Stat 475
Life Contingencies I

Chapter 3: Life tables and selection

Life tables

One of the traditional means of conveying information about a survival model is by using a **life table** (or **mortality table**).

This type of table assumes an initial cohort of individuals who are alive at age x_0.

- Often x_0 is 0, so that the table starts with newborns.
- The size of the initial cohort, ℓ_{x_0}, is called the **radix** of the table, and is completely arbitrary.

The life table gives the expected number of individuals alive (of the original ℓ_{x_0}) at each integral age up to ω (or some large age).

- The number of individuals expected to be alive at age x is ℓ_x.

Life tables — continued

Another quantity often found in life tables is the number of individuals, out of the ℓ_x that are expected to be alive at age x, that are expected to die prior to reaching age $x + 1$.

- This quantity is labelled d_x and can be calculated for each age x by using the equation $d_x = \ell_x - \ell_{x+1}$.
- The life table is fully defined by the column of ℓ_x values; d_x doesn't contain any new information — it's simply included for convenience.

Life tables may also tabulate other quantities for each integral value of x, such as q_x, p_x, and various other related items.

Life table calculations

Once we have a life table, we can use it to calculate the various probabilities we've been discussing.

Basic Relationship

$$ {}_t p_x = \frac{\ell_{x+t}}{\ell_x} $$

We often use this with $t = 1$ to get a one year survival rate, p_x.

We'll need additional information or assumptions to use this equation for non-integral values of t (more on this later).

More Relationships

$$ q_x = \frac{d_x}{\ell_x} \qquad {}_t q_x = \frac{\ell_x - \ell_{x+t}}{\ell_x} \qquad {}_{u|t} q_x = \frac{\ell_{x+u} - \ell_{x+u+t}}{\ell_x} $$

Life table example

x	ℓ_x	d_x
30	10,000.00	34.78
31	9,965.22	38.10
32	9,927.12	41.76
33	9,885.35	45.81
34	9,839.55	50.26
⋮	⋮	⋮

Table 3.1 from Dickson et al. (excerpted)

With the life table on the left:

1. Using the ℓ_x values, verify the column of d_x values.
2. Find ${}_3p_{30}$. [0.9885]
3. Find q_{30}. [0.0035]
4. Find ${}_{2|}q_{30}$. [0.0042]

Fractional Age Assumptions

When we used a continuous random variable to model the future lifetime (as in Chapter 2), we could find probabilities for periods of any length.

However, with a life table, ℓ_x is only specified for integer values of x. If we need values corresponding to fractional ages, we'll have to make some sort of assumption.

Three of the commonly used assumptions for fractional ages are:

- Uniform distribution of deaths (UDD)
- Constant force of mortality
- Balducci (hyperbolic)[1]

[1] You won't be responsible for this one for this class; it's also been taken off the MLC exam syllabus. I've included it in the notes only for completeness.

Uniform distribution of deaths (UDD) assumption

Under **UDD**, we have, for an integer x and for $0 \leq s < 1$:

UDD Assumption

$${}_s q_x \overset{UDD}{=} s\, q_x, \quad 0 \leq s < 1$$

From this we can derive fractional life table values:

Useful UDD Relationships

$$\ell_{x+s} \overset{UDD}{=} \ell_x - s\, d_x \overset{UDD}{=} (1-s)\, \ell_x + s\, \ell_{x+1}$$

We can also see that the density function within a year is constant:

Within-year Density for UDD

$${}_s p_x\, \mu_{x+s} \overset{UDD}{=} q_x$$

Constant force of mortality assumption

An alternative fractional age assumption posits a **constant force of mortality** (which we'll denote by μ_x^*) between integer ages:

Constant Force of Mortality Assumption

$$\mu_{x+s} \overset{CF}{=} \mu_x^*, \quad 0 \leq s < 1$$

We can find the value of this constant force of mortality:

Constant Force Relationship

$$\mu_x^* \overset{CF}{=} -\ln p_x \qquad \text{or equivalently,} \qquad p_x \overset{CF}{=} e^{-\mu_x^*}$$

Then we can calculate fractional year survival probabilities:

Fractional Year Survival Probabilities: Constant Force Assumption

$$_s p_x \overset{CF}{=} (p_x)^s$$

Balducci assumption

Another fractional age assumption, called the **Balducci** or hyperbolic assumption, assumes that the *inverse* of ℓ_{x+s} is a linear function of s:

Balducci Assumption

$$\frac{1}{\ell_{x+s}} \overset{Bal}{=} \frac{1}{\ell_x} - s\left(\frac{1}{\ell_x} - \frac{1}{\ell_{x+1}}\right), \quad 0 \leq s < 1$$

Under this assumption, we can derive formulas for various probabilities:

Balducci Relationships

$${}_s q_x \overset{Bal}{=} \frac{s\, q_x}{1-(1-s)q_x} \qquad {}_{1-s}q_{x+s} \overset{Bal}{=} (1-s)\, q_x$$

Fractional age assumption example

Using the life table above along with the _______________ fractional age assumption, calculate:

1. ${}_{0.3}p_{31}$ [0.9989]
2. ${}_{0.7}p_{30.6}$ [0.9975]
3. ${}_{2.9}p_{30.6}$ [0.9883]
4. ${}_{1.6|}q_{32}$ [0.00488]

Select and ultimate mortality models — notation

The life tables we have employed thus far are strictly based on attained age. To incorporate the effects of underwriting on mortality, we can consider **select and ultimate** models.

In order to accomodate select and ultimate mortality models, we will need some additional notation. For a person age $x + s$ who was selected at age x, we define the following symbols:

- $S_{[x]+s}(t)$ = survival function
- ${}_t p_{[x]+s}$ = probability of survival for another t years
- ${}_t q_{[x]+s}$ = probability of death within the next t years
- ${}_{u|t} q_{[x]+s}$ = deferred mortality probability
- $\mu_{[x]+s}$ = force of mortality

Select and ultimate mortality models — ultimate portion

The effect of the underwriting doesn't last forever — we usually assume some finite **select period** of d years.

Once a person is at or beyond the d year select period, they enter the **ultimate** part of the model.

From this point forward, mortality only depends on attained age and we can drop the more complicated notation. Then for $s \geq d$:

- ${}_tp_{[x-s]+s} = {}_tp_x$
- ${}_tq_{[x-s]+s} = {}_tq_x$
- ${}_{u|t}q_{[x-s]+s} = {}_{u|t}q_x$
- $\mu_{[x-s]+s} = \mu_x$

Select and ultimate mortality table example

Below is an excerpt of a select and ultimate mortality table with a 3 year select period (Table 3 from Jordan's Life Contingencies).

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	$x+3$
20	946,394	945,145	943,671	942,001	23
21	944,710	943,435	941,916	940,202	24
22	942,944	941,652	940,108	938,359	25
23	941,143	939,835	938,265	936,482	26
24	939,279	937,964	936,379	934,572	27
25	937,373	936,061	934,460	932,628	28
⋮	⋮	⋮	⋮	⋮	⋮

Select and ultimate mortality table example

Below is the same table as above, but in a different format.

Age x	Duration 0 $q_{[x]}$	Duration 1 $q_{[x-1]+1}$	Duration 2 $q_{[x-2]+2}$	Duration 3+ q_x
20	0.00132			
21	0.00135	0.00156		
22	0.00137	0.00161	0.00177	
23	0.00139	0.00164	0.00182	0.00191
24	0.00140	0.00167	0.00186	0.00196
25	0.00140	0.00169	0.00190	0.00200
⋮	⋮	⋮	⋮	⋮

Select and Ultimate Calculation Examples

Using the life table above, calculate:

1. ${}_{3}q_{[21]+2}$ [0.00577]

2. ${}_{2|3}q_{[20]+1}$ [0.00584]

3. ${}_{1.6}p_{[22]+2}$ under the UDD and CF assumptions. [0.99694]

Example

For all ages $x \geq 65$:

$$p_{[x]} = 0.999 \qquad p_{[x-1]+1} = 0.998 \qquad p_{[x-2]+2} = 0.997$$

x	ℓ_x
70	80556
71	79026
72	77410
73	75666
74	73802
75	71800

Find ${}_5p_{70}$, for select at ages 67-70. [0.891, 0.906, 0.923, 0.943]

You are given:

(i) $\mathring{e}_{30:\overline{40}|} = 27.692$

(ii) $S_0(t) = 1 - \frac{t}{\omega}, \quad 0 \le t \le \omega$

(iii) T_x is the future lifetime random variable for (x)

Calculate $Var(T_{30})$. [352.08]