the projective pointclasses.

The key technical idea behind the (pre-determinacy) results listed above on separation and reduction for the first two levels of the projective hierarchy was the notion of *sieve* (in French, *crible*). This construction first appeared in a paper of Lebesgue [Leb05], in which he proved the existence of Lebesgue-measurable sets which are not Borel. In Lebesgue's presentation, a **sieve** is an association of a closed subset F_r of the unit interval $[0, 1]$ to each rational number r in this interval. The sieve then represents the set of $x \in [0, 1]$ such that $\{r \mid x \in F_r\}$ is wellordered, under the usual ordering of the rationals. Using this approach, Luzin and Sierpiński [LS18, LS23] showed that $\utilde{\mathbf{\Sigma}}^1_1$ sets and $\utilde{\mathbf{\Pi}}^1_1$ sets are unions of $\aleph_1$ many Borel sets.

Much of the classical work of Luzin, Sierpiński, Kuratowski and Novikov mentioned here was redeveloped in the lightface context by Kleene [Kle43, Kle55C, Kle55B, Kle55A], who was unaware of their previous work. The two theories were unified primarily by Addison (for example, [Add59A]).

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[20]Members U,V of a pointclass Γ are **doubly universal** for Γ if for each pair A,B of members of Γ there exist an $x \in {}^\omega\omega$ such that $U_x = A$ and $V_x = B$. The non-selfdual projective pointclasses (*e.g.*, $\Sigma^1_1(a)$, $\Pi^1_1(a)$, $\Sigma^1_2(a)$, $\Pi^1_2(a)$, $\ldots$) all have doubly universal members.

While Blackwell's argument generalizes throughout the projective hierarchy, Moschovakis ([Mos67, Mos69B, Mos69C, Mos70A, Mos71A], see also [Mos09, pp. 202–206]) developed via the effective theory a generalization of the Luzin–Sierpiński approach (decomposing a set of reals into a wellordered sequence of simpler sets) which could be similarly propagated. Moschovakis's goal was to find a uniform approach to the theory of Π^1_1 and Σ^1_2; he was unaware of either Kuratowski's work or determinacy (personal communication). He extracted the following notions, for a given pointclass Γ: a Γ-**norm** for a set A is a function $\rho\colon A \to \mathrm{On}$ for which there exist relations $R^+ \in \Gamma$ and $R^- \in \breve{\Gamma}$ such that for any $y \in A$,

$$x \in A \wedge \rho(x) \leq \rho(y) \leftrightarrow R^+(x,y) \leftrightarrow R^-(x,y);$$

a pointclass Γ is said to have the **prewellordering property** if every $A \in \Gamma$ has a Γ-norm.[21] The prewellordering property was first explicitly formulated by Moschovakis in 1964; the definition just given is a reformulation due to Kechris. Kuratowski [Kur36] and Addison [Add59B] had shown that a variant of the property implies the reduction property; the same holds for the prewellordering property as defined by Moschovakis. Moschovakis applied Novikov's arguments to show that if Γ is a projective pointclass such that $\forall^1\Gamma \subseteq \Gamma$, and Γ has the prewellordering property, then so does the pointclass $\exists^1\Gamma$. Martin and Moschovakis independently completed the picture in 1968, proving what is now known as the First Periodicity Theorem.

Theorem 3.1 (First Periodicity Theorem). *Let Γ be an adequate pointclass and suppose that Δ_Γ-determinacy holds. Then for all $A \in \Gamma$, if A admits a Γ-norm, then $\forall^1 A$ admits a $\forall^1\exists^1\Gamma$-norm.*

Corollary 3.2 ([AM68, Mar68]). *Let Γ be an adequate pointclass closed under existential quantification over reals, and suppose that Δ_Γ-determinacy holds. If Γ satisfies the prewellordering property, then so does $\forall^1\Gamma$.*

Projective Determinacy (PD) is the statement that all projective sets of reals are determined. By the First Periodicity Theorem, under Projective Determinacy the following pointclasses have the prewellordering property, for any real a:

$$\Pi^1_1(a), \Sigma^1_2(a), \Pi^1_3(a), \Sigma^1_4(a), \Pi^1_5(a), \Sigma^1_6(a), \ldots$$

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[21] A **prewellordering** is a binary relation which is wellfounded, transitive and total. A function ρ from a set X to the ordinals induces a prewellording $\preceq$ on X by setting $a \preceq b$ if and only if $\rho(a) \leq \rho(b)$. Conversely, a prewellordering $\preceq$ on a set X induces a function ρ from X to the ordinals, where for each $a \in X$, $\rho(a)$ (the $\preceq$-**rank** of a) is the least ordinal α such that $\rho(b) < \alpha$ for all $b \in X$ such that $b \preceq a$ and $a \not\preceq b$. The range of ρ is called the **length** of $\preceq$.

By contrast (see [Kan03, pp. 409–410]), in $\mathbf{L}$ the pointclasses with the prewellordering property are

$$\Pi^1_1(a), \Sigma^1_2(a), \Sigma^1_3(a), \Sigma^1_4(a), \Sigma^1_5(a), \Sigma^1_6(a), \ldots$$

3.2. Scales. As noted above, the Axiom of Determinacy contradicts the Axiom of Choice, but it is consistent with, and even implies, certain weak forms of Choice. If X and Y are nonempty sets and A is a subset of the product $X \times Y$, a function f **uniformizes** A if the domain of f is the set of $x \in X$ such that there exists a $y \in Y$ with $(x, y) \in A$, and such that for each x in the domain of f, $(x, f(x)) \in A$. A consequence of the Axiom of Choice, **uniformization** is the statement that for every $A \subseteq \mathbb{R} \times \mathbb{R}$ there is a function f which uniformizes A.

Uniformization is not implied by AD, as it fails in $\mathbf{L}(\mathbb{R})$ whenever there are no uncountable wellordered sets of reals ([Sol78B]; see Section 3.3).

Uniformization was implicitly introduced by Jacques Hadamard [Had05], when he pointed out that the Axiom of Choice should imply the existence of functions on the reals which disagree everywhere with every algebraic function over the integers. Luzin [Luz30C] explicitly introduced the notion of uniformization and showed that such functions exist. He also announced several results on uniformization, including the fact that all Borel sets (but not all $\utilde{\boldsymbol{\Sigma}}^1_1$ sets) can be uniformized by $\utilde{\boldsymbol{\Pi}}^1_1$ functions. The result on Borel sets was proved independently by Sierpiński. Novikov [LN35] showed that every $\utilde{\boldsymbol{\Sigma}}^1_1$ set of pairs has a $\utilde{\boldsymbol{\Sigma}}^1_2$ uniformization.

A pointclass Γ is said to have the **uniformization property** if every set of pairs in Γ is uniformized by a function in Γ. Motokiti Kondô [Kon38] showed that the pointclasses $\utilde{\boldsymbol{\Pi}}^1_1$ and $\utilde{\boldsymbol{\Sigma}}^1_2$ have the uniformization property. The effective version of this result (*i.e.*, for Π^1_1 and Σ^1_2) was proved by Addison. In some sense this is as far as one can go in ZFC: Lévy [Lév65A] would show that consistently there exist Π^1_2 sets that cannot be uniformized by any projective function. Remarkably, Luzin [Luz25B] has predicted that the question of whether the projective sets are Lebesgue measurable and satisfy the perfect set property would never be solved.

After studying Kondô's proof, Moschovakis in 1971 isolated a property for sets of reals which induces uniformizations. Given a set A and an ordinal γ, a **scale** (or a γ-scale) on A into γ is a sequence of functions $\rho_n \colon A \to \gamma$ $(n \in \omega)$ such that whenever

- $\{x_i : i \in \omega\} \subseteq A$ and $\lim_{i \to \omega} x_i = x$, and
- the sequence $\langle \rho_n(x_i) : i \in \omega \rangle$ is eventually constant for each $n \in \omega$,

then $x \in A$ and, for every $n \in \omega$, $\rho_n(x)$ is less than or equal to the eventual value of $\langle \rho_n(x_i) : i \in \omega \rangle$. The scale is a **$\Gamma$-scale** if there exist $R^+ \in \Gamma$ and $R^- \in \breve{\Gamma}$ such that for all $y \in A$ and all $n \in \omega$,

$$x \in A \wedge \rho_n(x) \leq \rho_n(y) \leftrightarrow R^+(n, x, y) \leftrightarrow R^-(n, x, y).$$

A pointclass Γ has the **scale property** if every A in Γ has a Γ-scale. Moschovakis [Mos71A] proved the following three theorems about the scale property.

THEOREM 3.3. If Γ is an adequate pointclass, $A \in \Gamma$, and A admits a Γ-scale, then $\exists^1 A$ admits a $\exists^1\forall^1\Gamma$-scale.

THEOREM 3.4 (Second Periodicity Theorem). Suppose that Γ is an adequate pointclass such that Δ_Γ-determinacy holds. Then for all $A \in \Gamma$, if A admits a Γ-scale, then $\forall^1 A$ admits a $\forall^1\exists^1\Gamma$-scale.

THEOREM 3.5. Suppose that Γ is an adequate pointclass which is closed under integer quantification. Suppose that Γ has the scale property, and that Δ_Γ-determinacy holds. Then Γ has the uniformization property.

Kondô's proof of uniformization for $\undertilde{\mathbf{\Pi}}^1_1$ shows that $\Pi^1_1(a)$ has the scale property for every real a (see [Kan03, p. 419]). It follows that under $\undertilde{\mathbf{\Delta}}^1_{2n}$-determinacy, $\undertilde{\mathbf{\Pi}}^1_{2n+1}$ and $\undertilde{\mathbf{\Sigma}}^1_{2n+2}$ have the scale property, and every $\undertilde{\mathbf{\Pi}}^1_{2n+1}$ relation on the reals can be uniformized by a $\undertilde{\mathbf{\Pi}}^1_{2n+1}$ relation (and similarly for $\undertilde{\mathbf{\Sigma}}^1_{2n+2}$). Furthermore, under Projective Determinacy, for any real a, the projective pointclasses with the scale property are the same as those with the prewellordering property: $\Pi^1_1(a)$, $\Sigma^1_2(a)$, $\Pi^1_3(a)$, $\Sigma^1_4(a)$, $\Pi^1_5(a)$, $\Sigma^1_6(a)$, *etc.*

A **tree** on a set X is a collection of finite sequences from X closed under initial segments. Given sets X and Z, a positive integer k and a tree T on $X^k \times Z$, the **projection** of T, $\mathrm{p}[T]$, is the set of $x \in (X^\omega)^k$ such that for some $z \in Z^\omega$, $(x\restriction n, z\restriction n) \in T$ for all $n \in \omega$ (strictly speaking, this definition involves the identification of finite sequences of k-tuples with k-tuples of finite sequences). If one substitutes the Baire space ${}^\omega\omega$ for $\mathbb{R}$, Suslin's construction for analytic sets (see Section 2.2) essentially presents them as projections of trees on $\omega \times \omega$, modulo the representation of closed intervals. Many descriptive set theorists, starting perhaps with Luzin and Sierpiński [LS23], used trees to represent sets of reals, except that they converted these trees to linear orders via what is now known as the Kleene–Brouwer ordering (after [Bro24] and [Kle55C]). The explicit use of projections of trees as we have presented them here is due to Richard Mansfield [Man70]. As pointed out in [KM78B], given an ordinal γ, a γ-scale for a subset A of the Baire space naturally gives rise to a tree on $\omega \times \gamma$ such that $\mathrm{p}[T] = A$. Given a set Z, a subset of the Baire space is said to be Z-**Suslin** if it is the projection of a tree on $\omega \times Z$. Suslin's representation of analytic sets shows that a set is analytic if and only if it is ω-Suslin. Some authors use "Suslin" to mean "analytic". We will follow a different usage, however, and say that a subset of the Baire space is **Suslin** if is γ-Suslin for some ordinal γ.

Given a tree T on $\omega \times Z$ and a wellordering of Z, a member of $\mathrm{p}[T]$ can be found by following the so-called **leftmost** infinite branch through T

(similar to the proof of Kőnig's Lemma, one picks a path through the tree by taking the least next step which is the initial segment of an infinite path through the tree). In a similar manner, a tree on $(\omega \times \omega) \times \gamma$, for some ordinal γ, induces a uniformization of the projection of the tree.

3.3. The game quantifier. Given a Polish space $\mathfrak{X}$ and a set $B \subseteq \mathfrak{X} \times {}^{\omega}\omega$, we let $\Game B$ denote the set of $x \in \mathfrak{X}$ such that player I has a winning strategy in $G_\omega(B_x)$. If Γ is a pointclass, $\Game\Gamma$ is the class $\{\Game B \mid B \in \Gamma\}$. The following facts appear in [Mos09, pp. 245–246].

THEOREM 3.6. If Γ is an adequate pointclass then the following hold.

- $\Game\Gamma$ is adequate and closed under $\exists^0$ and $\forall^0$.
- $\exists^1\Gamma \subseteq \Game\Gamma$ and $\forall^1\Gamma \subseteq \Game\Gamma$.
- If $\mathsf{Det}(\Gamma)$ holds, then $\Game\Gamma \subseteq \forall^1\exists^1\Gamma$.

The First Periodicity Theorem can be stated more generally as the fact that if an adequate pointclass Γ has the prewellordering property, then so does $\Game\Gamma$, and the Second Periodicity Theorem can be similarly stated as saying that if an adequate pointclass Γ has the scale property, then so does $\Game\Gamma$ (see [Mos09, pp. 246,267]). The propagation of these properties through the projective pointclasses then follows from Theorem 3.6, given that they hold for $\undertilde{\mathbf{\Pi}}^1_1$ (and its variants).

Modifying the notion of Γ-scale by dropping the requirement that $\rho_n(x)$ is less than or equal to the eventual value of $\langle \rho_n(x_i) : i \in \omega \rangle$, one gets the notion of **Γ-semiscale**. Moschovakis's Third Periodicity Theorem [Mos73] concerns the definability of winning strategies and is stated using the game quantifier and the notion of semiscale.

THEOREM 3.7 (Third Periodicity Theorem). Suppose that Γ is an adequate pointclass, and that $\mathsf{Det}(\Gamma)$ holds. Fix $A \subseteq {}^{\omega}\omega$ in Γ, and suppose that A admits a Γ-semiscale and that player I has a winning strategy in the game $G_\omega(A)$. Then player I has a winning strategy coded by a subset of ω in $\Game\Gamma$.

One consequence the Third Periodicity Theorem in conjunction with Theorem 3.6 is the following [Mos73]: for any $n \in \omega$, if $\undertilde{\mathbf{\Sigma}}^1_{2n}$-determinacy holds, $A \subseteq \omega^\omega$ is $\Sigma^1_{2n}(a)$ for some real a and player I has a winning strategy in the game with payoff A, then player I has a winning strategy coded by a subset of ω in $\Delta^1_{2n+1}(a)$.

Let $\Game^1$ denote the game quantifier for **real games**, games of length ω where the players alternate playing real numbers. Then $\Game^1\undertilde{\mathbf{\Sigma}}^0_1$ defines the **inductive** sets of reals.[22] Moschovakis [Mos78] showed that the inductive sets have the scale property. Moschovakis [Mos83] showed that, assuming

[22]Formally, this definition requires a definable association of ω-sequences of reals to individual reals. Alternately, a set of reals is inductive if it is in $\Sigma_1^{J_{\kappa_{\mathbb{R}}}(\mathbb{R})}$, where J refers

the determinacy of all games with payoff in the class built from the inductive sets by the operations of projection and complementation, coinductive sets have scales in this class. Building on this work, Martin and Steel [MS83] showed that the pointclass $\utilde{\mathbf{\Sigma}}^2_1$ has the scale property in $\mathbf{L}(\mathbb{R})$. Kechris and Solovay had shown that if there is no wellordering of the reals in $\mathbf{L}(\mathbb{R})$, then there exists in $\mathbf{L}(\mathbb{R})$ a set of reals that cannot be uniformized, the set of pairs (x, y) such that y is not ordinal definable from x (*i.e.*, definable from x and some ordinals). This set is Π^2_1 in $\mathbf{L}(\mathbb{R})$.

The **Solovay Basis Theorem** says that if $P(A)$ is a Σ^2_1 relation on subsets of ${}^\omega\omega$ and there exists a witness to $P(A)$ in $\mathbf{L}(\mathbb{R})$, then there is a Δ^2_1 witness. This reflection result, along with the Martin–Steel theorem on scales in $\mathbf{L}(\mathbb{R})$, compensates in many circumstances for the fact that not every set of reals has a scale in $\mathbf{L}(\mathbb{R})$.

Steel [Ste83A] applied Jensen's fine structure theory [Jen72] to the study of scales in $\mathbf{L}(\mathbb{R})$, refining and unifying a great deal of work on scales and Suslin cardinals. Extending [MS83], he showed that for each positive ordinal α, determinacy for all sets of reals in $J_\alpha(\mathbb{R})$ implies that the pointclass $\Sigma_1^{J_\alpha(\mathbb{R})}$ has the scale property.

Martin [Mar83B] showed how to propagate the scale property using the game quantifier for integer games of fixed countable length (this subsumes propagation by the quantifier $\Game^1$), and Steel [Ste88, Ste08C] did the same for certain games of length ω_1.

3.4. Partially playful universes. The periodicity theorems showed that determinacy axioms imply structural properties for sets of reals beyond the classical regularity properties. It remained to show that these hypotheses were necessary. Towards this end, Moschovakis (see [Bec78]) identified for each integer n (under the assumption of $\utilde{\mathbf{\Delta}}^1_k$-determinacy, where k is the greatest even integer less than n) the smallest transitive Σ^1_n-correct model of ZF+Dependent Choices (DC) which contains all the ordinals (Joseph Shoenfield [Sho61] had shown that $\mathbf{L}$ is Σ^1_2-correct).[23] This model satisfies AC and $\utilde{\mathbf{\Delta}}^1_k$-determinacy and has a Σ^1_{n+1} wellordering of the reals. In this model, Π^1_i has the scale property for all odd $i \leq n$, and Σ^1_i has the scale property for all other positive integers i.

to Ronald Jensen's constructibility hierarchy and $\kappa_\mathbb{R}$ is the least κ such that $J_\kappa(\mathbb{R})$ is a model of Kripke–Platek set theory.

[23]The Axiom of Dependent Choices (DC) is the statement that if R is a binary relation on a nonempty set X, and if for each $x \in X$ there is a $y \in X$ such that xRy, then there exists an infinite sequence $\langle x_i : i < \omega\rangle$ such that $x_i R x_{i+1}$ for all $i \in \omega$. This statement is a weakening of the Axiom of Choice, sufficient to prove Kőnig's Lemma, the regularity of ω_1 and the wellfoundedness of ultrapowers by countably complete ultrafilters. See [Jec03].

Kechris and Moschovakis [KM78B] introduced the models $\mathbf{L}[T_{2n+1}]$, where T_{2n+1} denotes the tree for a Π^1_{2n+1}-scale for a complete Π^1_{2n+1} set. Moschovakis showed that $\mathbf{L}[T_1] = \mathbf{L}$, and conjectured that $\mathbf{L}[T_{2n+1}]$ is independent of the choice of complete set and scale when for all n. This conjecture was proved by Howard Becker and Kechris in [BK84].

Solovay [Sol66] showed that if $\mathbf{L} \cap \mathbb{R}$ is countable, then it is the largest countable Σ^1_2 set of reals (*i.e.*, a countable Σ^1_2 set which contains all other such sets). Kechris and Moschovakis [KM72] showed that for each positive integer n, if $\mathsf{Det}(\Delta^1_{2n})$ holds then there exists a largest countable Σ^1_{2n+2} set. The largest countable Σ^1_{2n} set came to be called C_{2n}. Kechris [Kec75B] showed that under Projective Determinacy there is for each integer n a largest countable Π^1_{2n+1} set, which he also called C_{2n+1}. The case $n = 0$ follows from $\mathsf{ZF}+\mathsf{DC}$ and was shown independently by David Guaspari, Kechris and Gerald Sacks [Sac76]. Kechris also showed that under Projective Determinacy there are no largest countable Σ^1_{2n+1} or Π^1_{2n} sets. It follows that under Projective Determinacy the lightface projective pointclasses with a largest countable set are the same as those in the zig-zag pattern above for the prewellording property and the scale property. Harrington and Kechris [HK81] showed (under the assumption that AD holds in $\mathbf{L}(\mathbb{R})$) that the reals of each $\mathbf{L}[T_{2n+1}]$ are exactly C_{2n+2}, for all integers n (the case $n = 1$ was due to Kechris and Martin).

Kechris showed (assuming Projective Determinacy) that each model $\mathbf{L}[\mathrm{C}_{2n}]$ satisfies $\mathsf{Det}(\utilde{\boldsymbol{\Delta}}^1_{2n-1})$ but not $\mathsf{Det}(\utilde{\boldsymbol{\Sigma}}^1_{2n-1})$, and has a Δ^1_{2n} wellordering of its reals. Martin would show that $\mathsf{Det}(\utilde{\boldsymbol{\Delta}}^1_{2n})$ implies $\mathsf{Det}(\utilde{\boldsymbol{\Sigma}}^1_{2n})$ for each positive integer n.

3.5. Wadge degrees. In 1968, William Wadge considered the following game, given two sets of reals A and B: player I builds a real x, player II builds a real y, and player II wins if $x \in A \leftrightarrow y \in B$. Determinacy for this class of games is known as **Wadge determinacy**. Given two sets of reals A, B, we say that $A \leq_{\mathrm{W}} B$ (A has **Wadge rank** less than or equal to B, or is **Wadge reducible** to B) if there is a continuous function f such that for all reals x, $x \in A$ if and only if $f(x) \in B$ (*i.e.*, such that $A = f^{-1}[B]$). Wadge determinacy implies that for any two sets of reals A, B, either $A \leq_{\mathrm{W}} B$ (in the case that player II has a winning strategy) or $\omega^\omega \setminus B \leq_{\mathrm{W}} A$ (in the case that player I does), from which it follows that for any two pointclasses closed under continuous preimages, either the two classes are dual (*i.e.*, a pair of the form Γ, $\breve{\Gamma}$) or one is contained in the other. Wadge showed that $\leq_{\mathrm{W}}$ is wellfounded on the Borel sets, and Martin, using an idea of Leonard Monk, extended this to all sets of reals under $\mathsf{AD}+\mathsf{DC}$ (see [Van78B]).

Wadge determinacy and the wellfoundedness of the Wadge hierarchy divide $\wp(\omega^\omega)$ into equivalence classes by Wadge reducibility and order these classes into a wellfounded hierarchy, where each level consists either of one selfdual equivalence class, or two non-selfdual classes, one consisting of all the complements of the members of the other. Wadge determinacy also implies that every non-selfdual adequate pointclass has a universal set (see [Van78B, p. 162]).

The discovery of Wadge determinacy led to further progress on separation and reduction. Robert Van Wesep [Van78A] proved that under AD, if Γ is a non-selfdual pointclass which is closed under continuous preimages, then Γ and $\breve{\Gamma}$ cannot both have the separation property. Kechris, Solovay and Steel [KSS81] showed that under AD+DC, if $\Gamma \subseteq \mathbf{L}(\mathbb{R})$ is nonselfdual boldface pointclass and Γ is closed under countable intersections and unions and either $\exists^1$ or $\forall^1$, but not complements, then either Γ or $\breve{\Gamma}$ has the prewellordering property. In 1981, Steel [Ste81B] showed that under AD, if Γ is a nonselfdual pointclass closed under continuous preimages, then either Γ or $\breve{\Gamma}$ has the separation property, and if one assumes in addition that Δ_Γ is closed under finite unions, then either Γ or $\breve{\Gamma}$ has the reduction property.

§4. Partition properties and the projective ordinals. A cardinal κ is **measurable** if there is a nonprincipal κ-complete ultrafilter on κ, where **κ-completeness** means closure under intersections of fewer than κ many elements. In ZFC measurable cardinals are strongly inaccessible. In 1967, Solovay (see [Jec03, p. 633] or [Kan03, p. 348]) showed that AD implies that the club filter on ω_1 is an ultrafilter, which implies that ω_1 is a measurable cardinal.[24] Ulam had shown that under ZFC there are stationary, co-stationary subsets of ω_1; Solovay's result shows the opposite under AD. Solovay also showed that under AD every subset of ω_1 is constructible from a real (*i.e.*, exists in $\mathbf{L}[a]$ for some real number a). Since the measurability of ω_1 implies that the sharp of each real exists, this gives another proof that the club filter on ω_1 is an ultrafilter, since for any real a, if $a^\#$ exists, then every subset of ω_1 in $\mathbf{L}[a]$ either contains or is disjoint from a tail of the a-indiscernibles below ω_1, which is a club set.

A **Turing degree** is a nonempty subset of $\wp(\omega)$ closed under equicomputability. A **cone** of Turing degrees is the set of all degrees above (or computing) a given degree.[25] Martin [Mar68] showed that under AD the cone measure on Turing degrees is an ultrafilter, *i.e.*, that every set of Turing degrees either contains or is disjoint from a cone. This important fact has a relatively short and simple proof: the two players collaborate to build a real, with the winner decided by whether the Turing degree of the

[24] A subset of an ordinal is **closed unbounded** (or **club**) if it is unbounded and closed in the order topology on the ordinals, and **stationary** if it intersects every club set. The **club filter** on an ordinal γ consists of all subsets of γ containing a club set.

[25] See [Soa87, Coo04] for more on the Turing degrees, including a more precise statement of their definition.

real falls inside the payoff set; the cone above the degree of any real coding a winning strategy must contain or be disjoint from the payoff set. Martin used this result to find a simpler proof of the measurability of ω_1. Solovay followed by showing that ω_2 is measurable as well. **Turing determinacy** is the restriction of AD to payoff sets closed under Turing equivalence. This form of determinacy is easily seen to suffice for Martin's result. In the early 1980s, Woodin would show that, in $\mathbf{L}(\mathbb{R})$, AD and Turing determinacy are equivalent.

Given an ordered set X and an ordinal β, $[X]^\beta$ denotes the set of subsets of X of ordertype β. Given ordinals α, β, δ, and γ, the expression $\alpha \to (\beta)^\gamma_\delta$ denotes the statement that for every function $f\colon [\alpha]^\gamma \to \delta$, there exists an $X \in [\alpha]^\beta$ such that f is constant on $[X]^\gamma$. Frank Ramsey [Ram30] proved that $\omega \to (\omega)^n_2$ holds for each positive $n \in \omega$ (this fact is known as **Ramsey's Theorem**). For infinitary partitions, Paul Erdős and András Hajnal [EH66] showed (in ZFC) that for any infinite cardinal κ there is a function $f\colon [\kappa]^\omega \to \kappa$ such that for every $X \in [\kappa]^\kappa$, the range of $f \restriction X$ is all of κ.

In 1968, Adrian Mathias [Mat68, Mat77] showed that $\omega \to (\omega)^\omega_2$ holds in Solovay's model from [Sol70], in which all sets of reals satisfy the regularity properties. A set $Y \subseteq [\omega]^\omega$ is said to be **Ramsey** if there exists an $X \in [\omega]^\omega$ such that either $[X]^\omega \subseteq Y$ or $[X]^\omega \cap Y = \varnothing$. The statement $\omega \to (\omega)^\omega_2$ is equivalent to the statement that every subset of $[\omega]^\omega$ is Ramsey. Prikry [Pri76] showed that under $\mathsf{AD}_\mathbb{R}$ (determinacy for games of perfect information of length ω for which the players play real numbers) every subset of $[\omega]^\omega$ is Ramsey. It follows from the main theorem of [MS83] that $\mathsf{AD} + \mathbf{V}{=}\mathbf{L}(\mathbb{R})$ implies that every such set is Ramsey. Whether AD alone suffices is still an open question.

In late 1968, Martin (see [Kan03, p. 392]) showed that AD implies $\omega_1 \to (\omega_1)^\omega_2$ (this implies for instance that the club filter on ω_1 is an ultrafilter). Kenneth Kunen then showed that AD implies that ω_1 satisfies the weak partition property, where a cardinal κ satisfies the **weak partition property** if $\kappa \to (\kappa)^\alpha_2$ holds for every $\alpha < \kappa$. Martin followed by showing that $\omega_1 \to (\omega_1)^{\omega_1}_2$, again under AD. The proof actually shows $\omega_1 \to (\omega_1)^{\omega_1}_{2^\omega}$ and $\omega_1 \to (\omega_1)^{\omega_1}_\alpha$ for every countable ordinal α. Martin and Paris (in an unpublished note [MP71], see [Kec78A]) showed that under AD+DC, ω_2 has the weak partition property.

Before continuing with this line of results, we briefly discuss the Coding Lemma and the projective ordinals.

4.1. Θ, the Coding Lemma and the projective ordinals. Following convention, we let Θ denote the least ordinal that is not a surjective image of $\mathbb{R}$. Under ZFC, $\Theta = \mathfrak{c}^+$, but under AD, Θ is a limit cardinal, as noted by Harvey Friedman (see [Kan03, p. 398]). This fact follows from a theorem known as the *Coding Lemma*, due to Moschovakis [Mos70A], extending earlier work of Friedman and Solovay.

Given a subset P of some Polish space, let $\boldsymbol{\Sigma}^1_1(P)$ denote the pointclass of sets which are Σ^1_1-definable using P and individual reals as parameters.

THEOREM 4.1 (Coding Lemma). Assume $\mathsf{ZF}+\mathsf{AD}$. Let $\preceq$ be a prewellordering of a set of reals X. Let ξ be the length of $\preceq$ and let A be a subset of ξ. Then there exists a $Y \subseteq X$ in $\boldsymbol{\Sigma}^1_1(\preceq)$ such that A is the set of $\preceq$-ranks of elements of Y.

As an immediate consequence, under AD, if $\xi < \Theta$, then there is a surjection from $\mathbb{R}$ onto $\wp(\xi)$ (furthermore, if $\alpha < \Theta^M$ for some wellfounded model M of ZF containing the reals, then such a surjection can be found in M). The proof of the Coding Lemma uses a version of Kleene's Recursion Theorem (first proved in [Kle38] for partial recursive functions on the integers), which can be stated as saying that given a suitable coding under which each real x codes a continuous partial function $\hat{x}$ (our notation) on the reals, for each two-variable continuous partial function g on the reals there is a real x such that $\hat{x}(w) = g(x, w)$ for all reals w.

If Γ is a pointclass, δ_Γ denotes the supremum of the lengths of the prewellorderings of the reals in Δ_Γ. The notation $\boldsymbol{\delta}^1_n$ is used to denote $\delta_{\boldsymbol{\Sigma}^1_n}$ (which is the same as $\delta_{\boldsymbol{\Pi}^1_n}$). The **projective ordinals** are the ordinals $\boldsymbol{\delta}^1_n$, for $n \in \omega \setminus \{0\}$. It follows from the results of [LS23] that $\boldsymbol{\Sigma}^1_1$ prewellorderings of the reals have countable length, and therefore that the ordinal $\boldsymbol{\delta}^1_1$ is equal to ω_1. Moschovakis [Mos70A] showed (under AD, using the Coding Lemma) that for each $n \in \omega$, $\boldsymbol{\delta}^1_{n+1}$ is a cardinal, and that $\boldsymbol{\delta}^1_{2n+1}$ is regular and (using just PD) strictly less than $\boldsymbol{\delta}^1_{2n+2}$. Martin showed (without AD) that $\boldsymbol{\delta}^1_2 \leq \omega_2$ (see [KM78B]); together these results show that under AD, $\boldsymbol{\delta}^1_2 = \omega_2$.

Kunen and Martin (see [KM78B]) independently established from $\mathsf{ZF}+\mathsf{DC}$ that every wellfounded κ-Suslin prewellordering has length less than κ^+ (this fact is sometimes called the **Kunen–Martin Theorem**). Moschovakis ([Mos70A]; see [Mos09, 4C.14]) showed (from PD) that any $\boldsymbol{\Pi}^1_{2n+1}$-norm on a complete $\boldsymbol{\Pi}^1_{2n+1}$ set has length $\boldsymbol{\delta}^1_{2n+1}$ (this result also uses Kleene's Recursion Theorem). By the scale property for $\boldsymbol{\Pi}^1_{2n+1}$ sets (under the assumption of DC + $\boldsymbol{\Delta}^1_{2n}$-determinacy, given $n \in \omega$ [Mos71A]), every $\boldsymbol{\Pi}^1_{2n+1}$ set (and thus every $\boldsymbol{\Sigma}^1_{2n+2}$ set) is $\boldsymbol{\delta}^1_{2n+1}$-Suslin, and, since $\boldsymbol{\delta}^1_{2n+1}$ is regular, every $\boldsymbol{\Sigma}^1_{2n+1}$ set is λ-Suslin for some $\lambda < \boldsymbol{\delta}^1_{2n+1}$. It follows that under the same hypothesis, $\boldsymbol{\delta}^1_{2n+2} \leq (\boldsymbol{\delta}^1_{2n+1})^+$, and under AD that $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$ for each $n \in \omega$.

Kechris [Kec74] proved (assuming AD) that $\boldsymbol{\delta}^1_{2n+1}$ is a successor cardinal (its predecessor is called λ_{2n+1}). It follows from his arguments, and those of the previous paragraph, that the pointclasses $\utilde{\mathbf{\Sigma}}^1_{2n+2}$ and $\utilde{\mathbf{\Sigma}}^1_{2n+1}$ are exactly the $\utilde{\boldsymbol{\delta}}^1_{2n+1}$-Suslin and λ_{2n+1}-Suslin sets respectively.

Given an ordinal λ, the λ-**Borel** sets of reals are those in the smallest class containing the open sets and closed under complements and well-ordered unions of length less than λ. Martin showed that if κ is a cardinal of uncountable cofinality, then all κ-Suslin sets are κ^+-Borel. He also showed (using AD+DC, the Coding Lemma and Wadge determinacy) that the $\utilde{\boldsymbol{\delta}}^1_{2n+1}$-Borel sets are $\utilde{\mathbf{\Delta}}^1_{2n+1}$, for each $n \in \omega$ (the reverse inclusion follows from the results of Moschovakis [Mos71A] mentioned above). Using this fact, Kechris proved (again, under AD) that λ_{2n+1} has cofinality ω. It follows (under AD) that $\utilde{\boldsymbol{\delta}}^1_{2n} < \utilde{\boldsymbol{\delta}}^1_{2n+1}$ for each $n \in \omega$, so that under AD the sequence $\langle \utilde{\boldsymbol{\delta}}^1_{n+1} : n \in \omega \rangle$ is a strictly increasing sequence of successor cardinals. Kunen [Kun71E] showed that $\utilde{\boldsymbol{\delta}}^1_n$ is regular for each positive $n \in \omega$.

Solovay noted that under AD, Θ is the Θth cardinal, and that under the further assumption of $\mathbf{V} = \mathbf{L}(\mathbb{R})$, Θ is regular (see [Kan03, p. 398]). He showed [Sol78B] that under DC, Θ has uncountable cofinality, and also that $\mathsf{ZFC} + \mathsf{AD}_{\mathbb{R}} + \mathrm{cf}(\Theta){>}\omega$ proves the consistency of ZF+$\mathsf{AD}_{\mathbb{R}}$, so that by Gödel's Second Incompleteness Theorem, if ZF+$\mathsf{AD}_{\mathbb{R}}$ is consistent, then so is $\mathsf{ZFC} + \mathsf{AD}_{\mathbb{R}} + \mathrm{cf}(\Theta){=}\omega$.[26] Kechris [Kec84], using the proof of the Third Periodicity Theorem and work of Martin, Moschovakis and Steel on scales [MMS82A], showed that DC follows from AD + $\mathbf{V}{=}\mathbf{L}(\mathbb{R})$. Woodin (see [Kec84]) strengthened Solovay's result that DC does not follow from AD by showing that, assuming AD + $\mathbf{V}{=}\mathbf{L}(\mathbb{R})$ there is an inner model of a forcing extension satisfying ZF+AD+$\neg\mathsf{AC}_\omega$ (DC directly implies AC_ω). Whether AD implies $\mathsf{DC}({}^\omega\omega)$ (DC for relations on ${}^\omega\omega$) is still open.

4.2. Partition properties and ultrafilters. Kunen in an unpublished note [Kun71F] proved that $\utilde{\boldsymbol{\delta}}^1_{2n} \to (\utilde{\boldsymbol{\delta}}^1_{2n})^\lambda_2$ for all positive $n \in \omega$ and $\lambda < \omega_1$, under AD. He also showed [Kun71G] (under the same hypothesis) that $\utilde{\boldsymbol{\delta}}^1_{2n} \to (\utilde{\boldsymbol{\delta}}^1_{2n})^{\utilde{\boldsymbol{\delta}}^1_{2n}}_2$ is false. Martin, in another unpublished note from 1971, showed that $\utilde{\boldsymbol{\delta}}^1_{2n+1} \to (\utilde{\boldsymbol{\delta}}^1_{2n+1})^\lambda_2$ for all positive $n \in \omega$ and $\lambda < \omega_1$, under AD.

While Erdős and Hajnal [EH58] had shown how to derive partition properties from measurable cardinals, Eugene Kleinberg proved the following result in the other direction, which shows (via $\lambda = \omega$) that $\utilde{\boldsymbol{\delta}}^1_n$ is measurable for each positive $n \in \omega$.[27]

Theorem 4.2 ([Kle70]). *If $\lambda < \kappa$, λ is regular, and $\kappa \to (\kappa)^{\lambda+\lambda}_2$ holds, then C^λ_κ is a normal ultrafilter over κ.*

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[26] The end of Section 6.2 continues this line of results.

[27] We let C^λ_κ denote the filter generated by the set of λ-closed unbounded subsets of κ. A filter is **normal** if every regressive function on a set in the filter is constant on a set in the filter.

In 1970, Kunen proved, using Martin's result on the cone measure on the Turing degrees, that under AD, any ω_1-complete filter on an ordinal $\lambda < \Theta$ can be extended to an ω_1-complete ultrafilter, and that every ultrafilter on an ordinal less than Θ is definable from ordinal parameters (see [Kan03, pp. 399–400]). Solovay [Sol78B] proved that under $\mathsf{AD}_{\mathbb{R}}$, there is a normal ultrafilter on $\wp_{\aleph_1}(\mathbb{R})$: for each $A \subseteq \wp_{\aleph_1}(\mathbb{R})$, consider the game where player I and player II collaborate to build a sequence $\langle s_i : i < \omega\rangle$ consisting of finite sets of reals, and player I wins if and only if $\bigcup\{s_i : i \in \omega\} \in A$.[28] This implies (again, under $\mathsf{AD}_{\mathbb{R}}$) that for each ordinal $\gamma < \Theta$ there is a normal ultrafilter on $\wp_{\aleph_1}\gamma$ (*i.e.*, that ω_1 is γ-**supercompact**). It is not known whether AD suffices for this result, though Harrington and Kechris [HK81] showed that if AD holds and γ is less than a Suslin cardinal, then there is a normal ultrafilter on $\wp_{\aleph_1}\gamma$.[29] Extending work of Becker [Bec81A] (who proved it in the case that γ is a Suslin cardinal), Woodin [Woo83B] showed that there is just one such ultrafilter for each $\gamma < \Theta$, if either $\mathsf{AD}_{\mathbb{R}}$ holds or AD holds and γ is below a Suslin cardinal. The end of Section 6.4 mentions more recent progress on these topics.

A cardinal κ is said to have the **strong partition property** if $\kappa \to (\kappa)^{\kappa}_{\mu}$ holds for every $\mu < \kappa$. As mentioned above, Martin showed that under AD, ω_1 has the strong partition property. In late 1977, Kechris adapted Martin's argument to show that under AD there exists a cardinal κ with the strong partition property such that the set of $\lambda < \kappa$ with the strong partition property is stationary below κ (see [Kan03, p. 432]). Pushing this further, Kechris, Kleinberg, Moschovakis and Woodin [KKMW81] showed (using a uniform version of the Coding Lemma) that AD implies that unboundedly many cardinals below Θ have the strong partition property and are stationary limits of cardinals with the strong partition property. They also showed that whenever λ is an ordinal below a cardinal with the strong partition property, all λ-Suslin sets are determined. Using work of Steel [Ste83A] and Martin [Mar83B], Kechris and Woodin [KW83] showed that in $\mathbf{L}(\mathbb{R})$, AD is equivalent to the assertion that Θ is a limit of cardinals with the strong partition property, and also to the statement that all Suslin sets are determined. James Henle, Mathias and Woodin [HMW85] later showed that the first equivalence does not follow from ZF+DC, since the existence of a nonprincipal ultrafilter on ω is consistent with Θ being a limit of cardinals with the strong partition property.

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[28] Given a cardinal κ and a set X, $\wp_\kappa X$ denotes the collection of subsets of X of cardinality less than κ. An ultrafilter U on $\wp_\kappa X$ is **normal** if for each $Y \in U$, if f is a regressive function on Y (*i.e.*, if $\mathrm{dom}(f) = Y$ and $f(A) \in A$ for all nonempty $A \in Y$) then f is constant on a set in U.

[29] An ordinal (necessarily a cardinal) κ is said to be **Suslin** if there is a set of reals which is κ-Suslin but not λ-Suslin for any $\lambda < \kappa$.

A key step in the proof of the Kechris–Woodin theorem was a transfer theorem extending results of Harrington and Martin (discussed in Section 5.3). Harrington and Martin had shown from ZF+DC that, for each real a, $\Pi^1_1(a)$-determinacy is equivalent to determinacy for the larger class $\bigcup_{\beta<\omega^2} \beta$-$\Pi^1_1(a)$. Kechris and Woodin showed, from the same hypothesis, that for all positive integers k, $\boldsymbol{\Delta}^1_{2k}$-determinacy is equivalent to $\Game^{(2k-1)} \bigcup_{\beta<\omega^2} \beta$-$\boldsymbol{\Pi}^1_1$-determinacy, where $\Game^{(2k-1)}$ indicates an application of $2k-1$ many instances of the game quantifier $\Game$. By Theorem 3.6, this means that $\boldsymbol{\Delta}^1_{2k}$-determinacy implies $\boldsymbol{\Pi}^1_{2k}$-determinacy. Martin had proved the lightface version in 1973 (see [KS85]). Later results of Woodin and Itay Neeman [Nee95] would show that $\boldsymbol{\Pi}^1_{n+1}$-determinacy is equivalent to $\Game^{(n)} \bigcup_{\beta<\omega^2} \beta$-$\boldsymbol{\Pi}^1_1$-determinacy for all $n \in \omega$.

4.3. Cardinals, uniform indiscernibles and the projective ordinals. A cardinal κ is **Ramsey** if for every function $f\colon [\kappa]^{<\omega} \to \{0,1\}$ (where $[\kappa]^{<\omega}$ denotes the finite subsets of κ) there exists $A \in [\kappa]^\kappa$ such that for each $n \in \omega$, $f\restriction[\kappa]^n$ is constant. Measurable cardinals are Ramsey, and if there exists a Ramsey cardinal then the sharp of each real number exists. Assuming the existence of a Ramsey cardinal, Martin and Solovay [MS69] showed that nonempty $\boldsymbol{\Sigma}^1_3$ subsets of the plane have $\boldsymbol{\Delta}^1_4$ uniformizations. As mentioned above, Lévy [Lév65A] had shown that ZFC does not suffice for this result. Martin and Solovay used an analysis of sharps for reals, and modeled their argument after the proof of the Kondô–Addison theorem. Mansfield [Man71] extended the Martin–Solovay analysis to show (using a measurable cardinal) that nonempty $\boldsymbol{\Pi}^1_2$ sets are uniformized by $\boldsymbol{\Pi}^1_3$ functions.

Given a positive ordinal α, u_α denotes the αth **uniform indiscernible**, the αth ordinal which is a Silver indiscernible for each real number. As bijections between ω and countable ordinals can be coded by reals, the first uniform indiscernible, u_1, is ω_1. It follows from the basic analysis of sharps that all uncountable cardinals are uniform indiscernibles, so $u_2 \leq \omega_2$. By applying the Kunen–Martin theorem inside models of the form $\mathbf{L}[a]$, for a a real number, and applying the basic analysis of sharps, Martin showed that $\boldsymbol{\delta}^1_2 = u_2$ if the sharp of every real exists (see [Kec78A]). Recall that by the results of Section 4.1, $\boldsymbol{\delta}^1_2 = \omega_2$, under AD.

Martin showed from ZF plus the assumption that the sharp of each real exists that every $\boldsymbol{\Sigma}^1_3$ set is u_ω-Suslin, and from AD that $u_\omega = \omega_\omega$ (see [Kan03, pp.203–204]). By the Kunen–Martin Theorem, then, AD implies that $\boldsymbol{\delta}^1_3 \leq \omega_{\omega+1}$. Solovay had shown that if the sharp of every real exists, then $u_{\xi+1}$ has the same cofinality as u_2, for every positive ordinal ξ (see [Kec78A]). Since $u_\omega = \omega_\omega$, it follows that each ω_n ($n \geq 2$) is of the form u_{k+1} for some positive integer k, and thus that each such ω_n has cofinality

ω_2. It follows that under AD+DC, $\underset{\sim}{\boldsymbol{\delta}}^1_3 = \omega_{\omega+1}$, since $\underset{\sim}{\boldsymbol{\delta}}^1_3$ is a regular cardinal, and therefore that $\underset{\sim}{\boldsymbol{\delta}}^1_4 = \omega_{\omega+2}$. Kunen and Solovay would then show that $u_n = \omega_n$ for all n satisfying $1 \leq n \leq \omega$.

In 1971, Kunen reduced the computation of $\underset{\sim}{\boldsymbol{\delta}}^1_5$ to the analysis of certain ultrapowers of $\underset{\sim}{\boldsymbol{\delta}}^1_3$ (see [Kec78A]; as part of his analysis, Kunen showed that $\underset{\sim}{\boldsymbol{\delta}}^1_3$ has the weak partition property, see [Sol78A]). The completion of this project was to take another decade. In the early 1980s, Martin proved new results analyzing these ultrapowers, and Steve Jackson, using joint work with Martin, computed $\underset{\sim}{\boldsymbol{\delta}}^1_5$. The following theorem [Jac88, Jac99] completes the calculation of the $\underset{\sim}{\boldsymbol{\delta}}^1_n$'s.

Theorem 4.3 (Jackson). *Assume* AD. *Then for* $n \geq 1$, $\underset{\sim}{\boldsymbol{\delta}}^1_{2n+1}$ *has the strong partition property and is equal to* $\omega_{w(2n-1)+1}$, *where* $w(1) = \omega$ *and* $w(m+1) = \omega^{w(m)}$ *in the sense of ordinal exponentiation.*

Jackson's proof of this theorem was over 100 pages long. Elements of his argument (as presented in [Jac99]) include the Kunen–Martin theorem, Kunen's $\underset{\sim}{\boldsymbol{\Delta}}^1_3$ coding for subsets of ω_ω [Sol78A], Martin's theorem that $\underset{\sim}{\boldsymbol{\Delta}}^1_{2n+1}$ is closed under intersections and unions of sequences of sets indexed by ordinals less than $\underset{\sim}{\boldsymbol{\delta}}^1_3$, and so-called homogeneous trees, a notion which traces back to [MS69] and a result of Martin discussed in the next section.

§5. Determinacy and large cardinals. As discussed above, a strongly inaccessible cardinal is an uncountable regular cardinal which is closed under cardinal exponentiation. If κ is strongly inaccessible, then $\mathbf{V}_\kappa$ is a model of ZFC, so that the existence of strongly inaccessible cardinals is not a consequence of ZFC. While there is no technical definition of **large cardinal**, a typical large cardinal notion (in the context of the Axiom of Choice) specifies a type of strongly inaccessible cardinal. Examples of this type include Ramsey cardinals, measurable cardinals, Woodin cardinals and supercompact cardinals. The **large cardinal hierarchy** orders large cardinals by **consistency strength**. That is, large cardinal notion A is below large cardinal notion B in the hierarchy if the existence of cardinals of type B implies the consistency of cardinals of type A. It is a striking empirical fact that the large cardinal hierarchy is linear, modulo open questions (the examples just given were listed in increasing order, for instance). Even more striking is the fact that many set-theoretic statements having no ostensible relationship to large cardinals are equiconsistent with some large cardinal notion.[30]

By results of Mycielski (discussed in Section 2.3), AD implies that ω_1 is strongly inaccessible in $\mathbf{L}$, which means that AD cannot be proved in ZFC. Moreover, Solovay's result that AD implies the measurability of ω_1

[30] [Kan03] is the standard reference for the large cardinal hierarchy.

implies that under AD, ω_1 (as computed in the full universe) is a measurable cardinal in certain inner models of AC, such as **HOD**.[31] As we shall see in this section, the relationship between large cardinals and determinacy runs in both directions: Various forms of determinacy imply the existence of models of ZFC containing large cardinals, and the existence of large cardinals can be used to prove the determinacy of certain definable sets of reals.

5.1. Measurable cardinals. Solovay [Sol66] showed in 1965 that if there exists a measurable cardinal then every uncountable $\boldsymbol{\Sigma}^1_2$ set of reals contains a perfect set. This result was proved independently by Mansfield (see [Sol66]). Martin [Mar70A] showed that in fact analytic determinacy follows from the existence of a Ramsey cardinal.

Roughly, the idea behind Martin's proof is that if A is the projection of a tree T on $\omega \times \omega$ and χ is a Ramsey cardinal, one can modify the original game for A to require the second player to play, in addition to his usual moves, a function $G^*\colon \omega^{<\omega} \to \chi$ witnessing (via the wellfoundedness of the ordinal χ) that the fragment of T corresponding to the real produced by the two players in their moves from the original game has no infinite branches, and thus that this real is not in the projection of T. This modified game is closed, and thus determined, by Gale–Stewart. If the second player has a winning strategy in the modified game, then he has a winning strategy in the original game by ignoring his extra moves. In general there is no reason that a winning strategy for the first player in the modified game will induce a winning strategy for the original game. However, if χ is a Ramsey cardinal, then there is uncountable $X \subseteq \chi$ such that, as long as the range of G^* is contained in X, the first player's strategy does not depend on the extra moves for the second player. Using this fact, the first player can convert his winning strategy in the modified game into a winning strategy in the original game. The notion of a determined (often closed) auxiliary game and a method for transferring strategies from the auxiliary game to the original game is the basis of many determinacy proofs.

Martin later proved the following refinement.

THEOREM 5.1. If the sharp of every real exists, then $\boldsymbol{\Pi}^1_1$-determinacy holds.

In the 1970s Kunen and Martin independently developed the notion of a **homogeneous** tree, following a line of ideas deriving from Martin's proof of $\boldsymbol{\Pi}^1_1$-determinacy (see [Kec81A]). Given a set Z and a cardinal κ, a tree on $\omega \times Z$ is said to be κ-**homogeneous** if for each $\sigma \in \omega^{<\omega}$ there is a κ-complete ultrafilter μ_σ on $Z^{|\sigma|}$ such that

[31] The inner model **HOD** (a model of ZFC) consists of all sets x such that every member of the transitive closure of $\{x\}$ is ordinal-definable (see [Jec03, Chapter 13]).

- for each $\sigma \in \omega^{<\omega}$, $\{z : (\sigma, z) \in T\} \in \mu_\sigma$;
- $\mathrm{p}[T]$ is the set of $x \in \omega^\omega$ such that the sequence $\langle \mu_{x\restriction i} : i \in \omega \rangle$ is countably complete.[32]

A tree is said to be **homogeneous** if it is $\aleph_1$-homogeneous. A set of reals is said to be **homogeneously Suslin** if it is the projection of a homogeneous tree. There are related notions of **weakly homogeneous tree** and **weakly homogeneously Suslin set** of reals, involving a more involved relationship with a set of ultrafilters. Though it was not the original definition, let us just say that a tree on a set of the form $\omega \times (\omega \times Z)$ is weakly homogeneous if and only if the corresponding tree on $(\omega \times \omega) \times Z$ is homogeneous, and note that a set of reals is weakly homogeneously Suslin if and only if it is the projection of a homogeneously Suslin set of pairs.

Martin's proof then shows the following.

THEOREM 5.2 (Martin). Homogeneously Suslin sets are determined.

The unfolding argument mentioned in Section 2.2 then shows that weakly homogeneously Suslin sets satisfy the regularity properties.

In retrospect, Martin's proof of analytic determinacy can be broken into two parts, the fact that homogeneously Suslin sets are determined, and the fact that if there is a Ramsey cardinal then $\boldsymbol{\Pi}^1_1$ sets are homogeneously Suslin.

The results of [MS69] can similarly be reinterpreted. If $\boldsymbol{\Pi}^1_1$ sets are homogeneously Suslin, then $\boldsymbol{\Sigma}^1_2$ sets are weakly homogeneously Suslin. The Martin–Solovay construction can be seen as a method for taking a γ-weakly homogeneous tree T (for some cardinal γ) and producing a tree S on $\omega \times \gamma'$, for some ordinal γ', projecting to the complement of the projection of T. From this follows that all $\boldsymbol{\Pi}^1_2$ sets, and thus all $\boldsymbol{\Sigma}^1_3$ sets, are projections of trees on the product of ω with some ordinal. More sophisticated arguments can be carried out from the existence of sharps, using the fact that sharps give ultrafilters over certain inner models.

5.2. Borel determinacy. In 1968, Friedman [Fri71B] showed that the Replacement axiom is necessary to prove Borel determinacy, even for sets invariant under Turing degrees (he also showed that analytic determinacy cannot hold in a forcing extension of $\mathbf{L}$). As refined by Martin, his results show (for each $\alpha < \omega_1$) that $\mathsf{ZFC} - \mathsf{PowerSet} - \mathsf{Replacement}$ + "the αth iteration of the power set of ${}^\omega\omega$ exists" does not prove the determinacy of all $\boldsymbol{\Sigma}^0_{1+\alpha+3}$ sets.

James Baumgartner mixed the method of Martin's $\boldsymbol{\Pi}^1_1$-determinacy proof with Davis's $\boldsymbol{\Sigma}^0_3$-determinacy proof to give a new proof of $\boldsymbol{\Sigma}^0_3$-determinacy in ZFC. Using a similar approach, Martin proved $\mathsf{Det}(\boldsymbol{\Sigma}^0_4)$ from the existence

[32] *i.e.*, for each sequence $\langle A_i : i \in \omega \rangle$ such that each $A_i \in \mu_{x\restriction i}$ there exists a $t \in Z^\omega$ such that $t\restriction i \in A_i$ for each i.

of a weakly compact cardinal,[33] and then Paris [Par72] proved it in ZFC. Paris noted at the end of his paper that his argument could be carried out without the power set axiom, assuming instead only that the ordinal ω_1 exists.

Andreas Blass [Bla75] and Mycielski (1967, unpublished) independently proved that $\mathsf{AD}_{\mathbb{R}}$ is equivalent to determinacy for integer games of length ω^2. The key idea in Blass's proof was to reduce determinacy in the given game to determinacy in another, auxiliary, game in such a way that one player's moves in the auxiliary game correspond to fragments of his strategy in the original game. Martin [Mar75] used this basic idea to prove Borel determinacy in 1974 (the auxiliary game was in fact an open game). In his [Mar85], Martin gave a short, inductive, proof of Borel determinacy, and introduced the notion of **unraveling** a set of reals—roughly, finding an association of the set to a clopen set in a larger domain with a map sending strategies in one game to strategies in the other. In his [Mar90], Martin extended this method to games of length ω played on any (possibly uncountable) set, with Borel payoff (in the corresponding sense). Neeman [Nee00, Nee06B] would unravel $\boldsymbol{\Pi}^1_1$ sets from the assumption of a measurable cardinal κ of Mitchell rank κ^{++} (proved to be an optimal hypothesis by Steel [Ste82B]; see [Jec03, pp. 357–360] for the definition of Mitchell rank). Complementing Friedman's theorem, Martin proved that for each $\alpha < \omega_1$, the determinacy of each Boolean combination of $\boldsymbol{\Sigma}^0_{\alpha+2}$ sets follows from $\mathsf{ZF} - \mathsf{PowerSet} - \mathsf{Replacement} + \Sigma_1\text{-}\mathsf{Replacement}$ + "the αth iteration of the power set of ${}^\omega\omega$ exists".

5.3. The difference hierarchy. Given a countable ordinal α and a real a, a set of reals X is said to be α-$\Pi^1_1(a)$ if there is wellordering of ω of length α recursive in a with corresponding rank function $R\colon \omega \to \alpha$ and a $\Pi^1_1(a)$ subset A of $\omega \times {}^\omega\omega$ such that

- for all $n, m \in \omega$, if $R(n) < R(m)$ then

$$\{x : (m,x) \in A\} \subseteq \{x : (n,x) \in A\}; \tag{3}$$

- X is the set of reals x for which the least ξ such that either $\xi = \alpha$ or $\xi < \alpha$ and $(R^{-1}(\xi), x) \notin A$ is odd.

This notation has its roots in [Hau08]. When a is itself recursive one writes α-Π^1_1. The union of the sets α-$\Pi^1_1(a)$ for all reals a is denoted α-$\boldsymbol{\Pi}^1_1$. The union of the sets α-$\boldsymbol{\Pi}^1_1$ for all $\alpha < \omega_1$ is denoted $\mathrm{Diff}(\boldsymbol{\Pi}^1_1)$. Note that $\mathrm{Diff}(\boldsymbol{\Pi}^1_1)$ is a proper subclass of $\boldsymbol{\Delta}^1_2$.

Friedman [Fri71A] extended Theorem 5.1 to show that $\mathsf{Det}(3\text{-}\boldsymbol{\Pi}^1_1)$ follows from the existence of the sharp of every real. Martin in 1975 then extended

[33] A cardinal κ is **weakly compact** if $\kappa \to (\kappa)^2_2$. Weakly compact cardinals are below the existence of $0^\#$ and above strongly inaccessible cardinals in the consistency strength hierarchy (see [Kan03, pp. 76,472]).

this result to show that the existence of $0^{\#}$ is equivalent to $\mathsf{Det}(\bigcup_{\beta<\omega^2}\beta\text{-}\Pi^1_1)$ (see [DuB90]). Harrington [Har78] then proved the converse to Theorem 5.1 by showing that $\mathsf{Det}(\Pi^1_1(a))$ implies the existence of $a^{\#}$, for each real a.

For the purposes of the next theorem, say that a model has α measurable cardinals and indiscernibles if there exists a set of ordertype α consisting of measurable cardinals of the model, and there exist uncountably many ordinal indiscernibles of the model above the supremum of these measurable cardinals. Martin proved the following theorem after Harrington's result.

Theorem 5.3. For any real a and any ordinal α recursive in a, the following are equivalent.

- $\mathsf{Det}(\bigcup_{\beta<\omega^2}(\omega^2\cdot\alpha+\beta)\text{-}\Pi^1_1(a))$.
- $\mathsf{Det}((\omega^2\cdot\alpha+1)\text{-}\Pi^1_1(a))$.
- There is an inner model of ZFC containing a and having α many measurable cardinals and indiscernibles.

Still, a large-cardinal consistency proof of $\mathsf{Det}(\utilde{\boldsymbol{\Delta}}^1_2)$, the hypothesis used by Addison and Martin in their extension of Blackwell's argument, remained beyond reach. John Green [Gre78] showed that $\mathsf{Det}(\Delta^1_2)$ implies the existence of an inner model with a measurable cardinal of Mitchell rank 1.

5.4. Larger cardinals. In Section 4 we defined a measurable cardinal to be a cardinal κ such that there exists a nonprincipal κ-complete ultrafilter on κ. Equivalently, under the Axiom of Choice, κ is measurable if and only if there is a nontrivial elementary embedding j from the full universe $\mathbf{V}$ into some inner model M whose critical point is κ, *i.e.*, such that κ is the least ordinal not mapped to itself by j. Many large cardinal notions can be expressed both in terms of ultrafilters and in terms of embeddings, though in the Choiceless context (without the corresponding form of Łoś's Theorem, see [Jec03, p. 159]) it is the definition in terms of ultrafilters which is relevant. For instance, a cardinal κ is **supercompact** if for each $\lambda>\kappa$ there exists a normal fine ultrafilter on $\wp_\kappa\lambda$.[34] Under the Axiom of Choice, κ is supercompact if and only if for every $\lambda>\kappa$ there is an elementary embedding j from $\mathbf{V}$ into an inner model M such that the critical point of j is κ and M is closed under sequences of length λ. Every supercompact cardinal is a limit of measurable cardinals. An even larger large cardinal notion is the huge cardinal, where an uncountable cardinal κ is **huge** if for some cardinal $\lambda>\kappa$ there is a κ-complete normal fine ultrafilter on $[\lambda]^\kappa$ (where "normal" and "fine" are defined in analogy with the supercompact case, see [Kan03, p. 331]). Under AC, κ is huge if and only if there is an elementary embedding $j\colon\mathbf{V}\to M$ with critical point κ such that M is closed under sequences of length $j(\kappa)$. The existence of huge cardinals does

[34]Given a cardinal κ and a set X, a collection U of subsets of $\wp_\kappa X$ is **fine** if it contains the collection of supersets of each element of $\wp_\kappa X$.

not imply the existence of supercompact cardinals, but it does imply their consistency.

Kunen [Kun71A] put a limit on the large cardinality hierarchy, showing in ZFC that there is no nontrivial elementary embedding from $\mathbf{V}$ into itself. A corollary of the proof is that for any elementary embedding j of $\mathbf{V}$ into any inner model M, if δ is the least ordinal above the critical point of j sent to itself by j, then $\mathbf{V}_{\delta+2} \nsubseteq M$. In 1978, Martin [Mar80] proved $\underset{\sim}{\boldsymbol{\Pi}}^1_2$-determinacy from the hypothesis **I2**, which states that for some ordinal δ there is a nontrivial elementary embedding of $\mathbf{V}$ into an inner model M with critical point less than δ such that $\mathbf{V}_\delta \subseteq M$ and $j(\delta) = \delta$.

In 1979, Woodin proved that for each $n \in \omega$, $\underset{\sim}{\boldsymbol{\Pi}}^1_{n+1}$ follows (in ZF) from the existence of an *n-fold strong rank-to-rank embedding*.[35] For $n = 1$, this is essentially the theorem of Martin just mentioned. For $n > 1$, these axioms are incompatible with the Axiom of Choice, by Kunen's theorem, though they are not known to be inconsistent with ZF.

In 1984, Woodin proved $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$ from **I0**, the statement that for some ordinal δ there is a nontrivial elementary embedding from $\mathbf{L}(\mathbf{V}_{\delta+1})$ into itself with critical point below δ, thus verifying Solovay's conjecture that $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$ would follow from large cardinals. I0 is one of the strongest large cardinal hypotheses not known to be inconsistent. The inner model program at the time had produced models for many measurable cardinals, hypotheses far short of I2, and so there was little hope of showing that I2 and I0 were necessary for these results.

New large cardinal concepts would prove to be the missing ingredient. Given an ideal I on a set X, forcing with the Boolean algebra given by the power set of X modulo I gives a $\mathbf{V}$-ultrafilter on the power set of X.[36] The ideal I is said to be **precipitous** if the ultrapower of $\mathbf{V}$ by this generic ultrafilter is wellfounded in all generic extensions. If the underlying set X is a cardinal κ, the ideal I is said to be **saturated** if the Boolean algebra $\wp(\kappa)/I$ has no antichains of cardinality κ^+.[37] If κ is a regular cardinal,

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[35] For positive $n \in \omega$, an **n-fold strong rank-to-rank embedding** is a sequence of elementary embeddings $j_1, \ldots, j_n$ such that for some cardinal λ,

- $j_i \colon \mathbf{V}_{\lambda+1} \to \mathbf{V}_{\lambda+1}$ whenever $1 \leq i \leq n$,
- $\kappa_\omega(j_i) < \kappa_\omega(j_{i+1})$ for all $i < n$,

where $\kappa_\omega(j)$ denotes the first fixed point of an elementary embedding j above the critical point.

[36] An **ideal** is a collection of sets closed under subsets and finite unions. Given a model M and a set X in M, an M-**ultrafilter** is a subset of $\wp(X) \cap M$ closed under supersets and finite intersections such that for every $A \subseteq X$ in M, exactly one of A and $X \setminus A$ is in U. Note that U does not need to be an element of M.

[37] An **antichain** in a partial order (or a Boolean algebra) is a set of pairwise incompatible elements. In the case of a Boolean algebra of the form $\wp(\kappa)/I$, an antichain is a collection of subsets of κ not in I which pairwise have intersection in I.

saturation of I implies precipitousness. Huge cardinals were invented by Kunen [Kun78], who used them to produce a saturated ideal on ω_1.

In early 1984, Matthew Foreman, Menachem Magidor and Shelah [FMS88] showed that if there exists a supercompact cardinal—a hypothesis much weaker than I0 or I2—then there is an ω_1-preserving forcing making the nonstationary ideal on ω_1 (NS_{ω_1}) saturated.

Foreman (see [For86]) and Magidor [Mag80] had earlier made a connection between generic elementary embeddings[38] and regularity properties for reals. Magidor [Mag80] in particular had shown that the Lebesgue measurability of Σ^1_3 sets followed from the existence of a generic elementary embedding with critical point ω_1 and wellfounded image model (the existence of such an embedding follows from the Foreman–Magidor–Shelah result mentioned above). Woodin noted that these arguments plus earlier work of his (see [Woo86]) could be used to extend this to Lebesgue measurability for all projective sets. Woodin also noted that arguments from [FMS88] could be used to prove the Lebesgue measurability of all sets of reals in $\mathbf{L}(\mathbb{R})$, if one could force to produce a saturated ideal on ω_1 without adding reals. Shelah then noted that techniques from [She98] could be modified to do just that. It followed then that the existence of a supercompact cardinal implies that all sets of reals in $\mathbf{L}(\mathbb{R})$ are Lebesgue measurable.

Woodin and Shelah then addressed the problem of weakening the hypotheses needed for the Lebesgue measurability of all projective sets of reals.[39] Woodin noted that a superstrong cardinal sufficed. Shelah then isolated a weaker notion now known as a **Shelah cardinal**, and showed that the existence of $n+1$ Shelah cardinals implies that $\utilde{\mathbf{\Sigma}}^1_{n+2}$ sets are Lebesgue measurable.

Definition 5.4. A cardinal κ is a **Shelah cardinal** if for every $f\colon \kappa \to \kappa$ there is an elementary embedding $j\colon \mathbf{V} \to N$ with critical point κ such that $\mathbf{V}_{j(f)(\kappa)} \subseteq N$.

Woodin noted that by modifying Shelah's definition one obtained a weaker, still sufficient, hypothesis, now known as a Woodin cardinal.

Definition 5.5. A cardinal δ is a **Woodin cardinal** if for each function $f\colon \delta \to \delta$ there exists an elementary embedding $j\colon \mathbf{V} \to M$ with critical point $\kappa < \delta$ closed under f such that $\mathbf{V}_{j(f)(\kappa)} \subseteq M$.

Woodin proved that the existence of n Woodin cardinals below a measurable cardinal implies the Lebesgue measurability of $\utilde{\mathbf{\Sigma}}^1_{n+2}$ sets, the same amount of measurability that would follow from $\utilde{\mathbf{\Pi}}^1_{n+1}$-determinacy. All of this work was done within a few weeks of the Foreman–Magidor–Shelah

[38] A **generic elementary embedding** is an elementary embedding of the universe $\mathbf{V}$ into some class model M which is definable in a forcing extension of $\mathbf{V}$.

[39] We follow the account in [Nee04].

result on the saturation of NS_{ω_1}. In [SW90] the hypothesis for the statement that all sets of reals in $\mathbf{L}(\mathbb{R})$ are Lebesgue measurable and have the property of Baire was reduced to the existence of ordertype $\omega+1$ many Woodin cardinals. The hypothesis was to be reduced even further.

Woodin extracted from the Foreman–Magidor–Shelah results a one-step forcing for producing generic elementary embeddings with critical point ω_1, and developed it into a general method, now known as the **stationary tower**. Using this he showed (by the fall of 1984, see his [Woo88]) that if there exists a supercompact cardinal (or a strongly compact cardinal), then every set of reals in $\mathbf{L}(\mathbb{R})$ is weakly homogeneously Suslin. (Steel and Woodin would show in 1990 that this conclusion in turn implies $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$.)

Steel had been working on the problem of finding inner models for supercompact cardinals. Inspired by the results of Foreman, Magidor, Shelah and Woodin, he begin to work on producing models for Woodin cardinals, and had some partial results by the spring of 1985, producing inner models with certain weak variants of Woodin cardinals. These models were generated by sequences of *extenders*, directed systems of ultrafilters which collectively generate elementary embeddings whose images contain more of $\mathbf{V}$ than possible for embeddings generated by a single ultrafilter. Special cases of extenders had appeared in Jensen's proof of the Covering Lemma. The general notion (which first appeared in [Dod82]) is Jensen's simplification of the notion of *hypermeasure*, which was introduced by Mitchell [Mit79]. Steel and Martin saw that the problem of building models with Woodin cardinals was linked to the problem of proving determinacy, and they set their sights on this problem in the late spring of 1985.

One key combinatorial problem related to elementary embeddings is whether infinite iterations of these embeddings produce wellfounded models. Kunen [Kun70] had shown that the answer was positive for iterations derived from a single ultrafilter. With extenders the situation was more complicated, as the iterations did not need to be linear but could produce trees of models with no rule for finding a path through the tree leading to a wellfounded model (indeed, this nonlinearity was essential, since otherwise the models would have simply definable wellorderings of their reals). The simplest such tree, a so-called **alternating chain**, is countably infinite and consists of two infinite branches. Martin and Steel saw that the issue of wellfoundedness for the direct limits along the two branches was linked. This observation led to the following theorem, proved in August of 1985.

Theorem 5.6 (Martin–Steel [MS89]). Suppose that λ is a Woodin cardinal and A is a λ^+-weakly homogeneously Suslin set of reals. Then for any $\gamma < \lambda$, ${}^{\omega}\omega \setminus A$ is γ-homogeneously Suslin.

It follows from this and the fact that analytic sets are homogeneously Suslin in the presence of a measurable cardinal that if there exist n Woodin cardinals below a measurable cardinal, then $\underset{\sim}{\boldsymbol{\Pi}}^1_{n+1}$ sets are determined, and that Projective Determinacy follows from the existence of infinitely many Woodin cardinals.

Combined with Woodin's application of the stationary tower mentioned above, the Martin–Steel theorem implied that $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$ follows from the existence of a supercompact cardinal. By the end of 1985, Woodin had improved the hypothesis to the existence of infinitely many Woodin cardinals below a measurable cardinal (see [Lar04]).

Theorem 5.7 (Woodin). If there exist infinitely many Woodin cardinals below a measurable cardinal, then AD holds in $\mathbf{L}(\mathbb{R})$.

In the spring of 1986, Martin and Steel [MaS94] produced **extender models** (i.e., models of the form $\mathbf{L}[\vec{E}]$, with $\vec{E}$ a sequence of extenders) with n Woodin cardinals and Δ^1_{n+2} wellorderings of the reals. Such a model necessarily has a Σ^1_{n+2} set which is not Lebesgue measurable, and fails to satisfy Π^1_{n+1}-determinacy.

Skipping ahead for a moment, let $(*)_n$ be the statement that for each real x there exists an iterable model M containing x and n Woodin cardinals plus the sharp of $\mathbf{V}^M_\delta$, for δ the largest of these Woodin cardinals. For odd n, the equivalence of $\underset{\sim}{\boldsymbol{\Pi}}^1_{n+1}$-determinacy and $(*)_n$ was proved by Woodin in 1989. That $(*)_n$ implies $\underset{\sim}{\boldsymbol{\Pi}}^1_{n+1}$-determinacy for all n was proved by Neeman [Nee95] in 1994. Roughly, Neeman's methods work by considering a modified game in which one player builds an iteration tree and makes moves in the image of the original game by the embeddings given by the tree. In 1995, Woodin proved that $\underset{\sim}{\boldsymbol{\Pi}}^1_{n+1}$-determinacy implies $(*)_n$ for even $n > 0$.

Woodin followed his Theorem 5.7 by determining the exact consistency strength of AD. The forward direction of Theorem 5.8 below (proved in [KW10]) shows from $\mathsf{ZF}+\mathsf{AD}$ that there exist infinitely many Woodin cardinals in an inner model of a forcing extension (**HOD** of the forcing extension with respect to certain parameters) of $\mathbf{V}$. The proof built on a sequence of results, starting with Solovay's theorem that AD implies that ω_1 is a measurable cardinal, which, as mentioned above, also shows that ω_1 (as defined in $\mathbf{V}$) is measurable in the inner model **HOD**. Becker (see [BM81]) had shown that, under AD, $\omega_1^{\mathbf{V}}$ is the least measurable in **HOD**. Becker, Martin, Moschovakis and Steel then showed that under $\mathsf{AD}+\mathbf{V}{=}\mathbf{L}(\mathbb{R})$, $\underset{\sim}{\boldsymbol{\delta}}^2_1$ is β-strong in **HOD**, where β is the least measurable cardinal greater than $\underset{\sim}{\boldsymbol{\delta}}^2_1$ in **HOD**.[40] In the 1980s, Woodin showed under the same hypothesis that

[40] The cardinal $\underset{\sim}{\boldsymbol{\delta}}^2_1$ is the supremum of the lengths of the $\underset{\sim}{\boldsymbol{\Delta}}^2_1$ prewellorderings of the reals; under $\mathsf{AD}+\mathbf{V}{=}\mathbf{L}(\mathbb{R})$ it is also the largest Suslin cardinal. A cardinal κ is **β-strong** if there is an elementary embedding $j\colon \mathbf{V} \to M$ with critical point κ such that $\mathbf{V}_\beta \subseteq M$, and **$<\delta$-strong** if it is β-strong for all $\beta < \delta$.

$\underset{\sim}{\boldsymbol{\delta}}^2_1$ is β-strong in **HOD** for every $\beta < \Theta$ (and that $\underset{\sim}{\boldsymbol{\delta}}^2_1$ is the least ordinal with this property), and that Θ is Woodin in **HOD**.

Theorem 5.8 (Woodin). The following are equiconsistent.

- ZF+AD.
- There exist infinitely many Woodin cardinals.

The following theorem illustrates the reverse direction of the equiconsistency (see [Ste09]). It can be seen as a special case of the Derived Model Theorem, discussed in Section 6.2. The partial order $\mathrm{Col}(\omega, {<}\delta)$ consists of all finite partial functions p from $\omega \times \delta$ to δ, with the requirement that $p(n, \alpha) \in \alpha$ for all (n, α) in the domain of p. The order is inclusion. If δ is a regular cardinal, then δ is the ω_1 of any forcing extension by $\mathrm{Col}(\omega, {<}\delta)$.

Theorem 5.9 (Woodin). Suppose that λ is a limit of Woodin cardinals, and $G \subseteq \mathrm{Col}(\omega, {<}\lambda)$ is $\mathbf{V}$-generic filter. Let $\mathbb{R}^* = \bigcup\{\mathbb{R}^{\mathbf{V}[G\restriction\alpha]} : \alpha < \lambda\}$. Then AD holds in $\mathbf{L}(\mathbb{R}^*)$.

The results of Section 5.3 illustrate the difficulties in proving the determinacy of Π^1_2 sets. Woodin resolved this problem in 1989. The forward direction of Theorem 5.10 is proved in [KW10]. The proof was inspired in part by a result of Kechris and Solovay [KS85], saying that in models of the form $\mathbf{L}[a]$ for $a \subseteq \omega$, Δ^1_2-determinacy implies the determinacy of all ordinal definable sets of reals. Standard arguments show that if Δ^1_2 determinacy holds, then it holds in $\mathbf{L}[x]$ for some real x. Woodin showed that if $\mathbf{V}$ is $\mathbf{L}[x]$ for some real x, and Δ^1_2-determinacy holds, then $\omega_2^{\mathbf{L}[x]}$ is a Woodin cardinal in **HOD**. Recall (from the end of Section 4.2) that Δ^1_2-determinacy and Π^1_2-determinacy are equivalent, by a result of Martin.

Theorem 5.10 (Woodin). The following are equiconsistent.

- ZFC+Det(Δ^1_2).
- ZFC+There exists a Woodin cardinal.

The following theorem illustrates the reverse direction. Its proof can be found in [Nee10, p. 1926]. The partial order $\mathrm{Col}(\omega, \delta)$ is the natural one for making δ countable : it consists of all finite partial functions from ω to δ, ordered by inclusion.

Theorem 5.11 (Woodin). If δ is a Woodin cardinal and $G \subseteq \mathrm{Col}(\omega, \delta)$ is a $\mathbf{V}$-generic filter, then Δ^1_2-determinacy holds in $\mathbf{V}[G]$.

§6. Later developments. In this final section we briefly review some of the developments that followed the results of the previous section. As discussed in the introduction, the set of topics presented here is by no means complete. The first subsection briefly introduces a regularity property for sets of reals which is induced by forcing-absoluteness. The second and third discuss forms of determinacy ostensibly stronger than AD, in models larger than $\mathbf{L}(\mathbb{R})$. The next subsection discusses applications of determinacy to the realm of AC, via producing models of AC by forcing over models of determinacy. In the last two we present some results which derive forms of determinacy from their ostensibly weak consequences, or from statements having no obvious relationship to determinacy. Many of the results of the last two subsections are applications of the study of canonical inner models for large cardinals.

6.1. Universally Baire sets. As discussed above in Sections 5.1 and 5.4, homogeneously Suslin and weakly homogeneously Suslin sets of reals played an important role in applications of large cardinals to regularity properties for sets of reals, as early as the 1969 results of Martin and Solovay. Qi Feng, Magidor, and Woodin [FMW92] introduced a related tree representation property for sets of reals. Given a cardinal κ, a set $A \subseteq \omega^\omega$ is κ-**universally Baire** if there exist trees S, T such that $\mathrm{p}[S] = A$ and S and T project to complements in every forcing extension by a partial order of cardinality less than or equal to κ.[41]

Woodin (see [Kan03, Lar04]) showed that if δ is a Woodin cardinal, then δ-universally Baire sets of reals are $<\delta$-weakly homogeneously Suslin. It follows from the arguments of [MS69] that if $A \subseteq \omega^\omega$ is κ^+-weakly homogeneously Suslin, then it is κ-universally Baire. Combining these facts with Theorem 5.6 gives the following.

THEOREM 6.1. If δ is a limit of Woodin cardinals, then the following are equivalent, for all sets of reals A.

- A is $<\delta$ homogeneously Suslin.
- A is $<\delta$ weakly homogeneously Suslin.
- A is $<\delta$-universally Baire.

Feng, Magidor, and Woodin showed that if $\delta_0 < \delta_1$ are Woodin cardinals, then every δ_1-universally Baire set is determined (this follows from Theorem 5.6 and the result of Woodin mentioned before the previous paragraph). Neeman later improved this, showing that if δ is a Woodin cardinal, then all δ-universally Baire sets are determined. In addition to the following theorem, Feng, Magidor and Woodin showed that $\mathsf{Det}(\boldsymbol{\Pi}^1_1)$ is equivalent to the statement that every $\boldsymbol{\Sigma}^1_2$ set of reals is universally Baire.

THEOREM 6.2 (Feng, Magidor, and Woodin [FMW92]). Assume $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$. Then the following are equivalent.

- $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$ holds in every forcing extension.
- Every set of reals in $\mathbf{L}(\mathbb{R})$ is universally Baire.

[41]The set A is $<\kappa$-**universally Baire** if it is γ-universally Baire for all $\gamma < \kappa$, and **universally Baire** if it is universally Baire for all κ.

Woodin's *Tree Production Lemma* is a powerful means for showing that sets of reals are universally Baire (see [Lar04]). Woodin's proof of Theorem 5.7 proceeded by applying the lemma to the set $\mathbb{R}^{\#}$. Informally, the lemma can be interpreted as saying that a set of reals A is δ-universally Baire if for every real r generic for a partial order in $\mathbf{V}_\delta$, either r is in the image of A for every $\mathbb{Q}_{<\delta}$-embedding[42] for which r is in the image model, or r is in the image of A for no such embedding.

THEOREM 6.3 (Tree Production Lemma). Suppose that δ is a Woodin cardinal. Let φ and ψ be binary formulas, and let x and y be arbitrary sets, and assume that the empty condition in the stationary tower $\mathbb{Q}_{<\delta}$ forces that for each real r,

$$M \models \psi(r, j(y)) \Leftrightarrow \mathbf{V}[r] \models \varphi(r, x), \tag{4}$$

where $j\colon \mathbf{V} \to M$ is the induced elementary embedding. Then $\{r : \psi(r, y)\}$ is $<\delta$-universally Baire.

6.2. AD^+ and $\mathsf{AD}_\mathbb{R}$. Moschovakis [Mos81] proved that under AD, if λ is less than Θ, A is a set of functions from ω to λ and A is Suslin and co-Suslin, then the game $G_\omega(A)$ is determined, where here the players play elements of λ. Woodin formulated the following axiom, which, assuming AD, holds in every inner model containing the reals whose sets of reals are all Suslin (in $\mathbf{V}$). A set of reals A is said to be ∞-Borel if there exist a set of ordinals S and binary formula φ such that $A = \{x \in \mathbb{R} : \mathbf{L}[x, S] \models \varphi(x, S)\}$. For example, a Suslin representation for a set of reals witnesses that the set ∞-Borel.

DEFINITION 6.4. AD^+ is the conjunction of the following statements.

- $\mathsf{DC}({}^\omega\omega)$.
- Every set of reals is ∞-Borel.
- If $\lambda < \Theta$ and $\pi\colon \lambda^\omega \to \omega^\omega$ is a continuous function, then $\pi^{-1}[A]$ is determined for every $A \subseteq \omega^\omega$.

It is an open question whether AD implies AD^+, though it is known that AD^+ holds in all models of AD of the form $\mathbf{L}(A, \mathbb{R})$, where A is a set of reals (some of the details of the argument showing this appear in [Jac10]). It is not known whether $\mathsf{AD}_\mathbb{R}$ implies AD^+, though AD^+ does follow from $\mathsf{AD}_\mathbb{R}$+DC.

The following consequences of AD^+ were announced in [Woo99].

THEOREM 6.5 (ZF+$\mathsf{DC}({}^\omega\omega)$). If AD^+ holds and $\mathbf{V} = \mathbf{L}(\wp(\mathbb{R}))$, then

- the pointclass Σ^2_1 has the scale property,
- every Σ^2_1 set of reals is the projection of a tree in $\mathbf{HOD}$,

[42] The partial order $\mathbb{Q}_{<\delta}$ is one form of Woodin's stationary tower, mentioned after Definition 5.5.

- every true Σ_1-sentence is witnessed by a $\underset{\sim}{\mathbf{\Delta}}^2_1$ set of reals.

Woodin's *Derived Model Theorem*, proved around 1986, gives a means of producing models of AD^+. The model $\mathbf{L}(\mathbb{R}^*, Hom^*)$ in the following theorem is said to be a **derived model** (over the ground model). A tree T is said to be **$<\lambda$-absolutely complemented** if there is a tree S such that $\mathrm{p}[T] = \mathbb{R} \setminus \mathrm{p}[S]$ in all forcing extensions by partial orders of cardinality less than λ.

Given an ordinal λ, $G \subseteq \mathrm{Col}(\omega, {<}\lambda)$ and $\alpha < \lambda$, we let $G{\restriction}\alpha$ denote $G \cap \mathrm{Col}(\omega, {<}\alpha)$. The model $\mathbf{V}(\mathbb{R}^*)$ in the following theorem can be defined as either $\bigcup_{\alpha \in \mathrm{Ord}} L(\mathbf{V}_\alpha, \mathbb{R}^*)$ or $\mathbf{HOD}^{V[G]}_{V \cup \mathbb{R}^*}$. Given a pointclass Γ, M_Γ denotes the collection of transitive sets x such that $\langle x, \in\rangle$ is isomorphic to $\langle \mathbb{R}/E, F/E\rangle$, for some $E, F \in \Gamma$ such that E is an equivalence relation on $\mathbb{R}$ and F is an E-invariant binary relation on $\mathbb{R}$. Models of the form $L(\Gamma, \mathbb{R}^*)$ below are called **derived models**. See [Ste09] for an earlier version of the theorem.

Theorem 6.6 (Derived Model Theorem; Woodin). Let λ be a limit of Woodin cardinals. Let $G \subseteq \mathrm{Col}(\omega, {<}\lambda)$ be a $\mathbf{V}$-generic filter. Let

- $\mathbb{R}^*$ be $\bigcup_{\alpha<\lambda} \mathbb{R}^{\mathbf{V}[G\restriction\alpha]}$;
- Hom^* be the collection of sets of the form $\mathrm{p}[T] \cap \mathbb{R}^*$, for T a $<\lambda$-absolutely complemented tree in $\mathbf{V}[G{\restriction}\alpha]$ for some $\alpha < \lambda$;
- Γ be the collection of sets of reals A in $\mathbf{V}[G]$ such that $\mathbf{L}(A, \mathbb{R}^*) \models \mathsf{AD}^+$.

Then

1. $\mathbf{L}(\Gamma, \mathbb{R}^*) \models \mathsf{AD}^+$.
2. Hom^* is the collection of Suslin, co-Suslin sets of reals in $\mathbf{L}(\Gamma, \mathbb{R}^*)$.
3. $M_\Gamma \prec_{\Sigma_1} \mathbf{L}(\Gamma, \mathbb{R}^*)$.

Woodin also showed that item (3) above is equivalent to AD^+, assuming $\mathsf{AD} + \mathbf{V}{=}\mathbf{L}(\wp(\mathbb{R}))$. The Derived Model Theorem has a converse, also due to Woodin, which says that all models of AD^+ arise in this fashion.

Theorem 6.7 (Woodin). Let M be a model of AD^+, and let Γ be the collection of sets of reals which are Suslin, co-Suslin in M. Then in a forcing extension of M there is an inner model N such that $\mathbf{L}(\Gamma, \mathbb{R}^*)$ is a derived model over N.

In unpublished work, Woodin has shown that over AD, $\mathsf{AD}_\mathbb{R}$ is equivalent to some of its ostensibly weak consequences (see [Woo99]). The implication from (2) to (1) in the following theorem is due independently to Martin. The implication from (1) to (2) relies heavily on work of Becker [Bec85]. Recall that Mycielski (see Section 2.3) showed that (1) implies (3); the implication from (2) to (3) is mentioned in Section 3.2.

Theorem 6.8 (Woodin). Assume $\mathsf{ZF}{+}\mathsf{DC}$. Then the following are equivalent.

1. $\mathsf{AD}_{\mathbb{R}}$.
2. AD+Every set of reals is Suslin.
3. AD+Uniformization.

Woodin would also produce models of $\mathsf{AD}_{\mathbb{R}}$ from large cardinals.

THEOREM 6.9 (Woodin). Suppose that there exists a cardinal δ of cofinality ω which is a limit of Woodin cardinals and $<\delta$-strong cardinals. Then there is a forcing extension in which there is an inner model containing the reals and satisfying $\mathsf{AD}_{\mathbb{R}}$.

Steel, using earlier work of Woodin, completed the equiconsistency with the following theorem.

THEOREM 6.10 (Steel). If $\mathsf{AD}_{\mathbb{R}}$ holds, then in a forcing extension there is a proper class model of ZFC in which there exists a cardinal δ of cofinality ω which is a limit of Woodin cardinals and $<\delta$-strong cardinals.

Recall from Section 4.1 that Θ is defined to be the least ordinal which is not a surjective image of the reals. Consideration of ordinal definable surjections gives the **Solovay sequence**, $\langle \vartheta_\alpha : \alpha \leq \Omega \rangle$. This sequence is defined by letting ϑ_0 be the least ordinal which is not the surjective image of an ordinal definable function on the reals, and, for each $\alpha < \Omega$, letting $\vartheta_{\alpha+1}$ be the least ordinal which is not a surjective image of $\wp(\vartheta_\alpha)$ via an ordinal definable function. Taking limits at limit stages and continuing until $\vartheta_\Omega = \Theta$ completes the definition. The consistency strength of AD^+ + "$\vartheta_\alpha = \Theta$" increases with α.

In $\mathbf{L}(\mathbb{R})$, $\vartheta_0 = \Theta$. Woodin proved that, assuming AD^+, $\mathsf{AD}_{\mathbb{R}}$ is equivalent to the assertion that the Solovay sequence has limit length. Woodin also showed, under the same assumption, ϑ_α is a Woodin cardinal in **HOD**, for all nonlimit $\alpha \leq \Omega$.

In unpublished work, Woodin showed that if it is consistent that there exists a Woodin limit of Woodin cardinals, then it is consistent that there exist sets of reals A and B such that the models $\mathbf{L}(A, \mathbb{R})$ and $\mathbf{L}(B, \mathbb{R})$ each satisfy AD but $\mathbf{L}(A, B, \mathbb{R})$ does not. Woodin also showed that in this case $\mathbf{L}(\Gamma, \mathbb{R}) \models \mathsf{AD}_{\mathbb{R}}$+$\mathsf{DC}$, where $\Gamma = \wp(\mathbb{R}) \cap \mathbf{L}(A, \mathbb{R}) \cap \mathbf{L}(B, \mathbb{R})$. Grigor Sargsyan showed that if there exist models $\mathbf{L}(A, \mathbb{R})$ and $\mathbf{L}(B, \mathbb{R})$ as above then there is a proper class model of $\mathsf{AD}_{\mathbb{R}}$ containing the reals in which Θ is regular.

6.3. Long games. As mentioned above, Blass [Bla75] and Mycielski showed that determinacy for games of length ω^2 is equivalent to $\mathsf{AD}_{\mathbb{R}}$. For each $n \in \omega$, determinacy for games of length $\omega + n$ is equivalent to AD (think of the game as being divided in two parts, where in the first part (of length ω) the players try to obtain a position from which they have a winning strategy in the second; the winning strategy in the second part can be coded by an integer, and thus uniformly chosen).

Martin and Woodin independently showed that $\mathsf{AD}_{\mathbb{R}}$ is equivalent to determinacy for games of length α for each countable $\alpha \geq \omega^2$. Determinacy for games of length $\omega \cdot 2$ easily gives uniformization. It follows from this and Theorem 6.8 that $\mathsf{AD}_{\mathbb{R}}$ is equivalent to determinacy for games of length α for each countable $\alpha \geq \omega \cdot 2$.

While AD does not imply uniformization, the Second Periodicity Theorem (Theorem 3.4) shows that PD implies the uniformization of projective sets. It follows that PD is equivalent to PD for games of length less than ω^2. As noted by Neeman [Nee05], the techniques from the Blass–Mycielski result above can be used to prove the determinacy of games of length ω^2 with analytic payoff from $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$ plus the existence of $\mathbb{R}^{\#}$.

Steel [Ste88] considered **continuously coded games**, games where each stage of the game is associated with an integer, and the game ends when an associated integer is repeated. Such a game must end after countably many rounds, but runs of the game can have any countable length. Steel proved that ZF + AD + DC + "every set of reals has a scale" + "ω_1 is $\wp(\mathbb{R})$-supercompact" implies the determinacy of all continuously coded games.

None of the results mentioned so far in this section involves proving determinacy directly from large cardinals. Instead they show that some form of determinacy for short games with complicated payoff implies determinacy for longer games with simpler payoff. Proving long game determinacy from large cardinals was pioneered, and extensively developed, by Neeman, who established a number of results on games of variable countable length, and even length ω_1 (see [Nee04, Nee05, Nee06A]). Neeman's techniques built on the proof of PD from Woodin cardinals by Martin and Steel, using iteration trees. In many cases, his proofs proceed from essentially optimal hypotheses. The proofs of many of these results reduced the determinacy of long games to the iterability of models containing large cardinals.

For example, given $C \subseteq \mathbb{R}^{<\omega_1}$, let $G_{\text{local}}(L, C)$ be the game where players player I and player II alternate playing natural numbers so as to define elements z_ξ of the Baire space. The game ends at the first γ such that γ is uncountable in $\mathbf{L}[z_\xi \colon \xi < \gamma]$, with player I winning if the sequence $\langle z_\xi : \xi < \gamma \rangle$ is in C. It follows from mild large cardinal assumptions (for instance, the existence of the sharp of every subset of ω_1) that γ must be countable.

Given a pointclass Γ, a set C consisting of countable sequences of reals is said to be Γ **in the codes** if the set of reals coding members of C (under a suitably definable coding) is in Γ.

Theorem 6.11 (Neeman). *Suppose that there exists a measurable cardinal above a Woodin limit of Woodin cardinals. Then the games $G_{\text{local}}(L, C)$ are determined for all C which are $\Game_\omega(<\omega^2\text{-}\Pi^1_1)$ in the codes.*

The preceding theorem is obtained by combining the results of [Nee04] and [Nee02A]. The proof proceeds by constructing an iterable class model M with a cardinal ϑ such that ϑ is a Woodin limit of Woodin cardinals in M and countable in $\mathbf{V}$ [Nee02A]. Using inner model theory, Neeman then transformed the iteration strategy of M into a winning strategy in $G_{\mathrm{local}}(L, C)$.

Adapting Kechris and Solovay's proof that Δ^1_2-determinacy implies the existence of a real x such that $\mathbf{L}[x]$ satisfies the determinacy of all ordinal definable sets of reals (discussed before Theorem 5.10), Woodin proved that the amount of determinacy in the conclusion of Theorem 6.11 implies that there exists a set $A \subseteq \omega_1$ such that in $\mathbf{L}[A]$, all games on integers of length ω_1 with payoff definable from reals and ordinals are determined (see [Nee04, Exercise 7F.15]). Larson and Shelah [LS08] showed that it is possible to force that some integer game of length ω_1 with definable payoff is undetermined.

We give one more result of Neeman, proving the determinacy of certain games of length ω_1. In Theorem 6.12 below, $\mathcal{L}^+$ is the language of set theory with one additional unary predicate. Given an integer k and a sequence $\bar{S}$ of stationary sets indexed by $[\omega_1]^{<k}$, $[\bar{S}]$ is the collection of increasing k-tuples $\langle \alpha_0, \ldots, \alpha_{k-1} \rangle$ from ω_1 such that each initial segment of length $j \leq k$ is in $S_{\langle \alpha_0, \ldots, \alpha_{j-1} \rangle}$. The game $G_{\omega_1, k}(\bar{S}, \varphi)$ is a game of length ω_1 in which the players collaborate to build a function $f \colon \omega_1 \to \omega_1$. Then player I wins if there is a club C such that

$$\langle L_{\omega_1}, r \rangle \models \varphi(\alpha_0, \ldots, \alpha_{k-1}) \tag{5}$$

for all $\langle \alpha_0, \ldots, \alpha_{k-1} \rangle \in [\bar{S}] \cap [C]^k$, and player II wins if there is a club C such that

$$\langle L_{\omega_1}, r \rangle \models \neg\varphi(\alpha_0, \ldots, \alpha_{k-1}) \tag{6}$$

for all $\langle \alpha_0, \ldots, \alpha_{k-1} \rangle \in [\bar{S}] \cap [C]^k$. Though there can be runs of the game for which neither player wins, determinacy for this game in the sense of Theorem 6.12 refers to the existence of a strategy for one player or the other that guarantees victory.

The model 0^W is the minimal iterable fine structural inner model M which has a top extender predicate whose critical point is Woodin in M. The existence of such a model is not known to follow from large cardinals.

The last part of the conclusion of Theorem 6.12 extends a result of Martin, who showed that for any recursive enumeration $\langle B_i : i < \omega \rangle$ of the $<\omega^2$-Π^1_1 sets, the set of i such that player I has a winning strategy in $G_\omega(B_i)$ is recursively isomorphic to $0^\#$.

THEOREM 6.12 (Neeman [Nee07A]). *Suppose that 0^W exists. Let $k < \omega$. Let $\bar{S}$ be a sequence of mutually disjoint stationary sets indexed by $[\omega_1]^{<k}$.*

Let φ be a $\mathcal{L}^+$ formula with k free variables. Then the game $G_{\omega_1,k}(\bar{S},\varphi)$ is determined. Furthermore, the winner of each such game depends only on φ and not on $\bar{S}$, and the set of φ for which the first player has a winning strategy is recursively equivalent to the canonical real coding 0^W.

If one allows the members of $\bar{S}$ all to be ω_1, then there are undetermined games of this type, as observed by Greg Hjorth (see [Nee07A]). If one allows the members of $\bar{S}$ all to be ω_1 and changes the winning condition for player player II to be simply the negation of the winning condition for player I then one can force from a strongly inaccessible limit of measurable cardinals that some game of this type is not determined [Lar05].

Given a set $A \subseteq {}^{<\omega_1}\omega$, $G_{\text{open}-\omega_1}(A)$ is the game of length ω_1 in which player I and player II collaborate to build a function from ω_1 to ω, with player I winning if some proper initial segment of the play is in A. The determinacy result in Theorem 6.12 includes the determinacy of all games $G_{\text{open}-\omega_1}(A)$ for sets A which are Π^1_1 in the codes. Combining Neeman's proof of Theorem 6.12 with his own theory of *hybrid strategy mice*, Woodin proved that if there exist proper class many Woodin limits of Woodin cardinals then AD^+ holds in the **Chang Model**, the smallest inner model of ZF containing the ordinals and closed under countable sequences.

6.4. Forcing over models of determinacy. Steel and Van Wesep [SVW82] showed that by forcing over a model of $\mathsf{AD}_{\mathbb{R}}$ + "Θ is regular" (the hypothesis they used was actually weaker) one can produce a model of ZFC in which NS_{ω_1} is saturated and $\underset{\sim}{\boldsymbol{\delta}}^1_2 = \omega_2$. This was the first consistency proof of either of these two statements with ZFC. Martin had conjectured that "$\forall n \in \omega\, \underset{\sim}{\boldsymbol{\delta}}^1_n = \aleph_n$" is consistent with ZFC, and this verified the conjecture for the case $n = 2$. Woodin [Woo83B] subsequently reduced the hypothesis to AD.

Shelah [She98] later showed that it was possible to force the saturation of NS_{ω_1} from a Woodin cardinal. Woodin [Woo99] proved that the saturation of NS_{ω_1} plus the existence of a measurable cardinal implies that $\underset{\sim}{\boldsymbol{\delta}}^1_2 = \omega_2$. Woodin then turned his proof into a general method for producing models of ZFC by forcing over models of determinacy. The most general form of this method, a partial order called $\mathbb{P}_{\max}$, consists roughly of a directed system containing all countable models of ZFC with a precipitous ideal on ω_1. In the presence of large cardinals, the resulting extension satisfies all forceable Π_2 sentences in $H(\omega_2)$, even with predicates for NS_{ω_1} and each set of reals in $\mathbf{L}(\mathbb{R})$. In this model, NS_{ω_1} is saturated and $\underset{\sim}{\boldsymbol{\delta}}^1_2 = \omega_2$. There are many variants of $\mathbb{P}_{\max}$. One of these variants, called $\mathbb{Q}_{\max}$, produces a model in which NS_{ω_1} is $\aleph_1$-dense (*i.e.*, $\wp(\omega_1)/\mathrm{NS}_{\omega_1}$ has a dense subset of cardinality $\aleph_1$; this implies saturation), from the assumption that AD holds in $\mathbf{L}(\mathbb{R})$. No other method is known for producing a model of ZFC in which NS_{ω_1} is $\aleph_1$-dense.

Steel [Ste95A] showed that under AD, $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})}$ is an extender model below Θ. Woodin then showed that the entire model $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})}$ is a model of the form $\mathbf{L}[\vec{E}, \Sigma]$, where $\vec{E}$ is a sequence of extenders and Σ is an iteration strategy corresponding to this sequence. Using this approach, Steel showed that for every regular $\kappa < \Theta$, the ω-club filter over κ is an ultrafilter in $\mathbf{L}(\mathbb{R})$. Woodin used this to show that ω_1 is $<\Theta$-supercompact in $\mathbf{L}(\mathbb{R})$. Previously it was known only that ω_1 is λ-supercompact for λ below the supremum of the Suslin cardinals (see the paragraph after Theorem 4.2).

Woodin also used the inner models approach to show that, in $\mathbf{L}(\mathbb{R})$, ω_1 is huge to κ for each measurable $\kappa < \Theta$, improving results of Becker. Neeman [Nee07B] used this approach to prove, for each $\lambda < \Theta$, the uniqueness of the normal ultrafilter on $\wp_{\aleph_1}\lambda$ witnessing the λ-supercompactness of ω_1. Previously this too was known only for $\lambda < \underset{\sim}{\boldsymbol{\delta}}^2_1$ (this is also discussed in the paragraph after Theorem 4.2). Neeman [Nee07B] and Woodin independently used this approach to show that, assuming $\mathsf{AD} + \mathbf{V}{=}\mathbf{L}(\mathbb{R})$, one could force without adding reals to obtain $\mathsf{ZFC} + \underset{\sim}{\boldsymbol{\delta}}^1_n = \omega_2$, for any $n \geq 3$. It is still unknown whether $\underset{\sim}{\boldsymbol{\delta}}^1_m$ can equal ω_n for any $m \geq n \geq 2$ (under ZFC).

6.5. Determinacy from its consequences. Woodin [Woo82] conjectured that Projective Determinacy follows from the statement that all projective sets are Lebesgue measurable, have the Baire property and can be uniformized by projective functions (all consequences of PD). This conjecture was refuted by Steel in 1997. If one requires the uniformization property for the scaled projective pointclasses, then the conjecture is still open. Woodin did prove the following version of the conjecture in the late 1990s, using work of Steel in inner model theory. Recall that AD implies the three statements below (see Sections 2.1 and 3.3).

THEOREM 6.13 (Woodin). *Assuming $\mathsf{ZF} + \mathsf{DC} + \mathbf{V}{=}\mathbf{L}(\mathbb{R})$, the Axiom of Determinacy follows from the conjunction of the following three statements.*

- *Every set of reals is Lebesgue measurable.*
- *Every set of reals has the property of Baire.*
- *Every Σ^2_1 subset of ${}^2({}^\omega\omega)$ can be uniformized.*

Woodin had proved another equivalence in the early 1980s.

THEOREM 6.14 (Woodin). *Assume $\mathsf{ZF} + \mathsf{DC} + \mathbf{V}{=}\mathbf{L}(\mathbb{R})$. Then the following are equivalent.*

- *AD.*
- *Turing determinacy.*

It is apparently an open question whether AD follows from $\mathsf{ZF} + \mathsf{DC} + \mathbf{V}{=}\mathbf{L}(\mathbb{R})$ plus either of (a) for every $\alpha < \Theta$ there is a surjection of ${}^\omega\omega$ onto $\wp(\alpha)$; (b) Θ is inaccessible.

Determinacy would turn out to be necessary for some of its earliest applications. For instance, Steel [Ste96] showed that $\boldsymbol{\Sigma}^1_3$-separation plus the existence of sharps for all reals implies $\boldsymbol{\Delta}^1_2$-determinacy. Applying related work of Steel, Hjorth [Hjo96A] showed that $\boldsymbol{\Pi}^1_2$-determinacy follows from Wadge determinacy for $\boldsymbol{\Pi}^1_2$ sets. Earlier, Harrington had shown that, for each real x, $\boldsymbol{\Pi}^1_1(x)$-Wadge determinacy implies that $x^\#$ exists. It is open whether Wadge determinacy for the projective sets implies PD.

6.6. Determinacy from other statements. Determinacy axioms such as PD and $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$ imply the consistency of ZFC (plus certain large cardinal statements) and so cannot be proved in ZFC. Empirically, however, these statements appear to follow from every natural statement of sufficient consistency strength. This includes a number of statements ostensibly having little relation to determinacy. In this section we give a few examples of this phenomenon. Most of these arguments use inner model theory, and our presentation relies heavily on [Sch10].

The following theorem shows, among other things, that in the presence of large cardinals, even mere forcing-absoluteness for the theory of $\mathbf{L}(\mathbb{R})$ implies $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$. The theorem is due to Steel and Woodin independently (see [Ste02]).

THEOREM 6.15. Suppose that κ is a measurable cardinal. Then the following are equivalent.

- For all partial orders $\mathbb{P} \in \mathbf{V}_\kappa$, the theory of $\mathbf{L}(\mathbb{R})$ is not changed by forcing with $\mathbb{P}$.
- For all partial orders $\mathbb{P} \in \mathbf{V}_\kappa$, AD holds in $\mathbf{L}(\mathbb{R})$ after forcing with $\mathbb{P}$.
- For all partial orders $\mathbb{P} \in \mathbf{V}_\kappa$, all sets of reals in $\mathbf{L}(\mathbb{R})$ are Lebesgue measurable after forcing with $\mathbb{P}$.
- For all partial orders $\mathbb{P} \in \mathbf{V}_\kappa$, there is no ω_1-sequence of reals in $\mathbf{L}(\mathbb{R})$ after forcing with $\mathbb{P}$.

A sequence $C = \langle C_\alpha : \alpha < \lambda \rangle$ (for some ordinal λ) is said to be **coherent** if each C_β is a club subset of β, and $C_\alpha = \alpha \cap C_\beta$ whenever α is a limit point of C_β. A **thread** of such a coherent sequence C is a club set $D \subseteq \lambda$ such that $C_\alpha = \alpha \cap D$ for all limit points α of D. The principle $\square(\lambda)$ says that there is a coherent sequence of length λ with no thread. The principle $\square_\kappa$ says that there is a coherent sequence C of length κ^+ such that the ordertype of C_α is at most κ, for each limit $\alpha < \lambda$ (in which case there cannot be a thread). These principles were isolated in the 1960s by Jensen [Jen72], who showed that $\square_\kappa$ holds in $\mathbf{L}$ for all infinite cardinals κ (see [Dev84, p. 141]).

Todorcevic [Tod84] showed that the Proper Forcing Axiom (PFA) implies that $\square(\kappa)$ fails for all cardinals κ of cofinality at least ω_2, from which it follows that $\square_\kappa$ fails for all uncountable cardinals. The failure of these square principles implies the failure of covering theorems for certain inner models, from which one can derive inner models with large cardinals. Using this general approach, Ernest Schimmerling [Sch95] proved that PFA implies $\underset{\sim}{\mathbf{\Delta}}^1_2$-determinacy. Woodin extended this proof to show that PFA implies PD.

In 1990, Woodin also showed that PFA plus the existence of a strongly inaccessible cardinal implies $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$. His proof introduced a technique known as the *core model induction*, an application of descriptive set theory and inner model theory. Roughly, the idea is to inductively work through the Wadge degrees to build canonical inner models which are correct for each Wadge class. The induction works through the gap structure highlighted in [Ste83A]. This general approach had previously been used by Kechris and Woodin [KW83] (see the end of Section 4.2).

Alessandro Andretta, Neeman, and Steel [ANS01] showed that PFA plus the existence of a measurable cardinal implies the existence of a model of $\mathsf{AD}_{\mathbb{R}}$ containing all the reals and ordinals. Steel [Ste05] showed that if $\square_\kappa$ fails for a singular strong limit cardinal κ, then AD holds in $\mathbf{L}(\mathbb{R})$. Building on Steel's work, Sargsyan produced a model of $\mathsf{AD}_{\mathbb{R}}+$"$\Theta$ is regular" from the same hypothesis.

The following theorem is due to Steel. Schimmerling [Sch07] had previously obtained PD from the same assumption.

Theorem 6.16. If $\kappa \geq \max\{\aleph_2, c\}$ and $\square(\kappa)$ and $\square_\kappa$ fail, then $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$ holds.

Todorcevic (see [Bek91]) and Boban Veličković [Vel92] showed that PFA implies that $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$. This gives another route towards showing that PFA implies that the AD holds in $\mathbf{L}(\mathbb{R})$. In May 2011, Andrés Caicedo, Larson, Sargsyan, Ralf Schindler, Steel and Martin Zeman showed that the hypothesis of Theorem 6.16 (with $\kappa = \aleph_2$) can be forced (using $\mathbb{P}_{\max}$) over a model of $\mathsf{AD}_{\mathbb{R}}$ in which Θ and some other member of the Solovay sequence are both regular.

Schimmerling and Zeman used the core model induction to prove the following theorem [SZ01]. They had previously derived Projective Determinacy from the failure of a weaker version of $\square_\kappa$ at a weakly compact cardinal; Woodin had then derived $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$ from the same hypothesis.

Theorem 6.17. If κ is a weakly compact cardinal and $\square_\kappa$ fails, then AD holds in $\mathbf{L}(\mathbb{R})$.

As discussed in Section 6.4, Woodin showed using a variation of $\mathbb{P}_{\max}$ that over a model of AD one can force to produce a model of ZFC in which the nonstationary ideal on ω_1 is $\aleph_1$-dense. Using the core model induction, he showed that the $\aleph_1$-density of NS_{ω_1} implies $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$.

Steel had previously shown, using inner models, that Projective Determinacy follows from CH plus the existence of a homogeneous ideal on ω_1 (a weaker assumption that the $\aleph_1$-density of NS_{ω_1}, which is in fact inconsistent with CH, by a theorem of Shelah). He had also shown [Ste96] that if NS_{ω_1} is saturated and there is a measurable cardinal, then $\underset{\sim}{\mathbf{\Delta}}^1_2$-determinacy holds. The hypothesis of the measurable cardinal was later removed in collaboration with Jensen.

Using the core model induction, Richard Ketchersid showed that if the restriction of NS_{ω_1} to some stationary set $S \subseteq \omega_1$ is $\aleph_1$-dense, and the restriction of the generic elementary embedding corresponding to forcing with $\wp(S)/\mathrm{NS}_{\omega_1}$ to each ordinal is an element of the ground model, then there is a model of $\mathsf{AD}^+ + \vartheta_0{<}\Theta$ containing the reals and the ordinals. Also using this method, Sargsyan would deduce the consistency of $\mathsf{AD}_{\mathbb{R}}$+"Θ is regular" from the same hypothesis. This gives an equiconsistency, as Woodin has shown how to force the hypothesis over a model of $\mathsf{AD}_{\mathbb{R}}$+"Θ is regular". In yet another application of the core model induction, Steel and Stuart Zoble [SZ] derived $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$ from a consequence of Martin's Maximum isolated by Todorcevic, known as the *Strong Reflection Principle* at ω_2.

We conclude with three more examples. Silver [Sil75] proved that if κ is a singular cardinal of uncountable cofinality and $2^\alpha = \alpha^+$ for club many $\alpha < \kappa$, then $2^\kappa = \kappa^+$. Gitik and Schindler (see [GSS06]) showed that if κ is a singular cardinal of uncountable cofinality and the set of $\alpha < \kappa$ for which $2^\alpha = \alpha^+$ is stationary and costationary, then PD holds. Schindler (in the same paper) showed that if $\aleph_\omega$ is a strong limit cardinal and $2^{\aleph_\omega} > \aleph_{\omega_1}$, then PD holds. It is not known whether either of these results can be strengthened to obtain $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$.

A cardinal κ is said to have the **Tree Property** if every tree of height κ with all levels of cardinality less than κ has a cofinal branch (*i.e.*, if there are no κ-Aronszajn trees). Foreman, Magidor and Schindler [FMS01] showed that if there exist infinitely many cardinals δ above the continuum such that the tree property holds at δ and at δ^+, then PD holds. The hypothesis of this statement had been shown consistent relative to the existence of infinitely many supercompact cardinals by James Cummings and Foreman [CF98]. It is not known whether the conclusion can be strengthened to $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$.

Finally, as mentioned in Section 2.3, Gitik showed that if there is a proper class of strongly compact cardinals, then there is a model of ZF in which all infinite cardinals have cofinality ω. Using the core model induction, Daniel Busche and Schindler [BS09] showed that this statement implies that PD holds, and that AD holds in the $\mathbf{L}(\mathbb{R})$ of a forcing extension of $\mathbf{HOD}$.

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REFERENCES

John W. Addison

[Add59A] *Separation principles in the hierarchies of classical and effective descriptive set theory*, ***Fundamenta Mathematicae***, vol. 46 (1959), pp. 123–135.

[Add59B] *Some consequences of the axiom of constructibility*, ***Fundamenta Mathematicae***, vol. 46 (1959), pp. 337–357.

John W. Addison and Yiannis N. Moschovakis

[AM68] *Some consequences of the axiom of definable determinateness*, ***Proceedings of the National Academy of Sciences of the United States of America***, (1968), no. 59, pp. 708–712.

Donald J. Albers and Gerald L. Alexanderson

[AA85] ***Mathematical people***, Birkhäuser, Boston, MA, 1985.

Alessandro Andretta, Itay Neeman, and John R. Steel

[ANS01] *The domestic levels of K^c are iterable*, ***Israel Journal of Mathematics***, vol. 125 (2001), pp. 157–201.

Stefan Banach and Alfred Tarski

[BT24] *Sur la décomposition des ensembles de points en parties respectivement congruentes*, ***Fundamenta Mathematicae***, vol. 6 (1924), pp. 244–277.

James Baumgartner, Donald A. Martin, and Saharon Shelah

[BMS84] *Axiomatic set theory. Proceedings of the AMS-IMS-SIAM joint summer research conference held in Boulder, Colo., June 19–25, 1983*, Contemporary Mathematics, vol. 31, Amer. Math. Soc., Providence, RI, 1984.

Howard S. Becker

[Bec78] *Partially playful universes*, in Kechris and Moschovakis [Cabal i], pp. 55–90, reprinted in [Cabal III], pp. ??–??

[Bec81A] AD *and the supercompactness of* $\aleph_1$, ***The Journal of Symbolic Logic***, vol. 46 (1981), pp. 822–841.

[Bec85] *A property equivalent to the existence of scales*, ***Transactions of the American Mathematical Society***, vol. 287 (1985), no. 2, pp. 591–612.

Howard S. Becker and Alexander S. Kechris

[BK84] *Sets of ordinals constructible from trees and the third Victoria Delfino problem*, in Baumgartner et al. [BMS84], pp. 13–29.

Howard S. Becker and Yiannis N. Moschovakis

[BM81] *Measurable cardinals in playful models*, in Kechris et al. [Cabal ii], pp. 203–214, reprinted in [Cabal III], pp. ??–??

Mohamed Bekkali

[Bek91] ***Topics in set theory: Lebesgue measurability, large cardinals, forcing axioms, rho-functions. notes on lectures by Stevo Todorčević***, Lecture Notes in Mathematics, vol. 1476, Springer-Verlag, Berlin, 1991.

David Blackwell

[Bla67] *Infinite games and analytic sets*, ***Proceedings of the National Academy of Sciences of the United States of America***, vol. 58 (1967), pp. 1836–1837.

[Bla69] *Infinite G_δ-games with imperfect information*, ***Polska Akademia Nauk. Instytut Matematyczny. Zastosowania Matematyki***, vol. 10 (1969), pp. 99–101.

Andreas Blass

[Bla75] *Equivalence of two strong forms of determinacy*, ***Proceedings of the American Mathematical Society***, vol. 52 (1975), pp. 373–376.

L. E. J. Brouwer

[Bro24] *Beweis dass jede volle Funktion gleichmässig stetig ist*, ***Koninklijke Akademie van Wetenschappen te Amsterdam. Proceedings of the Section of Sciences***, vol. 27 (1924), pp. 189–193.

Daniel Busche and Ralf Schindler

[BS09] *The strength of choiceless patterns of singular and weakly compact cardinals*, ***Annals of Pure and Applied Logic***, vol. 159 (2009), no. 1-2, pp. 198–248.

S. Barry Cooper

[Coo04] ***Computability theory***, Chapman & Hall/CRC, Boca Raton, FL, 2004.

James Cummings and Matthew Foreman

[CF98] *The tree property*, ***Advances in Mathematics***, vol. 133 (1998), no. 1, pp. 1–32.

Morton Davis

[Dav64] *Infinite games of perfect information*, ***Advances in game theory*** (Melvin Dresher, Lloyd S. Shapley, and Alan W. Tucker, editors), Annals of Mathematical Studies, vol. 52, Princeton University Press, 1964, pp. 85–101.

Keith J. Devlin

[Dev84] ***Constructibility***, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1984.

Anthony Dodd

[Dod82] ***The core model***, London Mathematical Society Lecture Note Series, vol. 61, Cambridge University Press, 1982.

Derrick Albert DuBose

[DuB90] *The equivalence of determinancy and iterated sharps*, ***The Journal of Symbolic Logic***, vol. 55 (1990), no. 2, pp. 502–525.

Paul Erdős and András Hajnal

[EH58] *On the structure of set mappings*, ***Acta Mathematica Academiae Scientiarum Hungaricae***, vol. 9 (1958), pp. 111–131.

[EH66] *On a problem of B. Jónsson*, ***Bulletin de l'Académie Polonaise des Sciences***, vol. 14 (1966), pp. 19–23.

Soloman Feferman and Azriel Lévy

[FL63] *Independence results in set theory by Cohen's method II*, ***Notices of the American Mathematical Society***, vol. 10 (1963), p. 593, abstract.

Qi Feng, Menachem Magidor, and W. Hugh Woodin

[FMW92] *Universally Baire sets of reals*, in Judah et al. [JJW92], pp. 203–242.

Matthew Foreman

[For86] *Potent axioms*, ***Transactions of the American Mathematical Society***, vol. 294 (1986), no. 1, pp. 1–28.

Matthew Foreman, Menachem Magidor, and Ralf-Dieter Schindler

[FMS01] *The consistency strength of successive cardinals with the tree property*, ***The Journal of Symbolic Logic***, vol. 66 (2001), no. 4, pp. 1837–1847.

Matthew Foreman, Menachem Magidor, and Saharon Shelah

[FMS88] *Martin's maximum, saturated ideals and nonregular ultrafilters. I*, ***Annals of Mathematics***, vol. 127 (1988), no. 1, pp. 1–47.

Harvey Friedman

[Fri71A] *Determinateness in the low projective hierarchy*, ***Fundamenta Mathematicae***, vol. 72 (1971), no. 1, pp. 79–95. (errata insert).

[Fri71B] *Higher set theory and mathematical practice*, ***Annals of Mathematical Logic***, vol. 2 (1971), no. 3, pp. 325–357.

David Gale and F. Stewart

[GS53] *Infinite games with perfect information*, ***Contributions to the theory of games, vol. 2***, Annals of Mathematics Studies, no. 28, Princeton University Press, 1953, pp. 245–266.

Moti Gitik

[Git80] *All uncountable cardinals can be singular*, ***Israel Journal of Mathematics***, vol. 35 (1980), no. 1-2, pp. 61–88.

Moti Gitik, Ralf Schindler, and Saharon Shelah

[GSS06] *PCF theory and Woodin cardinals*, ***Logic Colloquium '02***, Lecture Notes in Logic, vol. 27, Association for Symbolic Logic, 2006, pp. 172–205.

John Townsend Green

[Gre78] *Determinacy and the existence of large measurable cardinals*, ***Ph.D. thesis***, University of California, Berkeley, 1978.

Jacques Hadamard

[Had05] *Cinq letters sur la théorie des ensembles*, ***Bulletin de la Societé mathématique de France***, vol. 33 (1905), pp. 261–273.

András Hajnal

[Haj56] *On a consistency theorem connected with the generalized continuum problem*, ***Zeitschrift für Mathematische Logik und Grundlagen der Mathematik***, vol. 2 (1956), pp. 131–136.

[Haj61] *On a consistency theorem connected with the generalized continuum problem*, ***Acta Mathematica Academiae Scientiarum Hungaricae***, vol. 12 (1961), pp. 321–376.

Paul R. Halmos

[Hal50] ***Measure theory***, D. Van Nostrand Company, Inc., New York, 1950.

Leo A. Harrington

[Har78] *Analytic determinacy and* $0^{\#}$, ***The Journal of Symbolic Logic***, vol. 43 (1978), no. 4, pp. 685–693.

Leo A. Harrington and Alexander S. Kechris

[HK81] *On the determinacy of games on ordinals*, ***Annals of Mathematical Logic***, vol. 20 (1981), pp. 109–154.

Felix Hausdorff

[Hau08] *Grundzüge einer Theorie der geordneten Mengen*, ***Mathematische Annalen***, vol. 65 (1908), pp. 435–505.

[Hau14] *Bemerkung über den Inhalt von Punktmengen*, ***Mathematische Annalen***, vol. 75 (1914), pp. 428–434.

James Henle, A. R. D. Mathias, and W. Hugh Woodin

[HMW85] *A barren extension*, ***Methods in mathematical logic (Caracas, 1983)***, Lecture Notes in Mathematics, vol. 1130, Springer-Verlag, Berlin, 1985, pp. 195–207.

Gregory Hjorth

[Hjo96A] Π^1_2 *Wadge degrees*, ***Annals of Pure and Applied Logic***, vol. 77 (1996), no. 1, pp. 53–74.

Paul Howard and Jean E. Rubin

[HR98] ***Consequences of the Axiom of Choice***, Mathematical Surveys and Monographs, vol. 59, American Mathematical Society, 1998.

Stephen Jackson

[Jac88] AD *and the projective ordinals*, in Kechris et al. [Cabal iv], pp. 117–220, reprinted in [Cabal II], pp. 364–483.

[Jac99] ***A computation of*** $\underset{\sim}{\boldsymbol{\delta}}^1_5$, vol. 140, Memoirs of the AMS, no. 670, American Mathematical Society, July 1999.

[Jac10] *Structural consequences of* AD, in Kanamori and Foreman [KF10], pp. 1753–1876.

Thomas Jech

[Jec03] ***Set theory***, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, the third millennium edition, revised and expanded.

Ronald B. Jensen

[Jen72] *The fine structure of the constructible hierarchy*, ***Annals of Mathematical Logic***, vol. 4 (1972), pp. 229–308; erratum, p. 443.

H. Judah, W. Just, and W. Hugh Woodin

[JJW92] ***Set theory of the continuum***, MSRI publications, vol. 26, Springer-Verlag, 1992.

László Kalmár

[Kal1928–29] *Zur Theorie der abstrakten Spiele*, ***Acta Scientiarum Mathematicarum (Szeged)***, vol. 4 (1928–29), no. 1–2, pp. 65–85.

Akihiro Kanamori

[Kan95] *The emergence of descriptive set theory*, ***From Dedekind to Gödel (Boston, MA, 1992)***, Synthese Library, vol. 251, Kluwer Academic Publishers, Dordrecht, 1995, pp. 241–262.

[Kan03] ***The higher infinite***, second ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.

Akihiro Kanamori and Matthew Foreman

[KF10] ***Handbook of set theory***, Springer, 2010.

Alexander S. Kechris

[Kec74] *On projective ordinals*, ***The Journal of Symbolic Logic***, vol. 39 (1974), pp. 269–282.

[Kec75B] *The theory of countable analytical sets*, ***Transactions of the American Mathematical Society***, vol. 202 (1975), pp. 259–297.

[Kec78A] AD *and projective ordinals*, in Kechris and Moschovakis [Cabal i], pp. 91–132, reprinted in [Cabal II], pp. 304–345.

[Kec81A] *Homogeneous trees and projective scales*, in Kechris et al. [Cabal ii], pp. 33–74, reprinted in [Cabal II], pp. 270–303.

[Kec84] *The axiom of determinancy implies dependent choices in* $\mathbf{L}(\mathbb{R})$, ***The Journal of Symbolic Logic***, vol. 49 (1984), no. 1, pp. 161–173.

[Kec95] ***Classical descriptive set theory***, Graduate Texts in Mathematics, vol. 156, Springer, 1995.

Alexander S. Kechris, Eugene M. Kleinberg, Yiannis N. Moschovakis, and W. Hugh Woodin

[KKMW81] *The axiom of determinacy, strong partition properties, and nonsingular measures*, in Kechris et al. [Cabal ii], pp. 75–99, reprinted in [Cabal I], pp. 333–354.

Alexander S. Kechris, Benedikt Löwe, and John R. Steel

[Cabal I] *Games, scales, and Suslin cardinals: the Cabal seminar, volume I*, Lecture Notes in Logic, vol. 31, Cambridge University Press, 2008.

[Cabal II] *Wadge degrees and projective ordinals: the Cabal seminar, volume II*, Lecture Notes in Logic, vol. 37, Cambridge University Press, 2012.

[Cabal III] *Ordinal definability and recursion theory: the Cabal seminar, volume III*, Lecture Notes in Logic, vol. ??, Cambridge University Press, 2014.

Alexander S. Kechris, Donald A. Martin, and Yiannis N. Moschovakis

[Cabal ii] *Cabal seminar 77–79*, Lecture Notes in Mathematics, no. 839, Berlin, Springer, 1981.

[Cabal iii] *Cabal seminar 79–81*, Lecture Notes in Mathematics, no. 1019, Berlin, Springer, 1983.

Alexander S. Kechris, Donald A. Martin, and John R. Steel

[Cabal iv] *Cabal seminar 81–85*, Lecture Notes in Mathematics, no. 1333, Berlin, Springer, 1988.

Alexander S. Kechris and Yiannis N. Moschovakis

[KM72] *Two theorems about projective sets*, ***Israel Journal of Mathematics***, vol. 12 (1972), pp. 391–399.

[Cabal i] *Cabal seminar 76–77*, Lecture Notes in Mathematics, no. 689, Berlin, Springer, 1978.

[KM78B] *Notes on the theory of scales*, in *Cabal Seminar 76–77* [Cabal i], pp. 1–53, reprinted in [Cabal I], pp. 28–74.

Alexander S. Kechris and Robert M. Solovay

[KS85] *On the relative consistency strength of determinacy hypotheses*, ***Transactions of the American Mathematical Society***, vol. 290 (1985), no. 1, pp. 179–211.

Alexander S. Kechris, Robert M. Solovay, and John R. Steel

[KSS81] *The axiom of determinacy and the prewellordering property*, in Kechris et al. [Cabal ii], pp. 101–125, reprinted in [Cabal II], pp. 118–140.

Alexander S. Kechris and W. Hugh Woodin

[KW83] *Equivalence of partition properties and determinacy*, ***Proceedings of the National Academy of Sciences of the United States of America***, vol. 80 (1983), no. 6 i., pp. 1783–1786.

Stephen C. Kleene

[Kle38] *On notation for ordinal numbers*, ***The Journal of Symbolic Logic***, vol. 3 (1938), pp. 150–155.

[Kle43] *Recursive predicates and quantifiers*, ***Transactions of the American Mathematical Society***, vol. 53 (1943), pp. 41–73.

[Kle55A] *Arithmetical predicates and function quantifiers*, ***Transactions of the American Mathematical Society***, vol. 79 (1955), pp. 312–340.

[Kle55B] *Hierarchies of number-theoretic predicates*, ***Bulletin of the American Mathematical Society***, vol. 61 (1955), pp. 193–213.

[Kle55C] *On the forms of the predicates in the theory of constructive ordinals. II*, ***American Journal of Mathematics***, vol. 77 (1955), pp. 405–428.

Eugene M. Kleinberg

[Kle70] *Strong partition properties for infinite cardinals*, ***The Journal of Symbolic Logic***, vol. 35 (1970), pp. 410–428.

Peter Koellner and W. Hugh Woodin

[KW] *Foundations of set theory: The search for new axioms*, in preparation.

[KW10] *Large cardinals from determinacy*, in Kanamori and Foreman [KF10], pp. 1951–2119.

Motokiti Kondô

[Kon38] *Sur l'uniformization des complementaires analytiques et les ensembles projectifs de la seconde classe*, ***Japanese Journal of Mathematics***, vol. 15 (1938), pp. 197–230.

Dénes Kőnig

[Kőn27] *Über eine Schlusswiese aus dem Endlichen ins Unendliche*, ***Acta Scientiarum Mathematicarum (Szeged)***, vol. 3 (1927), no. 2–3, pp. 121–130.

Kenneth Kunen

[Kun70] *Some applications of iterated ultrapowers in set theory*, ***Annals of Mathematical Logic***, vol. 1 (1970), pp. 179–227.

[Kun71A] *Elementary embeddings and infinitary combinatorics*, ***The Journal of Symbolic Logic***, vol. 36 (1971), pp. 407–413.

[Kun71E] *A remark on Moschovakis' uniformization theorem*, circulated note, March 1971.

[Kun71F] *Some singular cardinals*, circulated note, September 1971.

[Kun71G] *Some more singular cardinals*, circulated note, September 1971.

[Kun78] *Saturated ideals*, ***The Journal of Symbolic Logic***, vol. 43 (1978), no. 1, pp. 65–76.

[Kun83] ***Set theory: An introduction to independence proofs***, Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland, Amsterdam, 1983, reprint of the 1980 original.

Kazimierz Kuratowski

[Kur36] *Sur les théorèmes de séparation dans las théorie des ensembles*, ***Fundamenta Mathematicae***, vol. 26 (1936), pp. 183–191.

Paul B. Larson

[Lar04] ***The stationary tower: Notes on a course by W. Hugh Woodin***, University Lecture Series, vol. 32, American Mathematical Society, Providence, RI, 2004.

[Lar05] *The canonical function game*, ***Archive for Mathematical Logic***, vol. 44 (2005), no. 7, pp. 817–827.

[Lar12] *A brief history of determinacy*, ***Sets and extensions in the twentieth century*** (Dov M. Gabbay, Akihiro Kanamori, and John Woods, editors), Handbook of the History of Logic, vol. 6, Elsevier, 2012, pp. 457–507.

Paul B. Larson. and Saharon Shelah

[LS08] *The stationary set splitting game*, ***Mathematical Logic Quarterly***, vol. 54 (2008), no. 2, pp. 187–193.

Henri Lebesgue

[Leb05] *Sur les fonctions représentables analytiquement*, ***Journal de Mathématiques Pures et Appliquées***, vol. 1 (1905), pp. 139–216.

[Leb18] *Remarques sur les théories de le mesure et de l'intégration*, ***Annales de l'École Normale supérieure***, vol. 35 (1918), pp. 191–250.

A. Lévy

[Lév57] *Indépendance conditionnelle de* $\mathbf{V} = \mathbf{L}$ *et d'axiomes qui se rattachent au système de M. Gödel*, ***Comptes rendus hebdomadaires des séances de l'Académie des Sciences***, vol. 245 (1957), pp. 1582–1583.

Azriel Lévy

[Lév60] *A generalization of Gödel's notion of constructibility*, ***The Journal of Symbolic Logic***, vol. 25 (1960), pp. 147–155.

[Lév65A] *Definability in axiomatic set theory. I*, ***Logic, methodology and philosophy of science. Proceedings of the 1964 International Congress*** (Amsterdam) (Yehoshua Bar-Hillel, editor), Studies in Logic and the Foundations of Mathematics, North-Holland, 1965, pp. 127–151.

[Lév65B] *A hierarchy of formulas in set theory*, ***Memoirs of the American Mathematical Society***, vol. 57 (1965), p. 76.

[Lév79] ***Basic set theory***, Springer-Verlag, Berlin, 1979.

Alain Louveau and Jean Saint-Raymond

[LSR87] *Borel classes and closed games: Wadge-type and Hurewicz-type results*, ***Transactions of the American Mathematical Society***, vol. 304 (1987), no. 2, pp. 431–467.

[LSR88B] *The strength of Borel Wadge determinacy*, in Kechris et al. [Cabal iv], pp. 1–30, reprinted in [Cabal II], pp. 74–101.

Nikolai Luzin

[Luz25A] *Les proprietes des ensembles projectifs*, ***Comptes rendus hebdomadaires des séances de l'Académie des Sciences***, vol. 180 (1925), pp. 1817–1819.

[Luz25B] *Sur les ensembles projectifs de M. Henri Lebesgue*, ***Comptes rendus hebdomadaires des séances de l'Académie des Sciences***, vol. 180 (1925), pp. 1318–1320.

[Luz25C] *Sur un problème de M. Emil Borel et les ensembles projectifs de M. Henri Lebesgue: les ensembles analytiques*, ***Comptes rendus hebdomadaires des séances de l'Académie des Sciences***, vol. 164 (1925), pp. 91–94.

[Luz27] *Sur les ensembles analytiques*, ***Fundamenta Mathematicae***, vol. 10 (1927), pp. 1–95.

[Luz30A] *Analogies entre les ensembles mesurables B et les ensembles analytiques*, ***Fundamenta Mathematicae***, vol. 16 (1930), pp. 48–76.

[Luz30C] *Sur le problème de M. J. Hadamard d'uniformisation des ensembles*, ***Comptes rendus hebdomadaires des séances de l'Académie des Sciences***, vol. 190 (1930), pp. 349–351.

Nikolai Luzin and Petr Novikov

[LN35] *Choix effectif d'un point dans un complemetaire analytique arbitraire, donne par un crible*, ***Fundamenta Mathematicae***, vol. 25 (1935), pp. 559–560.

Nikolai Luzin and Wacław Sierpiński

[LS18] *Sur quelques propriétés des ensembles (A)*, ***Bulletin de l'Académie des Sciences Cracovie, Classe des Sciences Mathématiques, Série A***, (1918), pp. 35–48.

[LS23] *Sur un ensemble non measurable B*, ***Journal de Mathématiques Pures et Appliqueées***, vol. 2 (1923), no. 9, pp. 53–72.

Menachem Magidor

[Mag80] *Precipitous ideals and* $\boldsymbol{\Sigma}^1_4$ *sets*, ***Israel Journal of Mathematics***, vol. 35 (1980), no. 1-2, pp. 109–134.

Richard Mansfield

[Man70] *Perfect subsets of definable sets of real numbers*, ***Pacific Journal of Mathematics***, vol. 35 (1970), no. 2, pp. 451–457.

[Man71] *A Souslin operation on* Π^1_2, ***Israel Journal of Mathematics***, vol. 9 (1971), no. 3, pp. 367–379.

Donald A. Martin

[Mar68] *The axiom of determinateness and reduction principles in the analytical hierarchy*, ***Bulletin of the American Mathematical Society***, vol. 74 (1968), pp. 687–689.

[Mar70A] *Measurable cardinals and analytic games*, ***Fundamenta Mathematicae***, vol. 66 (1970), pp. 287–291.

[Mar75] *Borel determinacy*, ***Annals of Mathematics***, vol. 102 (1975), no. 2, pp. 363–371.

[Mar80] *Infinite games*, ***Proceedings of the International Congress of Mathematicatians, Helsinki 1978*** (Helsinki) (Olli Lehto, editor), Academia Scientiarum Fennica, 1980, pp. 269–273.

[Mar83B] *The real game quantifier propagates scales*, in Kechris et al. [Cabal iii], pp. 157–171, reprinted in [Cabal I], pp. 209–222.

[Mar85] *A purely inductive proof of Borel determinacy*, ***Recursion theory (ithaca, n.y., 1982)***, Proceedings of Symposia in Pure Mathematics, vol. 42, American Mathematical Society, Providence, RI, 1985, pp. 303–308.

[Mar90] *An extension of Borel determinacy*, ***Annals of Pure and Applied Logic***, vol. 49 (1990), no. 3, pp. 279–293.

[Mar98] *The determinacy of Blackwell games*, ***The Journal of Symbolic Logic***, vol. 63 (1998), no. 4, pp. 1565–1581.

[Mar03] *A simple proof that determinacy implies Lebesgue measurability*, ***Università e Politecnico di Torino. Seminario Matematico. Rendiconti***, vol. 61 (2003), no. 4, pp. 393–397.

Donald A. Martin, Yiannis N. Moschovakis, and John R. Steel

[MMS82A] *The extent of definable scales*, ***Bulletin of the American Mathematical Society***, vol. 6 (1982), pp. 435–440.

Donald A. Martin, Itay Neeman, and Marco Vervoort

[MNV03] *The strength of Blackwell determinacy*, ***The Journal of Symbolic Logic***, vol. 68 (2003), no. 2, pp. 615–636.

Donald A. Martin and Jeff B. Paris

[MP71] $\mathsf{AD} \Rightarrow \exists$ *exactly* 2 *normal measures on* ω_2, circulated note, March 1971.

Donald A. Martin and Robert M. Solovay

[MS69] *A basis theorem for Σ^1_3 sets of reals*, ***Annals of Mathematics***, vol. 89 (1969), pp. 138–160.

Donald A. Martin and John R. Steel
[MS83] *The extent of scales in* $\mathbf{L}(\mathbb{R})$, in Kechris et al. [Cabal iii], pp. 86–96, reprinted in [Cabal I], pp. 110–120.
[MS89] *A proof of projective determinacy*, ***Journal of the American Mathematical Society***, vol. 2 (1989), pp. 71–125.
[MaS94] *Iteration trees*, ***Journal of the American Mathematical Society***, vol. 7 (1994), no. 1, pp. 1–73.

A. R. D. Mathias
[Mat68] *On a generalization of Ramsey's theorem*, ***Notices of the American Mathematical Society***, vol. 15 (1968), p. 931.
[Mat77] *Happy families*, ***Annals of Mathematical Logic***, vol. 12 (1977), no. 1, pp. 59–111.

R. Daniel Mauldin
[Mau81] ***The Scottish Book: Mathematics from the Scottish Café***, Birkhäuser, Boston, MA, 1981.

William J. Mitchell
[Mit79] *Hypermeasurable cardinals*, ***Logic Colloquium '78 (Mons, 1978)***, Studies in Logic and the Foundations of Mathematics, vol. 97, North-Holland, Amsterdam, 1979, pp. 303–316.

Yiannis N. Moschovakis
[Mos67] *Hyperanalytic predicates*, ***Transactions of the American Mathematical Society***, vol. 129 (1967), pp. 249–282.
[Mos69B] *Abstract first order computability I*, ***Transactions of the American Mathematical Society***, vol. 138 (1969), pp. 427–463.
[Mos69C] *Abstract first order computability II*, ***Transactions of the American Mathematical Society***, vol. 138 (1969), pp. 464–504.
[Mos70A] *Determinacy and prewellorderings of the continuum*, ***Mathematical logic and foundations of set theory. Proceedings of an international colloquium held under the auspices of the Israel Academy of Sciences and Humanities, Jerusalem, 11–14 November 1968*** (Y. Bar-Hillel, editor), Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam-London, 1970, pp. 24–62.
[Mos71A] *Uniformization in a playful universe*, ***Bulletin of the American Mathematical Society***, vol. 77 (1971), pp. 731–736.
[Mos73] *Analytical definability in a playful universe*, ***Logic, methodology, and philosophy of science IV*** (Patrick Suppes, Leon Henkin, Athanase Joja, and Gr. C. Moisil, editors), North-Holland, 1973, pp. 77–83.
[Mos78] *Inductive scales on inductive sets*, in Kechris and Moschovakis [Cabal i], pp. 185–192, reprinted in [Cabal I], pp. 94–101.
[Mos80] ***Descriptive set theory***, Studies in Logic and the Foundations of Mathematics, no. 100, North-Holland, Amsterdam, 1980.
[Mos81] *Ordinal games and playful models*, in Kechris et al. [Cabal ii], pp. 169–201, reprinted in [Cabal III], pp. ??–??
[Mos83] *Scales on coinductive sets*, in Kechris et al. [Cabal iii], pp. 77–85, reprinted in [Cabal I], pp. 102–109.
[Mos09] ***Descriptive set theory***, second ed., Mathematical Surveys and Monographs, vol. 155, American Mathematical Society, 2009.

JAN MYCIELSKI

[Myc64] *On the axiom of determinateness*, ***Fundamenta Mathematicae***, vol. 53 (1964), pp. 205–224.

[Myc66] *On the axiom of determinateness. II*, ***Fundamenta Mathematicae***, vol. 59 (1966), pp. 203–212.

JAN MYCIELSKI AND HUGO STEINHAUS

[MS62] *A mathematical axiom contradicting the axiom of choice*, ***Bulletin de l'Académie Polonaise des Sciences***, vol. 10 (1962), pp. 1–3.

JAN MYCIELSKI AND STANISŁAW ŚWIERCZKOWSKI

[MŚ64] *On the Lebesgue measurability and the axiom of determinateness*, ***Fundamenta Mathematicae***, vol. 54 (1964), pp. 67–71.

ITAY NEEMAN

[Nee95] *Optimal proofs of determinacy*, ***The Bulletin of Symbolic Logic***, vol. 1 (1995), no. 3, pp. 327–339.

[Nee00] *Unraveling $\boldsymbol{\Pi}^1_1$ sets*, ***Annals of Pure and Applied Logic***, vol. 106 (2000), no. 1-3, pp. 151–205.

[Nee02A] *Inner models in the region of a Woodin limit of Woodin cardinals*, ***Annals of Pure and Applied Logic***, vol. 116 (2002), no. 1-3, pp. 67–155.

[Nee04] ***The determinacy of long games***, de Gruyter Series in Logic and its Applications, vol. 7, Walter de Gruyter, Berlin, 2004.

[Nee05] *An introduction to proofs of determinacy of long games*, ***Logic Colloquium '01*** (Matthias Baaz, Sy-David Friedman, and Jan Krajíček, editors), Lecture Notes in Logic, vol. 20, Association for Symbolic Logic, 2005, pp. 43–86.

[Nee06A] *Determinacy for games ending at the first admissible relative to the play*, ***The Journal of Symbolic Logic***, vol. 71 (2006), no. 2, pp. 425–459.

[Nee06B] *Unraveling $\boldsymbol{\Pi}^1_1$ sets, revisited*, ***Israel Journal of Mathematics***, vol. 152 (2006), pp. 181–203.

[Nee07A] *Games of length ω_1*, ***Journal of Mathematical Logic***, vol. 7 (2007), no. 1, pp. 83–124.

[Nee07B] *Inner models and ultrafilters in $\mathbf{L}(\mathbb{R})$*, ***The Bulletin of Symbolic Logic***, vol. 13 (2007), no. 1, pp. 31–53.

[Nee10] *Determinacy in $\mathbf{L}(\mathbb{R})$*, in Kanamori and Foreman [KF10], pp. 1877–1950.

PETR NOVIKOV

[Nov35] *Sur la séparabilité des ensembles projectifs de seconde class*, ***Fundamenta Mathematicae***, vol. 25 (1935), pp. 459–466.

JOHN C. OXTOBY

[Oxt80] ***Measure and category***, second ed., Graduate Texts in Mathematics, vol. 2, Springer-Verlag, New York, 1980.

JEFF B. PARIS

[Par72] $\mathsf{ZF} \vdash \Sigma^0_4$ *determinateness*, ***The Journal of Symbolic Logic***, vol. 37 (1972), pp. 661–667.

KAREL PRIKRY

[Pri76] *Determinateness and partitions*, ***Proceedings of the American Mathematical Society***, vol. 54 (1976), pp. 303–306.

FRANK RAMSEY

[Ram30] *On a problem of formal logic*, ***Proceedings of the London Mathematical Society***, vol. 30 (1930), no. 2, pp. 2–24.

Gerald E. Sacks

[Sac76] *Countable admissible ordinals and hyperdegrees*, ***Advances in Mathematics***, vol. 20 (1976), no. 2, pp. 213–262.

Ernest Schimmerling

[Sch95] *Combinatorial principles in the core model for one Woodin cardinal*, ***Annals of Pure and Applied Logic***, vol. 74 (1995), no. 2, pp. 153–201.

[Sch07] *Coherent sequences and threads*, ***Advances in Mathematics***, vol. 216 (2007), no. 1, pp. 89–117.

[Sch10] *A core model toolbox and guide*, in Kanamori and Foreman [KF10], pp. 1685–1752.

Ernest Schimmerling and Martin Zeman

[SZ01] *Square in core models*, ***The Bulletin of Symbolic Logic***, vol. 7 (2001), no. 3, pp. 305–314.

Ulrich Schwalbe and Paul Walker

[SW01] *Zermelo and the early history of game theory*, ***Games and Economic Behavior***, vol. 34 (2001), no. 1, pp. 123–137.

Saharon Shelah

[She84] *Can you take Solovay's inaccessible away?*, ***Israel Journal of Mathematics***, vol. 48 (1984), no. 1, pp. 1–47.

[She98] ***Proper and improper forcing***, second ed., Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1998.

Saharon Shelah and W. Hugh Woodin

[SW90] *Large cardinals imply that every reasonably definable set of reals is Lebesgue measurable*, ***Israel Journal of Mathematics***, vol. 70 (1990), no. 3, pp. 381–394.

Joseph R. Shoenfield

[Sho61] *The problem of predicativity*, ***Essays on the foundations of mathematics*** (Y. Bar-Hillel et al., editors), Magnes Press, Jerusalem, 1961, pp. 132–139.

Wacław Sierpiński

[Sie24] *Sur une propriété des ensembles ambigus*, ***Fundamenta Mathematicae***, vol. 6 (1924), pp. 1–5.

[Sie25] *Sur une class d'ensembles*, ***Fundamenta Mathematicae***, vol. 7 (1925), pp. 237–243.

[Sie38] *Fonctions additives non complètement additives et fonctions non mesurables*, ***Fundamenta Mathematicae***, vol. 30 (1938), pp. 96–99.

Jack H. Silver

[Sil71C] *Some applications of model theory in set theory*, ***Annals of Mathematical Logic***, vol. 3 (1971), no. 1, pp. 45–110.

[Sil75] *On the singular cardinals problem*, ***Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1*** (Montreal, Que.), Canadian Mathematical Congress, 1975, pp. 265–268.

Robert I. Soare

[Soa87] ***Recursively enumerable sets and degrees***, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1987.

Robert M. Solovay

[Sol66] *On the cardinality of Σ^1_2 set of reals*, ***Foundations of Mathematics: Symposium papers commemorating the 60th birthday of Kurt Gödel*** (Jack J. Bulloff, Thomas C. Holyoke, and S. W. Hahn, editors), Springer-Verlag, 1966, pp. 58–73.

[Sol67A] *Measurable cardinals and the axiom of determinateness*, lecture notes prepared in connection with the Summer Institute of Axiomatic Set Theory held at UCLA, Summer 1967.

[Sol70] *A model of set-theory in which every set of reals is Lebesgue measurable*, ***Annals of Mathematics***, vol. 92 (1970), pp. 1–56.

[Sol78A] *A Δ^1_3 coding of the subsets of ω_ω*, in Kechris and Moschovakis [Cabal i], pp. 133–150, reprinted in [Cabal II], pp. 346–363.

[Sol78B] *The independence of* DC *from* AD, in Kechris and Moschovakis [Cabal i], pp. 171–184.

John R. Steel

[Ste81B] *Determinateness and the separation property*, ***The Journal of Symbolic Logic***, vol. 46 (1981), no. 1, pp. 41–44.

[Ste82B] *Determinacy in the Mitchell models*, ***Annals of Mathematical Logic***, vol. 22 (1982), no. 2, pp. 109–125.

[Ste83A] *Scales in* $\mathbf{L}(\mathbb{R})$, in Kechris et al. [Cabal iii], pp. 107–156, reprinted in [Cabal I], pp. 130–175.

[Ste88] *Long games*, in Kechris et al. [Cabal iv], pp. 56–97, reprinted in [Cabal I], pp. 223–259.

[Ste95A] $\mathbf{HOD}^{\mathbf{L}(\mathbb{R})}$ *is a core model below* Θ, ***The Bulletin of Symbolic Logic***, vol. 1 (1995), no. 1, pp. 75–84.

[Ste96] ***The core model iterability problem***, Lecture Notes in Logic, no. 8, Springer-Verlag, Berlin, 1996.

[Ste02] *Core models with more Woodin cardinals*, ***The Journal of Symbolic Logic***, vol. 67 (2002), no. 3, pp. 1197–1226.

[Ste05] PFA *implies* $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$, ***The Journal of Symbolic Logic***, vol. 70 (2005), no. 4, pp. 1255–1296.

[Ste08B] *Games and scales. Introduction to Part I*, in Kechris et al. [Cabal I], pp. 3–27.

[Ste08C] *The length-ω_1 open game quantifier propagates scales*, in Kechris et al. [Cabal I], pp. 260–269.

[Ste09] *The derived model theorem*, ***Logic Colloquium 2006***, Lecture Notes in Logic, vol. 19, Association for Symbolic Logic, 2009, pp. 280–327.

John R. Steel and Robert Van Wesep

[SVW82] *Two consequences of determinacy consistent with choice*, ***Transactions of the American Mathematical Society***, vol. 272 (1982), no. 1, pp. 67–85.

John R. Steel and Stuart Zoble

[SZ] *Determinacy from strong reflection*, in preparation.

M. Ya. Suslin

[Sus17] *Sur une définition des ensembles mesurables* B *sans nombres transfinis*, ***Comptes rendus hebdomadaires des séances de l'Académie des Sciences***, vol. 164 (1917), pp. 88–91.

Stevo Todorcevic

[Tod84] *A note on the proper forcing axiom*, in Baumgartner et al. [BMS84], pp. 209–218.

Stanisław Ulam

[Ula60] ***A collection of mathematical problems***, Interscience Tracts in Pure and Applied Mathematics, no. 8, Interscience Publishers, New York–London, 1960.

Robert Van Wesep

[Van78A] *Separation principles and the axiom of determinateness*, ***The Journal of Symbolic Logic***, vol. 43 (1978), no. 1, pp. 77–81.

[Van78B] *Wadge degrees and descriptive set theory*, in Kechris and Moschovakis [Cabal i], pp. 151–170, reprinted in [Cabal II], pp. 24–42.

Boban Veličković

[Vel92] *Forcing axioms and stationary sets*, ***Advances in Mathematics***, vol. 94 (1992), no. 2, pp. 256–284.

Giuseppe Vitali

[Vit05] *Sul problema della misura dei gruppi di punti di una retta*, ***Tipografia Gamberini e Parmeggiani***, (1905), pp. 231–235.

Jonh von Neumann and Oskar Morgenstern

[vNM04] ***Theory of games and economic behavior***, Princeton University Press, 2004, Reprint of the 1980 edition.

Stan Wagon

[Wag93] ***The Banach-Tarski paradox***, Cambridge University Press, 1993, corrected reprint of the 1985 original.

Philip Wolfe

[Wol55] *The strict determinateness of certain infinite games*, ***Pacific Journal of Mathematics***, vol. 5 (1955), pp. 841–847.

W. Hugh Woodin

[Woo82] *On the consistency strength of projective uniformization*, ***Proceedings of the Herbrand symposium (Marseilles, 1981)***, Studies in Logic and the Foundations of Mathematics, vol. 107, North-Holland, Amsterdam, 1982, pp. 365–384.

[Woo83B] *Some consistency results in* ZFC *using* AD, in Kechris et al. [Cabal iii], pp. 172–198.

[Woo86] *Aspects of determinacy*, ***Logic, methodology and philosophy of science, VII (Salzburg, 1983)***, Studies in Logic and the Foundations of Mathematics, vol. 114, North-Holland, Amsterdam, 1986, pp. 171–181.

[Woo88] *Supercompact cardinals, sets of reals, and weakly homogeneous trees*, ***Proceedings of the National Academy of Sciences of the United States of America***, vol. 85 (1988), no. 18, pp. 6587–6591.

[Woo99] ***The axiom of determinacy, forcing axioms, and the nonstationary ideal***, de Gruyter Series in Logic and its Applications, vol. 1, Walter de Gruyter, Berlin, 1999.

Ernst Zermelo

[Zer04] *Beweis, daß jede Menge wohlgeordnet werden kann*, ***Mathematische Annalen***, vol. 59 (1904), pp. 514–516.

[Zer13] *Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels*, ***Proceedings of the Fifth International Congress of Mathematicians***, vol. 2, 1913, pp. 501–504.

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