Math342(S. Zhang) **P7.1**

1 (7.1:1-8) Let the inner product be $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$. Find

$$\langle p, q\rangle,\ \|p\|,\ d(p,q),\ \text{a (nonzero) } h \perp q$$

where

$$p = x,\ q = 3x - 2.$$

ans:

Inner product: $\langle u, v\rangle \to R^1$
(1) $\langle u, v\rangle = \langle v, u\rangle$
(2) $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$,
(3) $\langle cu, v\rangle = c\langle u, v\rangle$
(4) $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0$ iff $u = 0$.
Recall: $f = 1 + x^2$, $g = 1 - x$. In last chapter, we define their coordinates under the standard basis:

$$[f]_B = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},\ [g]_B = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

When we define orthogonal projection:

$$\langle f, g\rangle = [f]_B \cdot [g]_B = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}^T \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 1.$$

Now, we usually use L^2 inner product:

$$\langle f, g\rangle = \int_0^1 fg dx = \int_0^1 (1 + x^2)(1 - x)dx$$
$$= \int_0^1 (1 + x^2 - x - x^3)dx$$
$$= 1 + \frac{1}{3} - \frac{1}{2} - \frac{1}{4} = \frac{1}{12}.$$

Definition: (1) Norm (length) $\|v\| = \sqrt{\langle v, v\rangle}$
(2) distance $d(u, v) = \|u - v\|$
(3) orthogonal $u \perp v$ if $\langle u, v\rangle = 0$.

$$\langle p, p\rangle = \int_0^1 x \cdot x dx = \frac{1}{3}$$
$$\|p\| = \sqrt{\langle p, p\rangle} = 1/\sqrt{3}$$
$$d(p, q) = \|p - q\|$$
$$\|p - q\|^2 = \langle p - q, p - q\rangle = \int_0^1 (-2x + 2)^2 dx$$
$$= \int_0^1 (4x^2 - 8x + 4)dx = \frac{3}{4} - 4 + 4 = \frac{3}{4}$$
$$d(p, q) = \|p - q\| = 2/\sqrt{3}$$
$$\langle p, q\rangle = \int_0^1 (3x^2 - 2x)dx = 1 - 1 = 0$$
$$p \perp q$$

Find a nonzero $h \perp q$:

$$h = \text{perp}_q p$$
$$= p - \frac{\langle p, q\rangle}{\langle q, q\rangle} q = p - 0 = x.$$

2 (7.1:25-28) Compute $\langle u, 3u + v\rangle$ and $\|3u + v\|$:

$$\|u\| = 10,\ \langle u, v\rangle = 6,\ \|v\| = 4$$

ans:

$$\langle u, 3u + v\rangle = 3\langle u, u\rangle + \langle u, v\rangle$$
$$= 3(10^2) + 6 = 306$$
$$\|3u + v\| = (9\|u\|^2 + 6\langle u, v\rangle + \|v\|^2)^{1/2}$$
$$= (900 + 36 + 16)^{1/2} = \sqrt{952}$$

3 (7.1:22) Sketch the unit circle in R^2 for the inner product. Show the coordinates of two/four vertices at least.

$$\langle \mathbf{u}, \mathbf{v}\rangle = 4u_1v_1 + u_1v_2 + u_2v_1 + 4u_2v_2.$$

ans: We first orthogonally diagonalize the quadratic form (matrix).

$$\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}^t A\mathbf{v}$$
$$= \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Find eigenvalues.

$$|A-\lambda I| = \begin{vmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix}$$
$$(4-\lambda)^2 - 1^2 = 0$$
$$\lambda = 5, 3.$$

For $\lambda = 5$, $(A-\lambda I)\mathbf{v} = 0$,

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \mathbf{v} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$
$$\mathbf{q}_1 = \mathbf{v}/||\mathbf{v}|| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda = 3$,

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \ \mathbf{v} = t\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
$$\mathbf{q}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

So

$$Q = (\mathbf{q}_1 \quad \mathbf{q}_2) = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

Then

$$Q^T A Q = D = \begin{pmatrix} 5 & \\ & 3 \end{pmatrix}$$
$$A = QDQ^T$$

On the unit circle,

$$\langle \mathbf{u}, \mathbf{u} \rangle = 1$$
$$\mathbf{u}^T A \mathbf{u} = (\mathbf{u}^T Q) D (Q^T \mathbf{u}) = \mathbf{w}^T D \mathbf{w}$$
$$= 5w_1^2 + 3w_2^2 = 1$$
$$D\mathbf{w} = \mathbf{u}$$

In the standard form:

$$\frac{w_1^2}{(1/\sqrt{5})^2} + \frac{w_2^2}{(1/\sqrt{3})^2} = 1$$

Because

$$\mathbf{u} = Q\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

For vertices of $\mathbf{w}$:

$$\mathbf{w} = \begin{pmatrix} 1/\sqrt{5} \\ 0 \end{pmatrix}, \ \mathbf{u} = Q\mathbf{w} = \begin{pmatrix} 1/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix},$$
$$\mathbf{w} = \begin{pmatrix} 0 \\ 1/\sqrt{3} \end{pmatrix}, \ \mathbf{u} = Q\mathbf{w} = \begin{pmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$$

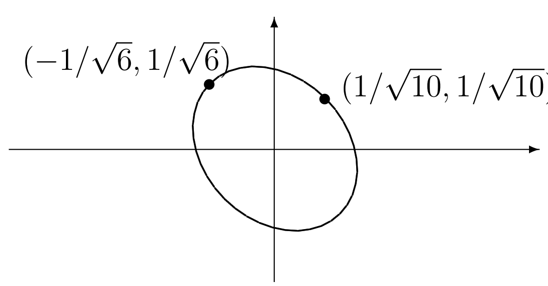
These are the vertices of the “unit circle”.

Because

$$\mathbf{w} = Q^T \mathbf{u} = \frac{1}{\sqrt{2}}\begin{pmatrix} u_1 + u_2 \\ -u_1 + u_2 \end{pmatrix}$$

we get the standard form:

$$\frac{((u_1+u_2)/\sqrt{2})^2}{(1/\sqrt{5})^2} + \frac{((-u_1+u_2)/\sqrt{2})^2}{(1/\sqrt{3})^2} = 1$$



4. (7.1:19-20,37-40) Apply the Gram-Schmidt process to $\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$ to obtain an orthognal basis (not under Euclidean inner product), under the following inner product:

$$\langle u, v \rangle = 2u_1v_1 + u_1v_2 + u_2v_1 + 2u_2v_2.$$

ans:

Projection/orthogonalization(Gram-Schmidt):

$$\text{proj}_g f = \frac{\langle f, g\rangle}{\langle g, g\rangle} g$$
$$\text{perp}_g f = f - \frac{\langle f, g\rangle}{\langle g, g\rangle} g.$$

Recall: $\mathbf{x}_1$, $\mathbf{x}_2$...

$$\mathbf{v}_1 = \mathbf{x}_1$$
$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1$$
$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2.$$

Orthonormal:

$$\mathbf{q}_1 = \frac{\mathbf{v}_1}{\sqrt{\mathbf{v}_1 \cdot \mathbf{v}_1}}$$
$$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\sqrt{\mathbf{v}_2 \cdot \mathbf{v}_2}}$$
$$\mathbf{q}_3 = \frac{\mathbf{v}_3}{\sqrt{\mathbf{v}_3 \cdot \mathbf{v}_3}}$$

Let

$$\langle \mathbf{u}_1, \mathbf{u}_2\rangle = \mathbf{u}^t A \mathbf{v}$$
$$= \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 3$$

We apply the Gram-Schmidt process to

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\mathbf{v}_2 = \mathbf{x}_2 - \text{proj}_{\mathbf{v}_1}\mathbf{x}_2$$
$$= \mathbf{x}_2 - \frac{\mathbf{x}_2^T A \mathbf{v}_1}{\mathbf{v}_1^T A \mathbf{v}_1}\mathbf{v}_1$$
$$= \mathbf{x}_2 - \frac{3}{2}\mathbf{v}_1 = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$$

Done.

Checking:

$$\mathbf{v}_1^T A \mathbf{v}_2 = 0$$
$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} = 0 \text{ (yes.)}$$