# Chapter 2

# Review of Forces and Moments

## 2.1 Forces

In this chapter we review the basic concepts of forces, and force laws. Most of this material is identical to material covered in EN030, and is provided here as a review. There are a few additional sections – for example forces exerted by a *damper* or *dashpot,* an *inerter,* and *interatomic* forces are discussed in Section 2.1.7.

### 2.1.1 Definition of a force

Engineering design calculations nearly always use classical (Newtonian) mechanics. In classical mechanics, the concept of a `force' is based on experimental observations that everything in the universe seems to have a preferred configuration – masses appear to attract each other; objects with opposite charges attract one another; magnets can repel or attract one another; you are probably repelled by your professor. But we don't really know *why* this is (except perhaps the last one).

The idea of a *force* is introduced to quantify the tendency of objects to move towards their preferred configuration. If objects accelerate very quickly towards their preferred configuration, then we say that there's a big force acting on them. If they don't move (or move at constant velocity), then there is no force. We can't see a force; we can only deduce its existence by observing its effect.

Specifically, forces are defined through Newton's laws of motion

Isaac Newton on a bad hair day

0. A `particle' is a small mass at some position in space.

1. When the sum of the forces acting on a particle is zero, its velocity is constant;

2. The sum of forces acting on a particle of constant mass is equal to the product of the mass of the particle and its acceleration;

3. The forces exerted by two particles on each other are equal in magnitude and opposite in direction.

The second law provides the definition of a force – if a mass $m$ has acceleration $\mathbf{a}$, the force $\mathbf{F}$ acting on it is

$$\mathbf{F} = m\mathbf{a}$$

Of course, there is a big problem with Newton's laws – what do we take as a fixed point (and orientation) in order to define acceleration? The general theory of relativity addresses this issue rigorously. But for engineering calculations we can usually take the earth to be fixed, and happily apply Newton's laws. In rare cases where the earth's motion is important, we take the stars far from the solar system to be fixed.

### 2.1.2 Causes of force

Forces may arise from a number of different effects, including

(i) Gravity;
(ii) Electromagnetism or electrostatics;
(iii) Pressure exerted by fluid or gas on part of a structure
(v) Wind or fluid induced drag or lift forces;
(vi) Contact forces, which act wherever a structure or component touches anything;
(vii) Friction forces, which also act at contacts.

Some of these forces can be described by universal laws. For example, gravity forces can be calculated using Newton's law of gravitation; electrostatic forces acting between charged particles are governed by Coulomb's law; electromagnetic forces acting between current carrying wires are governed by Ampere's law; buoyancy forces are governed by laws describing hydrostatic forces in fluids. Some of these universal force laws are listed in Section 2.6.

Some forces have to be measured. For example, to determine friction forces acting in a machine, you may need to measure the coefficient of friction for the contacting surfaces. Similarly, to determine aerodynamic lift or drag forces acting on a structure, you would probably need to measure its lift and drag coefficient experimentally. Lift and drag forces are described in Section 2.6. Friction forces are discussed in Section 12.

Contact forces are *pressures* that act on the small area of contact between two objects. Contact forces can either be measured, or they can be calculated by analyzing forces and deformation in the system of interest. Contact forces are very complicated, and are discussed in more detail in Section 8.

### 2.1.3 Units of force and typical magnitudes

In *SI units*, the standard unit of force is the Newton, given the symbol N.

The Newton is a *derived unit*, defined through Newton's second law of motion – a force of 1N causes a 1 kg mass to accelerate at 1 $ms^{-2}$.

The fundamental unit of force in the SI convention is kg m/s$^2$

In *US units*, the standard unit of force is the pound, given the symbol lb or lbf (the latter is an abbreviation for pound force, to distinguish it from pounds weight)

A force of 1 lbf causes a mass of 1 slug to accelerate at 1 ft/s$^2$

US units have a frightfully confusing way of representing mass – often the mass of an object is reported as *weight*, in lb or lbm (the latter is an abbreviation for pound mass). The weight of an object in lb is not mass at all – it's actually the gravitational *force* acting on the mass. Therefore, the mass of an object in slugs must be computed from its weight in pounds using the formula

$$m\,(\text{slugs}) = \frac{W\,(\text{lb})}{g\,(\text{ft/s}^2)}$$

where $g=32.1740$ ft/s$^2$ is the acceleration due to gravity.

A force of 1 lb(f) causes a mass of 1 lb(m) to accelerate at 32.1740 ft/s$^2$

The conversion factors from lb to N are

| | | |
|---|---|---|
| 1 lb | = | 4.448 N |
| 1 N | = | 0.2248 lb |

(www.onlineconversion.com is a handy resource, as long as you can tolerate all the hideous advertisements…)

As a rough guide, a force of 1N is about equal to the weight of a medium sized apple. A few typical force magnitudes (from `The Sizesaurus', by Stephen Strauss, Avon Books, NY, 1997) are listed in the table below

| Force | Newtons | Pounds Force |
|---|---|---|
| Gravitational Pull of the Sun on Earth | $3.5\times10^{22}$ | $7.9\times10^{21}$ |
| Gravitational Pull of the Earth on the Moon | $2\times10^{20}$ | $4.5\times10^{19}$ |
| Thrust of a Saturn V rocket engine | $3.3\times10^{7}$ | $7.4\times10^{6}$ |
| Thrust of a large jet engine | $7.7\times10^{5}$ | $1.7\times10^{5}$ |
| Pull of a large locomotive | $5\times10^{5}$ | $1.1\times10^{5}$ |
| Force between two protons in a nucleus | $10^{4}$ | $10^{3}$ |
| Gravitational pull of the earth on a person | $7.3\times10^{2}$ | $1.6\times10^{2}$ |
| Maximum force exerted upwards by a forearm | $2.7\times10^{2}$ | 60 |
| Gravitational pull of the earth on a 5 cent coin | $5.1\times10^{-2}$ | $1.1\times10^{-2}$ |
| Force between an electron and the nucleus of a Hydrogen atom | $8\times10^{-6}$ | $1.8\times10^{-8}$ |

### 2.1.4 Classification of forces: External forces, constraint forces and internal forces.

When analyzing forces in a structure or machine, it is conventional to classify forces as *external forces; constraint forces* or *internal forces*.

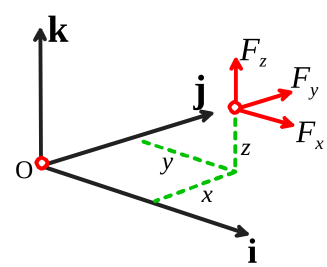
*External forces* arise from interaction between the system of interest and its surroundings.

Examples of external forces include gravitational forces; lift or drag forces arising from wind loading; electrostatic and electromagnetic forces; and buoyancy forces; among others. Force laws governing these effects are listed later in this section.

*Constraint forces* are exerted by one part of a structure on another, through joints, connections or contacts between components. Constraint forces are very complex, and will be discussed in detail in Section 8.

*Internal forces* are forces that act inside a solid part of a structure or component. For example, a stretched rope has a *tension force* acting inside it, holding the rope together. Most solid objects contain very complex distributions of internal force. These internal forces ultimately lead to structural failure, and also cause the structure to deform. The purpose of calculating forces in a structure or component is usually to deduce the internal forces, so as to be able to design stiff, lightweight and strong components. We will not, unfortunately, be able to develop a full theory of internal forces in this course – a proper discussion requires understanding of partial differential equations, as well as vector and tensor calculus. However, a brief discussion of internal forces in *slender members* will be provided in Section 9.

### 2.1.5 Mathematical representation of a force.




Force is a *vector* – it has a magnitude (specified in Newtons, or lbf, or whatever), and a *direction*.

A force is therefore always expressed mathematically as a vector quantity. To do so, we follow the usual rules, which are described in more detail in the vector tutorial. The procedure is

1. Choose basis vectors $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ or $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_2\}$ that establish three fixed (and usually perpendicular) directions in space;
2. Using geometry or trigonometry, calculate the force component along each of the three reference directions $(F_x,F_y,F_z)$ or $(F_1,F_2,F_3)$;
3. The vector force is then reported as

$$\mathbf{F}=F_x\mathbf{i}+F_y\mathbf{j}+F_z\mathbf{k}=F_1\mathbf{e}_1+F_2\mathbf{e}_2+F_3\mathbf{e}_3 \quad \text{(appropriate units)}$$

For calculations, you will also need to specify the point where the force acts on your system or structure. To do this, you need to report the *position vector of the point where the force acts on the structure*.

The procedure for representing a position vector is also described in detail in the vector tutorial. To do so, you need to:

1. Choose an origin
2. Choose basis vectors $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ or $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_2\}$ that establish three fixed directions in space (usually we use the same basis for both force and position vectors)

3. Specify the distance you need to travel along each direction to get from the origin to the point of application of the force $(r_x, r_y, r_z)$ or $(r_1, r_2, r_3)$
4. The position vector is then reported as

$$\mathbf{r} = r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k} = r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + r_3\mathbf{e}_3 \quad \text{(appropriate units)}$$

### 2.1.6 Measuring forces

Engineers often need to measure forces. According to the definition, if we want to measure a force, we need to get hold of a 1 kg mass, have the force act on it somehow, and then measure the acceleration of the mass. The magnitude of the acceleration tells us the magnitude of the force; the direction of motion of the mass tells us the direction of the force. Fortunately, there are easier ways to measure forces.

In addition to causing acceleration, forces cause objects to *deform* – for example, a force will stretch or compress a spring; or bend a beam. The deformation can be measured, and the force can be deduced.

The simplest application of this phenomenon is a spring scale. The change in length of a spring is proportional to the magnitude of the force causing it to stretch (so long as the force is not too large!)– this relationship is known as Hooke's law and can be expressed as an equation

$$k\delta = F$$

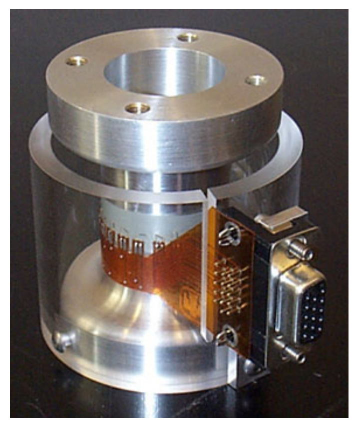
where the spring stiffness $k$ depends on the material the spring is made from, and the shape of the spring. The spring stiffness can be measured experimentally to calibrate the spring.

Spring scales are not exactly precision instruments, of course. But the same principle is used in more sophisticated instruments too. Forces can be measured precisely using a `force transducer' or `load cell' (A search for `force transducer' on any search engine will bring up a huge variety of these – a few are shown in the picture). The simplest load cell works much like a spring scale – you can load it in one direction, and it will provide an electrical signal proportional to the magnitude of the force. Sophisticated load cells can measure a force *vector*, and will record all three force components. Really fancy load cells measure both force vectors, and torque or moment vectors.

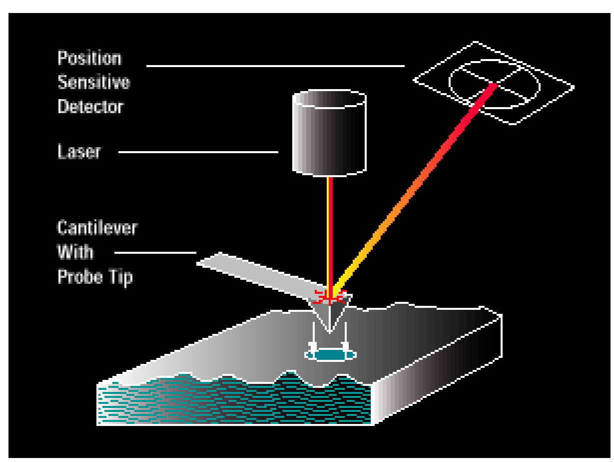

Simple force transducers capable of measuring a single force component. The instrument on the right is called a `proving ring' – there's a short article describing how it works at http://www.mel.nist.gov/div822/proving_ring.htm


A sophisticated force transducer produced by MTS systems, which is capable of measuring forces and moments acting on a car's wheel *in-situ.* The spec for this device can be downloaded at www.mts.com/downloads/SWIF2002_100-023-513.pdf.pdf

The basic design of all these load cells is the same – they measure (very precisely) the deformation in a part of the cell that acts like a very stiff spring. One example (from http://www.sandia.gov/isrc/Load_Cell/load_cell.html ) is shown on the right. In this case the `spring' is actually a tubular piece of high-strength steel. When a force acts on the cylinder, its length decreases slightly. The deformation is detected using `strain gages' attached to the cylinder. A strain gage is really just a thin piece of wire, which deforms with the cylinder. When the wire gets shorter, its electrical resistance decreases – this resistance change can be measured, and can be used to work out the force. It is possible to derive a formula relating the force to the change in resistance, the load cell geometry, and the material properties of steel, but the calculations involved are well beyond the scope of this course.



The most sensitive load cell currently available is the atomic force microscope (AFM) – which as the name suggests, is intended to measure forces between small numbers of atoms. This device consists of a very thin (about 1 $\mu m$) cantilever beam, clamped at one end, with a sharp tip mounted at the other. When the tip is brought near a sample, atomic interactions exert a force on the tip and cause the cantilever to bend. The bending is detected by a laser-mirror system. The device is capable of measuring forces of about 1 pN (that's $10^{-12}$ N!!), and is used to explore the properties of surfaces, and biological materials such as DNA strands and cell membranes. A nice article on the AFM can be found at http://www.di.com



*Selecting a load cell*

As an engineer, you may need to purchase a load cell to measure a force. Here are a few considerations that will guide your purchase.

1. How many force (and maybe moment) components do you need to measure? Instruments that measure several force components are more expensive…
2. Load capacity – what is the maximum force you need to measure?
3. Load range – what is the minimum force you need to measure?
4. Accuracy
5. Temperature stability – how much will the reading on the cell change if the temperature changes?
6. Creep stability – if a load is applied to the cell for a long time, does the reading drift?
7. Frequency response – how rapidly will the cell respond to time varying loads? What is the maximum frequency of loading that can be measured?
8. Reliability
9. Cost

### 2.1.7 Force Laws

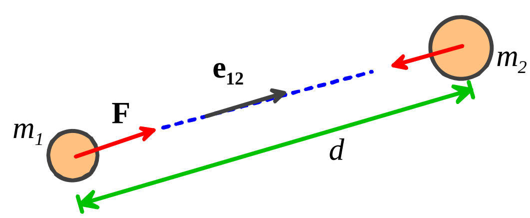
In this section, we list equations that can be used to calculate forces associated with

(i) Gravity
(ii) Forces exerted by linear springs
(iii) Electrostatic forces
(iv) Electromagnetic forces
(v) Hydrostatic forces and buoyancy
(vi) Aero- and hydro-dynamic lift and drag forces

### Gravitation

*Gravity forces acting on masses that are a large distance apart*

Consider two masses $m_1$ and $m_2$ that are a distance $d$ apart. Newton's law of gravitation states that mass $m_1$ will experience a force




$$\mathbf{F} = \frac{G m_1 m_2}{d^2} \mathbf{e}_{12}$$

where $\mathbf{e}_{12}$ is a unit vector pointing from mass $m_1$ to mass $m_2$, and $G$ is the Gravitation constant. Mass $m_2$ will experience a force of equal magnitude, acting in the opposite direction.

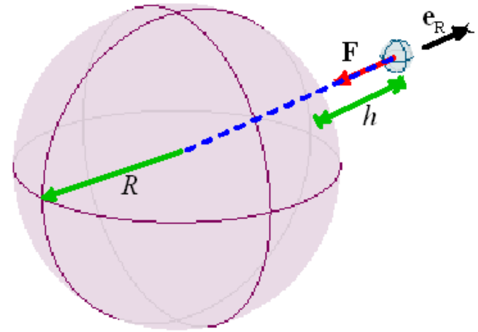
In SI units, $G = 6.673 \times 10^{-11} \mathrm{m}^3 \mathrm{kg}^{-1} \mathrm{s}^{-2}$

The law is strictly only valid if the masses are very small (*infinitely* small, in fact) compared with $d$ – so the formula works best for calculating the force exerted by one planet or another; or the force exerted by the earth on a satellite.

*Gravity forces acting on a small object close to the earth's surface*

For engineering purposes, we can usually assume that

1. The earth is spherical, with a radius $R$
2. The object of interest is small compared with $R$
3. The object's height $h$ above the earths surface is small compared to $R$




If the first two assumptions are valid, then one can show that Newton's law of gravitation implies that a mass $m$ at a height $h$ above the earth's surface experiences a force

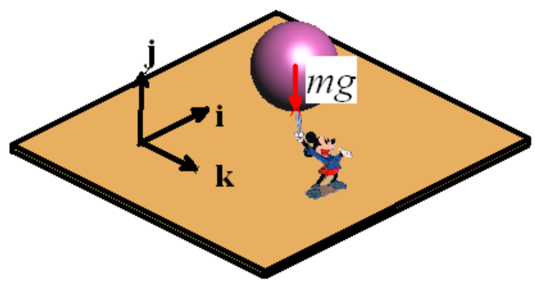
$$\mathbf{F} = -\frac{GMm}{(R+h)^2}\mathbf{e}_r$$

where $M$ is the mass of the earth; $m$ is the mass of the object; $R$ is the earth's radius, $G$ is the gravitation constant and $\mathbf{e}_r$ is a unit vector pointing from the center of the earth to the mass $m$. (Why do we have to show this? Well, the mass $m$ actually experiences a force of attraction towards every point inside the earth. One might guess that points close to the earth's surface under the mass would attract the mass more than those far away, so the earth would exert a larger gravitational force than a very small object with the same mass located at the earth's center. But this turns out not to be the case, as long as the earth is perfectly uniform and spherical).

If the third assumption ($h$<<$R$) is valid, then we can simplify the force law by setting

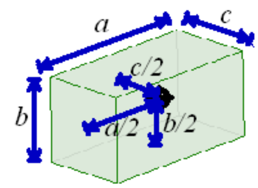
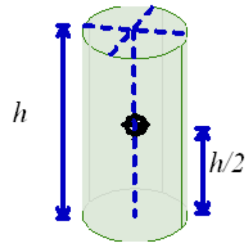
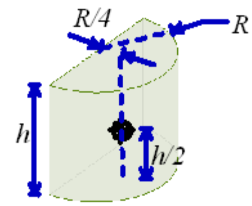
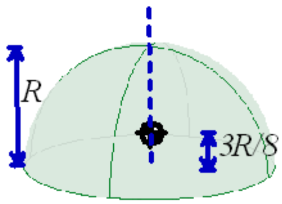
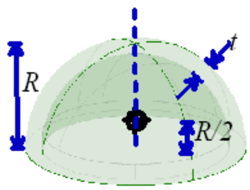
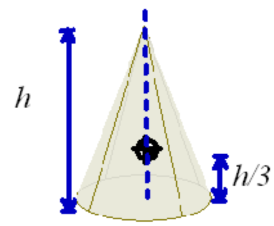
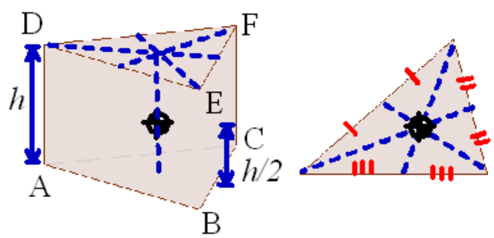
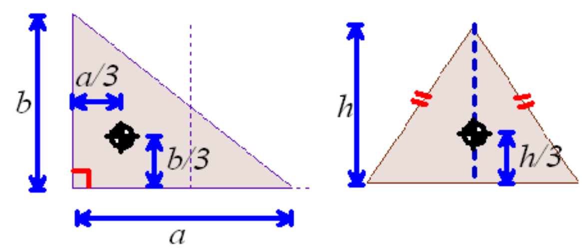
$$\frac{GM}{(R+h)^2} \approx g \Rightarrow \mathbf{F} = -mg\mathbf{j}$$

where g is a constant, and $\mathbf{j}$ is a `vertical' unit vector (i.e. perpendicular to the earth's surface).




In SI units $g = 9.81 ms^{-2}$.

The force of gravity acts at the *center of gravity* of an object. For most engineering calculations the center of gravity of an object can be assumed to be the same as its center of mass. For example, gravity would exert a force at the center of the sphere that Mickey is holding. The location of the center of mass for several other common shapes is shown below. The procedure for calculating center of mass of a complex shaped object is discussed in more detail in section 6.3.

| TABLE OF POSITIONS OF CENTER OF MASS FOR COMMON OBJECTS | | |
|---|---|---|
|   |   |   |
| Rectangular prism | Circular cylinder | Half-cylinder |
| | | |
|   |   |   |
| Solid hemisphere | Thin hemispherical shell | Cone |
| | | |
|   $\mathbf{r}_{CG} = (\mathbf{r}_A + \mathbf{r}_B + \mathbf{r}_C + \mathbf{r}_D + \mathbf{r}_E + \mathbf{r}_F)/6$ |   | |
| Triangular Prism | Thin triangular laminate | |

*Some subtleties about gravitational interactions*

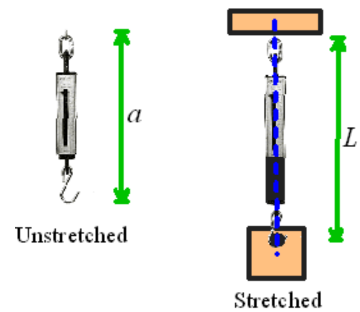
There are some situations where the simple equations in the preceding section don't work. Surveyors know perfectly well that the earth is no-where near spherical; its density is also not uniform. The earth's gravitational field can be quite severely distorted near large mountains, for example. So using the simple gravitational formulas in surveying applications (e.g. to find the `vertical' direction) can lead to large errors.

Also, the center of gravity of an object is *not* the same as its center of mass. Gravity is actually a distributed force. When two nearby objects exert a gravitational force on each other, every point in one body is attracted towards every point inside its neighbor. The distributed force can be replaced by a single, statically equivalent force, but the point where the equivalent force acts depends on the relative positions of the two objects, and is not generally a fixed point in either solid. One consequence of this behavior is that gravity can cause rotational accelerations, as well as linear accelerations. For example, the resultant force of gravity exerted on the earth by the sun and moon does not act at the center of mass of the earth. As a result, the earth *precesses* – that is to say, its axis of rotation changes with time.

**Forces exerted by springs**

A solid object (e.g. a rubber band) can be made to exert forces by stretching it. The forces exerted by a solid that is subjected to a given deformation depend on the shape of the component, the materials it is made from, and how it is connected to its surroundings. Solid objects can also exert *moments*, or *torques* – we will define these shortly. Forces exerted by solid components in a machine or structure are complicated, and will be discussed in detail separately. Here, we restrict attention to the simplest case: forces exerted by linear springs.

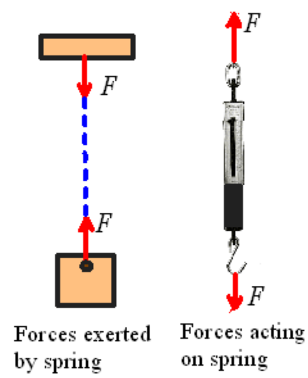
A spring scale is a good example of a linear spring. You can attach it to something at both ends. If you stretch or compress the spring, it will exert forces on whatever you connected to.




The forces exerted by the ends of the spring always act along the line of the spring. The magnitude of the force is (so long as you don't stretch the spring too much) given by the formula

$$F = k(L-a)$$

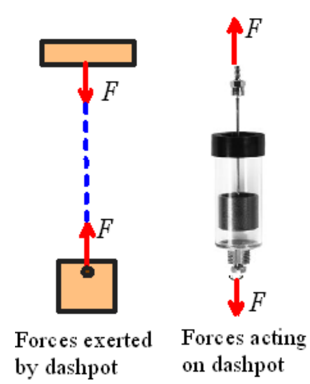
where $a$ is the un-stretched spring length; $L$ is the stretched length, and $k$ is the spring stiffness.




In the SI system, $k$ has units of N/m.

Note that when you draw a picture showing the forces exerted by a spring, you must always assume that the spring is stretched, so that the forces exerted by the spring are *attractive*. If you don't do this, your sign convention will be inconsistent with the formula $F = k(L-a)$, which assumes that a compressed spring ($L<a$) exerts a *negative* force.

**Forces exerted by dashpots**




A 'Dashpot' is somewhat like a spring except that it exerts forces that are proportional to the ***relative velocity*** of its two ends, instead of the relative displacement. The device is extremely useful for damping vibrations. The device usually consists of a plunger that forces air or fluid through a small orifice – the force required to expel the fluid is roughly proportional to the velocity of the plunger. For an example of a precision dashpot see http://www.airpot.com/beta/html/dashpot_defined.html

The forces exerted by the ends of the dashpot always act along the line of the dashpot. The magnitude of the force exerted by a fluid filled dashpot is given by the formula

$$F = \eta \frac{dL}{dt}$$

where $L$ is the length, and $\eta$ is the rate constant of the dashpot. Air filled dashpots are somewhat more complicated, because the compressibility of the air makes them behave like a combination of a dashpot and spring connected end-to-end.

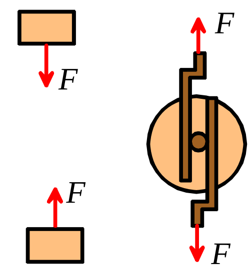
In the SI system, $\eta$ has units of Ns/m.

Note that when you draw a picture showing the forces exerted by a dashpot, you must always assume that the length of the dashpot is increasing, so that the forces exerted by the ends of the dashpot are *attractive*.

### Forces exerted by an 'Inerter'

The 'Inerter' is a device that exerts forces proportional to the relative *acceleration* of its two ends. It was invented in 1997 and used in secret by the McLaren Formula 1 racing team to improve the performance of their cars, but in 2008 was made broadly available (http://www.admin.cam.ac.uk/news/dp/2008081906)

The device is so simple that it is difficult to believe that it has taken over 100 years of vehicle design to think of it – but the secret is really in how to use the device to design suspensions than in the device itself. The device works by spinning a flywheel between two moving rods, as sketched in the figure.



Forces exerted by inerter

Forces acting on inerter

The forces exerted by the ends of the inerter always act along the line of the inerter. The magnitude of the force exerted by an inerter is given by the formula

$$F = \mu \frac{d^2 L}{dt^2}$$

where $L$ is the length, and $\mu$ is the inertia constant of the dashpot.

In the SI system, $\mu$ has units of Ns$^2$/m.

### Electrostatic forces

As an engineer, you will need to be able to design structures and machines that manage forces. Controlling gravity is, alas, beyond the capabilities of today's engineers. It's also difficult (but not impossible) to design a spring with a variable stiffness or unstretched length. But there are forces that you *can* easily control. Electrostatic and electromagnetic forces are among the most important ones.

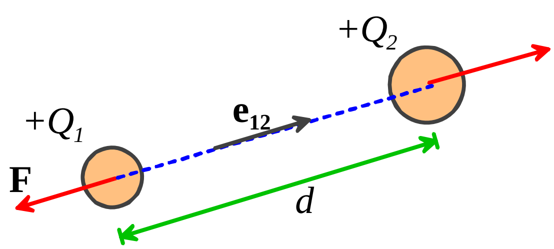
*Electrostatic* forces are exerted on, and by, charged objects. The concepts of electrical potential, current and charge are based on experiments. A detailed discussion of these topics is beyond the scope of this course (it will be covered in detail in EN51), but electromagnetic and electrostatic forces are so important in the design of engines and machines that the main rules governing forces in these systems will be summarized here.

*Electrostatic forces acting on two small charged objects that are a large distance apart*

*Coulomb's Law* states that if like charges $Q_1$ and $Q_2$ are induced on two particles that are a distance $d$ apart, then particle 1 will experience a force

$$\mathbf{F} = -\frac{Q_1 Q_2}{4\pi\varepsilon d^2}\mathbf{e}_{12}$$

(acting away from particle 2), where $\varepsilon$ is a fundamental physical constant known as the *Permittivity* of the medium surrounding the particles (like the Gravitational constant, its value must be determined by experiment).




In SI units, $Q_1, Q_2$ are specified in Coulombs, $d$ is in meters, and $\varepsilon$ is the permittivity of free space, with fundamental units $\text{Amperes}^2\text{kg}^{-1}\text{m}^{-3}$. Permittivity is more usually specified using derived units, in *Farads per meter*. The Farad is the unit of capacitance.

The value of $\varepsilon$ for air is very close to that of a vacuum. The permittivity of a vacuum is denoted by $\varepsilon_0$. In SI units its value is approximately $8.854185 \times 10^{-12}\ \text{Fm}^{-1}$

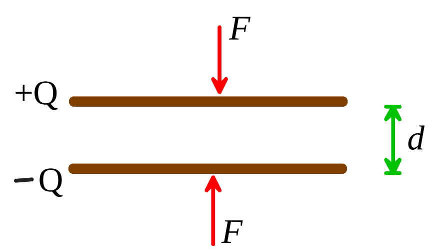
Like gravitational forces, the electrostatic forces acting on 3D objects with a general distribution of charge must be determined using complicated calculations. It's worth giving results for two cases that arise frequently in engineering designs:

*Forces acting between charged flat parallel plates*

Two parallel plates, which have equal and opposite charges $Q$ and are separated by a distance $d$, experience an attractive force with magnitude

$$F = Q^2/(2\varepsilon)$$

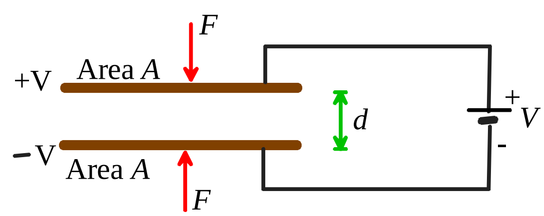
The force can be thought of as acting at the center of gravity of the plates.




Two parallel plates, which have area $A$, are separated by a distance $d$, and are connected to a power-supply that imposes an electrical potential difference $V$ across the plates, experience an attractive force with magnitude

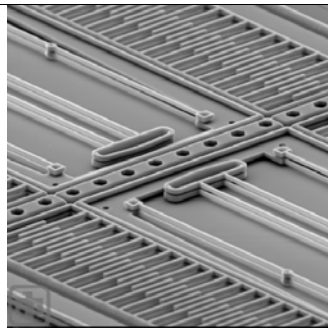
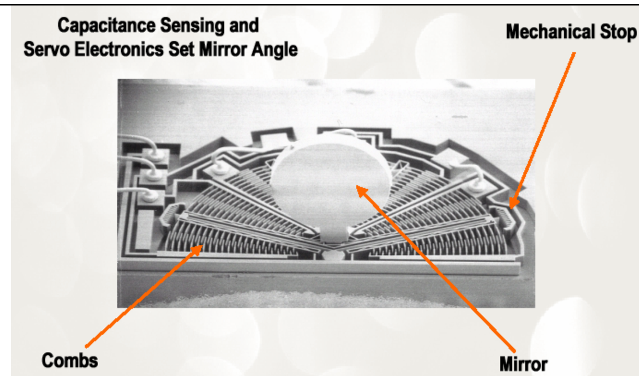
$$F = AV^2\varepsilon/(2d^2)$$

The force can be thought of as acting at the center of gravity of the plates.



*Applications of electrostatic forces*:

Electrostatic forces are small, and don't have many applications in conventional mechanical systems. However, they are often used to construct tiny motors for *micro-electro-mechanical systems* (MEMS). The basic idea is to construct a parallel-plate capacitor, and then to apply force to the machine by connecting the plates to a power-supply. The pictures below show examples of *comb drive* motors.

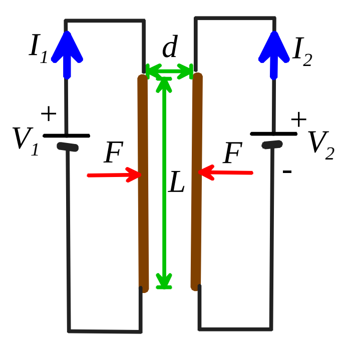
| | |
|---|---|
|  | <br> |
| An experimental comb drive MEMS actuator developed at Sandia National Labs, http://mems.sandia.gov/scripts/index.asp | A rotary comb drive actuator developed at iolon inc. Its purpose is to rotate the mirror at the center, which acts as an optical switch. |

The configurations used in practice are basically large numbers of parallel plate capacitors. A detailed discussion of forces in these systems will be deferred to future courses.

Electrostatic forces are also exploited in the design of oscilloscopes, television monitors, and electron microscopes. These systems generate charged particles (electrons), for example by heating a tungsten wire. The electrons are emitted into a strong electrostatic field, and so are subjected to a large force. The force then causes the particles to accelerate – but we can't talk about accelerations in this course so you'll have to take EN4 to find out what happens next…

## Electromagnetic forces

Electromagnetic forces are exploited more widely than electrostatic forces, in the design of electric motors, generators, and electromagnets.




*Ampere's Law* states that two long parallel wires which have length *L*, carry electric currents $I_1$ and $I_2$, and are a small distance *d* apart, will experience an attractive force with magnitude

$$F = \mu_0 I_1 I_2 L / (2\pi d)$$

where $\mu_0$ is a constant known as the *permeability* of free space.

In SI units, $\mu_0$ has fundamental units of $\mathrm{kg\,m}^{2}\mathrm{sec}^{-1}\,\mathrm{A}^{-2}$, but is usually specified in derived units of Henry per meter. The Henry is the unit of *inductance.*

The value of $\mu_0$ is *exactly* $4\pi \times 10^{-7}$ H/m

Electromagnetic forces between more generally shaped current carrying wires and magnets are governed by a complex set of equations. A full discussion of these physical laws is beyond the scope of this course, and will be covered in EN51.

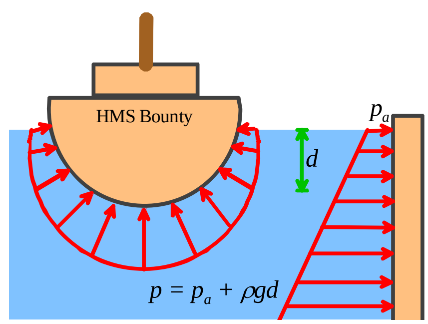
*Applications of electromagnetic forces*

Electromagnetic forces are widely exploited in the design of electric motors; force actuators; solenoids; and electromagnets. All these applications are based upon the principle that a current-carrying wire in a magnetic field is subject to a force. The magnetic field can either be induced by a permanent magnet (as in a DC motor); or can be induced by passing a current through a second wire (used in some DC motors, and all AC motors). The general trends of forces in electric motors follow Ampere's law: the force exerted by the motor increases linearly with electric current in the armature; increases roughly in proportion to the length of wire used to wind the armature, and depends on the geometry of the motor.

Two examples of DC motors – the picture on the right is cut open to show the windings. You can find more information on motors at http://my.execpc.com/~rhoadley/magmotor.htm

**Hydrostatic and buoyancy forces**

When an object is immersed in a stationary fluid, its surface is subjected to a *pressure*. The pressure is actually induced in the fluid by gravity: the pressure at any depth is effectively supporting the weight of fluid above that depth.




A pressure is a *distributed force*. If a pressure $p$ acts on a surface, a small piece of the surface with area $dA$ is subjected to a force

$$d\mathbf{F} = -p \; dA \; \mathbf{n}$$

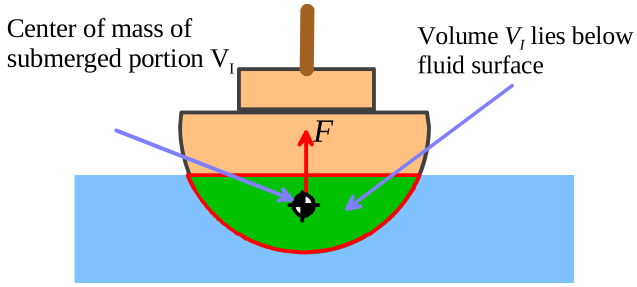
where **n** is a unit vector perpendicular to the surface. The total force on a surface must be calculated by integration. We will show how this is done shortly.

The pressure in a stationary fluid varies linearly with depth below the fluid surface

$$p = p_a + \rho g d$$

where $p_a$ is atmospheric pressure (often neglected as it's generally small compared with the second term); $\rho$ is the fluid density; $g$ is the acceleration due to gravity; and $d$ is depth below the fluid surface.

Archimedes' principle gives a simple way to calculate the resultant force exerted by fluid pressure on an immersed object.




The magnitude of the resultant force is equal to the weight of water displaced by the object. The direction is perpendicular to the fluid surface. Thus, if the fluid has mass density $\rho$, and a volume $V_I$ of the object lies below the surface of the fluid, the resultant force due to fluid pressure is

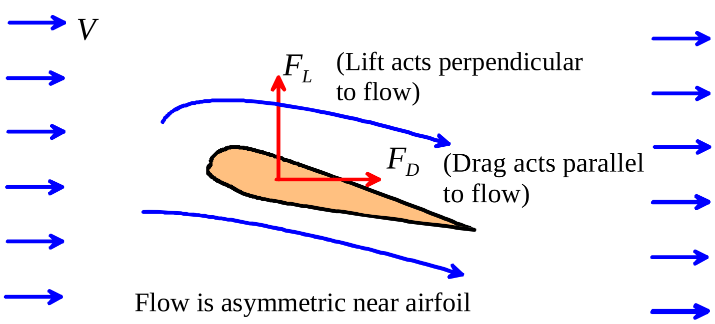
$$\mathbf{F} = \rho g V_I \mathbf{j}$$

The force acts at the *center of buoyancy* of the immersed object. The center of buoyancy can be calculated by finding the center of mass of the displaced fluid (i.e. the center of mass of the portion of the immersed object that lies below the fluid surface).

The buoyancy force acts in addition to gravity loading. If the object floats, the gravitational force is equal and opposite to the buoyancy force. The force of gravity acts (as usual) at the center of mass of the *entire* object.

### Aerodynamic lift and drag forces

Engineers who design large bridges, buildings, or fast-moving terrestrial vehicles, spend much time and effort in managing aero- or hydro-dynamic forces. Hydrodynamic forces are also of great interest to engineers who design bearings and car tires, since hydrodynamic forces can cause one surface to float above another, so reducing friction to very low levels.




In general, when air or fluid flow past an object (or equivalently, if the object moves through stationary fluid or gas), the object is subjected to two forces:

1. A *Drag force*, which acts parallel to the direction of air or fluid flow
2. A *Lift force,* which acts perpendicular to the direction of air or fluid flow.

The forces act at a point known as the center of lift of the object – but there's no simple way to predict where this point is.

The lift force is present only if airflow past the object is unsymmetrical (i.e. faster above or below the object). This asymmetry can result from the shape of the object itself (this effect is exploited in the design of airplane wings); or because the object is spinning (this effect is exploited by people who throw, kick, or hit balls for a living).

Two effects contribute to drag:

(1) Friction between the object's surface and the fluid or air. The friction force depends on the object's shape and size; on the speed of the flow; and on the *viscosity* of the fluid, which is a measure of the shear resistance of the fluid. Air has a low viscosity; ketchup has a high viscosity. Viscosity is often given the symbol $\eta$, and has the rather strange units in the SI system of $Nsm^{-2}$. In `American' units viscosity has units of `Poise' (or sometimes centipoises – that's $10^{-2}$ Poise). The conversion factor is $1P = 0.1Nsm^{-2}$. (Just to be confusing, there's another measure of viscosity, called *kinematic viscosity*, or *specific viscosity*, which is $\varsigma = \eta / \rho$, where $\rho$ is the mass density of the material. In this course we'll avoid using kinematic viscosity, but you should be aware that it exists!) Typical numbers are: Air: $\eta \approx 1.73 \times 10^{-5} Nsm^{-2}$ for a standard atmosphere (see http://users.wpi.edu/~ierardi/PDF/air_nu_plot.PDF for a more accurate number) ; Water, $\eta \approx 0.001Nsm^{-2}$; SAE40 motor oil $\eta \approx 0.5Nsm^{-2}$, ketchup $\eta \approx 60Nsm^{-2}$ (It's hard to give a value for the viscosity of ketchup, because it's thixotropic. See if you can find out what this cool word means – it's a handy thing to bring up if you work in a fast food restaurant.)

(2) Pressure acting on the objects surface. The pressure arises because the air accelerates as it flows around the object. The pressure acting on the front of the object is usually bigger than the pressure behind it, so there's a resultant drag force. The pressure drag force depends on the objects shape and size, the speed of the flow, and the fluid's mass density $\rho$.

Lift forces defy a simple explanation, despite the efforts of various authors to provide one. If you want to watch a fight, ask two airplane pilots to discuss the origin of lift in your presence. (Of course, you may not actually know two airplane pilots. If this is the case, and you still want to watch a fight, you could try http://www.wwe.com/, or go to a British soccer match). Lift is caused by a difference in pressure acting at the top and bottom of the object, but there's no simple way to explain the origin of this pressure difference. A correct explanation of the origin of lift forces can be found at http://www.grc.nasa.gov/WWW/K-12/airplane/right2.html (this site has some neat Java applets that calculate pressure and flow past airfoils). Unfortunately there are thousands more books and websites with incorrect explanations of lift, but you can find those for yourself (check out the explanation from the FAA!)

Lift and drag forces are usually quantified by defining a *coefficient of lift* $C_L$ and a *coefficient of drag* $C_D$ for the object, and then using the formulas

$$F_L = \left(\frac{1}{2}\rho V^2\right) C_L A_L \qquad\qquad F_D = \left(\frac{1}{2}\rho V^2\right) C_D A_D$$

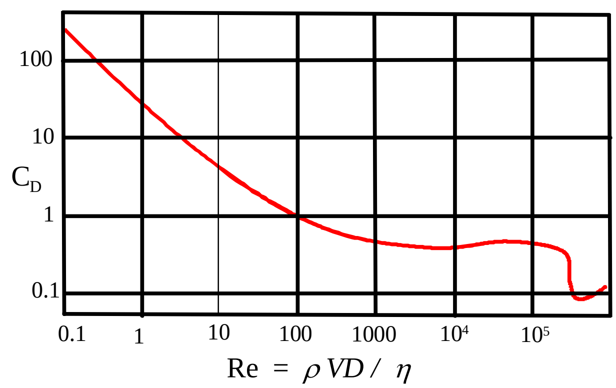
Here, $\rho$ is the air or fluid density, $V$ is the speed of the fluid, and $A_L$ and $A_D$ are measures of the area of the object. Various measures of area are used in practice – when you look up values for drag coefficients you have to check what's been used. The object's total surface area could be used. Vehicle manufacturers usually use the *projected frontal area* (equal to car height x car width for practical purposes) when reporting drag coefficient. $C_L$ and $C_D$ are dimensionless, so they have no units.

The drag and lift coefficients are not constant, but depend on a number of factors, including:

1. The shape of the object
2. The object's orientation relative to the flow (aerodynamicists refer to this as the `angle of attack')
3. The fluid's viscosity $\eta$, mass density $\rho$, flow speed $V$ and the object's size. Size can be quantified by $\sqrt{A_L}$ or $\sqrt{A_D}$; other numbers are often used too. For example, to quantify the drag force acting on a sphere we use its diameter *D*. Dimensional analysis shows that $C_D$ and $C_L$ can only depend on these factors through a dimensionless constant known as `Reynold's number', defined as

$$\mathrm{Re} = \frac{\rho V \sqrt{A}}{\eta}$$

For example, the graph on the right shows the variation of drag coefficient with Reynolds number for a smooth sphere, with diameter *D*. The projected area $A_D = \pi D^2/4$ was used to define the drag coefficient



The variation of drag coefficient with Reynolds number Re for a smooth sphere.

Many engineering structures and vehicles operate with Reynolds numbers in the range $10^3 - 10^6$, where drag coefficients are fairly constant (of order 0.01 - 0.5 or so). Lift coefficients for most airfoils are of order 1 or 2, but can be raised as high as 10 by special techniques such as blowing air over the wing)

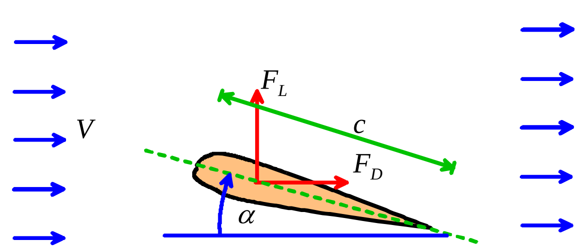
Lift and drag coefficients can be calculated approximately (you can buy software to do this for you, e.g. at http://www.hanleyinnovations.com/walite.html . Another useful resource is www.desktopaero.com/appliedaero ). They usually have to be measured to get really accurate numbers.

Tables of approximate values for lift and drag coefficients can be found at http://aerodyn.org/Resources/database.html

Lift and drag forces are of great interest to aircraft designers. Lift and drag forces on an airfoil are computed using the usual formula

$$F_L = \left(\frac{1}{2}\rho V^2\right) C_L A_W$$

$$F_D = \left(\frac{1}{2}\rho V^2\right) C_D A_W$$



The wing area $A_W = cL$ where *c* is the chord of the wing (see the picture) and *L* is its length, is used in defining both the lift and drag coefficient.

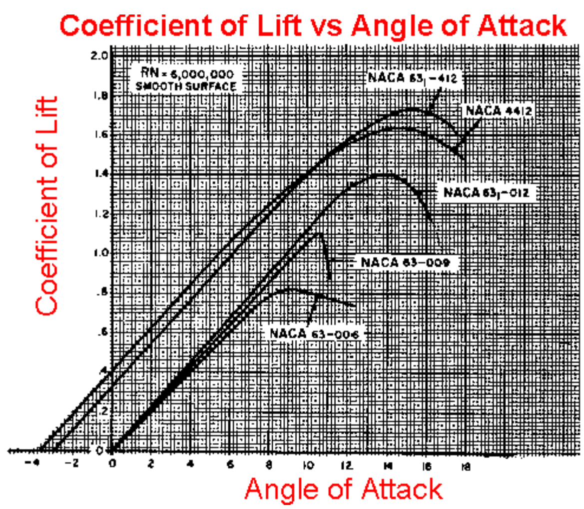
The variation of $C_L$ and $C_D$ with *angle of attack* $\alpha$ are crucial in the design of aircraft. For reasonable values of $\alpha$ (below stall - say less than 10 degrees) the behavior can be approximated by

$$C_L = k_L \alpha$$

$$C_D = k_{Dp} + k_{DI} \alpha^2$$

where $k_L$, $k_{Dp}$ and $k_{DI}$ are more or less constant for any given airfoil shape, for practical ranges of Reynolds number. The first term in the drag coefficient, $k_{Dp}$, represents *parasite drag* – due to viscous drag and some pressure drag. The second term $k_{DI}\alpha^2$ is called *induced drag*, and is an undesirable by-product of lift.

The graphs on the right, (taken from `Aerodynamics for Naval Aviators, H.H. Hurt, U.S. Naval Air Systems Command reprint') shows some experimental data for lift coefficient $C_L$ as a function of AOA (that's angle of attack, but you're engineers now so you have to talk in code to maximize your nerd factor. That's NF). The data suggest that $k_L \approx 0.1\deg^{-1}$, and in fact a simple model known as `thin airfoil theory' predicts that lift coefficient should vary by $2\pi$ per radian (that works out as 0.1096/degree)




The induced drag coefficient $k_{DI}$ can be estimated from the formula

$$k_{DI} = \frac{c^2 k_L^2}{\pi e A_W}$$

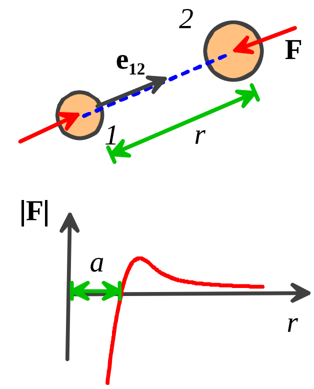
where $k_L \approx 0.1\deg^{-1}$ , $L$ is the length of the wing and *c* is its width; while *e* is a constant known as the `Oswald efficiency factor.' The constant *e* is always less than 1 and is of order 0.9 for a high performance wing (eg a jet aircraft or glider) and of order 0.7 for el cheapo wings.

The parasite drag coefficient $k_{Dp}$ is of order 0.05 for the wing of a small general aviation aircraft, and of order 0.005 or lower for a commercial airliner.

## Interatomic forces

Engineers working in the fields of nanotechnology, materials design, and bio/chemical engineering are often interested in calculating the motion of molecules or atoms in a system.

They do this using `Molecular Dynamics,' which is a computer method for integrating the equations of motion for every atom in the solid. The equations of motion are just Newton's law – **F**=m**a** for each atom – but for the method to work, it is necessary to calculate the forces acting on the atoms. Specifying these forces is usually the most difficult part of the calculation.




The forces are computed using empirical force laws, which are either determined experimentally, or (more often) by means of quantum-mechanical calculations. In the simplest models, the atoms are assumed to interact through ***pair forces***. In this case

- The forces exerted by two interacting atoms depends only on their relative positions, and is independent on the position of other atoms in the solid
- The forces act along the line connecting the atoms.
- The magnitude of the force is a function of the distance between them. The function is chosen so that (i) the force is repulsive when the atoms are close together; (ii) the force is zero at the equilibrium interatomic spacing; (iii) there is some critical distance where the attractive force has its maximum value (see the figure) and (iv) the force drops to zero when the atoms are far apart.

Various functions are used to specify the detailed shape of the force-separation law. A common one is the so-called 'Lennard Jones' function, which gives the force acting on atom (1) as

$$\mathbf{F}^{(1)} = -12E\left[\left(\frac{a}{r}\right)^{13} - \left(\frac{a}{r}\right)^{7}\right]\mathbf{e}_{12}$$

Here $a$ is the equilibrium separation between the atoms, and $E$ is the total bond energy – the amount of work required to separate the bond by stretching it from initial length $a$ to infinity.

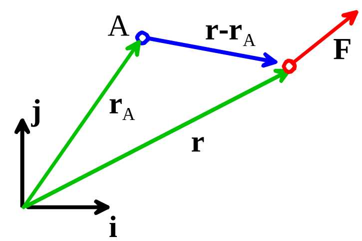
This function was originally intended to model the atoms in a Noble gas – like He or Ar, etc. It is sometimes used in simple models of liquids and glasses. It would not be a good model of a metal, or covalently bonded solids. In fact, for these materials pair potentials don't work well, because the force exerted between two atoms depends not just on the relative positions of the two atoms themselves, but also on the positions of other nearby atoms. More complicated functions exist that can account for this kind of behavior, but there is still a great deal of uncertainty in the choice of function for a particular material.

## 2.2 Moments

The *moment* of a force is a measure of its tendency to rotate an object about some point. The physical significance of a moment will be discussed later. We begin by stating the mathematical definition of the moment of a force about a point.

### 2.2.1 Definition of the moment of a force.

To calculate the moment of a force about some point, we need to know three things:




1. The force vector, expressed as components in a basis $(F_x, F_y, F_z)$, or better as $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$
2. The position vector (relative to some convenient origin) of the point where the force is acting $(x, y, z)$ or better $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
3. The position vector of the point (say point *A*) we wish to take moments about (you must use the same origin as for 2) $(x^{(A)}, y^{(A)}, z^{(A)})$ or $\mathbf{r}_A = x^{(A)}\mathbf{i} + y^{(A)}\mathbf{j} + z^{(A)}\mathbf{k}$

The **moment of F** about point A is then defined as

$$\mathbf{M}_A = (\mathbf{r} - \mathbf{r}_A) \times \mathbf{F}$$

We can write out the formula for the components of $\mathbf{M}_A$ in longhand by using the definition of a cross product

$$\begin{aligned}
\mathbf{M}_A &= [(x - x_A)\mathbf{i} + (y - y_A)\mathbf{j} + (z - z_A)\mathbf{k}] \times [F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}] \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ (x - x_A) & (y - y_A) & (z - z_A) \\ F_x & F_y & F_z \end{vmatrix} \\
&= \left\{(y - y_A)F_z - (z - z_A)F_y\right\}\mathbf{i} + \left\{(z - z_A)F_x - (x - x_A)F_z\right\}\mathbf{j} + \left\{(x - x_A)F_y - (y - y_A)F_x\right\}\mathbf{k}
\end{aligned}$$

The **moment of F about the origin** is a bit simpler

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$$

or, in terms of components

$$\begin{aligned}
\mathbf{M}_O &= [x\mathbf{i} + y\mathbf{j} + z\mathbf{k}] \times [F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}] \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} \\
&= \left\{yF_z - zF_y\right\}\mathbf{i} + \left\{zF_x - xF_z\right\}\mathbf{j} + \left\{xF_y - yF_x\right\}\mathbf{k}
\end{aligned}$$

**2.2.2 Resultant moment exerted by a force system**.

Suppose that $N$ forces $\mathbf{F}^{(1)},\mathbf{F}^{(2)}...\mathbf{F}^{(N)}$ act at positions $\mathbf{r}^{(1)},\ \mathbf{r}^{(2)},...\mathbf{r}^{(N)}$. The *resultant moment* of the force system is simply the sum of the moments exerted by the forces. You can calculate the resultant moment by first calculating the moment of each force, and then adding all the moments together (using vector sums).

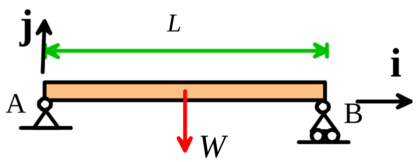
Just one word of caution is in order here – *when you compute the resultant moment, you must take moments about the same point for every force.*

Taking moments about a different point for each force and adding the result is meaningless!

**2.2.3 Examples of moment calculations using the vector formulas**

We work through a few examples of moment calculations

*Example 1:* The beam shown below is uniform and has weight $W$. Calculate the moment exerted by the gravitational force about points A and B.




We know (from the table provided earlier) that the center of gravity is half-way along the beam.
The force (as a vector) is

$$\mathbf{F} = -W\mathbf{j}$$

To calculate the moment about A, we take the origin at A. The position vector of the force relative to A is

$$\mathbf{r} = (L/2)\mathbf{i}$$

The moment about A therefore

$$\mathbf{M}_A = \mathbf{r}\times\mathbf{F} = (L/2)\mathbf{i}\times(-W)\mathbf{j} = -(WL/2)\mathbf{k}$$

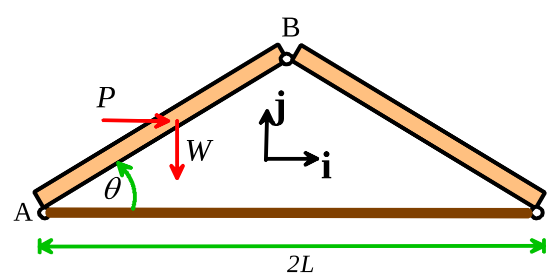
To calculate the moment about B, we take B as the origin. The position vector of the force relative to B is

$$\mathbf{r} = -(L/2)\mathbf{i}$$

Therefore

$$\mathbf{M}_B = \mathbf{r}\times\mathbf{F} = (-L/2)\mathbf{i}\times(-W)\mathbf{j} = (WL/2)\mathbf{k}$$

*Example 2*. Member AB of a roof-truss is subjected to a vertical gravitational force $W$ and a horizontal wind load $P$. Calculate the moment of the resultant force about B.




Both the wind load and weight act at the center of gravity. Geometry shows that the position vector of the CG with respect to B is

$$\mathbf{r} = (-L/2)\mathbf{i} - (L/2)\tan\theta\mathbf{j}$$

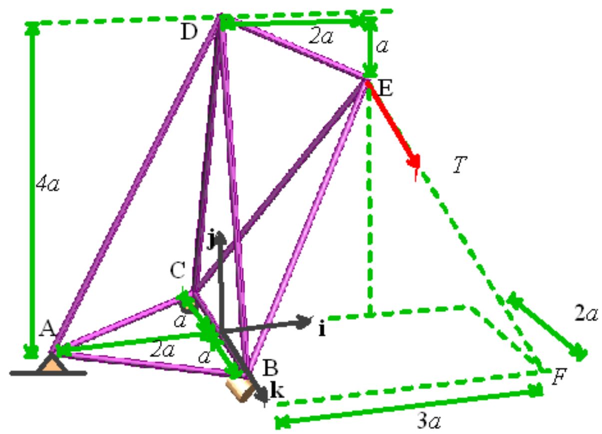
The resultant force is

$$\mathbf{F} = P\mathbf{i} - W\mathbf{j}$$

Therefore the moment about B is

$$\mathbf{M}_B = \mathbf{r}\times\mathbf{F} = \left[(-L/2)\mathbf{i} - (L/2)\tan\theta\,\mathbf{j}\right]\times\left[P\mathbf{i} - W\mathbf{j}\right]$$
$$= (L/2)\left\{W + P\tan\theta\right\}\mathbf{k}$$

*Example 3*. The structure shown is subjected to a force $T$ acting at $E$ along the line $EF$. Calculate the moment of $T$ about points A and D.




This example requires a lot more work. First we need to write down the force as a vector. We know the magnitude of the force is $T$, so we only need to work out its direction. Since the force acts along EF, the direction must be a unit vector pointing along $EF$. It's not hard to see that the vector $EF$ is

$$\overrightarrow{EF} = a\mathbf{i} - 3a\mathbf{j} + 2a\mathbf{k}$$

We can divide by the length of EF ($a\sqrt{14}$) to find a unit vector pointing in the correct direction

$$\mathbf{e}_{EF} = (\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})/\sqrt{14}$$

The force vector is

$$\mathbf{F} = T(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})/\sqrt{14}$$

Next, we need to write down the necessary position vectors

Force: $\mathbf{r} = 2a\mathbf{i} + 3a\mathbf{j}$

Point A: $\mathbf{r}_A = -2a\mathbf{i}$

Point D: $\mathbf{r}_D = 4a\mathbf{j}$

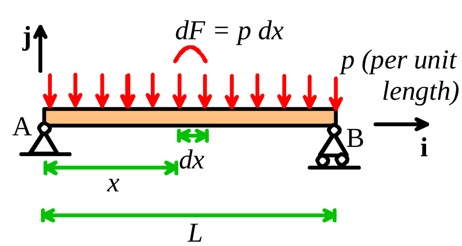
Finally, we can work through the necessary cross products

$$\begin{aligned}\mathbf{M}_A &= (\mathbf{r} - \mathbf{r}_A)\times\mathbf{F}\\ &= (4a\mathbf{i} + 3a\mathbf{j})\times T(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})/\sqrt{14}\\ &= \left(Ta/\sqrt{14}\right)\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & 3 & 0\\ 1 & -3 & 2\end{vmatrix}\\ &= Ta\left(6\mathbf{i} - 8\mathbf{j} - 15\mathbf{k}\right)/\sqrt{14}\end{aligned}$$

$$\begin{aligned}\mathbf{M}_D &= (\mathbf{r} - \mathbf{r}_D)\times\mathbf{F}\\ &= (2a\mathbf{i} - a\mathbf{j})\times T(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})/\sqrt{14}\\ &= \left(Ta/\sqrt{14}\right)\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -1 & 0\\ 1 & -3 & 2\end{vmatrix}\\ &= Ta\left(-2\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}\right)/\sqrt{14}\end{aligned}$$

Clearly, vector notation is very helpful when solving 3D problems!

*Example 4*. Finally, we work through a simple problem involving *distributed* loading. Calculate expressions for the moments exerted by the pressure acting on the beam about points A and B.



An arbitrary strip of the beam with length $dx$ is subjected to a force

$$d\mathbf{F} = -p\,dx\ \mathbf{j}$$

The position vector of the strip relative to A is

$$\mathbf{r} = x\mathbf{i}$$

The force acting on the strip therefore exerts a moment

$$d\mathbf{M}_A = x\mathbf{i} \times (-p\,dx)\,\mathbf{j} = -\,px\ dx\ \mathbf{k}$$

The total moment follows by summing (integrating) the forces over the entire length of the beam

$$\mathbf{M}_A = \int_0^L -p\,xdx\ \mathbf{k} = -(pL^2/2)\mathbf{k}$$

The position vector of the strip relative to B is

$$\mathbf{r} = (L-x)\mathbf{i}$$

The force acting on the strip exerts a moment

$$d\mathbf{M}_B = (L-x)\mathbf{i} \times (-p\,dx)\,\mathbf{j} = -\,p\,(L-x)\ dx\ \mathbf{k}$$

The total moment follows by summing (integrating) the forces over the entire length of the beam

$$\mathbf{M}_A = \int_0^L -p\,(L-x)dx\ \mathbf{k} = (pL^2/2)\mathbf{k}$$

### 2.2.4 The Physical Significance of a Moment

A force acting on a solid object has two effects: (i) it tends to accelerate the object (making the object's center of mass move); and (ii) it tends to cause the object to *rotate*.

1. *The moment of a force about some point quantifies its tendency to rotate an object about that point*.

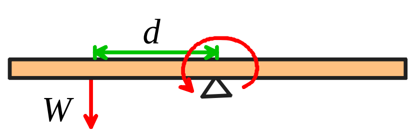
2. *The magnitude of the moment specifies the magnitude of the rotational force.*

3. *The direction of a moment specifies the axis of rotation associated with the rotational force, following the right hand screw convention.*

Let's explore these statements in more detail.

The best way to understand the physical significance of a moment is to think about the simple experiments you did with levers & weights back in kindergarten. Consider a beam that's pivoted about some point (e.g. a see-saw).

Hang a weight $W$ at some distance $d$ to the left of the pivot, and the beam will rotate (counter-clockwise)

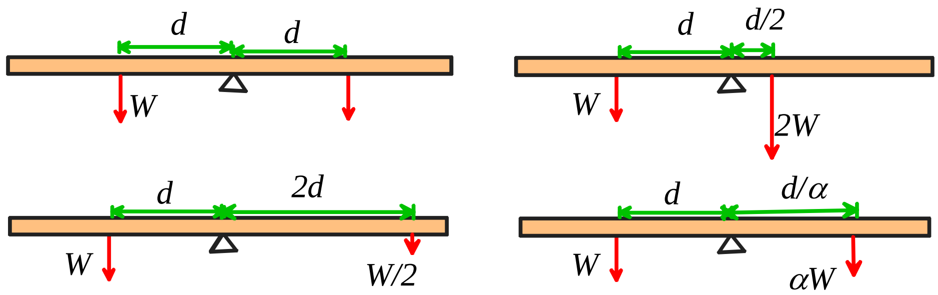


A force applied to a pivoted beam causes the beam to rotate

To stop the beam rotating, we need to hang a weight on the right side of the pivot. We could

(a) Hang a weight $W$ a distance $d$ to the right of the pivot
(b) Hang a weight $2W$ a distance $d/2$ to the right of the pivot

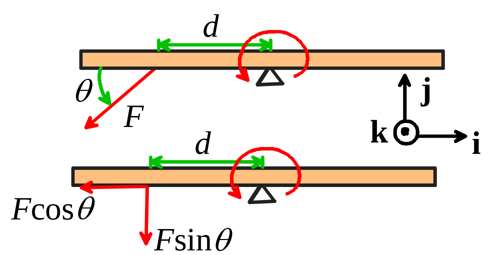
(c) Hang a weight *W/2* a distance *2d* to the right of the pivot
(d) Hang a weight $\alpha W$ a distance $d/\alpha$ to the right of the pivot.




Four ways to balance the beam

These simple experiments suggest that the turning tendency of a force about some point is equal to the distance from the point multiplied by the force. This is certainly consistent with $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$

To see where the cross product in the definition comes from, we need to do a rather more sophisticated experiment. Let's now apply a force *F* at a distance *d* from the pivot, but now instead of making the force act perpendicular to the pivot, let's make it act at some angle. Does this have a turning tendency *Fd?*




For a force applied at an angle, the turning tendency is $dF\sin\theta$

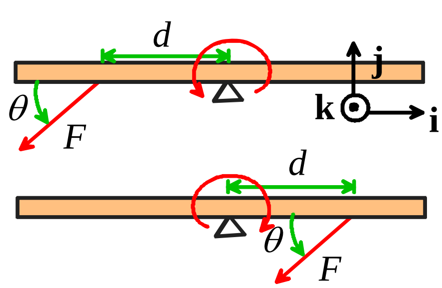
A little reflection shows that this cannot be the case. The force *F* can be split into two components – $F\sin\theta$ perpendicular to the beam, and $F\cos\theta$ parallel to it. But the component parallel to the beam will not tend to turn the beam. The turning tendency is only $dF\sin\theta$.

Let's compare this with $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$. Take the origin at the pivot, then

$$\mathbf{r} = -d\mathbf{i} \qquad \mathbf{F} = -F\cos\theta\mathbf{i} - F\sin\theta\mathbf{j} \quad \Rightarrow \quad \mathbf{r} \times \mathbf{F} = dF\sin\theta\mathbf{k}$$

so the magnitude of the moment correctly gives the magnitude of the turning tendency of the force. That's why the definition of a moment needs a *cross product*.

Finally we need to think about the significance of the *direction* of the moment. We can get some insight by calculating $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$ for forces acting on our beam to the right and left of the pivot




For the force acting on the left of the pivot, we just found

$$\mathbf{r} = -d\mathbf{i} \qquad \mathbf{F} = -F\cos\theta\mathbf{i} - F\sin\theta\mathbf{j} \quad \Rightarrow \quad \mathbf{r} \times \mathbf{F} = dF\sin\theta\mathbf{k}$$

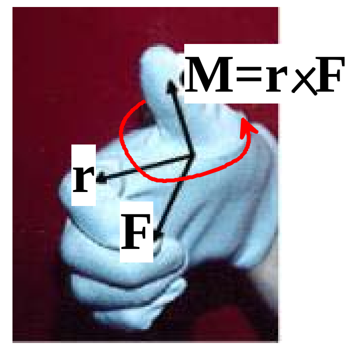
For the force acting on the right of the pivot

$$\mathbf{r} = d\mathbf{i} \qquad \mathbf{F} = -F\cos\theta\mathbf{i} - F\sin\theta\mathbf{j} \quad \Rightarrow \quad \mathbf{r} \times \mathbf{F} = -dF\sin\theta\mathbf{k}$$

Thus, the force on the left exerts a moment along the +**k** direction, while the force on the right exerts a moment in the –**k** direction.

Notice also that the force on the left causes counterclockwise rotation; the force on the right causes clockwise rotation. Clearly, the direction of the moment has something to do with the direction of the turning tendency.

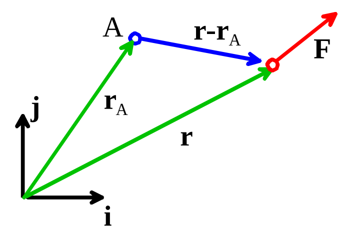
Specifically, *the direction of a moment specifies the axis associated with the rotational force, following the right hand screw convention.*




It's best to use the screw rule to visualize the effect of a moment – hold your right hand as shown, with the thumb pointing along the direction of the moment. Your curling fingers (moving from your palm to the finger tips) then indicate the rotational tendency associated with the moment. Try this for the beam problem. With your thumb pointing along +**k** (out of the picture), your fingers curl counterclockwise. With your thumb pointing along –**k**, your fingers curl clockwise.

### 2.2.5 A few tips on calculating moments

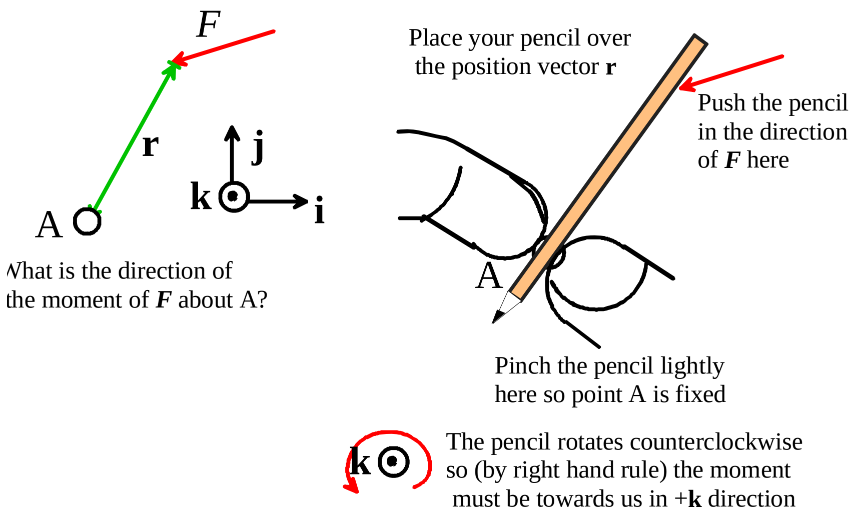
The safest way to calculate the moment of a force is to slog through the $\mathbf{M}_A = (\mathbf{r} - \mathbf{r}_A) \times \mathbf{F}$ formula, as described at the start of this section. As long as you can write down position vectors and force vectors correctly, and can do a cross product, it is totally fool-proof.




But if you have a good physical feel for forces and their effects you might like to make use of the following short cuts.

1. *The direction of a moment is always perpendicular to both* $(\mathbf{r} - \mathbf{r}_A)$ *and* **F**. *For 2D problems,* $(\mathbf{r} - \mathbf{r}_A)$ *and* **F** *lie in the same plane, so the direction of the moment must be perpendicular to this plane.*

Thus, a set of 2D forces in the {**i**,**j**} plane can only exert moments in the $\pm\mathbf{k}$ direction – this makes calculating moments in 2D problems rather simple; we just have to figure out whether the sign of a moment is positive or negative.




You can do a quick experiment to see whether the direction is +**k** or –**k.** Suppose you want to find the direction of the moment caused by **F** in the picture above about the point A. To do so,

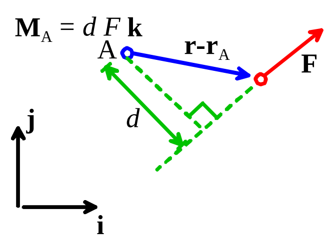
(i) Place your pencil on the page so that it lies on the line connecting A to the force.

(ii) Pinch the pencil lightly at A so it can rotate about A, but A remains fixed.

(iii) Push on the pencil in the direction of the force at B. If the pencil rotates counterclockwise, the direction of the moment of **F** about A is out of the picture (usually +**k)**. If it rotates clockwise, the direction of the moment is into the picture (–**k).** If it doesn't rotate, you're either holding the pencil in a death grip at A (then the experiment won't work) or else the force must be acting along the pencil – in this case the moment is zero.

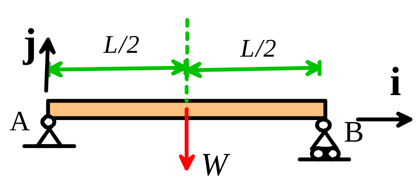
In practice you will soon find that you can very quickly tell the direction of a moment (in 2D, anyway) just by looking at the picture, but the experiment might help until you develop this intuition.

2. *The magnitude of a moment about some point is equal to the perpendicular distance from that point to the line of action of the force, multiplied by the magnitude of the force.*




Again, this trick is most helpful in 2D. Its use is best illustrated by example. Let's work through the simple 2D example problems again, but now use the short-cut.

***Example 1:*** The beam shown below is uniform and has weight $W$. Calculate the moment exerted by the gravitational force about points A and B.



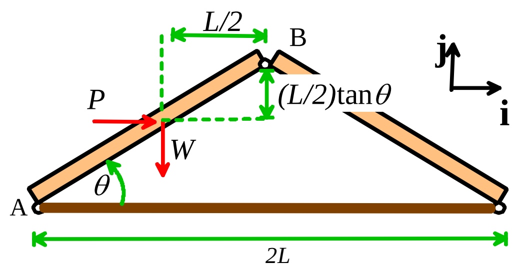

The perpendicular distance from a vertical line through the CG to A is $L/2$. The pencil trick shows that $W$ exerts a clockwise moment about A. Therefore

$$\mathbf{M}_A = -(LW/2)\mathbf{k}$$

Similarly, the perpendicular distance to B is *L/2*, and *W* exerts a counterclockwise moment about B. Therefore

$$\mathbf{M}_B = (LW/2)\mathbf{k}$$

***Example 2.*** Member AB of a roof-truss is subjected to a vertical gravitational force *W* and a horizontal wind load *P*. Calculate the moment of the resultant force about B.




The perpendicular distance from the line of action of *W* to B is *L/2*. *W* exerts a counterclockwise moment about B. Therefore *W* exerts a moment $\mathbf{M}_B = (L/2)W\mathbf{k}$

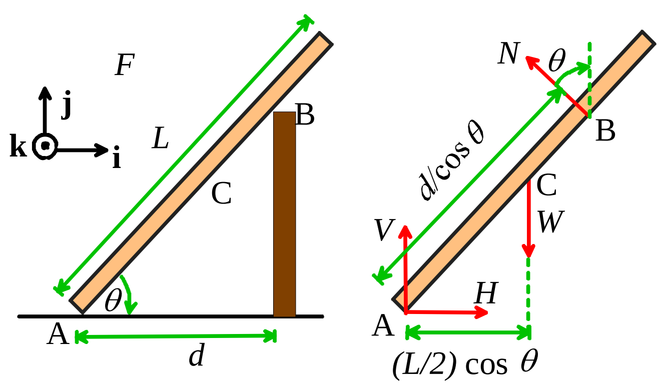
The perpendicular distance from the line of action of *P* to B is $(L/2)\tan\theta$. *P* also exerts a counterclockwise moment about B. Therefore

$\mathbf{M}_B = (L\tan\theta/2)P\mathbf{k}$

The total moment is

$$\mathbf{M}_B = (L/2)\{W + P\tan\theta\}\mathbf{k}$$

***Example 3:*** It is traditional in elementary statics courses to solve lots of problems involving ladders (oh boy! Aren't you glad you signed up for engineering?) . The picture below shows a ladder of length *L* and weight *W* resting on the top of a frictionless wall. Forces acting on the ladder are shown as well. Calculate the moments about point A of the reaction force at B (which acts perpendicular to the ladder) and the weight force at C (which acts at the center of gravity, half-way up the ladder).




The perpendicular distance from point A to the line along which *N* acts is $d/\cos\theta$. The pencil experiment (or inspection) shows that the direction of the moment of *N* about A is in the +**k** direction. Therefore the trick (perpendicular distance times force) gives

$$\mathbf{M}_A = (d/\cos\theta)\ N\ \mathbf{k} \qquad \text{(for the force acting at B)}$$

The perpendicular distance from point A to the line along which *W* is acting is $(L/2)\cos\theta$. The direction of the moment is –**k.** Therefore

$$\mathbf{M}_A = -(L/2)\cos\theta\ W\ \mathbf{k} \qquad \text{(for the weight force)}$$

Let's compare these with the answer we get using $\mathbf{M}_A = (\mathbf{r} - \mathbf{r}_A) \times \mathbf{F}$. We can take the origin to be at A to make things simple. Then, for the force at B

$$(\mathbf{r}_B - \mathbf{r}_A) = d\mathbf{i} + d\tan\theta\mathbf{j}$$
$$\mathbf{F}_B = -N\sin\theta\mathbf{i} + N\cos\theta\mathbf{j}$$
$$\Rightarrow \mathbf{M} = (\mathbf{r}_B - \mathbf{r}_A)\times\mathbf{F}_B = \left(dN\cos\theta + d\tan\theta N\sin\theta\right)\mathbf{k}$$
$$= \left(dN(\cos^2\theta + \sin^2\theta)/\cos\theta\right)\mathbf{k} = (dN/\cos\theta)\mathbf{k}$$

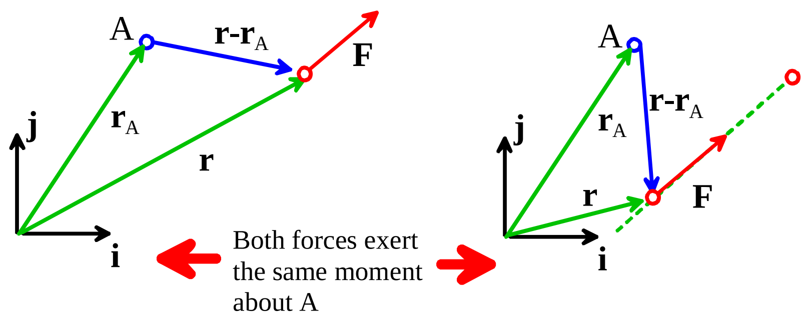
giving the same answer as before, but with a whole lot more effort!

Similarly, for the weight force

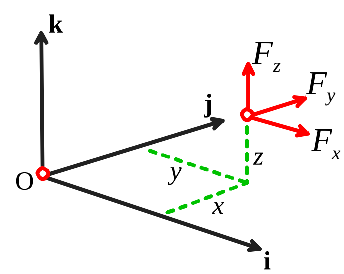
$$(\mathbf{r}_C - \mathbf{r}_A) = (L/2)\cos\theta\mathbf{i} + (L/2)\sin\theta\mathbf{j}$$
$$\mathbf{F}_C = -W\mathbf{j}$$
$$\Rightarrow \mathbf{M} = (\mathbf{r}_C - \mathbf{r}_A)\times\mathbf{F}_B = -(L/2)\cos\theta\; W\;\mathbf{k}$$

3. *The moment exerted by a force is unchanged if the force is moved in a direction parallel to the direction of the force.*

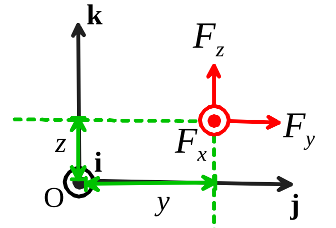
This is rather obvious in light of trick (2), but it's worth stating anyway.




4. *The component of moment exerted by a force about an axis through a point can be calculated by (i) finding the two force components perpendicular to the axis; then (ii) multiplying each force component by its perpendicular distance from the axis; and (iii) adding the contributions of each force component following the right-hand screw convention.*




The wording of this one probably loses you, so let's start by trying to explain what this means.




First, let's review what we mean by the *component of a moment about some axis*. The formula for the moment of a force about the origin is

$$\mathbf{M}_O = \left\{yF_z - zF_y\right\}\mathbf{i} + \left\{zF_x - xF_z\right\}\mathbf{j} + \left\{xF_y - yF_x\right\}\mathbf{k}$$

This has three components - $M_x = \left\{yF_z - zF_y\right\}$ about the $\mathbf{i}$ axis, $M_y = \left\{zF_x - xF_z\right\}$ about the $\mathbf{j}$ axis, and $M_z = \left\{xF_y - yF_x\right\}$ about the $\mathbf{k}$ axis.

The trick gives you a quick way to calculate *one* of the components. For example, let's try to find the **i** component of the moment about the origin exerted by the force shown in the picture.

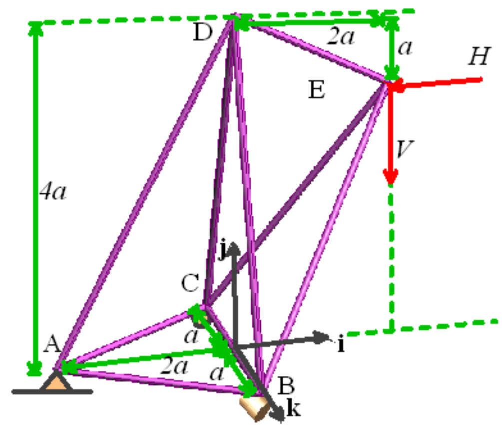
The rule says

(i) Identify the force components perpendicular to the **i** axis – that's $F_z$ and $F_y$ in this case;

(ii) Multiply each force component by its perpendicular distance from the axis. Drawing a view down the **i** axis is helpful. From the picture, we can see that $F_z$ is a distance *y* from the axis, and $F_y$ is a distance *z* from the axis. The two contributions we need are thus $yF_z$ and $zF_y$.

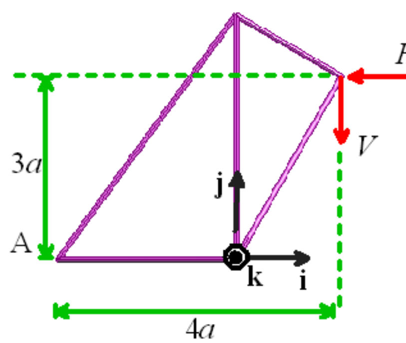
(iii) Add the two contributions according to the right hand screw rule. We know that each force component exerts a moment $\pm\mathbf{i}$ - we have to figure out which one is +**i** and which is –**i**. We can do the pencil experiment to figure this out – the answer is that $F_z$ exerts a moment along +**i,** while $F_y$ causes a moment along –**i.** So finally

$$M_x = \left\{ yF_z - zF_y \right\}.$$




*Example:* The structure shown is subjected to a vertical force *V* and horizontal force *H* acting at *E*. Calculate the **k** component of moment exerted about point A by the resultant force.

Our trick gives the answer immediately. First, draw a picture looking down the **k** axis




Clearly, the force *H* exerts a **k** component of moment $3aH\mathbf{k}$, while the force *V* exerts a **k** component of moment $-4aV\,\mathbf{k}$. The total **k** component of moment is

$$M_{Az} = a(3H - 4V)$$

This trick clearly can save a great deal of time. But to make use of it, you need excellent 3D visualization skills.

## 2.3 Force Couples, Pure Moments, Couples and Torques

We have seen that a force acting on a rigid body has two effects: (i) it tends to move the body; and (ii) it tends to rotate the body.

A natural question arises – is there a way to rotate a body without moving it? And is there a kind of force that causes *only* rotation without translation?

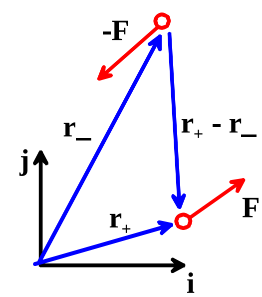
The answer to both questions is yes.

### 2.3.1 Force couples

A system of forces that exerts a resultant moment, but no resultant force, is called a *force couple*.

The simplest example of a force couple consists of two equal and opposite forces $+\mathbf{F}$ and $-\mathbf{F}$ acting some distance apart. Suppose that the force $-\mathbf{F}$ acts at position $\mathbf{r}_-$ while the force $+\mathbf{F}$ acts at position $\mathbf{r}_+$ The resultant moment is

$$\begin{aligned}\mathbf{M} &= \mathbf{r}_+ \times \mathbf{F} + \mathbf{r}_- \times (-\mathbf{F}) \\ &= (\mathbf{r}_+ - \mathbf{r}_-) \times \mathbf{F}\end{aligned}$$



Two equal and opposite forces exert a purely rotational force.

Of course, the vector $\mathbf{r}_+ - \mathbf{r}_-$ is just the vector from the point where $-\mathbf{F}$ acts to the point where $+\mathbf{F}$ acts. This gives a quick way to calculate the moment induced by a force couple:

*The moment induced by two equal and opposite forces is equal to the moment of one force about the point of action of the other*. It doesn't matter which force you use to do this calculation.

Note that a force couple

(i) Has zero resultant force

(ii) Exerts the same resultant moment about all points.

Its effect is to induce rotation without translation.

The effect of a force couple can therefore be characterized by a single vector moment **M**. The physical significance of **M** is equivalent to the physical significance of the moment of a force about some point. The direction of **M** specifies the axis associated with the rotational force. The magnitude of **M** specifies the intensity of the rotational force.

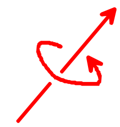
There are many practical examples of force systems that are best thought of as force-couple systems. They include

1. The forces exerted by your hand on a screw-driver
2. The forces exerted by the tip of a screw-driver on the head of a screw
3. The forces exerted by one part of a constant velocity joint on another
4. Drag forces acting on a spinning propeller

### 2.3.2 Pure moments, couples and torques.- Definition, Physical Interpretation, and Examples

A *pure moment* is a rotational force. Its effect is to induce rotation, without translation – just like a force couple.

Couples and torques are other names for a pure moment.

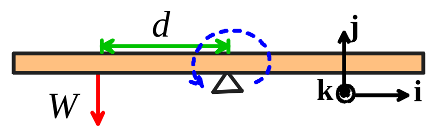


A 3D moment

A 2D moment

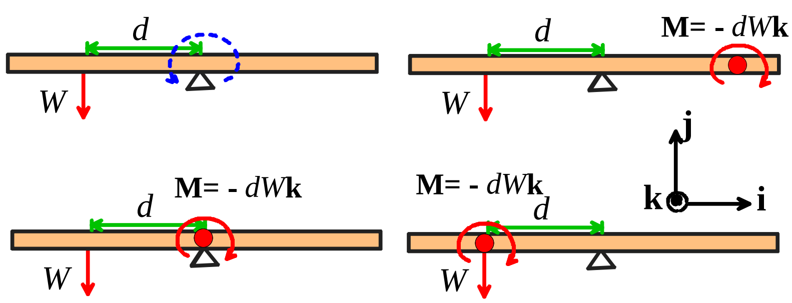
A pure moment is a vector quantity – it has magnitude and direction. The physical significance of the magnitude and direction of a pure moment are completely equivalent to the moment associated with a force couple system. The direction of a moment indicates the axis associated with its rotational force (following the right hand screw convention); the magnitude represents the intensity of the force. A moment is often denoted by the symbols shown in the figure.

The concept of a pure moment takes some getting used to. Its physical effect can be visualized by thinking about our beam-balancing problem again.




The picture above shows the un-balanced beam. We saw earlier that we can balance the beam again by adding a second force, which induces a moment equal and opposite to that of the force *W*.

We can also balance the beam by applying a *pure moment* to it. Since the moment of *W* is $dW\mathbf{k}$, a moment $\mathbf{M} = -dW\mathbf{k}$ applied anywhere on the beam would balance it.




You could even apply the moment to the left of the beam – even right on top of the force *W* if you like!

### 2.3.3 Units and typical magnitudes of moments

In the SI system, moments have units of Nm (Newton-meters).
In the US system, moments have units of ft-lb (foot pounds)
The conversion factor is 1 Nm = 0.738 ft lb; or 1 ft-lb = 1.356 Nm.
Typical magnitudes are:

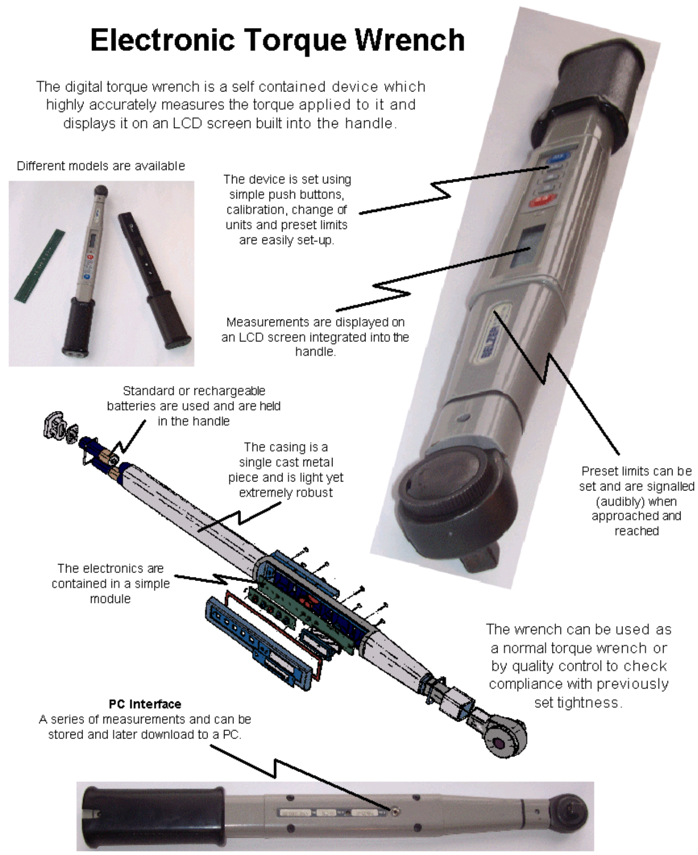
- Max torque exerted by a small Lego motor: 0.1 Nm
- Typical torque output of a typical car engine 300-600 Nm
- Breaking torque of a human femur: 140Nm

**2.3.4 Measuring Moments**

Just as you can buy a force transducer to measure forces, you can buy a force transducer that measures moments. We showed an example of a force-transducer attached to the wheel of a car during our earlier discussion of force transducers.

Another common moment-measuring system is a *torque-wrench*. (So then is Oprah a talk wench?) When you tighten the bolts on a precision machine, it's important to torque them correctly. If you apply too much torque, you will strip the thread. If you don't apply enough, the bolt will work itself loose during service.

You can buy a tool that measures the moment that you apply to a bolt while tightening it. The device may be mechanical, or electronic. An example (see [www.mac.ie/whatwedo/torquestory.asp](www.mac.ie/whatwedo/torquestory.asp) **)** is shown below.




**2.3.5 Engineering systems that exert torques**

There are many practical examples of moments, or torques, in engineering systems. For example,

(i) The driving axle on your car turns the wheels by exerting a moment on them.

(i) The drive shaft of any motor exerts a torque on whatever it's connected to. In fact, motors are usually rated by their torque capacity.

(ii) The purpose of a gearbox is to amplify or attenuate torque. You apply a torque to the input shaft, and get a bigger or smaller torque from the output shaft. To do this, the input and output shafts have to rotate at different speeds. There are also some clever gearboxes that allow you to *add* torques together – they are used in split-power variable speed transmissions, for example.

(iii) A *torque converter* serves a similar purpose to a gearbox. Unlike a gearbox, however, the input and output shafts don't rotate at the same speed. The output shaft can be stationary, exerting a large torque, while the input shaft rotates quickly under a modest torque. It is used as part of an automatic transmission system in a car.

(iv) Moments also appear as *reaction forces*. For example, the resistance you feel to turning the steering wheel of your car is caused by moments acting on the wheels where they touch the ground. The rolling resistance you feel when you ride your bike over soft ground or grass is also due to a moment acting where the wheel touches the ground.

(v) Moments appear as *internal forces* in structural members or components. For example, a beam will bend because of an internal moment whose direction is transverse to the direction of the beam. A shaft will twist because of an internal moment whose direction is parallel to the shaft. Just as an internal force causes points in a solid to move relative to each other, an internal moment causes points to *rotate* relative to each other.

## 2.4 Constraint and reaction forces and moments

Machines and structures are made up of large numbers of separate components. For example, a building consists of a steel frame that is responsible for carrying most of the weight of the building and its contents. The frame is made up of many separate beams and girders, connected to one another in some way. Similarly, an automobile's engine and transmission system contain hundreds of parts, all designed to transmit forces exerted on the engine's cylinder heads to the ground.

To analyze systems like this, we need to know how to think about the forces exerted by one part of a machine or structure on another.

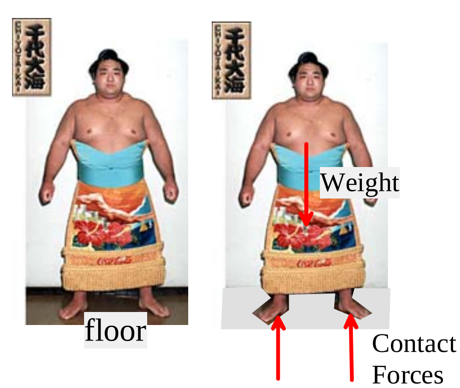
We do this by developing a set of *rules* that specify the forces associated with various types of joints and connections.

Forces associated with joints and connections are unlike the forces described in the preceding section. For all our preceding examples, (e.g. gravity, lift and drag forces, and so on) we always knew *everything* about the forces – magnitude, direction, and where the force acts.

In contrast, the rules for forces and moments acting at joints and contacts don't specify the forces completely. Usually (but not always), they will specify *where* the forces act; and they will specify that the forces and moments can only act along certain directions. The *magnitude* of the force is always unknown.

### 2.4.1 Constraint forces: overview of general nature of constraint forces

The general nature of a contact force is nicely illustrated by a familiar example – a person, standing on a floor (a Sumo wrestler was selected as a model, since they are particularly interested in making sure they remain in contact with a floor!). You know the floor exerts a force on you (and you must exert an equal force on the floor). If the floor is slippery, you know that the force on you acts perpendicular to the floor, but you can't make any measurements on the properties of the floor or your feet to determine what the force will be.




In fact, *the floor will always exert on your feet whatever force is necessary to stop them sinking through the floor*. (This is generally considered to be a good thing, although there are occasions when it would be helpful to be able to break this law).

We can of course deduce the magnitude of the force, by noting that since you *don't* sink through the floor, you are in equilibrium (according to Newton's definition anyway – you may be far from equilibrium mentally). Let's say you weight 300lb (if you don't, a visit to Dunkin Donuts will help you reach this weight). Since the only forces acting on you are gravity and the contact force, the resultant of the contact force must be equal and opposite to the force of gravity to ensure that the forces on you sum to zero. The magnitude of the total contact force is therefore 300lb. In addition, the resultant of the contact force must act along a line passing through your center of gravity, to ensure that the moments on you sum to zero.

From this specific example, we can draw the following general rules regarding contact and joint forces

(1) All contacts and joints impose *constraints* on the relative motion of the touching or connected components – that is to say, they allow only certain types of relative motion at the joint. (e.g. the floor imposes the constraint that your feet don't sink into it)
(2) Equal and opposite forces and moments act on the two connected or contacting objects. This means that for all intents and purposes, *a constraint force acts in more than one direction at the same time*. This is perhaps the most confusing feature of constraint forces.
(3) The direction of the forces and moments acting on the connected objects must be consistent with the allowable relative motion at the joint (detailed explanation below)
(4) The magnitude of the forces acting at a joint or contact is always unknown. It can *sometimes* be calculated by considering equilibrium (or for dynamic problems, the motion) of the two contacting parts (detailed explanation later).

Because forces acting at joints impose *constraints* on motion, they are often called *constraint forces*.

They are also called *reaction forces*, because the joints *react* to impose restrictions on the relative motion of the two contacting parts.

### 2.4.2 How to determine directions of reaction forces and moments at a joint

Let's explore the meaning of statement (3) above in more detail, with some specific examples.

In our discussion of your interaction with a slippery floor, we stated that the force exerted on you by the floor had to be perpendicular to the floor. How do we know this?

Because, according to (3) above, forces at the contact have to be consistent with the nature of relative motion at the contact or joint. If you stand on a slippery floor, we know

(1) You can slide freely in any direction parallel to the floor. That means there can't be a force acting parallel to the floor.
(2) If someone were to grab hold of your head and try to spin you around, you'd rotate freely; if someone were to try to tip you over, you'd topple. Consequently, there can't be any moment acting on you.
(3) You are prevented from sinking vertically into the floor. A force must act to prevent this.
(4) You can remove your feet from the floor without any resistance. Consequently, the floor can only exert a *repulsive* force on you, it can't attract you.

You can use similar arguments to deduce the forces associated with any kind of joint. Each time you meet a new kind of joint, you should ask

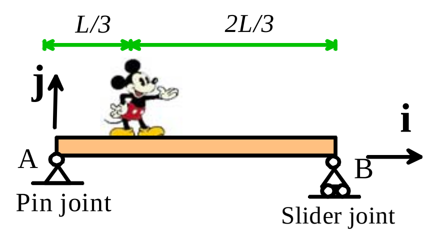
(1) Does the connection allow the two connected solids move relative to each other? If so, what is the direction of motion? There can be no component of reaction force along the direction of relative motion.
(2) Does the connection allow the two connected solids rotate relative to each other? If so, what is the axis of relative rotation? There can be no component of reaction moment parallel to the axis of relative rotation.
(3) For certain types of joint, a more appropriate question may be ‘Is it really healthy/legal for me to smoke this?’

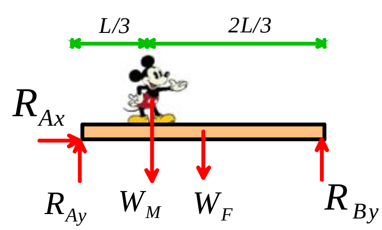
### 2.4.3 Drawing free body diagrams with constraint forces

When we solve problems with constraints, we are nearly always interested in analyzing forces in a structure containing many parts, or the motion of a machine with a number of separate moving components. Solving this kind of problem is not difficult – but it is very complicated because of the large number of forces involved and the large number of equations that must be solved to determine them. To avoid making mistakes, it is critical to use a systematic, and logical, procedure for drawing free body diagrams and labeling forces.

The procedure is best illustrated by means of some simple Mickey Mouse examples. When drawing free body diagrams yourself, you will find it helpful to consult Section 4.3.4 for the nature of reaction forces associated with various constraints.

***2D Mickey-mouse problem 1.*** The figure shows Mickey Mouse standing on a beam supported by a pin joint at one end and a slider joint at the other.



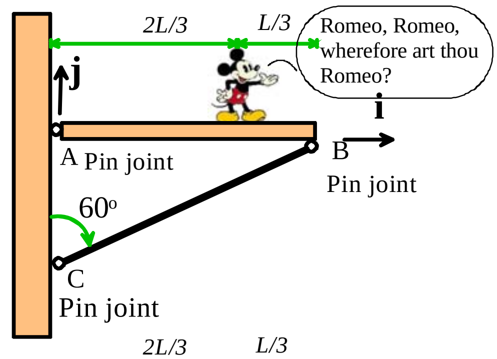

We consider Mickey and the floor together as the system of interest. We draw a picture of the system, isolated from its surroundings (disconnect all the joints, remove contacts, etc). In the picture, all the joints and connections are replaced by forces, following the rules outlined in the preceding section.

Notice how we’ve introduced variables to denote the unknown force components. It is sensible to use a convention that allows you to quickly identify both the position and direction associated with each variable. It is a good idea to use double subscripts – the first subscript shows *where* the force acts, the second shows its direction. Forces are *always* taken to be positive if they act along the positive *x*, *y* and *z* directions.

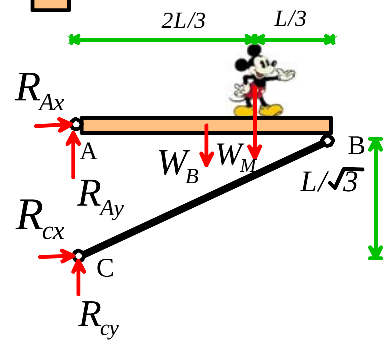
We’ve used the fact that A is a pin joint, and therefore exerts both vertical and horizontal forces; while B is a roller joint, and exerts only a vertical force. *Note that we always, always draw all admissible forces on the FBD, even if we suspect that some components may turn out later to be zero*. For example, it’s fairly clear that $R_{Ax}=0$ in this example, but it would be incorrect to leave off this force. This is especially important in dynamics problems where your intuition regarding forces is very often incorrect.

***2D Mickey Mouse problem 2*** Mickey mouse of weight $W_M$ stands on a balcony of weight $W_B$ as shown. The weight of strut CB may be neglected.

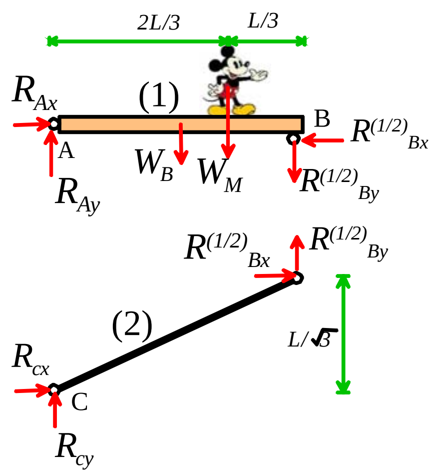
This time we need to deal with a structure that has two parts connected by a joint (the strut BC is connected to the floor AB through a pin joint). In cases like this you have a choice of (a) treating the two parts together as a single system; or (b) considering the strut and floor as two separate systems. As an exercise, we will draw free body diagrams for *both* here.




A free body diagram for the balcony and strut together is shown on the right. Note again the convention used to denote the reactions: the first label denotes the location of the force, the second denotes the direction. Both A and C are pin joints, and therefore exert both horizontal and vertical forces.



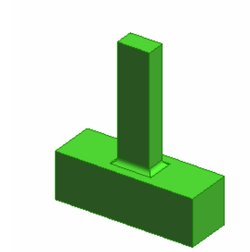

The picture shows free body diagrams for both components. Note the convention we've introduced to deal with the reaction force acting at B – it's important to use a systematic way to deal with forces exerted by one component in a system on another, or you can get hopelessly confused. The recommended procedure is




1. Label the components with numbers – here the balcony is (1) and the strut is (2)
2. Denote reaction forces acting between components with the following convention. In the symbol $R_{Bx}^{(1/2)}$, the superscript (1/2) denotes that the variable signifies the force exerted *by* component (1) *on* component (2) (it's easy to remember that (1/2) is 1 on 2). The subscript *Bx* denotes that the force acts at *B*, and it acts in the positive *x* direction.
3. The forces $R_{Bx}^{(1/2)}$, $R_{By}^{(1/2)}$ exerted by component (1) on component (2) are drawn in the positive *x* and *y* directions on the free body diagram for component (2).
4. The forces exerted by component (2) on component (1) are equal and opposite to $R_{Bx}^{(1/2)}$, $R_{By}^{(1/2)}$. They are therefore drawn in the *negative x and y direction*s on the free body diagram for component (1). You need to think of the reaction force components as acting in *two directions at the same time*. This is confusing, but that's the way life is.

### 2.4.4 Reaction Forces and Moments associated with various types of joint

**Clamped, or welded joints**

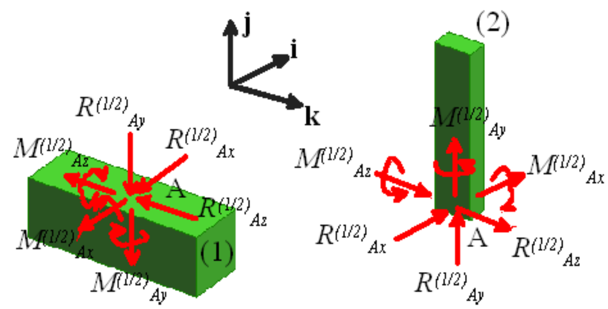


No relative motion or rotation is possible.

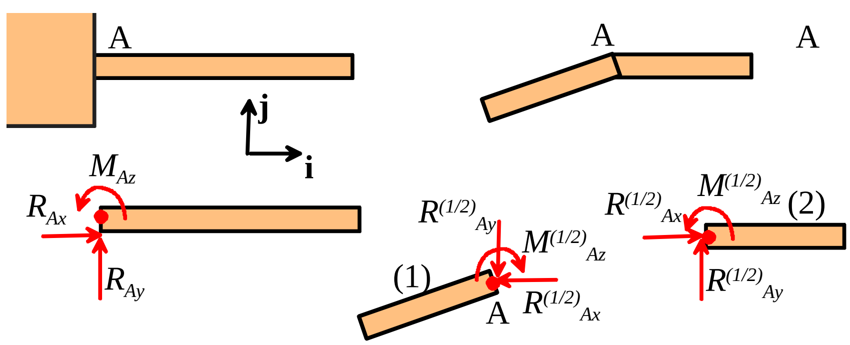
*Reaction forces:* No relative motion is possible in any direction. Three components of reaction force must be present to prevent motion in all three directions.

*Reaction moments:* No relative rotation is possible about any axis. Three components of moment must be present to prevent relative rotation.

The figure shows reaction forces acting on the two connected components. The forces and moments are labeled according to the conventions described in the preceding section.



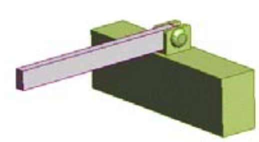

2D versions of the clamped joint are shown below



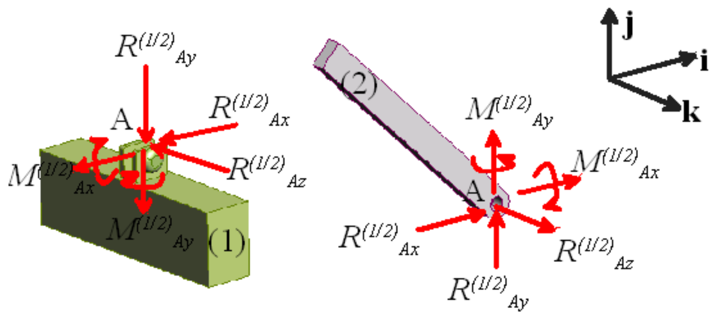

**Pinned joint**.

A pinned joint is like a door hinge, or the joint of your elbow. It allows rotation about one axis, but prevents all other relative motion.

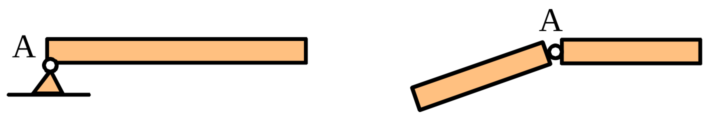
*Reaction forces:* No relative motion is possible at the joint. There must be 3 components of reaction force acting to prevent motion.

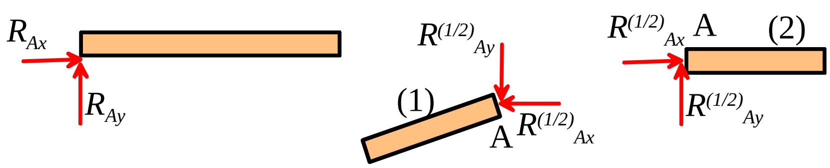


*Reaction moments:* Relative rotation is possible about *one* axis (perpendicular to the hinge) but is prevented about axes perpendicular to the hinge. There must be two components of moment acting at the joint.

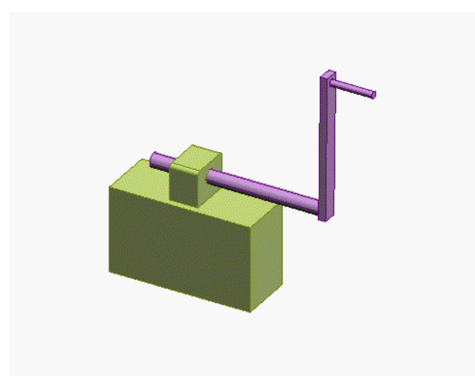



2D pinned joints are often represented as shown in the picture below




**Roller and journal bearings**

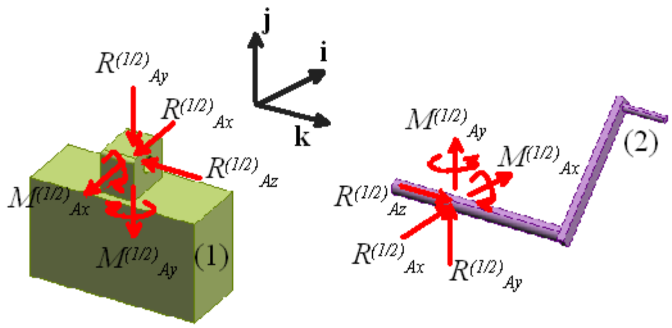


Bearings are used to support rotating shafts. You can buy many different kinds of bearing, which constrain the shaft in different ways. We'll look at a couple of different ones.

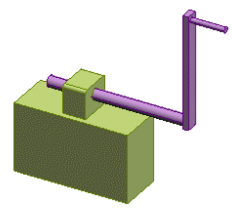
*Example 1:* The bearing shown below is like a pin joint: it allows rotation about one axis, but prevent rotation about the other two, and prevents all relative displacement of the shaft.

*Reaction forces:* No relative motion is possible at this kind of bearing. There must be 3 components of reaction force.

*Reaction moments:* Relative rotation is allowed about one axis (parallel to the shaft), but prevented about the other two. There must be two components of reaction moment.

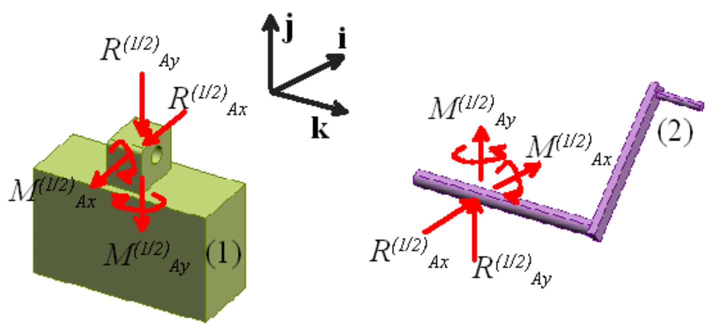



*Example2*: Some types of bearing allow the shaft both to rotate, and to slide through the bearing as shown below



*Reaction forces:* No relative motion is possible transverse to the shaft, but the shaft can slide freely through the bearing. There must be 2 components of reaction force.

*Reaction moments:* Relative rotation is allowed about one axis (parallel to the shaft), but prevented about the other two. There must be two components of reaction moment.

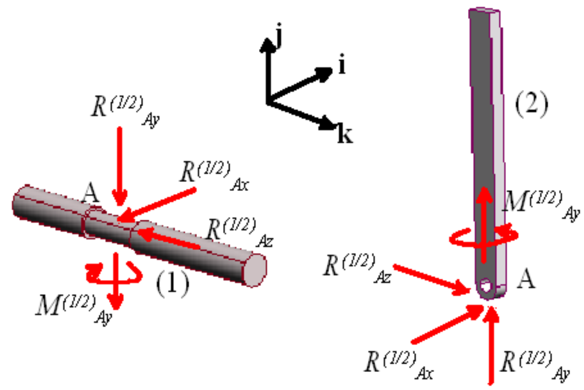



Roller bearings don't often appear in 2D problems. When they do, they look just like a pinned joint.

**Swivel joint:** Like a pinned joint, but allows rotation about two axes. There must be 3 components of reaction force, and 1 component of reaction moment.

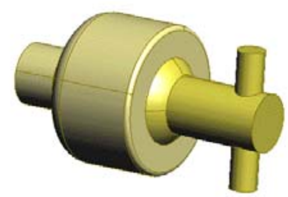
*Reaction forces*: All relative motion is prevented by the joint. There must be three components of reaction force.

*Reaction moments*: rotation is permitted about two axes, but prevented about the third. There must be one component of reaction force present.



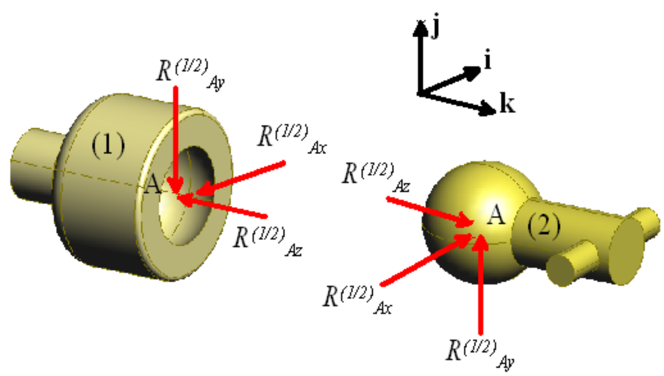

Swivel joints don't often appear in 2D problems. When they do, they look just like a pinned joint.

**Ball and socket joint** Your hip joint is a good example of a ball and socket joint. The joint prevents motion, but allows your thigh to rotate freely relative to the rest of your body.



*Reaction forces*: Prevents any relative motion. There must be three components of reaction force.

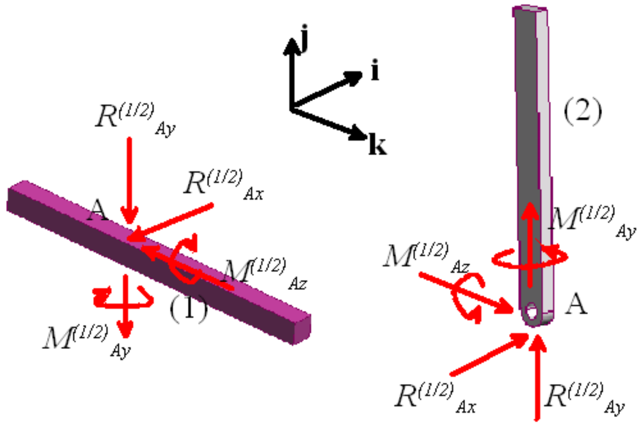
*Reaction moments*. Allows free rotation about all 3 axes. No reaction moments can be present.



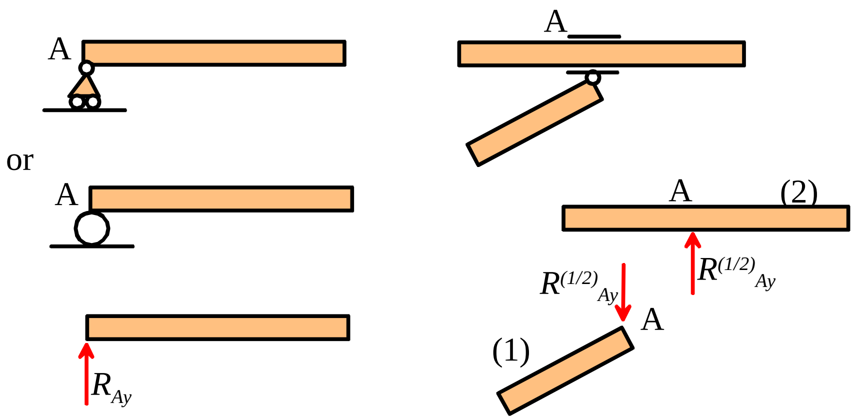

Ball joints don't often appear in 2D problems. When they do, they look just like a pinned joint.

**Slider with pin joint** Allows relative motion in one direction, and allows relative rotation about one axis

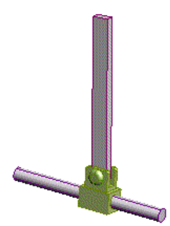
*Reaction forces*: Motion is prevented in two directions, but allowed in the third. There must be two components of reaction force, acting along directions of constrained motion.
*Reaction moments:* Relative rotation is prevented about two axes, but allowed about a third. There must be two components of reaction moment.




2D slider joints are often represented as shown in the picture below

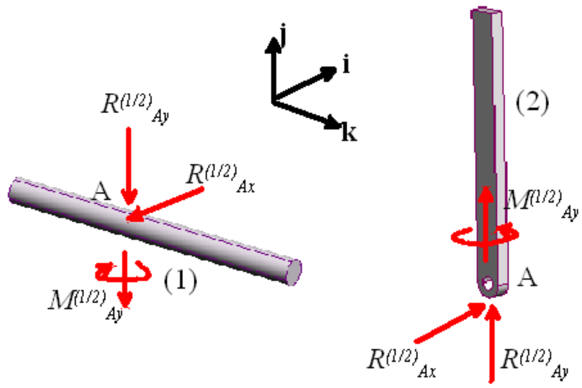



**Slider with swivel joint:** Similar to a swivel joint, but allows motion in one direction in addition to rotation about two axes.



*Reaction forces*: Relative motion is prevented in two directions, but allowed in the third. There must be two components of reaction force acting to prevent motion.

*Reaction moments*: Rotation is permitted around two axes, but prevented around the third. There must be one component of reaction moment.




In 2D, a slider with swivel looks identical to a slider with a pin joint.

### 2.4.5 Contact Forces

Contacts are actually a bit more complex than our glib discussion of your interaction with a slippery floor might suggest.

The nature of the forces acting at a contact depends on three things:

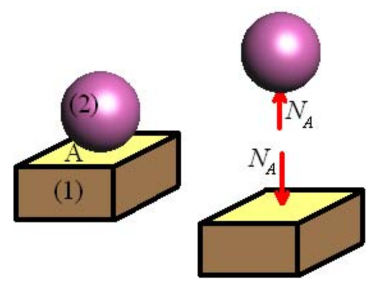
(1) Whether the contact is lubricated, i.e. whether friction acts at the contact
(2) Whether there is significant rolling resistance at the contact
(3) Whether the contact is *conformal*, or *nonconformal*.

A detailed discussion of friction forces will be left until later. For now, we will consider only two limiting cases (a) fully lubricated (frictionless) contacts; and (b) ideally rough (infinite friction) contacts.

Rolling resistance will not be considered at all in this course.

*Forces acting at frictionless nonconformal contacts*

A contact is said to be *nonconformal* if the two objects initially touch at a point. The contact between any two convex surfaces is always non-conformal. Examples include contact between two balls, a ball and a flat surface, or contact between two non-parallel cylinders.

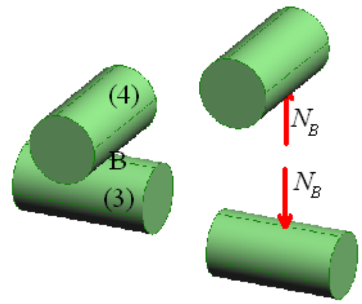



The simplest approximation to the forces acting at a non-conformal lubricated contact states that

(1) Each solid is subjected to a force at the contact point;
(2) The forces between the two solids are *repulsive;* this requires $N_A > 0$ and $N_B > 0$ for both examples illustrated in the picture;
(3) The direction of the forces are along the line connecting the centers of curvature of the two contacting surfaces;
(4) The moments acting on each solid at the contact point are negligible.

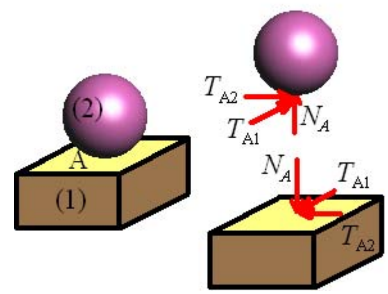
Three rules that help to establish the direction of frictionless contact forces are:

(1) When one of the two contacting surfaces is *flat*, the force must act perpendicular to the flat surface;
(2) When two solids contact along sharp edges, the contact force must be perpendicular to both edges.
(3) When two curved surfaces contact, the reaction force acts along a line joining the centers of curvature of the two objects.



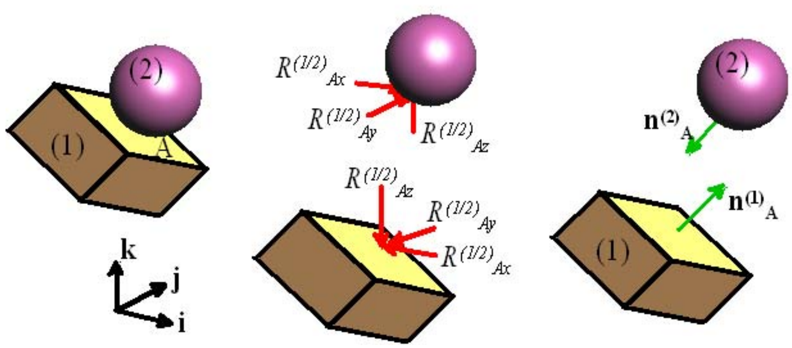

*Forces acting at rough (infinite friction) nonconformal contacts*

A rough nonconformal contact behaves somewhat like a pinned joint. There can be no relative motion of the contacting surfaces, therefore there must be three components of reaction force acting on both contacting solids. Unlike a pin joint, however, the contact can only sustain a repulsive normal force. This means that the components of force shown in the picture must satisfy $N_A \geq 0$. If the normal force is zero (eg when the two surfaces are about to separate), the tangential forces $T_{A1} = T_{A2} = 0$ as well.




The contacting solids can rotate freely relative to one another. Therefore there must be no moment acting on the contacting solids at the point of contact.

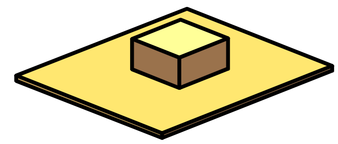
Usually the forces acting at a rough contact are represented by components acting perpendicular and parallel to the contacting surfaces, as shown in the picture above. If you do this, it's easy to enforce the $N_A \geq 0$ constraint. But if it's more convenient, you can treat the contact just like a pin joint, and express the reaction forces in any arbitrary basis, as shown in the picture below.



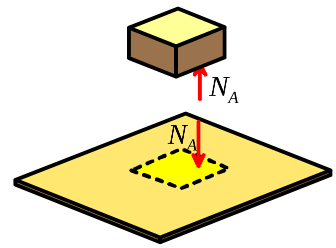

There's a minor disadvantage to doing this – it's not easy to check whether the normal force between the surface is repulsive. You can do it using vectors – for the picture shown the normal force is repulsive if

$$\mathbf{R}_A^{(1/2)} \cdot \mathbf{n}_A^{(1)} \geq 0$$

*Forces acting at frictionless conformal contacts*



A contact is said to be *conformal* if two objects initially contact over a finite area. Examples include contact between the face of a cube and a flat surface; contact between the flat end of a cylinder and a flat surface; or a circular pin inside a matching circular hole.



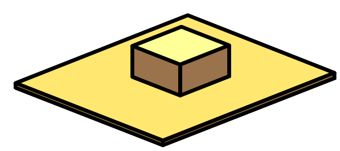

Two conformal solids are actually subjected to a *pressure* over the area where they are in contact. It’s really hard to calculate the pressure distribution (you have to model the deformation of the two contacting solids), so instead we replace the pressure by a statically equivalent force.

If the two contacting surfaces are *flat*, then

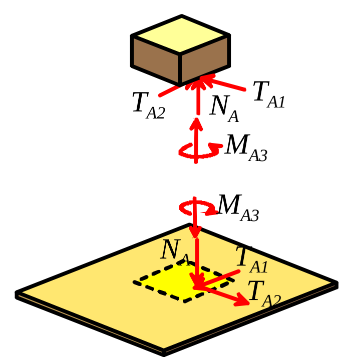
(1) The reaction force can be modeled as a single force, with no moment
(2) The force can act anywhere within the area of contact (its actual position is determined by force and moment balance)
(3) The force must be perpendicular to the two surfaces
(4) The force acts to repel the two solids.

You can of course make really weird conformal contacts – like a jigsaw connection – that can completely prevent both relative translation and rotation of the contacting solids. In this case the contact behaves just like a clamped joint.

*Forces acting at ideally rough (infinite friction) conformal contacts*



No relative motion can occur at the contact. There must therefore be three components of force acting on each solid.




The forces can act anywhere within the area of contact – (its actual position is determined by force and moment balance)

The component of force acting normal to the surface must be repulsive.

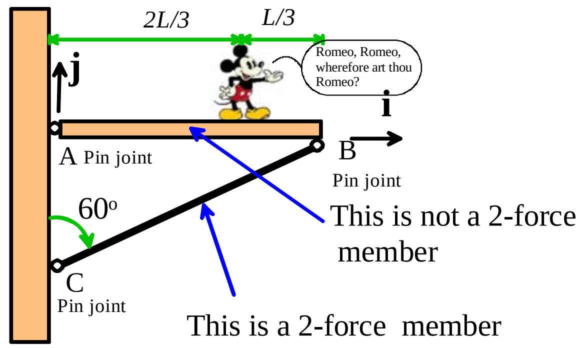
No relative rotation of the two solids can occur. A moment must act about an axis perpendicular to the contact to prevent relative rotation about this axis.

### 2.4.6 Some short-cuts for drawing free body diagrams in systems containing components with negligible mass

The safest procedure in solving any statics or dynamics problem is to set up and solve equations of motion for every different part of the structure or machine. There are two particularly common structural or machine elements that can be treated using short-cuts. These are (i) Two force members in a structure; and (ii) A freely rotating wheel in a machine.

**A *Two-force member*** is a component or structural member that

1. is connected only to two ball-and socket type joints (in 3D) or pin joints (in 2D).
2. has negligible weight




We've seen an example already in one of the Mickey Mouse examples – it's shown again in the picture to remind you. Member BC is a two-force member, because its weight is negligible, and it has only two pin joints connecting it to other members. Member AB is *not* a two-force member – partly because it's weight is not negligible, but also because Mickey exerts a force on the member.

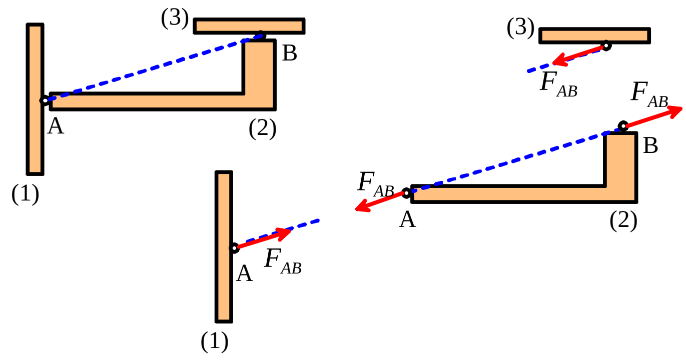
The following rules are very helpful

- *Only one component of reaction force acts at the joints on a 2-force member*
- *The reaction force component acts along a line connecting the two joints.*

It's trivial to show this – if forces act on a body at only two points, and the body is in static equilibrium, then the forces *have* to be equal and opposite, and must also act along the same line, to ensure that both forces and moments are balanced.

A generic 2 force member is shown in the figure. Note that a 2-force member doesn't have to be straight, though it often is.

By convention, a positive reaction force is normally taken to *pull* at each end of the member, as shown. Equal and opposite reaction forces must act on whatever is connected to the two force member.

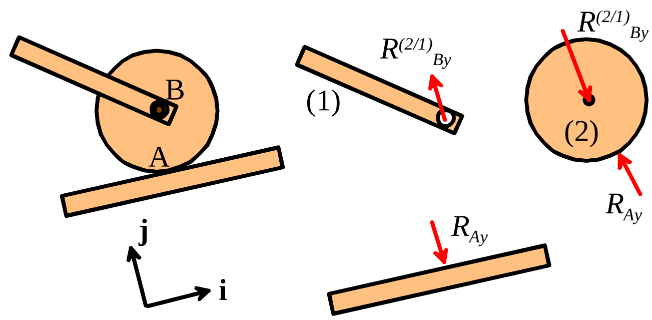


**Forces on a freely rotating wheel with negligible weight:** Wheels are so ubiquitous that it's worth developing a short-cut to deal with them. The picture shows a generic 2D wheel, mounted onto an axle with a frictionless bearing. The contact between wheel and ground is assumed to be ideally rough (infinite friction).

The following trick is helpful

*For a freely rotating 2D wheel, there is only one component of reaction force at the contact between the ground and the wheel.*

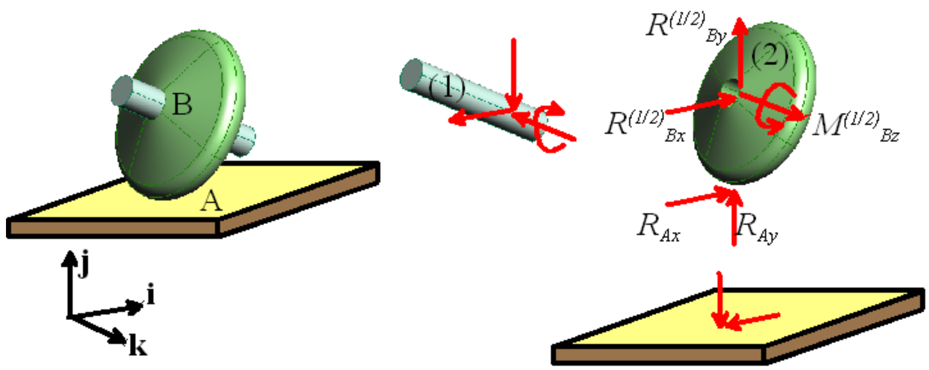
The picture shows a free body diagram for a 2D wheel mounted on a frictionless bearing.



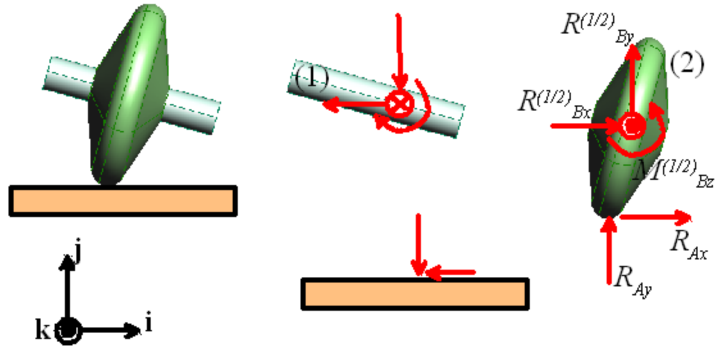

Since only two forces act on the wheel (the force at the axle, and the contact force), it behaves just like a 2 force member. The two forces must be equal and opposite, and must act along the same line. Moreover, the contact force must satisfy $R_{Ay} > 0$.

*For a freely rotating 3D wheel, there are 2 components of reaction force acting at the contact between the wheel and ground. One component acts perpendicular to the ground; the other acts parallel to the ground and perpendicular to the direction of motion of the wheel (i.e. parallel to the projection of the wheel's axle on the ground) .*

The picture below shows all the forces and moments acting on a freely rotating 3D wheel. The reactions that act on the axle are also shown.




A view from in front of the wheel shows the directions of the forces and moments more clearly




The forces and moments shown are the *only nonzero components of reaction force*.

The missing force and moment components can be shown to be zero by considering force and moment balance for the wheel. The details are left as an exercise.

Finally, a word of caution.

**You can only use these shortcuts if:**

1. **The wheel's weight in negligible;**
2. **The wheel rotates freely (no bearing friction, and nothing driving the wheel);**
3. **There is only one contact point on the wheel.**

If any of these conditions are violated you must solve the problem by applying all the proper reaction forces at contacts and bearings, and drawing a separate free body diagram for the wheel.

## 2.5 Friction Forces

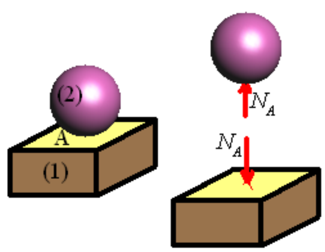
Friction forces act wherever two solids touch. It is a type of contact force – but rather more complicated than the contact forces we've dealt with so far.

It's worth reviewing our earlier discussion of contact forces. When we first introduced contact forces, we said that the nature of the forces acting at a contact depends on three things:

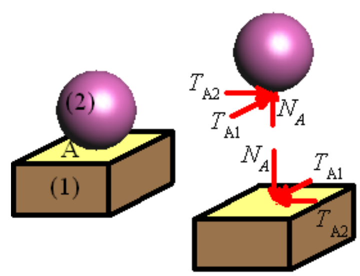
(4) Whether the contact is lubricated, i.e. whether friction acts at the contact
(5) Whether there is significant rolling resistance at the contact
(6) Whether the contact is *conformal*, or *nonconformal*.

We have so far only discussed two types of contact (a) fully lubricated (frictionless) contacts; and (b) ideally rough (infinite friction) contacts.

Remember that for a frictionless contact, only one component of force acts on the two contacting solids, as shown in the picture on the left below. In contrast, for an ideally rough (infinite friction) contact, three components of force are present as indicated on the figure on the right.

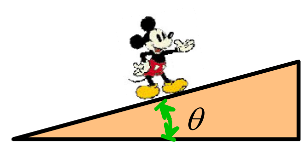



(a) Reaction forces at a frictionless contact

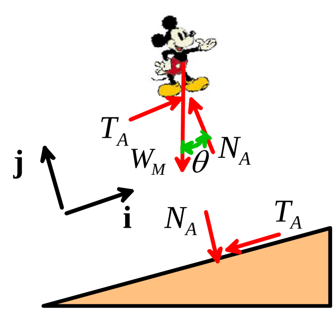



(b) Reaction forces at an ideally rough contact

All real surfaces lie somewhere between these two extremes. The contacting surfaces will experience both a normal and tangential force. The normal force must be repulsive, but can have an arbitrary magnitude. The tangential forces can act in any direction, but their magnitude is limited. If the tangential forces get too large, the two contacting surfaces will *slip* relative to each other.




This is why it's easy to walk up a dry, rough slope, but very difficult to walk up an icy slope. The picture below helps understand how friction forces work. The picture shows the big MM walking up a slope with angle $\theta$, and shows the forces acting on M and the slope. We can relate the normal and tangential forces acting at the contact to Mickey's weight and the angle $\theta$ by doing a force balance




Omitting the tedious details, we find that

$$T_A = W_M \sin\theta$$

$$N_A = W_M \cos\theta$$

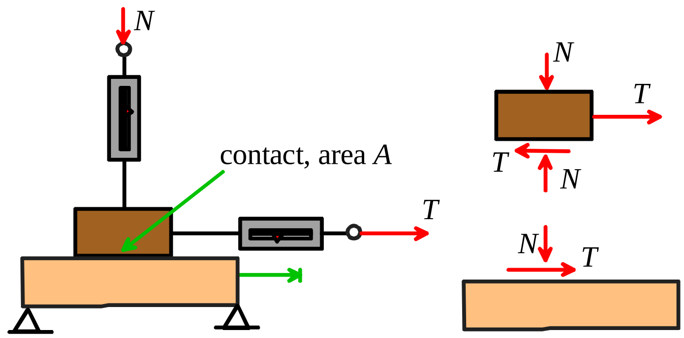
Note that a tangential force $T_A = W_M \sin\theta$ must act at the contact. If the tangential force gets too large, then Mickey will start to slip down the slope.

When we do engineering calculations involving friction forces, we always want to calculate the forces that will cause the two contacting surfaces to slip. Sometimes (e.g. when we design moving machinery) we are trying to calculate the forces that are needed to *overcome* friction and keep the parts moving. Sometimes (e.g. when we design self-locking joints) we need to check whether the contact can safely support tangential force without sliding.

### 2.5.1 Experimental measurement of friction forces

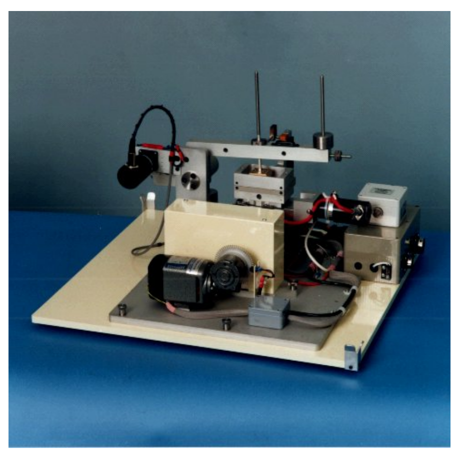
To do both these calculations, we need to know how to determine the critical tangential forces that cause contacting surfaces to slip. The critical force must be determined experimentally. Leonardo da Vinci was apparently the first person to do this – his experiments were repeated by Amontons and Coulomb about 100 years later. We now refer to the formulas that predict friction forces as Coulomb's law or Amonton's law (you can choose which you prefer!).

The experiment is conceptually very simple – it's illustrated in the figure.



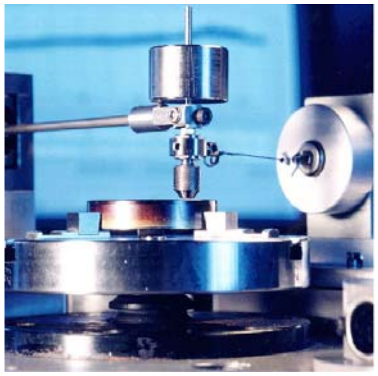
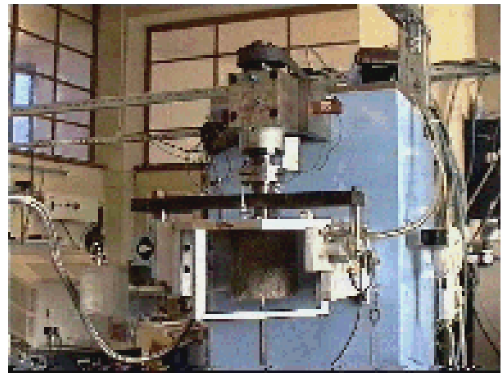

We put two solids in contact, and push them together with a normal force $N$. We then try to slide the two solids relative to each other by applying a tangential force $T$. The forces could be measured by force transducers or spring scales. A simple equilibrium calculation shows that, as long as the weight of the components can be neglected, the contacting surfaces must be subject to a normal force $N$ and a tangential force $T$.

In an experiment, a normal force would first be applied to the contact, and then the tangential force would be increased until the two surfaces start to slip. We could measure the critical tangential force as a function of $N$, the area of contact $A$, the materials and lubricants involved, the surface finish, and other variables such as temperature.



You can buy standard testing equipment for measuring friction forces – one configuration is virtually identical to the simple experiment described above – a picture (from http://www.plint-tribology.fsnet.co.uk/cat/at2/leaflet/te75r.htm ) is shown below. This instrument is used to measure friction between polymeric surfaces.

There are many other techniques for measuring friction. One common configuration is the `pin on disk' machine. Two examples are shown below. The picture on the left is from **www.ist.fhg.de/leistung/gf4/ qualitaet/bildgro4.html** , and shows details of the pin and disk. The picture on the right, from **www.ulg.ac.be/tribolog/ test.htm** shows a pin on disk experiment inside an environmental chamber. In this test, a pin is pressed with a controlled force onto the surface of a rotating disk. The force required to hold the pin stationary is measured.

Another test configuration consists of two disks that are pressed into contact and then rotated with different speeds. The friction force can be deduced by measuring the torque required to keep the disks moving. An example (from http://www.ms.ornl.gov/htmlhome/mituc/te53.htm ) is shown below

If you want to see a real friction experiment visit Professor Tullis' lab at Brown (you don't actually have to go there in person; he has a web site with very detailed descriptions of his lab) – he measures friction between rocks, to develop earthquake prediction models.

A friction experiment must answer two questions:

(i) What is the critical tangential force that will cause the surfaces to start to slide? The force required to initiate sliding is known as the *static friction* force.

(ii) If the two surfaces do start to slip, what tangential force is required to *keep* them sliding? The force required to maintain steady sliding is referred to as the *kinetic friction* force.

We might guess that the critical force required to cause sliding could depend on

(i) The area of contact between the two surfaces
(ii) The magnitude of the normal force acting at the contact
(iii) Surface roughness
(iv) The nature of the crud on the two surfaces
(v) What the surfaces are made from

We might also guess that once the surfaces start to slide, the tangential force needed to maintain sliding will depend on the sliding velocity, in addition to the variables listed.

In fact, experiments show that

(i) The critical force required to initiate sliding between surfaces is independent of the area of contact. This is very weird. In fact, when Coulomb first presented this conclusion to the Academy Francaise, he was thrown out of the room, because the academy thought that the strength of the contact should increase in proportion to the contact area. We'll discuss why it doesn't below.

(ii) The critical force required to initiate sliding between two surfaces is proportional to the normal force. If the normal force is zero, the contact can't support *any* tangential force. Doubling the normal force will double the critical tangential force that initiates slip.

(iii) Surface roughness has a very modest effect on friction. Doubling the surface roughness might cause only a few percent change in friction force.

(iv) The crud on the two surfaces has a big effect on friction. Even a little moisture on the surfaces can reduce friction by 20-30%. If there's a thin layer of grease on the surfaces it can cut friction by a factor of 10. If the crud is *removed*, friction forces can be huge, and the two surfaces can seize together completely.

(v) Friction forces depend quite strongly on what the two surfaces are made from. Some materials like to bond with each other (metals generally bond well to other metals, for example) and so have high friction forces. Some materials (e.g. Teflon) don't bond well to other materials. In this case friction forces will be smaller.

(v) If the surfaces start to slide, the tangential force often (but not always) drops slightly. Thus, kinetic friction forces are often a little lower than static friction forces. Otherwise, kinetic friction forces behave just like static friction – they are independent of contact area, are proportional to the normal force, etc.

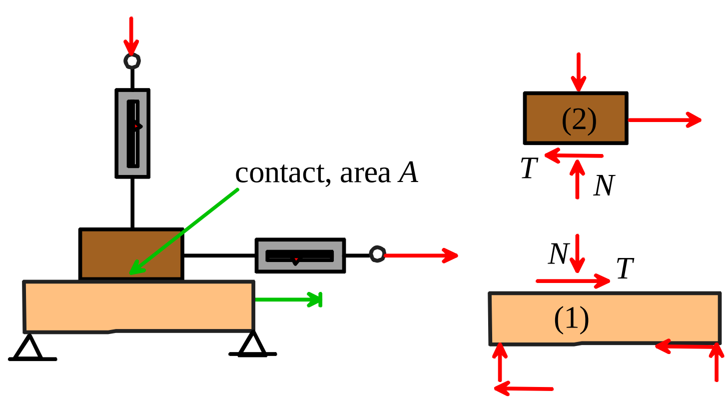
(vi) The kinetic friction force usually (but not always) decreases slightly as the sliding speed increases. Increasing sliding speed by a factor of 10 might drop the friction force by a few percent.

Note that there are some exceptions to these rules. For example, friction forces acting on the tip of an atomic force microscope probe will behave completely differently (but you'll have to read the scientific literature to find out how and why!). Also, rubbers don't behave like most other materials. Friction forces between rubber and other materials don't obey all the rules listed above.

### 2.5.2 Definition of friction coefficient: the Coulomb/Amonton friction law

A simple mathematical formula known as the Coulomb/Amonton friction law is used to describe the experimental observations listed in the preceding section.

*Friction forces at 2D contacts*




Friction forces at a 2D contact are described by the following laws:

(i) If the two contacting surfaces do not slide, then

$$|T| < \mu N$$

(ii) The two surfaces will start to slip if

$$|T| = \mu N$$

(iii) If the two surfaces are sliding, then

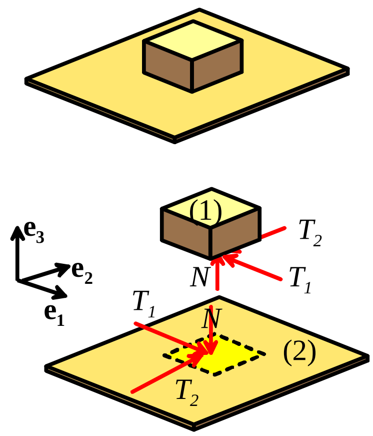
$$T = \pm \mu N$$

The sign in this formula must be selected so that $T$ opposes the direction of slip.

In all these formulas, $\mu$ is called the `coefficient of friction' for the two contacting materials. For most engineering contacts, $0 < \mu < 1$. Actual values are listed below.

Probably we need to explain statement (iii) in more detail. Why is there a $\pm$? Well, the picture shows the tangential force $T$ acting to the right on body (1) and to the left on body (2). If (1) is stationary and (2) moves to the right, then this is the correct direction for the force and we'd use $T = +\mu N$. On the other hand, if (1) were stationary and (2) moved to the left, then we'd use $T = -\mu N$ to make sure that the tangential force acts so as to oppose sliding.

*Friction forces at 3D contacts*




3D contacts are the same, but more complicated. The tangential force can have *two components*. To describe this mathematically, we introduce a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ with $\mathbf{e}_1, \mathbf{e}_2$ in the plane of the contact, and $\mathbf{e}_3$ normal to the contact. The tangential force $\mathbf{T}^{(1/2)}$ exerted by body (1) on body (2) is then expressed as components in this basis

$$\mathbf{T}^{(1/2)} = T_1 \mathbf{e}_1 + T_2 \mathbf{e}_2$$

(i) If the two contacting surfaces do not slide, then

$$\sqrt{T_1^2 + T_2^2} < \mu N$$

(ii) The two surfaces will start to slip if

$$\sqrt{T_1^2 + T_2^2} = \mu N$$

(iii) If the two surfaces are sliding, then

$$\mathbf{T}^{(1/2)} = \mu N \frac{\mathbf{v}_{12}}{|\mathbf{v}_{12}|}$$

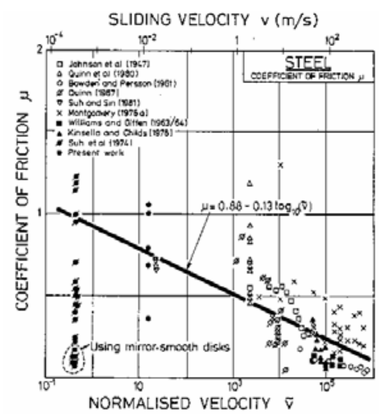
where $\mathbf{T}^{(1/2)}$ denotes the tangential force exerted by body (1) on body (2), and $\mathbf{v}_{12}$ is the relative velocity of body (1) with respect to body (2) at the point of contact. The relative velocity can be computed from the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ of the two contacting solids, using the equation $\mathbf{v}_{12} = \mathbf{v}_1 - \mathbf{v}_2$.

### 2.5.3 Experimental values for friction coefficient

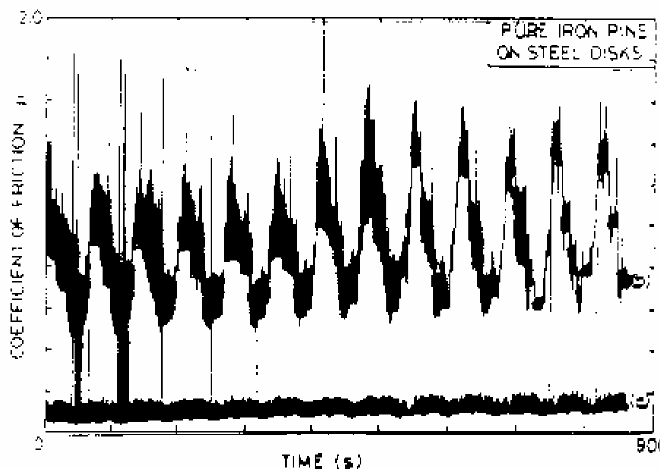
The table below (taken from `Engineering Materials' by Ashby and Jones, Pergammon, 1980) lists rough values for friction coefficients for various material pairs.

| Material | Approx friction coefficient |
|---|---|
| Clean metals in air | 0.8-2 |
| Clean metals in wet air | 0.5-1.5 |
| Steel on soft metal (lead, bronze, etc) | 0.1-0.5 |
| Steel on ceramics (sapphire, diamond, ice) | 0.1-0.5 |
| Ceramics on ceramics (eg carbides on carbides) | 0.05-0.5 |
| Polymers on polymers | 0.05-1.0 |
| Metals and ceramics on polymers (PE, PTFE, PVC) | 0.04-0.5 |
| Boundary lubricated metals (thin layer of grease) | 0.05-0.2 |
| High temperature lubricants (eg graphite) | 0.05-0.2 |
| Hydrodynamically lubricated surfaces (full oil film) | 0.0001-0.0005 |

These are rough guides only – friction coefficients for a given material can by highly variable. For example, Lim and Ashby (Cambridge University Internal Report CUED/C-mat./TR.123 January 1986) have catalogued a large number of experimental measurements of friction coefficient for steel on steel, and present the data graphically as shown below. You can see that friction coefficient for steel on steel varies anywhere between 0.0001 to 3.




Friction coefficient can even vary significantly during a measurement. For example, the picture below (from Lim and Ashby, Acta Met 37 3 (1989) p 767) shows the time variation of friction coefficient during a pin-on-disk experiment.



### 2.5.4 Static and kinetic friction

Many introductory statics textbooks define two different friction coefficients. One value, known as the *coefficient of static friction* and denoted by $\mu_s$, is used to model static friction in the equation giving the condition necessary to initiate slip at a contact

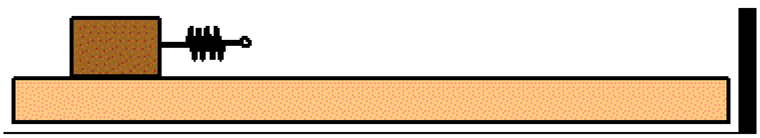
$$|T| < \mu_s N$$

A second value, known as the *coefficient of kinetic friction*, and denoted by $\mu_k$, is used in the equation for the force required to maintain steady sliding between two surfaces

$$T = \pm \mu_k N$$

I don't like to do this (I'm such a rebel). It is true that for some materials the static friction force can be a bit higher than the kinetic friction force, but this behavior is by no means universal, and in any case the difference between $\mu_k$ and $\mu_s$ is very small (of the order of 0.05). We've already seen that $\mu$ can vary far more than this for a given material pair, so it doesn't make much sense to quibble about such a small difference.

The real reason to distinguish between static and kinetic friction coefficient is to provide a simple explanation for *slip-stick oscillations* between two contacting surfaces. Slip-stick oscillations often occur when we try to do the simple friction experiment shown below.



If the end of the spring is moved steadily to the right, the block sticks for a while until the force in the spring gets large enough to overcome friction. At this point, the block jumps to the right and then sticks again, instead of smoothly following the spring. If $\mu$ were constant, then this behavior would be impossible. By using $\mu_s > \mu_k$, we can explain it. But if we're not trying to model slip-stick oscillations, it's much more sensible to work with just one value of $\mu$.

In any case, there's a much better way to model slip-stick oscillations, by making $\mu$ depend on the velocity of sliding. Most sophisticated models of slip-stick oscillations (e.g. models of earthquakes at faults) do this.

### 12.6 The microscopic origin of friction forces

Friction is weird. In particular, we need to explain

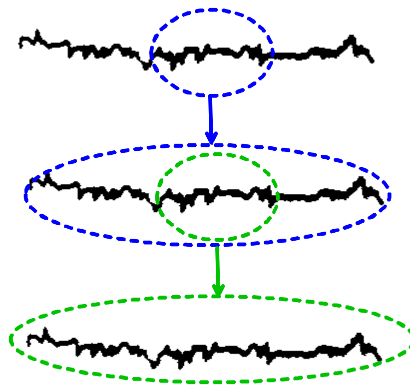
(i) why friction forces are independent of the contact area
(ii) why friction forces are proportional to the normal force.

Coulomb grappled with these problems and came up with an incorrect explanation. A truly satisfactory explanation for these observations was only found 20 years or so ago.

To understand friction, we must take a close look at the nature of surfaces. Coulomb/Amonton friction laws are due to two properties of surfaces:

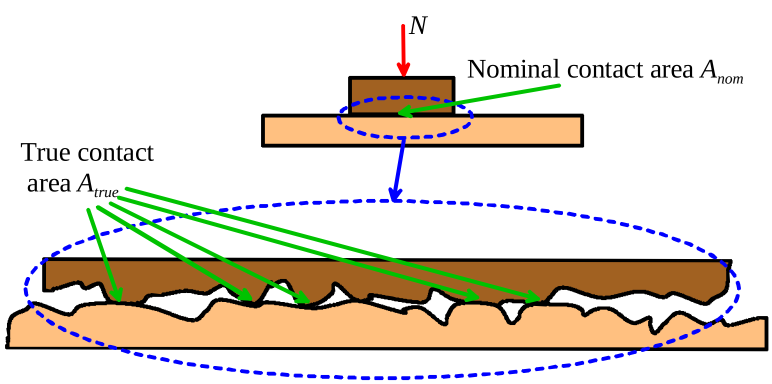
(1) All surfaces are rough;
(2) All surfaces are covered with a thin film of oxide, an adsorbed layer of water, or an organic film.

Surface roughness can be controlled to some extent – a cast surface is usually very rough; if the surface is machined the roughness is somewhat less; roughness can be reduced further by grinding, lapping or polishing the surfaces. But you can't get rid of it altogether. Many surfaces can be thought of as having a *fractal* geometry. This means that the roughness is statistically self-similar with length scale – as you zoom in on the surface, it always looks (statistically) the same (more precisely the surfaces are *self-affine*. When you zoom in, it looks like the surface got stretched vertically – surfaces are rougher at short wavelengths than at long ones).



Of course no surface can be truly fractal: roughness can't be smaller than the size of an atom and can't be larger than the component; but most surfaces look fractal over quite a large range of lengths. Various statistical measures are used to quantify surface roughness, but a discussion of these parameters is beyond the scope of this course.

Now, visualize what the contact between two rough surfaces looks like. The surfaces will only touch at high spots (these are known in the trade as `asperities') on the two surfaces. Experiments suggest that there are huge numbers of these contacts (nobody has really been able to determine with certainty how many there actually are). The asperity tips are squashed flat where they contact, so that there is a finite total area of contact between the two surfaces. However, the *true* contact area (at asperity tips) is much smaller than the *nominal* contact area.




The true contact area can be estimated by measuring the surface roughness, and then calculating how the surfaces deform when brought into contact. At present there is some uncertainty as to how this should be done – this is arguably the most important unsolved problem in the field. The best estimates we have today all agree that:

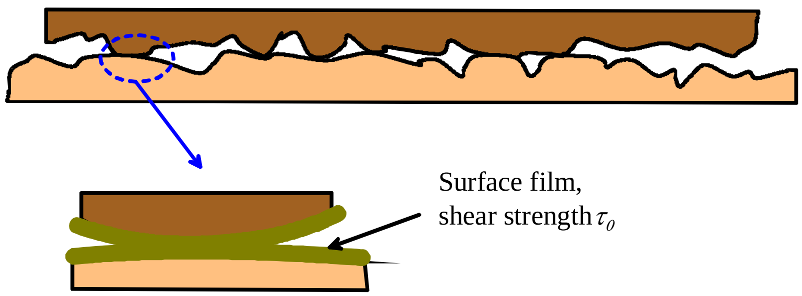
*The true area of contact between two rough surfaces is proportional to the normal force pressing them together*.

$$A_{true} = CN$$

At present, there is no way to measure or calculate the contact $C$ accurately.

This is true for all materials (except for rubbers, which are so compliant that the true contact area is close to the nominal contact area), and is just a consequence of the statistical properties of surface roughness. The reason that the true contact area increases in proportion to the load is that as the surfaces are pushed into contact, the *number* of asperity contacts increases, but the average size of the contacts remains the same, because of the fractal self-similarity of the two surfaces.

Finally, to understand the cause of the Coulomb/Amonton friction law, we need to visualize what happens when two rough surfaces slide against each other.




Each asperity tip is covered with a thin layer of oxide, adsorbed water, or grease. It's possible to remove this film in a lab experiment – in which case friction behavior changes dramatically and no longer follows Coulomb/Amonton law – but for real engineering surfaces it's always present.

The film usually has a low mechanical strength. It will start to deform, and so allow the two asperities to slide past each other, when the tangential force per unit area acting on the film reaches the shear strength of the film $\tau_0$.

The tangential friction force due to shearing the film on the surface of all the contacting asperities is therefore

$$T = \tau_0 A_{true}$$

Combining this with the earlier result for the true contact area gives

$$T = \tau_0 CN$$
$$\Rightarrow \mu = \tau_0 C$$

Thus, the friction force is proportional to the normal force. This simple argument also explains why friction force is independent of contact area; why it is so sensitive to surface films, and why it can be influenced (albeit only slightly) by surface roughness.