Credit Valuation Adjustment Calculation

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ABSTRACT

This paper presents a new model for computing credit valuation adjustment (CVA). The model takes correlated and potentially simultaneous defaults into account. As such, it can naturally capture wrong/right way risk. Empirically, we find evidence that wrong way risk has a material effect on credit valuation adjustment.

Key Words: credit value adjustment, wrong way risk, right way risk, credit risk modeling,

Banks are required by the International Accounting Standard (IAS) 39 to provide a fair-value adjustment due to counterparty risk in 2000. But this requirement received a little attention until the recent financial crises in which the profit and loss (P&L) swings due to CVA changes were measured in billions of dollars.

CVA allows institutions to quantify counterparty risk as a single measurable P&L number, It also allows banks to dynamically manage, price and hedge counterparty risk. The benefits of CVA are widely acknowledged. Many banks have set up internal credit risk trading desks to manage counterparty risk on derivatives.

The earlier works on CVA are mainly focused on one-way CVA that assumes that only one counterparty is defaultable and the other one is default-free. A trend that has become increasingly relevant and popular has been to consider the two-way nature of counterparty credit risk.

CVA, by definition, is the difference between the risk-free portfolio value and the true (or risky or defaultable) portfolio value that takes into account the possibility of a counterparty's default.

In general, risky valuation can be classified into two categories: the *default time approach* (DTA) and the *default probability approach* (DPA).

Although the DTA is very intuitive, it has the disadvantage that it explicitly involves the default time. We are very unlikely to have complete information about a firm's default point, which is often inaccessible. Usually, valuation under the DTA is performed via Monte Carlo simulation.

The DPA relies on the probability distribution of the default time rather than the default time itself. Sometimes the DPA yields simple closed form solutions.

The current popular CVA methodology is first derived using DTA and then discretized over a time grid in order to yield a feasible solution. The discretization, however, is inaccurate. In fact, this model has never been rigorously proved. Since CVA is used for financial accounting and pricing, its accuracy is essential

In this paper, we present a model for computing CVA. Our study shows that the pricing process of a defaultable contract normally has a backward recursive nature if its payoff could be positive or negative.

An intuitive way of understanding these backward recursive behaviors is that we can think of that any contingent claim embeds two default options. In other words, when entering an OTC derivatives transaction, one party grants the other party an option to default and, at the same time, also receives an option to default itself. In theory, default may occur at any time. Therefore, the default options are American style options that normally require a backward induction valuation.

Wrong way risk occurs when exposure to a counterparty is adversely correlated with the credit quality of that counterparty, while right way risk occurs when exposure to a counterparty is positively correlated with the credit quality of that counterparty. Since this new model allows us to incorporate correlated and potentially simultaneous defaults into risky valuation, it can naturally capture wrong/right way risk.

The rest of this paper is organized as follows: Section 2 discusses one-way risky valuation and one-way CVA. Section 2 elaborates two-way risky valuation and two-way CVA. Section 3 presents numerical results. The conclusions are given in Section 4. . All proofs and a practical framework that embraces netting agreements, margining agreements and wrong/right way risk are contained in the appendices.

1. One-way CVA

We consider a filtered probability space (Ω , $\mathcal{F}$, $\{\mathcal{F}_t\}_{t\geq 0}$, $\mathcal{P}$) satisfying the usual conditions, where Ω denotes a sample space; $\mathcal{F}$ denotes a σ -algebra; $\mathcal{P}$ denotes a probability measure; $\{\mathcal{F}_t\}_{t\geq 0}$ denotes a filtration.

The stopping (or default) time τ of a firm is modeled as a Cox arrival process (also known as a doubly stochastic Poisson process) whose first jump occurs at default and is defined as,

$$\tau = \inf\left\{t : \int_0^t h(s, \Phi_s) ds \geq \Delta\right\} \quad (1)$$

where $h(t)$ or $h(t, \Phi_t)$ denotes the stochastic hazard rate or arrival intensity dependent on an exogenous common state Γ_t, and Δ is a unit exponential random variable independent of Φ_t.

It is well-known that the survival probability from time t to s in this framework is defined by

$$p(t,s) := P(\tau > s \mid \tau > t, Z) = \exp\left(-\int_t^s h(u) du\right) \quad (2a)$$

The default probability for the period (t, s) in this framework is defined by

$$q(t,s) := P(\tau \leq s \mid \tau > t, Z) = 1 - p(t,s) = 1 - \exp\left(-\int_t^s h(u) du\right) \quad (2b)$$

The risk free value of the financial contract is given by

$$V^F(t) = E\left[D(t,T) X_T \mid \mathcal{F}_t\right] \quad (3a)$$

where

$$D(t,T) = \exp\left[-\int_t^T r(u) du\right] \quad (3b)$$

where $E\{\bullet \mid \mathcal{F}_t\}$ denotes the expectation conditional on the $\mathcal{F}_t$, $D(t,T)$ denotes the risk-free discount factor at time t for the maturity T and $r(u)$ denotes the risk-free short rate at time u ($t \leq u \leq T$).

Risky valuation can be generally classified into two categories: the *default time approach* (DTA) and the *default probability (intensity) approach* (DPA).

The DTA involves the default time explicitly. The value of this defaultable contract is the discounted expectation of all the payoffs and is given by

$$V(t) = E\left[\left(D(t,T) X_T 1_{\tau > T} + D(t,\tau) \varphi V(\tau) 1_{\tau \leq T}\right) \mid \mathcal{F}_t\right] \quad (4)$$

where 1_Y is an indicator function that is equal to one if Y is true and zero otherwise.

The DPA relies on the probability distribution of the default time rather than the default time itself. We divide the time period (*t*, *T*) into *n* very small time intervals (Δt) and assume that a default may occur only at the end of each very small period. In our derivation, we use the approximation $\exp(y) \approx 1 + y$ for very small *y*. The survival and the default probabilities for the period (t , $t + \Delta t$) are given by

$$\hat{p}(t) := p(t, t + \Delta t) = \exp(-h(t)\Delta t) \approx 1 - h(t)\Delta t \tag{5a}$$

$$\hat{q}(t) := q(t, t + \Delta t) = 1 - \exp(-h(t)\Delta t) \approx h(t)\Delta t \tag{5b}$$

The binomial default rule considers only two possible states: default or survival. Under a risk-neutral measure, the value of the asset at *t* is the expectation of all the payoffs discounted at the risk-free rate and is given by

$$V(t) = E\left\{\exp(-r(t)\Delta t)\left[\hat{p}(t) + \varphi(t)\hat{q}(t)\right]V(t + \Delta t)\middle|\mathcal{F}_t\right\} \approx E\left\{\exp(-y(t)\Delta t)V(t + \Delta t)\middle|\mathcal{F}_t\right\} \tag{6}$$

where $y(t) = r(t) + h(t)(1 - \varphi(t)) = r(t) + c(t)$ denotes the risky rate and $c(t) = h(t)(1 - \varphi(t))$ is called the (short) credit spread.

Similarly, we have

$$V(t + \Delta t) = E\left\{\exp(-y(t + \Delta t)\Delta t)V(t + 2\Delta t)\middle|\mathcal{F}_{t+\Delta t}\right\} \tag{7}$$

Note that $\exp(-y(t)\Delta t)$ is $\mathcal{F}_{t+\Delta t}$ -measurable. By definition, an $\mathcal{F}_{t+\Delta t}$ -measurable random variable is a random variable whose value is known at time $t + \Delta t$. Based on the *taking out what is known* and *tower* properties of conditional expectation, we have

$$\begin{aligned} V(t) &= E\left\{\exp(-y(t)\Delta t)V(t + \Delta t)\middle|\mathcal{F}_t\right\} \\ &= E\left\{\exp(-y(t)\Delta t)E\left[\exp(-y(t + \Delta t)\Delta t)V(t + 2\Delta t)\middle|\mathcal{F}_{t+\Delta t}\right]\middle|\mathcal{F}_t\right\} \\ &= E\left\{\exp\left(-\textstyle\sum_{i=0}^{1} y(t + i\Delta t)\Delta t\right)V(t + 2\Delta t)\middle|\mathcal{F}_t\right\} \end{aligned} \tag{8}$$

By recursively deriving from *t* forward over *T* and taking the limit as Δt approaches zero, the risky value of the asset can be expressed as

$$V(t) = E\left\{\exp\left[-\int_t^T y(u)du\right]V(T)\middle|\mathcal{F}_t\right\} \tag{9}$$

In theory, a default may happen at any time, i.e., a risky contract is continuously defaultable. This Continuous Time Risky Valuation Model is accurate but sometimes complex and expensive. For simplicity, people sometimes prefer the Discrete Time Risky Valuation Model that assumes that a default may only happen at some discrete times. A natural selection is to assume that a default may occur only on the payment dates. From now on, we will focus on the discrete setting only, but many of the points we make are equally applicable to the continuous setting.

For a derivative contract, usually its payoff may be either an asset or a liability to each party. Thus, we further relax the assumption and suppose that X_T may be positive or negative.

In the case of $X_T > 0$, the survival value is equal to the payoff X_T and the default payoff is a fraction of the payoff φX_T. Whereas in the case of $X_T \leq 0$, the contract value is the payoff itself, because the default risk of party *B* is irrelevant for one-way risky valuation in this case. Therefore, we have

The one-way risky value of the single-payment contract in a discrete-time setting is given by

$$V(t) = E\left[F(t,T)X_T \middle| \mathcal{F}_t\right] \tag{10a}$$

where

$$F(t,T) = D(t,T)\left[1 - 1_{X_T \geq 0}\, q(t,T)\left(1 - \varphi(T)\right)\right] \tag{10b}$$

Formula (10) can be easily extended from one-period to multiple-periods. Suppose that a defaultable contract has *m* cash flows. Let the *m* cash flows be represented as $X_1, \ldots, X_m$ with payment dates $T_1, \ldots, T_m$. Each cash flow may be positive or negative. We have the following proposition.

The one-way risky value of the multiple-payment contract is given by

$$V(t) = \sum_{i=1}^{m} E\left[\left(\prod_{j=0}^{i-1} F(T_j, T_{j+1})\right) X_i \middle| \mathcal{F}_t\right] \tag{11a}$$

where $t = T_0$ *and*

$$F(T_j, T_{j+1}) = D(T_j, T_{j+1})\left[1 - 1_{(X_{j+1}+V(T_{j+1}))\geq 0}\, q(T_j, T_{j+1})\left(1-\varphi(T_{j+1})\right)\right] \tag{11b}$$

The risky valuation in Formula (11) has a backward nature. The coupled valuation behavior allows us to capture wrong/right way risk properly where counterparty credit quality and market prices may be correlated. This type of problem can be best solved by working backwards in time, with the later risky value feeding into the earlier ones, so that the process builds on itself in a recursive fashion, which is referred to as *backward induction*.

The similarity between American style financial options and American style default options is that both require a backward recursive valuation procedure. The difference between them is in the optimal strategy. The American financial option seeks an optimal value by comparing the exercise value with the continuation value, whereas the American default option seeks an optimal discount factor based on the option value in time.

The one-way CVA, by definition, can be expressed as

$$CVA(t) = V^F(t) - V(t) = \sum_{i=1}^{m} E\left[\left(D(t,T_i) - \prod_{j=0}^{i-1} F(T_j, T_{j+1})\right) X_i \middle| \mathcal{F}_t\right] \tag{12}$$

Formula (11) provides a general form for pricing a one-way defaultable contract. Applying it to a particular situation in which we assume that all the payoffs are nonnegative, we derive the following corollary:

If all the payoffs are nonnegative, the risky value of the multiple-payments contract is given by

$$V(t) = \sum_{i=1}^{m} E\left[\left(\prod_{j=0}^{i-1} \overline{F}(T_j, T_{j+1})\right) X_i \middle| \mathcal{F}_t\right] \tag{13a}$$

where $t = T_0$ *and*

$$\overline{F}(T_j, T_{j+1}) = D(T_j, T_{j+1})\left[1 - q(T_j, T_{j+1})\left(1-\varphi(T_{j+1})\right)\right] \tag{13b}$$

The proof of this corollary is easily obtained according to Formula (11) by setting $\left(X_{j+1} + V(T_{j+1})\right) \geq 0$, since the value of the contract at any time is also nonnegative.

$$CVA(t) = V^F(t) - V(t) = \sum_{i=1}^{m} E\left[D(t,T_i)\left(1 - \prod_{j=0}^{i-1}\left(1 - q(T_j, T_{j+1})(1-\varphi(T_{j+1}))\right)\right) X_i \middle| \mathcal{F}_t\right] \tag{14}$$

2. Two-way CVA

Default correlation can be positive or negative. The effect of positive correlation is usually called contagion, whereas the latter is referred to as competition effect.

Two counterparties are denoted as *A* and *B*. The binomial default rule considers only two possible states: default or survival. Therefore, the default indicator Y_j for party *j* (*j*=*A*, *B*) follows a Bernoulli distribution, which takes value 1 with default probability q_j and value 0 with survival probability p_j, i.e., $P\{Y_j = 0\} = p_j$ and $P\{Y_j = 1\} = q_j$.

Consider a pair of random variables (Y_A, Y_B) that has a bivariate Bernoulli distribution. The joint probability representations are given by

$$p_{00} := P(Y_A = 0, Y_B = 0) = p_A p_B + \sigma_{AB} \tag{15a}$$

$$p_{01} := P(Y_A = 0, Y_B = 1) = p_A q_B - \sigma_{AB} \tag{15b}$$

$$p_{10} := P(Y_A = 1, Y_B = 0) = q_A p_B - \sigma_{AB} \tag{15c}$$

$$p_{11} := P(Y_A = 1, Y_B = 1) = q_A q_B + \sigma_{AB} \tag{15d}$$

where $E(Y_j) = q_j$, $\sigma_j^2 = p_j q_j$, $\sigma_{AB} := E\left[(Y_A - q_A)(Y_B - q_B)\right] = \rho_{AB}\sigma_A\sigma_B = \rho_{AB}\sqrt{q_A p_A q_B p_B}$ where ρ_{AB} denotes the default correlation coefficient and σ_{AB} denotes the default covariance.

Table 1. Payoffs of a two-wayly defaultable contract

This table displays all possible payoffs at time *T*. In the case of $X_T > 0$, there are a total of four possible states at time *T:*

State	$Y_A = 0, Y_B = 0$	$Y_A = 1, Y_B = 0$	$Y_A = 0, Y_B = 1$	$Y_A = 1, Y_B = 1$
Comments	*A* & *B* survive	*A* defaults, *B* survives	*A* survives, *B* defaults	*A* & *B* default
Probability	p_{00}	p_{10}	p_{01}	p_{11}

Payoff	$X_T > 0$	X_T	$\overline{\varphi}_B X_T$	$\varphi_B X_T$	$\varphi_{AB} X_T$
	$X_T < 0$	X_T	$\varphi_A X_T$	$\overline{\varphi}_A X_T$	$\varphi_{AB} X_T$

At time *T*, there are a total of four ($2^2 = 4$) possible states shown in Table 1. The risky value of the contract is the discounted expectation of the payoffs and is given by the following proposition.

The two-way risky value of the single-payment contract is given by

$$V(t) = E\left[K(t,T)X_T \mid \mathcal{F}_t\right] = E\left[D(t,T)\left(1_{X_T \geq 0} k_B(t,T) + 1_{X_T < 0} k_A(t,T)\right)X_T \mid \mathcal{F}_t\right] \quad (16a)$$

where

$$\begin{aligned} k_B(t,T) = {} & p_B(t,T)p_A(t,T) + \varphi_B(T)q_B(t,T)p_A(t,T) + \overline{\varphi}_B(T)p_B(t,T)q_A(t,T) \\ & + \varphi_{AB}(T)q_B(t,T)q_A(t,T) + \sigma_{AB}(t,T)\left(1 - \varphi_B(T) - \overline{\varphi}_B(T) + \varphi_{AB}(T)\right) \end{aligned} \quad (16b)$$

$$\begin{aligned} k_A(t,T) = {} & p_B(t,T)p_A(t,T) + \varphi_A(T)q_A(t,T)p_B(t,T) + \overline{\varphi}_A(T)p_A(t,T)q_B(t,T) \\ & + \varphi_{AB}(T)q_B(t,T)q_A(t,T) + \sigma_{AB}(t,T)\left(1 - \varphi_A(T) - \overline{\varphi}_A(T) + \varphi_{AB}(T)\right) \end{aligned} \quad (16c)$$

Using a similar derivation as in Formula (11), we can easily extend Proposition 3 from one-period to multiple-periods. Suppose that a defaultable contract has *m* cash flows. Let the *m* cash flows be represented as X_i with payment dates T_i , where *i* = 1,…,*m*. Each cash flow may be positive or negative. The two-way risky value of the multiple-payment contract is given by

The two-way risky value of the multiple-payment contract is given by

$$V(t) = \sum_{i=1}^{m} E\left[\left(\prod_{j=0}^{i-1} K(T_j, T_{j+1})\right) X_i \mid \mathcal{F}_t\right] \quad (17a)$$

where $t = T_0$ *and*

$$K(T_j, T_{j+1}) = D(T_j, T_{j+1})\left(1_{(X_{j+1} + V(T_{j+1})) \geq 0} k_B(T_j, T_{j+1}) + 1_{(X_{j+1} + V(T_{j+1})) < 0} k_A(T_j, T_{j+1})\right) \quad (17b)$$

where $k_A(T_j, T_{j+1})$ *and* $k_B(T_j, T_{j+1})$ *are defined in Proposition 3.*

Formula (17) says that the pricing process of a multiple-payment contract has a backward nature since there is no way of knowing which risk-adjusted discounting rate should be used without knowledge of the future value. Only on the maturity date, the value of the contract and the decision strategy are clear. Therefore, the evaluation must be done in a backward fashion, working from the final payment date towards the present. This type of valuation process is referred to as backward induction.

The two-way CVA of the multiple-payment contract can be expressed as

$$CVA(t) = V^F(t) - V(t) = \sum_{i=1}^{m} \left\{ E\left[D(t,T_i) X_i \middle| \mathcal{F}_t \right] - E\left[\left(\prod_{j=0}^{i-1} K(T_j, T_{j+1}) \right) X_i \middle| \mathcal{F}_t \right] \right\} \tag{18}$$

3. Numerical Results

In this section, we study the impact of margin agreements on CVA. The testing portfolio consists of a number of interest rate and equity derivatives. The number of simulation scenarios (or paths) is 20,000. The time buckets are set weekly. If the computational requirements exceed the system limit, one can reduce both the number of scenarios and the number of time buckets.

The rationale is that the calculation becomes less accurate due to the accumulated error from simulation discretization, and inherited errors from calibration of the underlying models, such as those due to the change of macro-economic climate. The collateral margin period of risk is assumed to be 14 days (2 weeks).

The results are presented in the following tables. Table 2 illustrates that if party *A* has an infinite collateral threshold $H_A = \infty$ i.e., no collateral requirement on *A*, the CVA value increases while the threshold H_B increases. Table 3 shows that if party *B* has an infinite collateral threshold $H_B = \infty$, the CVA value actually decreases while the threshold H_A increases. This reflects the two-way impact of the collaterals on the CVA. The impact is mixed in Table 4 when both parties have finite collateral thresholds.

Table 2. The impact of collateral threshold H_B on the CVA

This table shows that given an infinite H_A, the CVA increases while H_B increases

Collateral Threshold H_B	10.1 Mil	15.1 Mil	20.1 Mil	Infinite (∞)
CVA	19,550.91	20,528.65	21,368.44	22,059.30

Table 3. The impact of collateral threshold H_A on the CVA

This table shows that given an infinite H_B, the CVA decreases while H_A increases, where H_B denotes the collateral threshold of party *B* and H_A denotes the collateral threshold of party *A*.

Collateral Threshold H_A	10.1 Mil	15.1 Mil	20.1 Mil	Infinite (∞)
CVA	28,283.64	25,608.92	23,979.11	22,059.30

Table 4. The impact of the both collateral thresholds on the CVA

The CVA may increase or decrease while both collateral thresholds change

Collateral Threshold H_B	10.1 Mil	15.1 Mil	20.1 Mil	Infinite (∞)
Collateral Threshold H_A	10.1 Mil	15.1 Mil	20.1 Mil	Infinite (∞)
CVA	25,752.98	22,448.45	23,288.24	22,059.30

Wrong way risk occurs when exposure to a counterparty is adversely correlated with the credit quality of that counterparty, while right way risk occurs when exposure to a counterparty is positively correlated with the credit quality of that counterparty.

To capture wrong/right way risk, we need to determine the dependency between counterparties and to correlate the credit spreads or hazard rates with the other market risk factors, e.g. equities, commodities, etc., in the scenario generation.

The impact of the correlation on the CVA is show in Table 5. The results say that the CVA increases when the absolute value of the negative correlation increases.

Table 5. The impact of wrong way risk on the CVA

This table shows that the CVA increases while the negative correlation ρ increases in the absolute value.

Correlation ρ	0	-50%	-100%
CVA	165.15	205.95	236.99

4. Conclusion

This article presents a framework for pricing risky contracts. We find that the valuation of risky assets and their CVAs, in most situations, has a backward recursive nature and requires a backward induction valuation. An intuitive explanation is that two counterparties implicitly sell each other an option to default when entering into an OTC derivative transaction.

If we assume that a default may occur at any time, the default options are American style options. If we assume that a default may only happen on the payment dates, the default options are Bermudan style options. Both Bermudan and American options require backward induction valuations.

Based on our theory, we propose a novel cash-flow-based framework (see appendix) for calculating two-way CVA at the counterparty portfolio level. Numerical results show that these credit mitigation techniques and wrong/right way risk have significant impacts on CVA.

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