

# Stat 475

## Life Contingencies I

### Chapter 5: Life annuities

Here we're going to consider the valuation of **life annuities**.

- A life annuity is regularly (e.g., continuously, annually, monthly, etc.) spaced series of payments, which are usually based on the survival of the policyholder.
- These annuities are important for retirement plans, pensions, structured settlements, life insurance, and in many other contexts.

Many of the ideas and notation used in the valuation of life annuities borrow from annuities-certain.

- As with life insurance, the life-contingent nature of the payments means that the present value of benefits is a *random variable* rather than a fixed number.

# Review of annuities-certain

Recall the actuarial symbols for the present value of  $n$ -year annuities-certain:

$$\text{Annuity-immediate: } a_{\overline{n}|} = \sum_{k=1}^n v^k = \frac{1 - v^n}{i}$$

$$\text{Annuity-due: } \ddot{a}_{\overline{n}|} = \sum_{k=0}^{n-1} v^k = \frac{1 - v^n}{d}$$

$$\text{Continuous annuity: } \bar{a}_{\overline{n}|} = \int_0^n e^{-\delta t} dt = \frac{1 - v^n}{\delta}$$

$$m^{\text{th}} \text{ly annuity-immediate: } a_{\overline{n}|}^{(m)} = \sum_{k=1}^{nm} \frac{1}{m} v^{k/m} = \frac{1 - v^n}{i^{(m)}}$$

Increasing annuity-immediate:

$$(Ia)_{\overline{n}|} = \sum_{k=1}^n k v^k = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$$

Decreasing annuity-immediate:

$$(Da)_{\overline{n}|} = \sum_{k=1}^n (n+1-k) v^k = \frac{n - a_{\overline{n}|}}{i}$$

For the corresponding accumulated values, we replace  $a$  by  $s$  in each case. For example:

$$a_{\overline{n}|}(1+i)^n = s_{\overline{n}|}$$

# Whole life annuity-due

Consider an annuity that pays ( $x$ ) an amount of \$1 on an annual basis for as long as ( $x$ ) is alive, with the first payment occurring immediately.

- This type of life annuity is known as a **whole life annuity-due**.

We'll denote the random variable representing the PV of this annuity benefit by  $Y$ :

$$Y = 1 + v + v^2 + \dots + v^{K_x} = \ddot{a}_{\overline{K_x+1}|} \quad (1)$$

The pmf of  $Y$  is given by:

$$P(Y = \ddot{a}_{\overline{k}|}) = {}_{k-1}q_x \quad \text{for } k = 1, 2, 3, \dots$$

# Whole life annuity-due EPV

The EPV of this annuity is denoted by  $\ddot{a}_x$  and we can use our general strategy to find this EPV:

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_k p_x$$

We could also take the expectation of the expression given in (1):

## Important Relation

$$\ddot{a}_x = E[Y] = \frac{1 - A_x}{d} \quad \text{so that} \quad 1 = d \ddot{a}_x + A_x$$

This relationship actually holds between the underlying random variables, not just the expected values.

# Whole life annuity-due EPV, variance, and recursion

Finally, we could calculate the EPV using the standard formula for the expectation of a discrete RV:

$$\ddot{a}_x = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|} v^k q_x$$

We can find the variance of  $Y$  in a similar manner:

$$V[Y] = \frac{{}^2A_x - (A_x)^2}{d^2}$$

(Note that we **cannot** find the second moment of  $Y$  by computing the first moment at double the force of interest.)

Further, we have the recursion  $\ddot{a}_x = 1 + v p_x \ddot{a}_{x+1}$

# Term annuity-due

Now consider an annuity that pays ( $x$ ) an amount of \$1 on an annual basis for up to  $n$  years, so long as ( $x$ ) is alive, with the first payment occurring immediately.

- This type of life annuity is known as a **term annuity-due**.

For this type of annuity, PV of the benefit is:

$$Y = \ddot{a}_{\overline{\min(K_x+1, n)}|} = \begin{cases} \ddot{a}_{\overline{K_x+1}|} & \text{if } K_x \leq n - 1 \\ \ddot{a}_{\overline{n}|} & \text{if } K_x \geq n \end{cases}$$

In this case, the EPV is denoted by  $\ddot{a}_{x:\overline{n}|}$

EPV for Term Annuity-Due

$$\ddot{a}_{x:\overline{n}|} = \frac{1 - A_{x:\overline{n}|}}{d} = \sum_{t=0}^{n-1} v^t {}_t p_x = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|} {}_k|q_x + n p_x \ddot{a}_{\overline{n}|}$$





## Example

Suppose that a person currently age 65 wants to purchase a 3-year term annuity due with annual payments of \$50,000 each. Given that

$$p_{65} = 0.95 \quad p_{66} = 0.91 \quad p_{67} = 0.87 \quad i = 7\%$$

calculate the expected value and variance of the present value of this annuity benefit. [132,146.91; 503,557,608.1]

Now suppose that a life insurance company sells this type of annuity to 100 such people, all age 65, all with the same payment amounts. Assuming these people are all independent, calculate the expected value and variance of the total present value of all of the annuity benefits. [13,214,691; 50,355,760,810]

# Whole life annuity-immediate

Now we'll turn to consider life annuities-immediate, starting with a whole life annuity-immediate.

- This type of annuity pays  $(x)$  an amount of \$1 on an annual basis for as long as  $(x)$  is alive, with the first payment occurring at age  $x + 1$ .

We'll denote the random variable representing the PV of this annuity benefit by  $Y^*$ :

$$Y^* = v + v^2 + \dots + v^{K_x} = a_{\overline{K_x}|}$$

with pmf

$$P(Y^* = a_{\overline{k}|}) = {}_k|q_x \quad \text{for } k = 0, 1, 2, 3, \dots$$

## Whole life annuity-immediate (continued)

We have the relationship  $Y^* = Y - 1$ , where  $Y$  is the PV of the whole life annuity-due and  $Y^*$  is the corresponding PV of the corresponding whole life annuity-immediate.

- The only difference between these two annuities is the payment at time 0.
- From this relationship, we can obtain expressions for the EPV and variance of  $Y^*$

$$a_x = \ddot{a}_x - 1 \qquad V[Y^*] = V[Y] = \frac{{}^2A_x - (A_x)^2}{d^2}$$

We also have the recursion relation:  $a_x = v p_x + v p_x a_{x+1}$

# Term annuity-immediate

Now consider an annuity that pays  $(x)$  an amount of \$1 on an annual basis for up to  $n$  years, so long as  $(x)$  is alive, with the first payment occurring at age  $x + 1$ .

- This type of life annuity is known as a **term annuity-immediate**.

For this type of annuity, PV of the benefit is:

$$Y^* = a_{\overline{\min(K_x, n)}|} = \begin{cases} a_{\overline{K_x}|} & \text{if } K_x \leq n \\ a_{\overline{n}|} & \text{if } K_x > n \end{cases}$$

In this case, the EPV is denoted by  $a_{x:\overline{n}|}$

EPV for Term Annuity-Due

$$a_{x:\overline{n}|} = \sum_{t=1}^n v^t {}_t p_x = \ddot{a}_{x:\overline{n}|} + {}_n E_x - 1$$

# Example

Assume that  ${}_{20}E_x = 0.35$ ,  ${}_{20}q_x = 0.3$ , and  $A_{x:\overline{20}|}^1 = 0.2$ .

Find:

①  $A_{x:\overline{20}|}$  [0.55]

②  $\ddot{a}_{x:\overline{20}|}$  [13.21]

③  $a_{x:\overline{20}|}$  [12.56]

Your company currently offers a whole life annuity product that pays the annuitant 12,000 at the beginning of each year. A member of your product development team suggests enhancing the product by adding a death benefit that will be paid at the end of the year of death.

Using a discount rate,  $d$ , of 8%, calculate the death benefit that minimizes the variance of the present value random variable of the new product. [150,000]

# Whole life continuous annuity

Next we'll consider continuous life annuities, starting with a whole life continuous annuity.

- This type of annuity pays ( $x$ ) continuously at a rate of \$1 per year for as long as ( $x$ ) is alive, starting now.

Letting  $Y$  denote the random variable representing the PV of this annuity benefit, we have:  $Y = \bar{a}_{\overline{T_x}|}$

The EPV for this annuity is denoted  $\bar{a}_x \dots$

EPV for Whole Life Continuous Annuity

$$\bar{a}_x = \int_0^{\infty} e^{-\delta t} {}_t p_x dt = \frac{1 - \bar{A}_x}{\delta}$$

$\dots$  and the variance is  $V[Y] = \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{\delta^2}$



# Term continuous annuity

We can also have a continuous annuity that pays for a maximum of  $n$  years, so long as  $(x)$  is alive.

- This type of life annuity is known as an  **$n$ -year term continuous annuity**.

For this type of annuity, PV of the benefit is:

$$Y = \bar{a}_{\min(T_x, n)|} = \begin{cases} \bar{a}_{\overline{T_x}|} & \text{if } T_x \leq n \\ \bar{a}_{\overline{n}|} & \text{if } T_x > n \end{cases}$$

In this case, the EPV is denoted by  $\bar{a}_{x:\overline{n}|}$

EPV for  $n$ -year Term Continuous Annuity

$$\bar{a}_{x:\overline{n}|} = \int_0^n e^{-\delta t} {}_t p_x dt = \frac{1 - \bar{A}_{x:\overline{n}|}}{\delta}$$



# Example

Suppose that the survival model for  $(x)$  is described by the exponential model with  $\mu_x = \mu \forall x$ .

Let  $Y$  be the PV of a continuous whole life annuity payable at a rate of 1 per year to  $(x)$ .

- ① Find  $\frac{d}{d\mu} \bar{a}_x \left[ -\frac{1}{(\delta+\mu)^2} \right]$
- ② Assuming that  $\delta = 0.08$  and  $\mu = 0.04$ , find the expected value and variance of  $Y$ . [8.333; 13.889]
- ③ Find the probability that  $Y < 5$ . [0.2254]

For a continuous whole life annuity of 1 on  $(x)$ :

- $T_x$  is the future lifetime random variable for  $(x)$ .
- The forces of interest and mortality are constant and equal.
- $\bar{a}_x = 12.50$

Calculate the standard deviation of  $\bar{a}_{\overline{T_x}|}$ . [7.217]

# SOA Practice Problem #79

For a group of individuals all age  $x$ , you are given:

- 30% are smokers and 70% are nonsmokers.
- The constant force of mortality for smokers is 0.06 at all ages.
- The constant force of mortality for nonsmokers is 0.03 at all ages.
- $\delta = 0.08$

Calculate  $Var\left(\bar{a}_{\overline{T_x}|}\right)$  for an individual chosen at random from this group. [14.1]

## $m^{\text{th}}$ ly life annuities

We can also analyze life annuities that pay on an  $m^{\text{th}}$ ly basis.

First consider an annuity that pays an amount of 1 per year, paid in installments of amount  $\frac{1}{m}$  at the beginning of each  $m^{\text{th}}$  of a year, so long as  $(x)$  lives.

The PV of this annuity is  $Y = \ddot{a}_{\overline{K_x^{(m)} + \frac{1}{m}}|}^{(m)} = \frac{1 - v^{K_x^{(m)} + \frac{1}{m}}}{d^{(m)}}$

The EPV of  $Y$  is given by

$$E[Y] = \ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}} = \sum_{k=0}^{\infty} \frac{1}{m} v^{\frac{k}{m}} \frac{k}{m} p_x$$

We can also consider the “immediate” version:  $a_x^{(m)} = \ddot{a}_x^{(m)} - \frac{1}{m}$



## $m^{\text{th}}$ ly term annuities

Now consider an annuity that pays an amount of 1 per year, paid in installments of amount  $\frac{1}{m}$  at the beginning of each  $m^{\text{th}}$  of a year, so long as  $(x)$  lives, but for a maximum of  $n$  years.

The PV of this annuity is

$$Y = \ddot{a}_{\min(K_x^{(m)} + \frac{1}{m}, n)}^{(m)} = \frac{1 - v^{\min(K_x^{(m)} + \frac{1}{m}, n)}}{d^{(m)}}$$

The EPV of  $Y$  is given by

$$E[Y] = \ddot{a}_{x:\overline{n}|}^{(m)} = \frac{1 - A_{x:\overline{n}|}^{(m)}}{d^{(m)}} = \sum_{k=0}^{mn-1} \frac{1}{m} v^{\frac{k}{m}} \frac{k}{m} p_x$$

We can also consider the “immediate” version:

$$a_{x:\overline{n}|}^{(m)} = \ddot{a}_{x:\overline{n}|}^{(m)} + nE_x \frac{1}{m} - \frac{1}{m}$$



# Deferred annuities

As we did with life insurance, we can consider the notion of deferring annuity benefits.

- In this case, the first annuity payment would occur at some point in the future (rather than at the beginning or end of the first year).

We denote this deferral in the “usual” manner, so that, for example, the EPV of a whole life annuity due for  $(x)$ , deferred  $u$  years, would be denoted by

$${}_u|\ddot{a}_x = \sum_{k=u}^{\infty} v^k {}_k p_x$$

The same idea also applies to the continuous, immediate, and  $m^{\text{th}}$ ly life annuity forms.

## Deferred annuities (continued)

As we did with life insurance, we can use the idea of deferral to construct or deconstruct various types of annuities.

This leads to various relationships among the annuities and their EPVs:

$${}_u| \ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{u}|}$$

$${}_u| \ddot{a}_x = {}_uE_x \ddot{a}_{x+u}$$

$$\ddot{a}_{x:\overline{n}|} = \ddot{a}_x - {}_nE_x \ddot{a}_{x+n}$$

$$\ddot{a}_{x:\overline{n}|} = \sum_{u=0}^{n-1} {}_u| \ddot{a}_{x:\overline{1}|}$$

Again, there are analogous relationships for the other annuity payment forms.

# Example

You are given that:

$$\ddot{a}_{30:\overline{10}|} = 7.79064 \quad \ddot{a}_{40:\overline{10}|} = 7.78144 \quad \ddot{a}_{50} = 15.1511$$

$${}_{10}p_{30} = 0.99611 \quad {}_{10}p_{40} = 0.99233 \quad i = 0.06$$

Find:

- ①  ${}_{10}E_{30}$  [0.5562]
- ②  $\ddot{a}_{30:\overline{20}|}$  [12.1187]
- ③  $\ddot{a}_{30}$  [16.7884]



For a 20-year deferred whole life annuity-due of 1 per year on (45), you are given:

- $l_x = 10(105 - x)$ ,  $0 \leq x \leq 105$
- $i = 0$

Calculate the probability that the sum of the annuity payments actually made will exceed the EPV at issue of the annuity. [0.450]

# Guaranteed annuities

We can also consider annuities in which some of the payments are certain, rather than being contingent on the policyholder being alive at the time of payment.

- That is, some of the payments may be **guaranteed** rather than life contingent.
- This creates a sort of annuity that's a hybrid between the ones we've studied in this chapter and annuities-certain.
- Any payments made after the annuitant dies would go to a beneficiary.
- This reduces the risk of a very poor return, but there's a cost associated with it...

For example, consider a whole life annuity due that will pay for a minimum of  $n$  years, even if death occurs within the first  $n$  years.

- This is sometimes called a “life and  $n$ -year certain” annuity.

The PV of this annuity benefit would be

$$Y = \begin{cases} \ddot{a}_{\overline{n}|} & \text{if } K_x \leq n - 1 \\ \ddot{a}_{\overline{K_x+1}|} & \text{if } K_x \geq n \end{cases}$$

The EPV of an whole life annuity due that guarantees the first  $n$  payments is denoted by  $\ddot{a}_{\overline{x:\overline{n}|}}$

We can relate this EPV to other annuity EPVs:

$$\ddot{a}_{\overline{x:\overline{n}|}} = \ddot{a}_{\overline{n}|} + {}_nE_x \ddot{a}_{x+n}$$

# Example

A person age 65 has accumulated a sum of \$100,000 in her retirement account. She wishes to use this money to purchase an level payment annuity, with the first payment occurring today.

Assume that:

$$i = 6\% \quad {}_{10}p_{65} = 0.71623 \quad \ddot{a}_{65} = 9.89693 \quad \ddot{a}_{75} = 7.21702$$

- ① What payment amount could she afford if she purchases a whole life annuity due? [10,104.14]
- ② What payment amount could she afford if she purchases a whole life annuity due with the first 10 years guaranteed? [9,356.23]

At interest rate  $i$ :

- $\ddot{a}_x = 5.6$
- The EPV of a 2-year certain and life annuity-due of 1 on  $(x)$  is  $\ddot{a}_{\overline{x:2}|} = 5.6459$
- $e_x = 8.83$
- $e_{x+1} = 8.29$

Calculate  $i$ . [0.07886]

# Actuarial accumulated annuity values

Thus far, we've mostly considered find the actuarial value of life annuities at the time of purchase (age  $x$  or time 0).

We might also consider valuing life annuities at other points in time.

For example, the **actuarial accumulated value** of an  $n$  year term (life) annuity-due, one year after the final payment, would be

$$\ddot{s}_{x:\overline{n}|} = \frac{\ddot{a}_{x:\overline{n}|}}{{}_nE_x} \quad \text{so that} \quad \ddot{a}_{x:\overline{n}|} = {}_nE_x \ddot{s}_{x:\overline{n}|}$$

We can also write analogous relationships for the other payment forms (continuous, immediate,  $m^{\text{th}}$ ly).

We can also consider annuities with non-level payment patterns.

For example, we can use our general EPV strategy to find the EPV of an arithmetically increasing  $n$ -year term life annuity-due:

$$(I\ddot{a})_{x:\overline{n}|} = \sum_{t=0}^{n-1} v^t (t+1) {}_t p_x$$

The continuous version would be derived analogously:

$$(\bar{I}\bar{a})_{x:\overline{n}|} = \int_0^n t e^{-\delta t} {}_t p_x dt$$

# Example

Consider a  $n$ -year term annuity due that pays amount  $n$  at time 0,  $n - 1$  at time 1,  $\dots$ , and 1 and time  $n - 1$ . (Each payment is made if and only if the policyholder is alive at the time of the payment.)

- 1 Give the pmf for the present value of benefits for this annuity.
- 2 Write the EPV of the annuity as a summation.
  - Note that this EPV would be denoted  $(D\ddot{a})_{x:\overline{n}|}$
- 3 Write this EPV in terms of increasing and level annuity EPVs.



# Fractional age assumptions

By using the fractional age relationships we developed for life insurances, we can find corresponding formulas for annuities. For example, under UDD we have:

$$\ddot{a}_x^{(m)} \stackrel{UDD}{=} \alpha(m) \ddot{a}_x - \beta(m)$$

where

$$\alpha(m) = \frac{id}{i^{(m)} d^{(m)}} \quad \text{and} \quad \beta(m) = \frac{i - i^{(m)}}{i^{(m)} d^{(m)}}$$

Letting  $m \rightarrow \infty$  yields:  $\bar{a}_x \stackrel{UDD}{=} \left( \frac{id}{\delta^2} \right) \ddot{a}_x - \left( \frac{i - \delta}{\delta^2} \right)$

And for an  $n$ -year term annuity, we have

$$\ddot{a}_{x:\overline{n}|}^{(m)} \stackrel{UDD}{=} \alpha(m) \ddot{a}_{x:\overline{n}|} - \beta(m) (1 - {}_nE_x)$$



# Woolhouse's Formula

Another approach to approximating EPVs of  $m^{\text{th}}$ ly and continuous life annuities is by using **Woolhouse's formula**. For an  $m^{\text{th}}$ ly whole life annuity, the EPV approximation is given by

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2} (\delta + \mu_x)$$

Letting  $m \rightarrow \infty$  yields:  $\bar{a}_x \approx \ddot{a}_x - \frac{1}{2} - \frac{1}{12} (\delta + \mu_x)$

And for an  $n$ -year term life annuity, we have

$$\ddot{a}_{x:\overline{n}|}^{(m)} \approx \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m} (1 - v^n {}_n p_x) - \frac{m^2-1}{12m^2} (\delta + \mu_x - v^n {}_n p_x (\delta + \mu_{x+n}))$$

and

$$\bar{a}_{x:\overline{n}|} \approx \ddot{a}_{x:\overline{n}|} - \frac{1}{2} (1 - v^n {}_n p_x) - \frac{1}{12} (\delta + \mu_x - v^n {}_n p_x (\delta + \mu_{x+n}))$$



# SOA Practice Problem #7

For an annuity payable semiannually, you are given:

- Deaths are uniformly distributed over each year of age.
- $q_{69} = 0.03$
- $i = 0.06$
- $1000\bar{A}_{70} = 530$

Calculate  $\ddot{a}_{69}^{(2)}$ . [8.5957]

# SOA Practice Problem #11

For a group of individuals all age  $x$ , 30% are smokers (s), 70% are nonsmokers (ns),

- $\delta = 0.10$
- $\bar{A}_x^s = 0.444$
- $\bar{A}_x^{ns} = 0.286$
- $T_x$  is the future lifetime of ( $x$ )
- $\text{Var} \left[ \bar{a}_{\overline{T_x}|}^s \right] = 8.818$
- $\text{Var} \left[ \bar{a}_{\overline{T_x}|}^{ns} \right] = 8.503$

Calculate  $\text{Var} \left[ \bar{a}_{\overline{T_x}|} \right]$  for an individual chosen at random from this group. [9.1217]