MATH 210B: HOMEWORK 6

Problem 55. Let K/F be a finite Galois extension with group G. Let M be an intermediate extension corresponding to a subgroup $H < G$.

(a) Show that G acts on the set $\mathrm{Hom}_F(M, K)$, which is in bijection with $\mathrm{Hom}_F(M, \overline{F})$, for any fixed algebraic closure $\overline{F}$.
(b) Show that this G-set is isomorphic to G/H.
(c) When H is normal in G, why does this generalise to the Galois correspondence $G/H \cong \mathrm{Gal}(M/F)$?

Problem 56. Consider the field $A = \mathbb{C}[X_1, \ldots, X_n]$, polynomials over the complex numbers in n independent variables. The symmetric group S_n acts on A by permuting the variables. Let T be a variable independent from X_i. The polynomial

$$F(T) = (T - X_1) \cdots (T - X_n)$$

is invariant under the action of S_n.

(a) Calculate the coefficient of T^d in $F(T)$. These are denoted s_d and called the *elementary symmetric polynomials* in n variables.
(b) Prove that s_d is a homogeneous polynomial of degree d.

Problem 57. We define the *weight* of a monomial $T_1^{v_1} \cdots T_n^{v_n}$ is the sum $\sum i \cdot v_i$. The weight of a polynomial is defined to be the maximum weight of its monomials.

Suppose that $f(X) \in A$ is fixed by S_n and $\deg f = d$. Prove that there exists a polynomial $g(T_1, \ldots, T_n)$ of weight $\leq d$ such that

$$f(X) = g(s_1, \ldots, s_n).$$

Further, prove that if f is homogeneous of degree d, then every monomial in g is of weight d.

Problem 58.

(a) Write the symmetric polynomial $X_1^2 + \cdots + X_n^2 \in A$ in terms of the elementary symmetric polynomials.
(b) Write the symmetric polynomial $X_1^3 + X_2^3 + X_3^3 \in \mathbb{C}[X_1, X_2, X_3]$ in terms of the elementary symmetric polynomials.

Problem 59. Prove that the elementary symmetric polynomials are algebraically independent over $\mathbb{C}$. That is, prove that there exists no polynomial $p(T) \in \mathbb{C}[T_1, \ldots, T_n]$ such that $p(s_1, \ldots, s_n) = 0$.

Problem 60. Consider the field of rational functions $K = \mathbb{C}(X_1, \ldots, X_n)$.

(a) Prove that $S_n \subset \mathrm{Aut}(K)$, where again the symmetric group acts by permuting the variables.

(b) Prove that the fixed field of S_n is $\mathbb{C}(s_1, \ldots, s_n)$, and that $K/\mathbb{C}(s_1, \ldots, s_n)$ is a Galois extension.

Problem 61.

(a) Prove that there exists a Galois extension $K/\mathbb{Q}$ with Galois group $\mathbb{Z}/p^k\mathbb{Z}$ for any prime p and any $k \geq 1$.

(b) Let A be a finite abelian group. Prove that there exists a Galois extension $K/\mathbb{Q}$ such that $\mathrm{Gal}(K/\mathbb{Q}) \cong A$. Hint: for any $n > 1$, there exists infinitely many primes p such that $p \equiv 1 \mod n$.

Problem 62. Show that the discriminant of the polynomial T^2+bT+c is $\Delta = b^2-4c$.

Problem 63. Show that the discriminant of the polynomial $T^3 + pT + q$ is $\Delta = -4p^3 - 27q^2$.

Problem 64. Reprove that the Galois group of $T^4 + 7$ in $\mathbb{Q}[T]$ is D_8, the dihedral group of the square. Describe all subfields of its splitting field using the Galois correspondence.

Problem 65. Let p be an odd prime.

(a) Calculate the Galois group of $T^p - 1$ over $\mathbb{Q}$, and prove that its splitting field K cannot be embedded in $\mathbb{R}$.

(b) Find an element $\alpha \in K$ such that $\mathbb{Q}(\alpha)$ is the subfield of K fixed by complex conjugation.