ECE 107: Electromagnetism

Set 4: Maxwell's equations

Instructor: Prof. Vitaliy Lomakin

Department of Electrical and Computer Engineering
University of California, San Diego, CA 92093

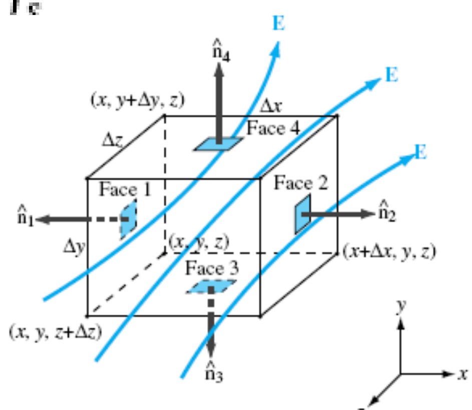
Vector analysis

Divergence

$$\nabla \cdot \mathbf{E} \triangleq \text{div } \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\text{div } \mathbf{E} \triangleq \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{E} \cdot d\mathbf{s}}{\Delta v}$$

$$\int_v \nabla \cdot \mathbf{E} \, dv = \oint_s \mathbf{E} \cdot d\mathbf{s} \qquad \text{(divergence theorem).}$$

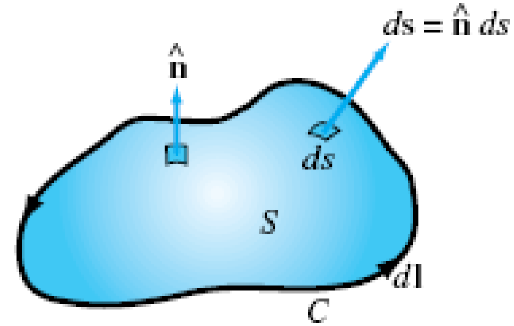


Curl

$$\nabla \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

$$\nabla \times \mathbf{B} = \text{curl } \mathbf{B} \triangleq \lim_{\Delta s \to 0} \frac{1}{\Delta s} \left[\hat{\mathbf{n}} \oint_C \mathbf{B} \cdot d\mathbf{l} \right]_{\max}$$

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \oint_C \mathbf{B} \cdot d\mathbf{l} \qquad \text{(Stokes's theorem),}$$



Differential Maxwell's equations (1)

Time domain Maxwell's equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$ Faraday's law

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$ Ampere's law

$$\nabla \cdot \mathbf{D} = \rho$$ Gauss's law

$$\nabla \cdot \mathbf{B} = 0$$ Gauss's law-magnetic

Basic quantities:

$\mathbf{E}(t,\mathbf{r})-$	electric field	V/m
$\mathbf{H}(t,\mathbf{r})-$	magnetic field	A/m
$\mathbf{D}(t,\mathbf{r})-$	electric flux density	C/m^2
$\mathbf{B}(t,\mathbf{r})-$	magnetic flux density	webers$/m^2$
$\mathbf{J}-$	electric current density	A/m^2
$\rho-$	charge density	C/m^3

Differential Maxwell's equations (2)

Frequency domain Maxwell's equations $\partial/\partial t \to j\omega$:

$\nabla \times \tilde{\mathbf{E}} = -j\omega\tilde{\mathbf{B}}$ Faraday's law

$\nabla \times \tilde{\mathbf{H}} = j\omega\tilde{\mathbf{D}} + \tilde{\mathbf{J}}$ Ampere's law

$\nabla \cdot \tilde{\mathbf{D}} = \tilde{\rho}$ Gauss's law

$\nabla \cdot \tilde{\mathbf{B}} = 0$ Gauss's law-magnetic

Basic quantities:

$\tilde{\mathbf{E}}(\omega,\mathbf{r})-$ electric field (phasor) V/m

$\tilde{\mathbf{H}}(\omega,\mathbf{r})-$ magnetic field (phasor) A/m

$\tilde{\mathbf{D}}(\omega,\mathbf{r})-$ electric flux density (phasor) C/m^2

$\tilde{\mathbf{B}}(\omega,\mathbf{r})-$ magnetic flux density (phasor) webers/m^2

$\mathbf{J}-$ electric current density (phasor) A/m^2

$\rho-$ charge density (phasor) C/m^3

Differential Maxwell's equations (3)

- **Analogy with TL equations** $(\partial/\partial t, \omega \neq 0)$
  - **Assume lossless case** $\Rightarrow \mathbf{B} = \mu_0 \mathbf{H},\ \mathbf{D} = \varepsilon_0 \mathbf{E}$
  - **Time domain**

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\xrightarrow{\{\mathbf{E},\mathbf{H}\} = \underbrace{\{v,i\}(t,z)}_{\text{voltage}} \underbrace{\{\psi_v,\psi_i\}(\mathbf{r})}_{\text{cross-sectional distribution}}}$$

$$-\frac{\partial v(t,z)}{\partial z} = L' \frac{\partial i(t,z)}{\partial t}$$

$$-\frac{\partial i(t,z)}{\partial z} = C' \frac{\partial v(t,z)}{\partial t}$$

  - **Frequency domain**

$$\nabla \times \tilde{\mathbf{E}} = -j\omega \tilde{\mathbf{B}}$$

$$\nabla \times \tilde{\mathbf{H}} = j\omega \tilde{\mathbf{D}} + \tilde{\mathbf{J}}$$

$$\xrightarrow{\{\tilde{\mathbf{E}},\tilde{\mathbf{H}}\} = \underbrace{\{\tilde{V},\tilde{I}\}(\omega,z)}_{\text{voltage}} \underbrace{\{\psi_v,\psi_i\}(\mathbf{r})}_{\text{cross-sectional distribution}}}$$

$$-\frac{d\tilde{V}(\omega,z)}{dz} = j\omega L' \tilde{I}(\omega,z)$$

$$-\frac{d\tilde{I}(\omega,z)}{dz} = j\omega L' \tilde{V}(\omega,z)$$

Differential Maxwell's equations (4)

- **Statics**
  - $\partial/\partial t = \omega = 0 \Rightarrow$ **two uncoupled sets of (static) equations are obtained for** $\mathbf{E}\,\&\,\mathbf{D}$ **and** $\mathbf{H}\,\&\,\mathbf{B}$
  - **Electrostatics**

$$\nabla \times \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{D} = \rho$$

  - **Magnetostatics**

$$\nabla \times \mathbf{H} = \mathbf{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

Integral Maxwell's equations (1)

Time domain Maxwell's equations:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{s}$$

Faraday's law (Stock's theorem to FD)

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \frac{d}{dt} \iint_S \mathbf{D} \cdot d\mathbf{s} + I$$

Ampere's law (Stock's theorem to FD)

$$\oiint_S \mathbf{D} \cdot d\mathbf{s} = Q$$

Gauss's law (Gauss' theorem to FD)

$$\oiint_S \mathbf{B} \cdot d\mathbf{s} = 0$$

Gauss's law-magnetic (Gauss' theorem to FD)

Basic quantities:

$I(t) -$ total current $\quad A$

$Q(t) -$ total charge $\quad C$

Integral Maxwell's equations (2)

Frequency domain Maxwell's equations:

$$\oint_C \tilde{\mathbf{E}} \cdot d\mathbf{l} = -j\omega \iint_S \tilde{\mathbf{B}} \cdot d\mathbf{s}$$
Faraday's law (Stock's theorem to FD)

$$\oint_C \tilde{\mathbf{H}} \cdot d\mathbf{l} = j\omega \iint_S \tilde{\mathbf{D}} \cdot d\mathbf{s} + \tilde{I}$$
Ampere's law (Stock's theorem to FD)

$$\oiint_S \tilde{\mathbf{D}} \cdot d\mathbf{s} = \tilde{Q}$$
Gauss's law (Gauss' theorem to FD)

$$\oiint_S \tilde{\mathbf{B}} \cdot d\mathbf{s} = 0$$
Gauss's law-magnetic (Gauss' theorem to FD)

Basic quantities:

$\tilde{I}(\omega)-$ total current A

$\tilde{Q}(\omega)-$ total charge C

Integral Maxwell's equations (3)

- **Statics**
  - **Electrostatics**

$$\oint_C \tilde{\mathbf{E}} \cdot d\mathbf{l} = 0$$

$$\oiint_S \tilde{\mathbf{D}} \cdot d\mathbf{s} = \tilde{Q}$$

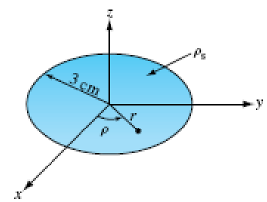
  - **Magnetostatics**

$$\oint_C \tilde{\mathbf{H}} \cdot d\mathbf{l} = \tilde{I}$$

$$\oiint_S \tilde{\mathbf{B}} \cdot d\mathbf{s} = 0$$

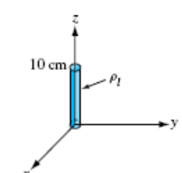
Charges

- **Charge densities**
  - **Volume charge density** ρ_v
    - $$\rho_v = \lim_{\Delta v \to 0} \frac{\Delta q}{\Delta v} = \frac{dq}{dv} \qquad (\text{C/m}^3),$$
    - **Average charge per (physically large) unit volume**
  - **Surface charge density** ρ_s
    - $$\rho_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} = \frac{dq}{ds} \qquad (\text{C/m}^2).$$
    - **Average charge per unit surface**



(b) Surface charge distribution

  - **Line charge density** ρ_l
    - $$\rho_l = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl} \qquad (\text{C/m})$$
    - **Average charge per unit length**



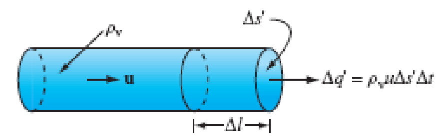
(a) Line charge distribution

  - **Total charge**

$$Q = \int_v \rho_v \, dv \quad \text{or/and} \quad Q = \int_s \rho_s \, ds \quad \text{or/and} \quad Q = \int_l \rho_l \, dl$$

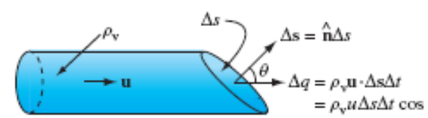
Currents

- **Current densities**
  - **Volume charge density**
    - **Charge through a tube cross-section**



$$\Delta q' = \rho_v \, \Delta v = \rho_v \, \Delta l \, \Delta s' = \rho_v u \, \Delta s' \, \Delta t.$$

    - **Charge through a generalized cross-section**



$$\Delta q = \rho_v \mathbf{u} \cdot \Delta \mathbf{s} \, \Delta t,$$

    - **Current density**

$$\Delta I = \frac{\Delta q}{\Delta t} = \rho_v \mathbf{u} \cdot \Delta \mathbf{s} = \mathbf{J} \cdot \Delta \mathbf{s},$$

$$\mathbf{J} = \rho_v \mathbf{u} \qquad (\mathrm{A/m^2})$$

  - **Surface current density**
    - **thin current sheet (thickness** $d \to 0$ **)** $\Rightarrow \mathbf{J}_s = \lim_{d \to 0} d\mathbf{J} \; (A/m)$
  - **Total current** $I = \int_s \mathbf{J} \cdot \hat{\mathbf{n}} \, ds$ or/and $I = \int_l \mathbf{J}_s \cdot \hat{\mathbf{n}} \, dl$

Vector wave equation

Lossless case: $\mathbf{D}=\varepsilon_0\mathbf{E},\quad \mathbf{B}=\mu_0\mathbf{H},\quad \rho=0,\quad \mathbf{J}=0$

$$\nabla\times\mathbf{E}=-\mu_0\frac{\partial\mathbf{H}}{\partial t}\longrightarrow\nabla\times\left(\nabla\times\mathbf{E}\right)=-\frac{\partial}{\partial t}\left(\nabla\times\mathbf{H}\right)$$

use $$\nabla\times\left(\nabla\times\mathbf{E}\right)=\nabla\underbrace{\left(\nabla\cdot\mathbf{E}\right)}_{0}-\nabla^2\mathbf{E}\;=-\nabla^2\mathbf{E}$$

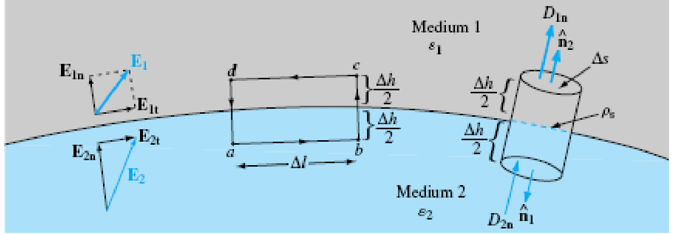
$$-\nabla^2=-\mu_0\frac{\partial}{\partial t}\left(\nabla\times\mathbf{H}\right)$$

use $$\nabla\times\mathbf{H}=\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

$$\nabla^2\mathbf{E}-\mu_0\varepsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2}=0\underbrace{\Rightarrow}_{\text{for 1D}}\quad\frac{\partial^2\mathbf{E}}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=0$$

Electric field boundary conditions (1)

- **Boundary conditions for any dissimilar media**
  - **Tangential components**



$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -d/dt\iint_S \mathbf{B}\cdot d\mathbf{s} \qquad \Delta h \to 0$$

$$\Rightarrow \int_a^b \mathbf{E}_2\cdot d\mathbf{l} + \int_c^d \mathbf{E}_1\cdot d\mathbf{l} + \underbrace{\int_a^d \mathbf{E}\cdot d\mathbf{l}}_{\sim\Delta h\to 0} + \underbrace{\int_c^b \mathbf{E}\cdot d\mathbf{l}}_{\sim\Delta h\to 0} = \underbrace{-\frac{d}{dt}\iint_S \mathbf{B}\cdot d\mathbf{s}}_{\sim\Delta h\to 0}$$

$$E_{2t}\,\Delta l - E_{1t}\,\Delta l = 0 \rightarrow \boxed{E_{1t} = E_{2t}} \Rightarrow \boxed{\hat{\mathbf{n}}_2\times(\mathbf{E}_1-\mathbf{E}_2)=0}$$

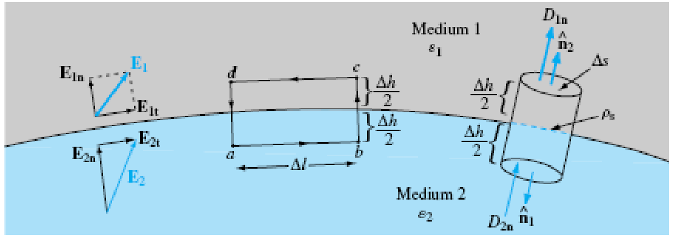
  - **Normal components**

$$\oint_S \mathbf{D}\cdot d\mathbf{s} = \int_{\text{top}} \mathbf{D}_1\cdot\hat{\mathbf{n}}_2\,ds + \int_{\text{bottom}} \mathbf{D}_2\cdot\hat{\mathbf{n}}_1\,ds = Q = \rho_s\,\Delta s$$

$$\boxed{\hat{\mathbf{n}}_2\cdot(\mathbf{D}_1-\mathbf{D}_2)=\rho_s}$$

Electric field boundary conditions (2)

- **Boundary conditions for dielectric media**
  - **Tangential components**



$$\boxed{E_{1t} = E_{2t}} \Rightarrow \boxed{\hat{\mathbf{n}}_2 \times (\mathbf{E}_1 - \mathbf{E}_2) = 0}$$

  - **Normal components**

$$\boxed{\hat{\mathbf{n}}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = 0}$$

Electric field boundary conditions (3)

- **Dielectric-conductor boundary conditions**
  - **Assume medium 2 is perfect conductor**

$\Rightarrow \quad \mathbf{E}_2 = \mathbf{D}_2 = 0,$

$$\Rightarrow \quad \begin{aligned} E_{1t} &= D_{1t} = 0, \\ D_{1n} &= \varepsilon_1 E_{1n} = \rho_s. \end{aligned}$$

$$\Rightarrow \quad \boxed{\begin{aligned} \hat{\mathbf{n}}_2 \times \mathbf{E}_1 &= 0 \\ \hat{\mathbf{n}}_2 \cdot \mathbf{D}_1 &= \rho_s \end{aligned}}$$

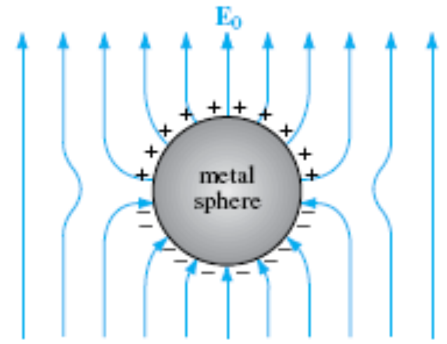


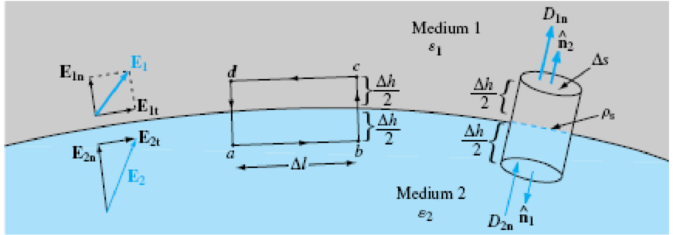
Figure 4-21: Metal sphere placed in an external electric field $\mathbf{E}_0$.

Magnetic field boundary conditions

- **Boundary conditions for any dissimilar media**
  - **Tangential components**

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = \int_a^b \mathbf{H}_2\cdot d\mathbf{l} + \int_c^d \mathbf{H}_1\cdot d\mathbf{l} = I,$$

$$H_{2t}\,\Delta l - H_{1t}\,\Delta l = J_s\,\Delta l$$



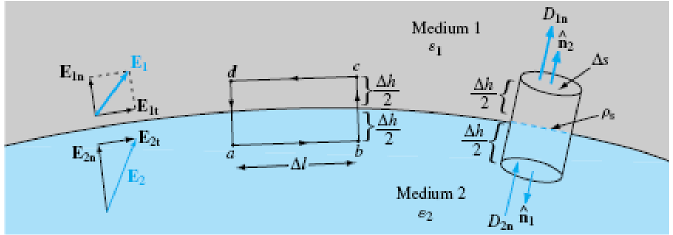
$$\boxed{H_{2t} - H_{1t} = J_s} \qquad \boxed{\hat{\mathbf{n}}_2 \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s}$$

  - **Normal components**

$$\oint_S \mathbf{B}\cdot d\mathbf{s} = 0 \qquad \boxed{\hat{\mathbf{n}}_2\cdot(\mathbf{B}_1 - \mathbf{B}_2) = 0}$$

Magnetic field boundary conditions

- **Boundary conditions for dielectric media**
  - **Tangential components**



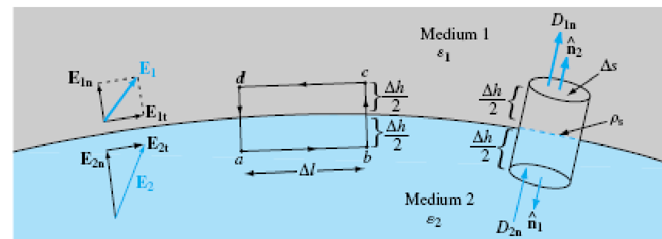
$$\hat{\mathbf{n}}_2 \times (\mathbf{H}_1 - \mathbf{H}_2) = 0$$

  - **Normal components**

$$\hat{\mathbf{n}}_2 \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$$

Magnetic field boundary conditions

- **Boundary conditions for perfect conductors**
  - **Tangential components**



$$\hat{\mathbf{n}}_2 \times \mathbf{H}_1 = \mathbf{J}_s$$

  - **Normal components**

$$\hat{\mathbf{n}}_2 \cdot \mathbf{B}_1 = 0$$

Complete set of differential MEs

Maxwell's equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$ Faraday's law

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$ Ampere's law

$$\nabla \cdot \mathbf{D} = \rho$$ Gauss's law

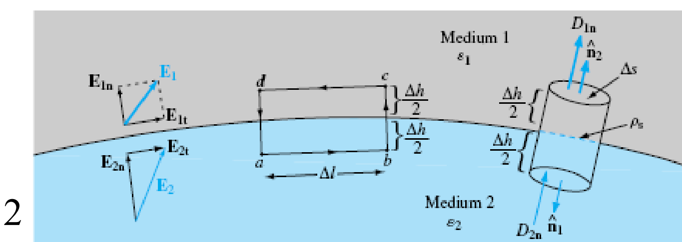
$$\nabla \cdot \mathbf{B} = 0$$ Gauss's law-magnetic

Boundary conditions:

$$\hat{\mathbf{n}}_{12} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0 \qquad \hat{\mathbf{n}}_{12} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s$$

$$\hat{\mathbf{n}}_{12} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \rho_s \qquad \hat{\mathbf{n}}_{12} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0$$

$\hat{\mathbf{n}}_{12} = \hat{\mathbf{n}}_2$ – normal to the interface from medium 1 to medium 2



Constitutive relations:

$$\mathbf{D} = \varepsilon \mathbf{E}$$

$$\mathbf{B} = \mu \mathbf{H}$$