Divorcing Made Easy

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Abstract. We discuss the proportionally fair allocation of a set of indivisible items to k agents. We assume that each agent specifies only a ranking of the items from best to worst. Agents do not specify their valuations of the items. An allocation is proportionally fair if all agents believe that they have received their fair share of the value according to how they value the items.

We give simple conditions (and a fast algorithm) for determining whether the agents rankings give sufficient information to determine a proportionally fair allocation. An important special case is a divorce situation with two agents. For such a divorce situation, we provide a particularly simple allocation rule that should have applications in the real world.

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1 Introduction

Let us consider the purely fictional situation where after twenty years of marriage, the purely fictional characters Ivana and Donald decide to divorce. Which of them will get the purely fictional family estate in Connecticut? Who will get the purely fictional family home in Palm Beach? The purely fictional hotel in Honolulu? The purely fictional hotel in Chicago? The one in Toronto? And the ones in Tampa, Fort Lauderdale, and Atlanta? Well, they clearly will have to agree on a good way of dividing their property. The division is very easy to implement, if one of the partners gets nothing whereas the other one gets everything, that is, if the division is implemented according to the well-known *"Don't get mad, get everything"* rule. The division becomes substantially more difficult, if they both are entitled to 50 percent of their common property; and the analysis of such situations is the topic of this paper.

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Let us start with a highly simplified situation (essentially taken from the second chapter of Brams & Taylor, 1999) where the property of the divorcing couple Ivana and Donald only consists of the following four items: a four-bedroom house, a retirement account (pension), a portfolio of investments, and the custody of their three children. Ivana and Donald rank these four items in the following way:

Rank	Ivana	Donald
1	Pension	House
2	House	Investments
3	Investments	Custody
4	Custody	Pension

Brams & Taylor suggest several allocation protocols that are based on alternately choosing items, and they point out some of the trouble that can arise from strategic, irrational, or revengeful behavior. (And of course we do expect Ivana and Donald to behave strategically, irrationally, and revengefully!) But let us note that this particular example with four items actually allows a straightforward allocation that is *proportionally fair*, which in this setting means that both parties believe that they have received at least half of the aggregate value of the items according to how they value the items: Ivana could get the pension and the investments, and Donald could get the house and custody. Ivana believes that this allocation gives her at least half of the value since she prefers the pension to the house and the investments to the custody. Similarly Donald believes that this allocation gives him at least half of the value since he prefers the house to the investments and custody to the pension. This is *independent of the exact financial or emotional values that are responsible for their rankings.* Hence in this particular example, Ivana and Donald can settle peacefully.

Here is a second, more troublesome scenario for Ivana and Donald:

Rank	Ivana	Donald
1	Custody	Custody
2	House	House
3	Investments	Investments
4	Pension	Pension

Without querying more information on the precise values that Ivana and Donald respectively assign to their items, it is not possible to find an allocation that is guaranteed to be proportionally fair. For instance in case both partners value custody strictly higher than house plus investments plus pension together, whichever partner doesn't get the custody would view the allocation as unfair. Intuitively the first divorce scenario allows a simple solution because Ivana and Donald *disagree* on their rankings; the second divorce scenario does not allow a simple solution because Ivana and Donald perfectly *agree* on their rankings.

Ivana and Donald's problem deals with the fair allocation of *indivisible* items. Fair divisions of continuously *divisible* items fall into the closely related area of cake cutting; see for instance Brams & Taylor (1996) and Robertson & Webb (1998). A recurrent theme in cake cutting is that the *right management of disagreement will make all involved parties happier.* Steinhaus (1948) attributes this crucial observation on cake division to Bronisław Knaster: *"It may be stated incidentally that if there are two (or more) partners with different estimations, there exists a division giving to everybody more than his due part. This fact disproves the common opinion that differences in estimations make fair division difficult."* Dubins & Spanier (1961) gave a non-constructive existence proof for Knaster's observation, whereas Woodall (1986) later provided a constructive algorithmic proof for this result.

Consider one final scenario for Ivana and Donald:

Rank	Ivana	Donald
1	Custody	House
2	House	Custody
3	Investments	Investments
4	Pension	Pension

In this scenario, there is disagreement on the rankings. But in this case the disagreement is insufficient to determine an allocation that is guaranteed to be proportionally fair. If Ivana for instance is allocated custody and the investments, then Donald might be unhappy since he assigns equally high values to custody, house and investments, but no value to the pension. And if Ivana for instance gets custody and the pension, then she herself might be unhappy for similar reasons.

Contributions of This Paper. In this paper we consider the proportionally fair allocation of a set of indivisible items to k agents. We assume that each agent specifies only an ordinal ranking of the items from best to worst. Agents do not specify their valuations of the items. An allocation is proportionally fair if all agents believe that they have received their fair share of the value according to how they personally value the items. We give a more precise formulation of the problem in Section 2. In Section 3 we give simple conditions for determining whether there is sufficient disagreement in the agents rankings to determine a proportionally fair allocation; in Section 4 we translate our insights into a fast recognition algorithm.

An important special case is a divorce situation with two agents (as in the purely fictional example with Ivana and Donald). Our main result in Section 5 is an extremely simple allocation rule that solves the classical common-property divorce situation in which each spouse is entitled to at least half of the joint assets. This rule could serve as a first step in real-world divorce settlement negotiations. If our rule detects an allocation, then in principle both parties should be satisfied with this allocation. If our rule fails to detect an allocation, then one would have to resort to other (more elaborate) division mechanisms as discussed by Brams & Taylor (1999); and if everything else fails, there always remains the possibility of lengthy court battles.

We stress that our approach only asks every agent to provide their personal *ordinal ranking* of the items, but does not require to have precise cardinal information on *how much* they value every particular item. Of course, ordinal information is much easier to provide than cardinal information.

2 Formal Definition of the Allocation Problem

Consider a set I of n indivisible items $1, \ldots, n$ that are to be allocated to k agents $1, \ldots, k$. Every agent j is entitled to a proportion a_j of these items; we assume that $0 < a_j < 1$ and that all these proportions add up to 1, that is, that $\sum_{j=1}^{k} a_j = 1$ holds. The most interesting special case is certainly when the proportions are equal, that is when each $a_j = \frac{1}{k}$. We write $i_1 \prec_j i_2$ (or equivalently $i_2 \succ_j i_1$) to denote that agent j values item i_2 strictly higher than item i_1. The preferences of agent j are summarized in his ordinal ranking π^j of the n items with

$$\pi^j(1) \succ_j \pi^j(2) \succ_j \cdots\cdots \succ_j \pi^j(n).$$

Agent j *lexicographically prefers* an item set $I_1 \subseteq I$ to another item set $I_2 \subseteq I$, if $I_1 - I_2$ contains an item that he ranks better than all the items in $I_2 - I_1$.

Next consider a valuation $v : I \to \mathbb{R}$ that assigns non-negative real values $v(1), \ldots, v(n)$ to the items $1, \ldots, n$ (these values could for instance be measured in dollars, or they could be based on emotions). We assume that valuations are additive on the subsets of I; hence the value of a subset $J \subseteq I$ is defined as $v(J) = \sum_{i \in J} v(i)$. An *allocation* is a partition of the item set I into disjoint subsets $I_1, \ldots, I_k$ where subset I_j is given to agent j. If $v(J) \geq a_j v(I)$, then subset $J \subseteq I$ is *proportionally fair* for agent j under valuation v. A *proportionally fair allocation* is an allocation that is proportionally fair to all k agents. A valuation v is *compatible* with the ordinal ranking of agent j, if $i_1 \prec_j i_2$ implies $v(i_1) \leq v(i_2)$ for all items i_1 and i_2. Here is the first central definition of this paper.

Definition 1. *A subset $J \subseteq I$ of items is* ordinally acceptable *for agent j, if it is proportionally fair for every valuation v that is compatible with j's ordinal ranking.*

If there are three items $1, 2, 3$ that an agent with proportional entitlement $\frac{1}{2}$ ranks $1 \succ 2 \succ 3$, then the item set $\{1\}$ will be ordinally acceptable for him under the compatible valuation $v(1) = 100$ and $v(2) = v(3) = 1$. Note however that item set $\{1\}$ is not ordinally acceptable for him, since under another compatible valuation $v'(1) = 100$ and $v'(2) = v'(3) = 99$ the set would not be proportionally fair for him.

Here is the second central definition of this paper.

Definition 2. *An allocation* $I_1, \ldots, I_k$ *is* ordinally fair*, if for every agent* j *the set* I_j *is ordinally acceptable.*

An ordinally fair allocation is the cheapest and simplest way of reaching a compromise that is ordinally acceptable for all participating agents. In the following sections, we discuss how to recognize whether a particular situation allows such an ordinally fair allocation.

3 Combinatorial Characterizations

In this section we derive combinatorial results on ordinally acceptable item sets and ordinally fair allocations. Our first result is a purely combinatorial characterization of ordinally acceptable item sets whose statement does not use item valuations.

Lemma 1. *A subset* $J \subseteq I$ *of items is ordinally acceptable for agent* j*, if and only if for every* p *with* $1 \le p \le n$ *the following condition is satisfied:*

$$|J \cap \{\pi^j(1), \ldots, \pi^j(p)\}| \;\ge\; a_j p. \tag{1}$$

Proof. To simplify the presentation, we will assume throughout the proof that the ordinal ranking of agent j is $1 \succ_j 2 \succ_j 3 \succ_j \cdots \succ_j n$ and hence $\pi^j(i) = i$ holds for all i. Furthermore, we will only consider valuations v that satisfy $0 \le v(i) \le 1$ for all items $i \in I$; this can be done without loss of generality since the values $v(i)$ can be scaled and normalized.

Now consider a valuation v as a geometric point in n-dimensional Euclidean space. Which points in $\mathbb{R}^n$ correspond to valuations that are compatible with agent j's ordinal ranking? First, every coordinate i with $1 \le i \le n$ must satisfy $v(i) \ge 0$ and $v(i) \le 1$. Secondly, every coordinate i with $1 \le i \le n-1$ must satisfy $v(i) \ge v(i+1)$. Hence we are dealing with a convex compact subset V of the n-dimensional unit cube that is the intersection of the halfspaces bounded by these $3n-1$ hyperplanes. The extreme points of the polytope V are the intersection points of n-element subsets of these hyperplanes, that is, the $n+1$ points $E_0, \ldots, E_n$, where the first p coordinates of point E_p are 1 and the remaining $n-p$ coordinates are 0. To see that E_p is an extreme point of V, note that E_p is the unique point in V maximizing $\sum_{i=1}^{p} v(i)$. To see that there are no other extreme points in V consider a linear objective $\sum_{i=1}^{n} \beta_i v(i)$, and a candidate extreme point $\hat{v}$. If there are a $\hat{v}(i)$ and $\hat{v}(i+1)$, where neither is 0 or 1, then one can increase and decrease the value of this objective, while maintaining feasibility, by increasing and decreasing $\hat{v}(i)$ and $\hat{v}(i+1)$ by some identical small amount.

Next consider an item set $J \subseteq I$. Under which valuations $v \in V$ will set J be ordinally acceptable for agent j? By definition, these are the valuations v that satisfy the linear inequality $\sum_{i \in J} v(i) \ge \alpha_j \sum_{i \in I} v(i)$ and hence are contained in the closed halfspace $H \subseteq \mathbb{R}^n$ that underlies this inequality. Set J is ordinally acceptable for agent j if and only if the polytope V is entirely contained in H, which is the case if and only if all the extreme points E_p with $0 \leq p \leq n$ of the polytope are contained in H. Now the origin E_0 is trivially contained in H, and the statement $E_p \in H$ for $1 \leq p \leq n$ is equivalent to condition (1). □

The following lemma yields a second combinatorial characterization of ordinally acceptable item sets.

Lemma 2. *Let $J \subseteq I$ be a set of m items, and let $r_1 < r_2 < \cdots < r_m$ denote the ranks of the items in J in the ranking of agent j (in other words, this set J consists of the r_1-most favorite item, the r_2-most favorite item, ..., and the r_m-most favorite item for agent j). Then set J is ordinally acceptable for agent j, if and only if the following two conditions hold: First*

$$a_j n \ \leq \ m, \tag{2}$$

and secondly all ℓ with $1 \leq \ell \leq m$ satisfy

$$r_\ell \ \leq \ (\ell - 1)\frac{1}{a_j} + 1. \tag{3}$$

Proof. We use the characterization in Lemma 1. For the if-statement, assume that set J satisfies (2) and (3), and note that this implies $r_1 = 1$. Consider some p with $1 \leq p \leq n$. If $r_{\ell-1} \leq p \leq r_\ell - 1$ with $2 \leq \ell \leq m$, then (3) implies

$$|J \cap \{\pi^j(1), \ldots, \pi^j(p)\}| \ = \ \ell - 1 \ \geq \ a_j(r_\ell - 1) \ \geq \ a_j p,$$

which yields (1). If $r_m \leq p \leq n$, then (2) implies

$$|J \cap \{\pi^j(1), \ldots, \pi^j(p)\}| \ = \ m \ \geq \ a_j n \ \geq \ a_j p,$$

which again yields (1). For the only-if-statement, assume that J is ordinally acceptable. Then by setting $p = 1$, respectively by setting $p = r_\ell - 1$ with $2 \leq \ell \leq m$, condition (1) implies condition (3). Finally (2) follows by using $p = n$ in (1). □

The statements in Lemma 1 and Lemma 2 put severe constraints on situations that allow ordinally fair allocations. The following lemma shows that in ordinally fair allocations the a_j's must be rational numbers of a very special form, and that the numerators of these rational numbers a priori determine the number of items that are allocated to every agent.

Lemma 3. *Assume that there exists an ordinally fair allocation where agent j ($j = 1, \ldots, k$) receives b_j items. Then $a_j = b_j/n$ holds for $j = 1, \ldots, k$.*

Proof. Consider an ordinally fair allocation that allocates b_j items to agent j. Then condition (2) in Lemma 2 implies $b_j \geq a_j n$. Since all proportions a_j add up to a total of 1, this leads to

$$n \ = \ \sum_{j=1}^{k} b_j \ \geq \ n \sum_{j=1}^{k} a_j \ = \ n.$$

This implies that every inequality in fact is an equality, and consequently that $b_j = a_j n$ holds for all j. □

4 An Efficient Algorithmic Characterization

In this section, we provide a fast algorithm that recognizes situations that allow ordinally fair allocations, and that computes such an allocation whenever one exists. The main idea is to translate the problem into an equivalent matching problem in a bipartite graph; see for instance Lovász & Plummer (1986).

Consider an instance of the allocation problem with n agents. We assume that the numbers $b_j = a_j n$ ($1 \le j \le k$) are integers, as otherwise by Lemma 3 an ordinally fair allocation cannot exist. For every agent $j = 1, \ldots, k$ and for $\ell = 1, \ldots, b_j$, let $I(j, \ell)$ denote the $(\ell - 1)/a_j + 1$ highest ranked items in the ordinal ranking of j. According to condition (3) in Lemma 2, the ℓ-best item assigned to agent j should be from this set $I(j, \ell)$. We create a bipartite graph $(X \cup Y, E)$ with vertex set $X \cup Y$ and a set E of edges between X and Y.

- For every item $i \in I$, there is a corresponding vertex $x(i)$ in X.
- For every agent $j = 1, \ldots, k$ and for every $\ell = 1, \ldots, b_j$, there is a corresponding vertex $y(j, \ell)$ in Y. Intuitively speaking, this vertex encodes the ℓ-th item that is assigned to agent j.
- For every agent $j = 1, \ldots, k$, for every $\ell = 1, \ldots, b_j$, and for every item $i \in I(j, \ell)$, there is a corresponding edge in E that connects vertex $x(i)$ to vertex $y(j, \ell)$.

By Lemma 2, some fixed item set J is ordinally acceptable for some fixed agent j, if and only if the bipartite graph contains a matching M_j between the vertices $x(i)$ with $i \in J$ and the vertices $y(j, \ell)$ with $1 \le \ell \le b_j$. Furthermore, an allocation $I_1, \ldots, I_k$ is ordinally fair if such matchings M_j exist for $j = 1, \ldots, k$. Then the union of all matchings M_j with $1 \le j \le k$ forms a perfect matching between the vertex sets X and Y, that is, a subset of the edges that touches every vertex in $X \cup Y$ exactly once.

Lemma 4. *A ordinally fair allocation exists if and only if the corresponding bipartite graph possesses a perfect matching.* □

It is well-known that perfect matchings can be detected and computed in polynomial time; see Lovász & Plummer (1986). Whence we arrive at the theoretical main result of this paper.

Theorem 1. *It is possible in polynomial time to determine if an ordinally fair allocation exists, and if so, to find one.* □

Although Theorem 1 fully settles the problem from the mathematical point of view, its applicability in real world scenarios may be limited: First, the behavior of perfect matching algorithms are quite intricate, and hence will be hard to understand and impossible to reproduce for the litigant parties (who in a real world scenario will most likely be mathematically illiterate). Secondly, the resulting ordinally fair allocation is not uniquely determined. The bipartite graph might have many distinct perfect matchings, and the perfect matching algorithm will simply pick and output one of them. This type of behavior may not be tolerable as part of a legal negotiation process.

As a partial way out of this dilemma, we could try and make the agents choose their items in alternating turns. A turn of agent j would consist in picking the most desired item in his ranking π^j that (i) has not been allocated in any of the earlier turns, and that (ii) yields a partial allocation that still can be extended to an ordinally fair allocation. Unfortunately, this approach does not resolve the first limitation (since the perfect matching algorithm still plays a major role in the process), and it does not remove but only shift the headache in the second limitation to the choice of the alternating turn sequence. And finding an appropriate turn sequence is a challenging task on its own. In the following section, we offer a full remedy for the important special case with $k = 2$ agents who both are entitled to a proportion of $\frac{1}{2}$.

5 A Simple Allocation Rule for Divorce Situations

In this section we consider a divorce situation where a set I with n items has to be allocated to two agents (called husband [Donald] and wife [Ivana]) with entitled proportions $a_1 = a_2 = \frac{1}{2}$. For $1 \leq \ell \leq n$ we denote by H_ℓ and W_ℓ the set of the ℓ most desired items in the rankings of husband and wife, respectively. By Lemma 3, we will throughout assume that n is an even number.

The TRUMP rule

For $\ell := 1$ to $n/2$ do

> Let x be the unallocated item in $H_{2\ell-1}$ that the wife likes least
> Allocate x to the husband
>
> Let y be the unallocated item in $W_{2\ell-1}$ that the husband likes least
> Allocate y to the wife

Fig. 1. An allocation rule for two divorcing agents with entitled proportions $\frac{1}{2}$

Figure 1 presents our allocation rule called TRUMP. The naming of our rule is inspired by certain trick-taking card games where a trump card automatically prevails over all other cards and wins the trick. Similarly our rule prevails over all other rules (in the cases where it succeeds!). To start the analysis of TRUMP, we note that TRUMP might get stuck during the ℓth round of the loop, if all the items in set $H_{2\ell-1}$ or $W_{2\ell-1}$ have already been allocated in earlier rounds. In such a case we say that TRUMP fails. Otherwise the rule succeeds, and then at termination has allocated $n/2$ of the items to the husband and the remaining $n/2$ items to the wife.

Lemma 5. *For every divorce situation with an even number n of items, the following three statements are pairwise equivalent:*

(i) The TRUMP *rule succeeds.*
(ii) There exists an ordinally fair allocation.
(iii) $H_{2\ell-1} \neq W_{2\ell-1}$ *holds for* $\ell = 1, \ldots, n/2$.

Proof. We show that (i) implies (ii) implies (iii) implies (i).

First assume (i). Then the allocation computed by TRUMP assigns $n/2$ items to the husband and $n/2$ items to the wife, and hence satisfies condition (2) in Lemma 2. Since the ℓth item ($1 \leq \ell \leq n/2$) assigned to husband and wife belongs to their $2\ell - 1$ top-ranked items, the computed allocation also satisfies condition (3). Then Lemma 2 yields that the computed allocation is ordinally acceptable for husband and wife, and hence (ii) holds.

Next assume that (ii) holds, and consider some fixed ordinally fair allocation. Let ℓ be an integer from the range $1 \leq \ell \leq n/2$. According to condition (3) in Lemma 2, the ordinally fair allocation gives at least ℓ items from $H_{2\ell-1}$ to the husband and at least ℓ items from $W_{2\ell-1}$ to the wife. This implies $|H_{2\ell-1} \cup W_{2\ell-1}| \geq 2\ell$, and makes $H_{2\ell-1} = W_{2\ell-1}$ impossible. This yields (iii).

Finally assume (iii). Consider the moment in time when TRUMP enters the loop for the ℓth time ($1 \leq \ell \leq n/2$). Up to this moment husband and wife each have received $\ell - 1$ items. Since only $2\ell - 2$ items have been allocated, there exists at least one eligible element x among the $2\ell - 1$ items in $H_{2\ell-1}$, and hence TRUMP cannot get stuck while selecting x. Next, let z denote the element in $H_{2\ell-1}$ that the wife likes the least. From $H_{2\ell-1} \neq W_{2\ell-1}$ we conclude $z \notin W_{2\ell-1}$. If TRUMP has not assigned item z to the husband in one of the earlier rounds, it must assign z to the husband in the current round. In any case, at the moment when item y is to be selected for the wife, at most $2\ell - 2$ of the up to now $2\ell - 1$ allocated items are in $W_{2\ell-1}$. Hence TRUMP can also not get stuck while selecting y. This yields (i). □

The above lemma and its proof imply the main theorem of this section.

Theorem 2. *Whenever a divorce situation allows an ordinally fair allocation, the* TRUMP *rule succeeds in finding one.* □

Since TRUMP always assigns to the husband those items that the wife wants the least, the highly ranked items in the wife's ranking will remain available for the wife. And by symmetry, the highly ranked items in the husband's ranking will remain available for the husband. These observations suggest that there should be no other ordinally fair allocation that makes both husband and wife happier. The following lemma makes this intuition mathematically precise.

Lemma 6. *Consider a divorce situation for which* TRUMP *computes the ordinally fair allocation H and W. Then there does not exist any other ordinally fair allocation H' and W' in which the husband lexicographically prefers H' to H and in which the wife lexicographically prefers W' to W.*

Proof. Suppose otherwise. Let x_0 denote the item in $H' - H = W - W'$ ranked highest by the husband, and let y_0 denote the item in $W' - W = H - H'$ ranked highest by the wife. Then by the definition of lexicographical preference the husband prefers x_0 to y_0, whereas the wife prefers y_0 to x_0.

By symmetry, we may assume that Trump allocates y_0 to the husband before it assigns x_0 to the wife. Then at the moment when Trump allocates y_0, also item x_0 would be eligible. Since the wife ranks y_0 above x_0, Trump would not allocate item y_0 at that moment. □

Finally, we note that Trump does not treat husband and wife in a perfectly symmetric fashion: In every round the husband receives his item x before the wife does receive her item y, and hence x is not an eligible option for the wife. Consequently if husband and wife switch places, the output of Trump might change. Assume that the husband ranks $\langle 1, 2, 3, 4\rangle$ whereas the wife ranks $\langle 4, 2, 3, 1\rangle$. Then Trump gives $\{1, 3\}$ to the husband. But if husband and wife would switch places, the husband would receive the set $\{1, 2\}$, which he lexicographically prefers to $\{1, 3\}$.

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