An introduction to ludics

Lionel Vaux

Institut de Mathématiques de Luminy, Marseille, France

LOCI Symposium:
Rebuilding logic and rethinking language in interaction terms
21 July 2011

Ludics in a few words

Ludics:

- erases the distinction between syntax and semantics;
- allows to rebuild logic from the sole notion of interaction.

The basic artifact of ludics is the *design*:

- designs are abstract representations of linear logic proofs;
- designs rely on an alternation of polarities in proofs;
- designs retain only the information relevant for *local* interaction;
- designs needs not represent correct proofs.

Linear Logic

- Girard, 80's
- classical logic: negation is involutive
- takes cut elimination in sequent calculus seriously
- drops structural rules

A quick reminder

- a sequent is a pair of lists: $A_1, \dots, A_n \vdash B_1, \dots, B_p$
- it "means" $A_1 \wedge \cdots \wedge A_n \Rightarrow B_1 \vee \cdots \vee B_p$
- the cut rule is $\dfrac{A \vdash B \qquad B \vdash C}{A \vdash C}$
- cut elimination gives proofs without detours, which have good properties
- up to De Morgan laws, we can restrict to sequents $\vdash B_1, \dots, B_p$ and the cut becomes $\dfrac{\vdash A, B \qquad \vdash \neg B, C}{\vdash A, C}$
- provable sequents admit cut free proofs

Linear logic: rules

	multiplicative	additive
$\wedge$:	$\dfrac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\,(\otimes)$	$\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\,(\&)$
$\vee$:	$\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,(\parr)$	$\dfrac{\vdash \Gamma, A_i}{\vdash \Gamma, A_1 \oplus A_2}\,(\oplus_i)$

Linear logic: rules

$$\frac{}{\vdash X^\perp, X}\ (\mathrm{ax})$$

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$$\frac{\vdash \Gamma, A \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\ (\mathrm{cut})$$

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$$\frac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ (\text{cut})$$

$$A, B := X \mid X^{\perp} \mid A \parr B \mid A \otimes B \mid A \mathbin{\&} B \mid A \oplus B$$

$$(A \parr B)^{\perp} = A^{\perp} \otimes B^{\perp}$$

$$(A \mathbin{\&} B)^{\perp} = A^{\perp} \oplus B^{\perp}$$

Linear logic: cut elimination

A multiplicative cut ($\parr / \otimes$):

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \vdash \Gamma, A\end{matrix} \qquad \begin{matrix}\vdots \\ \vdash \Gamma', B\end{matrix}}{\vdash \Gamma, \Gamma', A \otimes B}\,(\otimes) \qquad \dfrac{\begin{matrix}\vdots \\ \vdash \Delta, A^{\perp}, B^{\perp}\end{matrix}}{\vdash \Delta, A^{\perp} \parr B^{\perp}}\,(\parr)}{\vdash \Gamma, \Gamma', \Delta}\,(\text{cut})$$

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reduces to

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Linear logic: cut elimination

An additive cut ($\&/\oplus$):

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Linear logic: cut elimination

Identity:

$$\dfrac{\dfrac{}{\vdash A, \underline{A^\perp}}\,(\mathrm{ax}) \qquad \begin{array}{c}\vdots \\ \vdash \underline{A}, \Gamma\end{array}}{\vdash A, \Gamma}\,(\mathrm{cut})$$

reduces to

$$\begin{array}{c}\vdots \\ \vdash A, \Gamma\end{array}$$

Linear logic: cut elimination

Bureaucracy: e.g.,

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \vdash \Gamma, A, B\end{matrix}}{\vdash \Gamma, A \parr B}(\parr) \qquad \dfrac{\begin{matrix}\vdots \\ \vdash \Delta, A^{\perp} \otimes B^{\perp}, C\end{matrix}}{\vdash \Delta, A^{\perp} \otimes B^{\perp}, C \oplus D}(\oplus_1)}{\vdash \Gamma, \Delta, C \oplus D}(\mathrm{cut})$$

Linear logic: cut elimination

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Focusing

Reversibility

The connectives $\parr$ and $\&$ are *reversible*:
from the conclusion and active formula, one can recover the premises.

During proof search, one can always perform reversible rules.
We thus divide connectors between two classes: $\parr$ and $\&$ are *negative*, and $\otimes$ and $\oplus$ are *positive*.
Positive connectors are *not* reversible but:

Focusing

Every provable sequent admits a focused cut-free proof.

A cut-free proof is *focused* if:

- each time we decompose a formula using an introduction rule, we focus on its subformulas, as long as they have the same polarity;
- if a sequent contains a negative formula, we first apply negative rules.

Synthetic connectives: rules

Up to focusing and the distributivity isomorphism $A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$, we obtain:

- one negative (reversible) rule:

$$\frac{\left(\vdash (P_{i,j})_{j \in J_i}, \Gamma\right)_{i \in I}}{\vdash \&_{i \in I} \parr_{j \in J_i} P_{i,j}, \Gamma} \, (-)$$

- one positive rule:

$$\frac{(\vdash N_{i_0,j}, \Gamma_j)_{j \in J_{i_0}}}{\vdash \bigoplus_{i \in I} \bigotimes_{j \in J_i} N_{i,j}, \Gamma} \, (+, i_0)$$

with $\Gamma = \sum_{j \in J_{i_0}} \Gamma_j$.

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Plus axiom and cut.

Synthetic connectives: cut elimination

$$\dfrac{\dfrac{\left(\begin{array}{c}\pi_j\\ \vdots\\ \vdash P^{\perp}_{i_0,j},\Gamma_j\end{array}\right)_{j\in J_{i_0}}}{\vdash \bigoplus_{i\in I}\bigotimes_{j\in J_i} P^{\perp}_{i,j},\Gamma}\,(+,i_0) \qquad \dfrac{\left(\begin{array}{c}\rho_{i,j}\\ \vdots\\ \vdash (P_{i,j})_{j\in J_i},\Delta\end{array}\right)_{i\in I}}{\vdash \&_{i\in I}\invamp_{j\in J_i} P_{i,j},\Delta}\,(-)}{\vdash \Gamma,\Delta}\,(\mathrm{cut})$$

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$$
\dfrac{
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\qquad
\dfrac{\left(\begin{array}{c}\rho_{i,j} \\ \vdots \\ \vdash (P_{i,j})_{j \in J_i}, \Delta\end{array}\right)_{i = i_0}}{\vdash \underline{\&_{i\in I} \invamp_{j \in J_i} P_{i,j}}, \Delta}\,(-)
}{\vdash \Gamma, \Delta}\,(\text{cut})
$$

Synthetic connectives: cut elimination

$$\dfrac{\dfrac{\left(\begin{array}{c}\pi_j\\ \vdots\\ \vdash P_{i_0,j}^{\perp}, \Gamma_j\end{array}\right)_{j\in J_{i_0}}}{\vdash \bigoplus_{i\in I}\bigotimes_{j\in J_i} P_{i,j}^{\perp}, \Gamma}(+, i_0) \qquad \dfrac{\left(\begin{array}{c}\rho_{i,j}\\ \vdots\\ \vdash (P_{i,j})_{j\in J_i}, \Delta\end{array}\right)_{i=i_0}}{\vdash \&_{i\in I}\,⅋_{j\in J_i} P_{i,j}, \Delta}(-)}{\vdash \Gamma, \Delta}(\text{cut})$$

reduces to

$$\dfrac{\left(\begin{array}{c}\pi_j\\ \vdots\\ \vdash P_{i_0,j}^{\perp}, \Gamma_j\end{array}\right)_{j\in J_{i_0}} \qquad \begin{array}{c}\rho_{i_0,j}\\ \vdots\\ \vdash (P_{i_0,j})_{j\in J_{i_0}}, \Delta\end{array}}{\vdash (\Gamma_j)_{j\in J_{i_0}}, \Delta}(\text{cut}) \times \# J_{i_0}$$

Loci

Ludics founds logic on the interaction between proofs: cut-elimination between A and $A^\perp$.

To enable this dialogue without preconception:

- Ludics forgets about the meaning of formulas. Sequents only retain information on the location of subformulas: the *locus*.
- It introduces a generic "dummy" proof: the *daimon*. The essential point of interaction is that both parties should reach an agreement: one must give up, using the daimon.

Definition

An address (or locus) is a finite list of natural numbers.
A sequent is a pair $\Lambda \vdash \Delta$ where Λ holds at most one formula.
If $\Lambda = \emptyset$ the sequent is positive, otherwise it is negative.

Designs

. . . as abstract proof trees (dessins)

daimon

$$\frac{}{\vdash \Delta}\ (\maltese)$$

negative rule

$$\frac{\left(\vdash (\xi i)_{i \in I}\,, \Delta_I\right)_{I \in \mathcal{N}}}{\xi \vdash \Delta}\ (-, \xi, \mathcal{N})$$

where $\mathcal{N} \subseteq \mathfrak{P}_f(\mathbf{N})$ and each $\Delta_I \subseteq \Delta$.

positive rule

$$\frac{(\xi i \vdash \Delta_i)_{i \in I}}{\vdash \xi, \Delta}\ (+, \xi, I)$$

where I is finite, $\bigcup \Delta_i \subseteq \Delta$, and $\Delta_i \cap \Delta_j = \emptyset$ for all $i \neq j$.

Proofs as designs

$$\dfrac{\dfrac{\begin{matrix}\vdots & \qquad & \vdots \\ \vdash P, Q, S & \qquad & \vdash R, S\end{matrix}}{\vdash (P \parr Q) \,\&\, R, S}\,(-)}{\vdash T \oplus ((P \parr Q) \,\&\, R) \oplus U, S}\,(+, \{2\})$$

becomes

$$\dfrac{\dfrac{\begin{matrix}\vdots & \qquad & \vdots \\ \vdash \xi 21, \xi 22, \sigma & \qquad & \vdash \xi 23, \sigma\end{matrix}}{\xi 2 \vdash \sigma}\,(-, \xi 2, \{\{1,2\},\{3\}\})}{\vdash \xi, \sigma}\,(+, \xi, \{2\})$$

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- Designs as dessins (trees) actually retain irrelevant information about the context of rules: compare

$$\dfrac{\dfrac{\dfrac{}{\vdash \xi 12}\;(\maltese)}{\xi 1 \vdash \sigma}\;(-, \xi 1, \{\{2\}\})}{\vdash \xi, \sigma}\;(+, \xi, \{1\}) \quad \text{with} \quad \dfrac{\dfrac{\dfrac{}{\vdash \xi 12}\;(\maltese)}{\xi 1 \vdash}\;(-, \xi 1, \{\{2\}\})}{\vdash \xi, \sigma}\;(+, \xi, \{1\})$$

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  One can introduce a further level of abstraction to fix this: designs as strategies (desseins).
  Intuitively: desseins = sets of branches in a dessin.

Interaction: cut nets

Definition

A cut net is a non empty set of designs s.t.:

- addresses in conclusions are either disjoint or identical;
- each address appears in at most two conclusions, and then with opposite polarities: this is a *cut*;
- the graph with conclusions as vertices and cuts as arrows is connected and acyclic.

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In particular there is exactly one design without a cut on the left: its conclusion is the main sequent and its last rule the main rule.

Interaction: cut elimination as normalization

The case of closed nets: all addresses are cuts

The main design D is then necessarily positive.

- The main rule is (✠): normalization immediately ends and results in ✠.
- The main rule is $(+, \xi, I)$: then ξ is a cut, with the negative address of another design E, whose last rule is $(-, \xi, \mathcal{N})$.
  - if $I \notin \mathcal{N}$, normalization fails;
  - otherwise, for all $i \in I$, we consider the subdesign D_i of D with conclusion $(\xi i \vdash \cdots)$, and the subdesign E' of E with conclusion $(\vdash \xi I, \cdots)$: we replace D with the D_i's and E with E'. We normalize the net obtained as the component of E'.

The general case

When none of the above cases applies, we normalize above the main rule (*cf.* commutative cuts in sequent calculus).

Example

Start from a net made of two designs:

$$
\dfrac{\dfrac{\dfrac{\vdots}{\xi 1 \vdash} \qquad \dfrac{\vdots}{\xi 2 \vdash \sigma 31}}{\vdash \xi, \sigma 31} (+, \xi, \{1,2\})}{\dfrac{\qquad}{\sigma 3 \vdash \xi} (-, \sigma 3, \{\{1\}\})} \qquad \dfrac{\vdots}{\sigma 7 \vdash}
$$

$$
\dfrac{\sigma 3 \vdash \xi \qquad \sigma 7 \vdash}{\vdash \xi, \sigma} (+, \sigma, \{3,7\})
$$

$$
\dfrac{\dfrac{\vdots}{\vdash \xi 0, \tau} \qquad \dfrac{\vdots}{\vdash \xi 1, \xi 2, \tau} \qquad \dfrac{\vdots}{\vdash \xi 3, \tau}}{\xi \vdash \tau} (-, \xi, \{\{0\}, \{1,2\}, \{3\}\})
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$$\dfrac{\begin{array}{c}\vdots\\ \vdash \xi 0, \tau\end{array} \qquad \begin{array}{c}\vdots\\ \vdash \xi 1, \xi 2, \tau\end{array} \qquad \begin{array}{c}\vdots\\ \vdash \xi 3, \tau\end{array}}{{\color{red}\xi} \vdash \tau}\,(-,\xi,\{\{0\},\{1,2\},\{3\}\})$$

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We reached a genuine cut.

Example

It remains to normalize a cut net made of three designs:

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \xi 1 \vdash\end{matrix} \qquad \begin{matrix}\vdots \\ \xi 2 \vdash \sigma 31\end{matrix}}{\sigma 3 \vdash}(-, \sigma 3, \{\{1\}\}) \qquad \begin{matrix}\vdots \\ \sigma 7 \vdash\end{matrix}}{\vdash \sigma}(+, \sigma, \{3, 7\})$$

$$\begin{matrix}\vdots \\ \vdash \xi 1, \xi 2, \tau\end{matrix}$$

Fax

There are no axioms, because there are no formulas.
Instead there is a generic η-expansion, given by the fax design $\mathfrak{F}_{\xi,\xi'}$:

$$
\dfrac{\cdots \qquad \dfrac{\cdots \qquad \begin{array}{c}\mathfrak{F}_{\xi' i,\xi i}\\ \vdots \\ \xi' i \vdash \xi i\end{array} \qquad \cdots}{\vdash \xi', (\xi i)_{i \in I}}\,(+,\xi',I) \qquad \cdots}{\xi \vdash \xi'}\,(-,\xi,\mathfrak{P}_f(\mathbf{N}))
$$

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The axiom $P \oplus Q \vdash P \oplus Q$ becomes:

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{F}_{\xi' 1,\xi 1}\\ \vdots \\ \xi' 1 \vdash \xi 1\end{matrix}}{\vdash \xi 1, \xi'}\,(+, \xi', \{1\}) \qquad \dfrac{\begin{matrix}\mathfrak{F}_{\xi' 1,\xi 1}\\ \vdots \\ \xi' 2 \vdash \xi 2\end{matrix}}{\vdash \xi 2, \xi'}\,(+, \xi', \{2\})}{\xi \vdash \xi'}\,(-, \xi, \{\{1\}, \{2\}\})$$

Fax

There are no axioms, because there are no formulas.
Instead there is a generic η-expansion, given by the fax design $\mathfrak{F}_{\xi,\xi'}$:

$$\dfrac{\cdots \qquad \dfrac{\cdots \qquad \begin{array}{c}\mathfrak{F}_{\xi' i,\xi i}\\ \vdots \\ \xi' i \vdash \xi i\end{array} \qquad \cdots}{\vdash \xi', (\xi i)_{i\in I}}\,(+,\xi',I) \qquad \cdots}{\xi \vdash \xi'}\,(-,\xi,\mathfrak{P}_f(\mathbf{N}))$$

Normalizing a design D of conclusion $\xi' \vdash \Gamma$ with $\mathfrak{F}_{\xi,\xi'}$ results in a relocalized design D', with conclusion $\xi \vdash \Gamma$.

Rebuilding logic: orthogonality

Definition

Let D be a design with conclusion $\Lambda \vdash \Gamma$ and for all $\xi \in \Lambda \cup \Gamma$, let E_ξ be a designs of conclusion $\vdash \xi$ or $\xi \vdash$ so that $N = \{D\} \cup \{E_\xi \mid \xi \in \Lambda \cup \Gamma\}$ is a closed cut net.
We say D is orthogonal to (E_ξ) if N normalizes to the daimon.

Rebuilding logic: behaviours

Definition

Let $\mathbf{D}$ be a set of designs with the same conclusion: we write $\mathbf{D}^{\perp\perp}$ for its bidual.

We say $\mathbf{D}$ is a behaviour if $\mathbf{D} = \mathbf{D}^{\perp\perp}$.

Rebuilding logic: behaviours

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Let $\mathbf{D}$ be a set of designs with the same conclusion: we write $\mathbf{D}^{\perp\perp}$ for its bidual.

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Behaviours are the ludics counterpart of formulas.

Rebuilding logic: additives

- Any intersection of behaviours is a behaviour.
- It does not necessarily hold for union: write $\bigsqcup \mathbf{D}_i = (\bigcup \mathbf{D}_i)^{\perp\perp}$.
- If $\mathbf{D}_1 \cap \mathbf{D}_2 = \emptyset$, $\mathbf{D}_1 \sqcup \mathbf{D}_2 = \mathbf{D}_1 \cup \mathbf{D}_2$.

Fact

$\bigcap$ and $\bigsqcup$ provide *locative* interpretations of $\&$ and $\bigoplus$.
To recover the usual connectives, we should introduce some more structure.

Rebuilding logic: multiplicatives

The basic idea is to introduce a binary operation on positive designs:
if the first (positive) actions of D and D' are I and J,
we form a new design $D \odot D'$ with first action $I \cup J$,
and branches selected among those of D and D'.

Fact

Several choices for $\odot$ are possible, with interesting properties.
Setting $\mathbf{D} \otimes \mathbf{D}' = \{D \odot D' \mid D \in \mathbf{D}, D' \in \mathbf{D}'\}^{\perp\perp}$ provides a locative interpretation of tensor.
We recover $\parr$ by duality.

What is missing from this talk?

Almost everything :-)

- the good notion of designs (desseins);
- beautiful theorems (associativity, separation, stability, . . .);
- the notion of truth;
- completeness theorems;
- *etc.*

(...)