



Optimal non-dissipative discontinuous Galerkin methods for Maxwell's equations in Drude metamaterials



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ABSTRACT

Simulation of electromagnetic wave propagation in metamaterials leads to more complicated time domain Maxwell's equations than the standard Maxwell's equations in free space. In this paper, we develop and analyze a non-dissipative discontinuous Galerkin (DG) method for solving the Maxwell's equations in Drude metamaterials. Previous discontinuous Galerkin methods in the literature for electromagnetic wave propagation in metamaterials were either non-dissipative but sub-optimal, or dissipative and optimal. Our method uses a different and simple choice of numerical fluxes, achieving provable non-dissipative stability and optimal error estimates simultaneously. We prove the stability and optimal error estimates for both semi- and fully discrete DG schemes, with the leap-frog time discretization for the fully discrete case. Numerical results are given to demonstrate that the DG method can solve metamaterial Maxwell's equations effectively.

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1. Introduction

Since the first successful construction of metamaterials with both negative permittivity and negative permeability in 2000 by Smith et al. [1], there has been a growing interest in the study of metamaterials and its applications in invisibility cloak design (e.g. [2,3]) and sub-wavelength imaging (e.g. [4,5]). Numerical simulation of wave propagation in metamaterials plays a very important role in these investigations. Description of electromagnetic wave propagation in metamaterials leads to more complicated time domain Maxwell's equations than the standard Maxwell's equations in free space. Compared to many excellent works obtained for solving Maxwell's equations in the free space (see [6–8] and references therein), developing efficient and rigorous numerical methods for metamaterial Maxwell's equations needs much more efforts. The most recent progress in this direction can be found in the monograph by Hao and Mittra [9] on the finite-difference time-domain (FDTD) methods for modeling metamaterials, and in the monograph by Li and Huang [10] on the finite-element time-domain (FETD) methods for metamaterials.

The discontinuous Galerkin (DG) method was initially introduced in 1973 by Reed and Hill [11] for solving a neutron transport equation. Because of a few nice features of the DG methods (such as local solvability, flexibility in h - p adaptivity, and efficiency in parallel implementation), various DG methods have been proposed for solving different partial differential equations since then. The work of Cockburn and Shu [12,13] on Runge–Kutta DG (RKDG) methods for solving linear and nonlinear time dependent hyperbolic partial differential equations (PDEs), in which the spatial discretization is by DG and the time discretization is by Runge–Kutta methods (other time discretization methods are of course also possible, such

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as the leap frog method adopted in this paper), has facilitated the rapid advance and application of DG methods. Also relevant to our schemes studied in this paper, especially the choice of the alternating fluxes, we should mention the so-called local discontinuous Galerkin (LDG) method, which was introduced by Cockburn and Shu [14] for solving time-dependent convection–diffusion systems and was used in [15] for solving second-order hyperbolic equations. For more details on the algorithm design, analysis, implementation and application of DG and LDG schemes for solving time-dependent PDEs, readers can consult the review articles [16,17] and the references therein.

Maxwell’s equations play a very important role in describing wave propagation phenomena in various media. It is no surprise that many DG methods have already been developed for time-harmonic Maxwell’s equations in free space (e.g., [18–21]), time-dependent Maxwell’s equations in both free space (e.g., [22–24,7]) and dispersive media (e.g., [25–27]). The recently developed metamaterials also promoted some study of DG methods for solving Maxwell’s equations in metamaterials (e.g., [28–31]). Among these methods, there are non-dissipative ones, e.g. [30], and optimal ones, e.g. [31]. However, our new method in this paper appears to be the only one to attain both optimal convergence rate and non-dissipation. Recently Xing et al. [15,32] and Chung et al. [33,34] proposed optimal and non-dissipative discontinuous Galerkin methods for wave equations. Our approach is similar to the LDG method of Xing et al. [15,32] in terms of the choices of the numerical fluxes in the scheme and the projection in the error estimate proof.

The rest of the paper is organized as follows. In Section 2, we first present the governing equations for wave propagation in Drude metamaterials. In Section 3, we present the semi-discrete DG method, and prove its stability and the optimal error estimate in the L^2 norm. Then in Section 4, we propose a fully discrete DG method with leap-frog type time discretization. Detailed numerical stability analysis is carried out. The optimal convergence result is stated, but the lengthy technical proof is skipped as it mostly follows the line of the proof of the semi-discrete case. Section 5 is devoted to numerical experiments that demonstrate the optimal convergence rates of the proposed DG method. Finally, we conclude the paper in Section 6.

2. The governing equations

Consider the metamaterial Drude model [10] in the domain Ω :

$$\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H} - \mathbf{J} \tag{2.1}$$

$$\mu_0 \frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \mathbf{E} - \mathbf{K} \tag{2.2}$$

$$\frac{1}{\epsilon_0 \omega_{pe}^2} \frac{\partial \mathbf{J}}{\partial t} + \frac{\Gamma_e}{\epsilon_0 \omega_{pe}^2} \mathbf{J} = \mathbf{E} \tag{2.3}$$

$$\frac{1}{\mu_0 \omega_{pm}^2} \frac{\partial \mathbf{K}}{\partial t} + \frac{\Gamma_m}{\mu_0 \omega_{pm}^2} \mathbf{K} = \mathbf{H} \tag{2.4}$$

supplemented with the perfect conduct (PEC) boundary condition

$$\mathbf{n} \times \mathbf{E} = \mathbf{0} \quad \text{on } \partial\Omega, \tag{2.5}$$

and the initial conditions

$$\mathbf{E}(\mathbf{x}, 0) = \mathbf{E}_0(\mathbf{x}), \quad \mathbf{H}(\mathbf{x}, 0) = \mathbf{H}_0(\mathbf{x}), \quad \mathbf{J}(\mathbf{x}, 0) = \mathbf{J}_0(\mathbf{x}), \quad \mathbf{K}(\mathbf{x}, 0) = \mathbf{K}_0(\mathbf{x}), \tag{2.6}$$

where \mathbf{n} denotes the outward unit normal vector of $\partial\Omega$, $\mathbf{E}_0(\mathbf{x})$, $\mathbf{H}_0(\mathbf{x})$, $\mathbf{J}_0(\mathbf{x})$ and $\mathbf{K}_0(\mathbf{x})$ are some given proper functions. Here ϵ_0 is the vacuum permittivity, μ_0 is the vacuum permeability, $\omega_{pe} \geq 0$ and $\omega_{pm} \geq 0$ are the electric and magnetic plasma frequencies respectively, Γ_e and Γ_m are the electric and magnetic damping frequencies respectively, $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{H}(\mathbf{x}, t)$ are the electric and magnetic fields respectively, and $\mathbf{J}(\mathbf{x}, t)$ and $\mathbf{K}(\mathbf{x}, t)$ are the induced electric and magnetic currents respectively.

To avoid the technicality of the proof for 3D problems, below we only consider the transverse-electric mode with respect to z in two dimensions (2-D), i.e., the so-called TE_z mode, which involves only fields $\mathbf{E} = (E_x, E_y)$, $\mathbf{H} = H_z := H$, $\mathbf{J} = (J_x, J_y)$, $\mathbf{K} = K_z$, and the curls $\nabla \times \mathbf{E} = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}$ and $\nabla \times \mathbf{H} = (\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x})'$. Here the subindices x , y and z denote the components in the x , y and z directions, respectively. More specifically, the governing equations of the TE_z Drude model can be written as:

$$\epsilon_0 \frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y} - J_x, \tag{2.7}$$

$$\epsilon_0 \frac{\partial E_y}{\partial t} = -\frac{\partial H_z}{\partial x} - J_y, \tag{2.8}$$

$$\mu_0 \frac{\partial H_z}{\partial t} = -\frac{\partial E_y}{\partial x} + \frac{\partial E_x}{\partial y} - K_z, \tag{2.9}$$

$$\frac{1}{\epsilon_0 \omega_{pe}^2} \frac{\partial J_x}{\partial t} + \frac{\Gamma_e}{\epsilon_0 \omega_{pe}^2} J_x = E_x, \tag{2.10}$$

$$\frac{1}{\epsilon_0 \omega_{pe}^2} \frac{\partial J_y}{\partial t} + \frac{\Gamma_e}{\epsilon_0 \omega_{pe}^2} J_y = E_y, \tag{2.11}$$

$$\frac{1}{\mu_0 \omega_{pm}^2} \frac{\partial K_z}{\partial t} + \frac{\Gamma_m}{\mu_0 \omega_{pm}^2} K_z = H_z. \tag{2.12}$$

For simplicity, we consider solving (2.7)–(2.12) on a rectangular type physical domain $\Omega = [a, b] \times [c, d]$, which is discretized by a possibly non-uniform grid

$$a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \dots < x_{N_x + \frac{1}{2}} = b, \quad c = y_{\frac{1}{2}} < y_{\frac{3}{2}} < \dots < y_{N_y + \frac{1}{2}} = d.$$

The time domain $[0, T]$ is discretized into $N_t + 1$ uniform intervals by discrete times $0 = t_0 < t_1 < \dots < t_{N_t + 1} = T$, where $t_n = n \cdot \tau$, and the time step size $\tau = \frac{T}{N_t + 1}$. For simplicity, we define the rectangular cells $K_{i,j} = I_i \times J_j$, where $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, $i = 1, \dots, N_x$, and $J_j = [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$, $j = 1, \dots, N_y$. The mesh sizes are denoted by $h_i^x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$, $h_j^y = y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}$, with $h^x = \max_{1 \leq i \leq N_x} h_i^x$, $h^y = \max_{1 \leq j \leq N_y} h_j^y$, and $h = \max(h^x, h^y)$ the maximal mesh size. We also assume that the mesh is regular, i.e. $h_i^x \geq C \cdot h$ and $h_j^y \geq C \cdot h$, $\forall i, j$, for some C independent of h .

The finite element space V_h^k is chosen as the space of tensor products of piecewise polynomials of degree at most k in each variable on every element $K_{i,j}$, i.e.,

$$V_h^k = \{v : v|_K \in Q_k(K), \forall K \in \mathcal{T}_h\}, \tag{2.13}$$

where \mathcal{T}_h denotes the Cartesian grid on Ω described above with mesh size h , and Q_k is the space of tensor products of one-dimensional polynomials of degree up to k .

For the corresponding variable u we denote its numerical solution u_h , which belongs to the finite element space V_h^k . Note that functions in V_h^k are allowed to have discontinuities across element interfaces. Furthermore we denote by $u_h(x_{i+\frac{1}{2}}^+, y)$ (or $(u_h)_{i+\frac{1}{2}, y}^+$, $u_h^+(x_{i+\frac{1}{2}}, y)$) and $u_h(x_{i+\frac{1}{2}}^-, y)$ (or $(u_h)_{i+\frac{1}{2}, y}^-$, $u_h^-(x_{i+\frac{1}{2}}, y)$) the limit values of u_h at $x_{i+\frac{1}{2}}$ from the right cell $K_{i+1,j}$ and from the left cell $K_{i,j}$, respectively. $u_h(x, y_{j+\frac{1}{2}}^+)$ (or $(u_h)_{x, j+\frac{1}{2}}^+$, $u_h^+(x, y_{j+\frac{1}{2}})$) and $u_h(x, y_{j+\frac{1}{2}}^-)$ (or $(u_h)_{x, j+\frac{1}{2}}^-$, $u_h^-(x, y_{j+\frac{1}{2}})$) are defined similarly. We denote by $\|\cdot\|$ the L^2 norm over the domain Ω .

Finally, we would like to remark that the corresponding PEC boundary condition (2.5) in 2-D becomes:

$$E_x(x, y, t)|_{y=c, d} = 0, \quad E_y(x, y, t)|_{x=a, b} = 0. \tag{2.14}$$

3. The semi-discrete DG method

The DG method for (2.7)–(2.12) can be formulated as follows: find $E_{xh}, E_{yh}, H_{zh}, J_{xh}, J_{yh}, K_{zh} \in C^1([0, T]; V_h^k)$ such that

$$\epsilon_0 \int_{K_{i,j}} \frac{\partial E_{xh}}{\partial t} \phi - \int_{I_i} \left((\hat{H}_{zh} \phi^-)_{x, j+\frac{1}{2}} - (\hat{H}_{zh} \phi^+)_{x, j-\frac{1}{2}} \right) dx + \int_{K_{i,j}} H_{zh} \frac{\partial \phi}{\partial y} + \int_{K_{i,j}} J_{xh} \phi = 0, \tag{3.1}$$

$$\epsilon_0 \int_{K_{i,j}} \frac{\partial E_{yh}}{\partial t} \psi + \int_{J_j} \left((\hat{H}_{zh} \psi^-)_{i+\frac{1}{2}, y} - (\hat{H}_{zh} \psi^+)_{i-\frac{1}{2}, y} \right) dy - \int_{K_{i,j}} H_{zh} \frac{\partial \psi}{\partial x} + \int_{K_{i,j}} J_{yh} \psi = 0, \tag{3.2}$$

$$\begin{aligned} \mu_0 \int_{K_{i,j}} \frac{\partial H_{zh}}{\partial t} \chi + \int_{J_j} \left((\hat{E}_{yh} \chi^-)_{i+\frac{1}{2}, y} - (\hat{E}_{yh} \chi^+)_{i-\frac{1}{2}, y} \right) dy - \int_{K_{i,j}} E_{yh} \frac{\partial \chi}{\partial x} \\ - \int_{I_i} \left((\hat{E}_{xh} \chi^-)_{x, j+\frac{1}{2}} - (\hat{E}_{xh} \chi^+)_{x, j-\frac{1}{2}} \right) dx + \int_{K_{i,j}} E_{xh} \frac{\partial \chi}{\partial y} + \int_{K_{i,j}} K_{zh} \chi = 0, \end{aligned} \tag{3.3}$$

$$\frac{1}{\epsilon_0 \omega_{pe}^2} \int_{K_{i,j}} \frac{\partial J_{xh}}{\partial t} u_1 + \frac{\Gamma_e}{\epsilon_0 \omega_{pe}^2} \int_{K_{i,j}} J_{xh} u_1 - \int_{K_{i,j}} E_{xh} u_1 = 0, \tag{3.4}$$

$$\frac{1}{\epsilon_0 \omega_{pe}^2} \int_{K_{i,j}} \frac{\partial J_{yh}}{\partial t} u_2 + \frac{\Gamma_e}{\epsilon_0 \omega_{pe}^2} \int_{K_{i,j}} J_{yh} u_2 - \int_{K_{i,j}} E_{yh} u_2 = 0, \tag{3.5}$$

$$\frac{1}{\mu_0 \omega_{pm}^2} \int_{K_{i,j}} \frac{\partial K_{zh}}{\partial t} v + \frac{\Gamma_m}{\mu_0 \omega_{pm}^2} \int_{K_{i,j}} K_{zh} v - \int_{K_{i,j}} H_{zh} v = 0, \tag{3.6}$$

for all test functions $\phi, \psi, \chi, u_1, u_2, v \in V_h^k$. $\hat{H}_{zh}, \hat{E}_{yh}, \hat{E}_{xh}$ are the cell boundary terms obtained from integration by parts, and they are the so-called numerical fluxes. These numerical fluxes are functions on cell boundaries and should be designed carefully to ensure numerical stability or energy conservation, and optimal rate of convergence, for different PDEs. Motivated by the LDG schemes for diffusion equations and second order wave equations [14,15], here we choose the simple but elegant alternating fluxes:

$$\hat{E}_{xh} \left(x, y_{j+\frac{1}{2}} \right) = E_{xh}^+ \left(x, y_{j+\frac{1}{2}} \right) \quad \forall j = 1, \dots, N_y - 1, \tag{3.7}$$

$$\hat{E}_{xh} \left(x, y_{\frac{1}{2}} \right) = \hat{E}_{xh} \left(x, y_{N_y+\frac{1}{2}} \right) = 0, \tag{3.8}$$

$$\hat{E}_{yh} \left(x_{i+\frac{1}{2}}, y \right) = E_{yh}^+ \left(x_{i+\frac{1}{2}}, y \right) \quad \forall i = 1, \dots, N_x - 1, \tag{3.9}$$

$$\hat{E}_{yh} \left(x_{\frac{1}{2}}, y \right) = \hat{E}_{yh} \left(x_{N_x+\frac{1}{2}}, y \right) = 0, \tag{3.10}$$

$$\hat{H}_{zh} \left(x, y_{j+\frac{1}{2}} \right) = H_{zh}^- \left(x, y_{j+\frac{1}{2}} \right) \quad \forall j = 1, \dots, N_y, \tag{3.11}$$

$$\hat{H}_{zh} \left(x, y_{\frac{1}{2}} \right) = H_{zh}^+ \left(x, y_{\frac{1}{2}} \right) + c_0 \left[\left[E_{xh} \left(x, y_{\frac{1}{2}} \right) \right] \right], \tag{3.12}$$

$$\hat{H}_{zh} \left(x_{i+\frac{1}{2}}, y \right) = H_{zh}^- \left(x_{i+\frac{1}{2}}, y \right) \quad \forall i = 1, \dots, N_x, \tag{3.13}$$

$$\hat{H}_{zh} \left(x_{\frac{1}{2}}, y \right) = H_{zh}^+ \left(x_{\frac{1}{2}}, y \right) - c_0 \left[\left[E_{yh} \left(x_{\frac{1}{2}}, y \right) \right] \right], \tag{3.14}$$

where c_0 is a constant independent of mesh size h , and the jumps $\left[\left[E_{xh} \left(x, y_{\frac{1}{2}} \right) \right] \right] = E_{xh}^+ \left(x, y_{\frac{1}{2}} \right) - 0$, $\left[\left[E_{yh} \left(x_{\frac{1}{2}}, y \right) \right] \right] = E_{yh}^+ \left(x_{\frac{1}{2}}, y \right) - 0$. Here we use the standard notation $\left[\left[\phi \right] \right] = \phi^+ - \phi^-$ for jumps on cell boundaries.

Remark 3.1. We will see later why the jump terms in (3.12) and (3.14) are necessary at the physical boundary, in the proof of optimal error estimates (Theorem 3.2), see also e.g. [35]. Note also that when $c_0 = \frac{1}{2}$, (3.12) and (3.14) coincide with the standard upwind fluxes, which are:

$$\begin{aligned} \hat{H}_{zh} \left(x, y_{\frac{1}{2}} \right) &= \frac{1}{2} \left(H_{zh}^+ \left(x, y_{\frac{1}{2}} \right) + H_{zh}^- \left(x, y_{\frac{1}{2}} \right) \right) + \frac{1}{2} \left[\left[E_{xh} \left(x, y_{\frac{1}{2}} \right) \right] \right] \\ \hat{H}_{zh} \left(x_{\frac{1}{2}}, y \right) &= \frac{1}{2} \left(H_{zh}^+ \left(x_{\frac{1}{2}}, y \right) + H_{zh}^- \left(x_{\frac{1}{2}}, y \right) \right) - \frac{1}{2} \left[\left[E_{yh} \left(x_{\frac{1}{2}}, y \right) \right] \right] \end{aligned}$$

where the undefined $H_{zh}^- \left(x, y_{\frac{1}{2}} \right)$ and $H_{zh}^- \left(x_{\frac{1}{2}}, y \right)$ are replaced by $H_{zh}^+ \left(x, y_{\frac{1}{2}} \right)$ and $H_{zh}^+ \left(x_{\frac{1}{2}}, y \right)$.

3.1. The stability analysis

In this subsection, we present the stability analysis for our scheme.

First, let us look at the stability for the governing equations. Multiplying the governing equations (2.7)–(2.12) by $E_x, E_y, H_z, J_x, J_y, K_z$, respectively, then integrating over the space and time domain $\Omega \times [0, t]$, summing up the resultants, and using the 2D PEC boundary condition (2.14), we can easily obtain the energy identity (cf. [36]):

$$\begin{aligned} & \left[\epsilon_0 (\|E_x\|^2 + \|E_y\|^2) + \mu_0 \|H_z\|^2 + \frac{1}{\epsilon_0 \omega_{pe}^2} (\|J_x\|^2 + \|J_y\|^2) + \frac{1}{\mu_0 \omega_{pm}^2} \|K_z\|^2 \right] (t) \\ & + \int_0^t \left[\frac{2\Gamma_e}{\epsilon_0 \omega_{pe}^2} (\|J_x\|^2 + \|J_y\|^2) + \frac{2\Gamma_m}{\mu_0 \omega_{pm}^2} \|K_z\|^2 \right] (s) ds \\ & = \left[\epsilon_0 (\|E_x\|^2 + \|E_y\|^2) + \mu_0 \|H_z\|^2 + \frac{1}{\epsilon_0 \omega_{pe}^2} (\|J_x\|^2 + \|J_y\|^2) + \frac{1}{\mu_0 \omega_{pm}^2} \|K_z\|^2 \right] (0). \end{aligned} \tag{3.15}$$

Below we will show that our proposed semi-discrete DG method satisfies a similar energy identity as that given in the continuous level (3.15).

Theorem 3.1. *The semi-discrete DG method (3.1)–(3.6) with alternating fluxes (3.7)–(3.14) satisfies the following energy identity: For any $t \geq 0$,*

$$\left[\epsilon_0 (\|E_{xh}\|^2 + \|E_{yh}\|^2) + \mu_0 \|H_{zh}\|^2 + \frac{1}{\epsilon_0 \omega_{pe}^2} (\|J_{xh}\|^2 + \|J_{yh}\|^2) + \frac{1}{\mu_0 \omega_{pm}^2} \|K_{zh}\|^2 \right] (t)$$

$$\begin{aligned}
 & + \int_0^t \left[\frac{2\Gamma_e}{\epsilon_0\omega_{pe}^2} (\|J_{xh}\|^2 + \|J_{yh}\|^2) + \frac{2\Gamma_m}{\mu_0\omega_{pm}^2} \|K_{zh}\|^2 \right] (s) ds \\
 & + \int_0^t \left[c_0 \int_a^b (E_{xh}^+)_{x, \frac{1}{2}}^2 dx + c_0 \int_c^d (E_{yh}^+)_{\frac{1}{2}, y}^2 dy \right] (s) ds \\
 = & \left[\epsilon_0 (\|E_{xh}\|^2 + \|E_{yh}\|^2) + \mu_0 \|H_{zh}\|^2 + \frac{1}{\epsilon_0\omega_{pe}^2} (\|J_{xh}\|^2 + \|J_{yh}\|^2) + \frac{1}{\mu_0\omega_{pm}^2} \|K_{zh}\|^2 \right] (0). \tag{3.16}
 \end{aligned}$$

Proof. Taking $\phi = E_{xh}$, $\psi = E_{yh}$, $\chi = H_{zh}$, $u_1 = J_{xh}$, $u_2 = J_{yh}$, $v = K_{zh}$ in (3.1)–(3.6), respectively, adding the resultants together, we have

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \int_{K_{i,j}} \left[\epsilon_0 (|E_{xh}|^2 + |E_{yh}|^2) + \mu_0 |H_{zh}|^2 + \frac{1}{\epsilon_0\omega_{pe}^2} (|J_{xh}|^2 + |J_{yh}|^2) + \frac{1}{\mu_0\omega_{pm}^2} |K_{zh}|^2 \right] \\
 & + \int_{K_{i,j}} \left[\frac{\Gamma_e}{\epsilon_0\omega_{pe}^2} (|J_{xh}|^2 + |J_{yh}|^2) + \frac{\Gamma_m}{\mu_0\omega_{pm}^2} |K_{zh}|^2 \right] \\
 & + sum_I + sum_J + \int_{K_{i,j}} \left(H_{zh} \frac{\partial E_{xh}}{\partial y} + E_{xh} \frac{\partial H_{zh}}{\partial y} \right) - \int_{K_{i,j}} \left(H_{zh} \frac{\partial E_{yh}}{\partial x} + E_{yh} \frac{\partial H_{zh}}{\partial x} \right) = 0, \tag{3.17}
 \end{aligned}$$

where we denote the boundary integral terms

$$\begin{aligned}
 sum_I & = - \int_{I_i} \left((\hat{H}_{zh} E_{xh}^-)_{x, j+\frac{1}{2}} - (\hat{H}_{zh} E_{xh}^+)_{x, j-\frac{1}{2}} \right) dx - \int_{I_i} \left((\hat{E}_{xh} H_{zh}^-)_{x, j+\frac{1}{2}} - (\hat{E}_{xh} H_{zh}^+)_{x, j-\frac{1}{2}} \right) dx, \\
 sum_J & = \int_{J_j} \left((\hat{H}_{zh} E_{yh}^-)_{i+\frac{1}{2}, y} - (\hat{H}_{zh} E_{yh}^+)_{i-\frac{1}{2}, y} \right) dy + \int_{J_j} \left((\hat{E}_{yh} H_{zh}^-)_{i+\frac{1}{2}, y} - (\hat{E}_{yh} H_{zh}^+)_{i-\frac{1}{2}, y} \right) dy.
 \end{aligned}$$

From the choice of fluxes given above and integration by parts, we can simplify sum_I , sum_J as follows:

$$\sum_{1 \leq j \leq N_y} \left(sum_I + \int_{K_{i,j}} \left(H_{zh} \frac{\partial E_{xh}}{\partial y} + E_{xh} \frac{\partial H_{zh}}{\partial y} \right) \right) = c_0 \int_{I_i} (E_{xh}^+)_{x, \frac{1}{2}}^2 dx \tag{3.18}$$

$$\sum_{1 \leq i \leq N_x} \left(sum_J - \int_{K_{i,j}} \left(H_{zh} \frac{\partial E_{yh}}{\partial x} + E_{yh} \frac{\partial H_{zh}}{\partial x} \right) \right) = c_0 \int_{J_j} (E_{yh}^+)_{\frac{1}{2}, y}^2 dy. \tag{3.19}$$

Hence, summing up (3.17) with respect to both $1 \leq i \leq N_x$ and $1 \leq j \leq N_y$, using (3.18) and (3.19), and integrating over the time domain $[0, t]$, we conclude the proof. \square

Remark 3.2. Our scheme is non-dissipative for periodic boundary conditions. More specifically, if we have periodic boundary conditions:

$$E_x(x, c, t) = E_x(x, d, t), \quad E_y(a, y, t) = E_y(b, y, t), \tag{3.20}$$

$$H_z(x, c, t) = H_z(x, d, t), \quad H_z(a, y, t) = H_z(b, y, t), \tag{3.21}$$

instead of the PEC conditions (2.14), we can accordingly simplify the flux definitions in (3.8), (3.10), (3.12), and (3.14) as:

$$\hat{E}_{xh} \left(x, y_{\frac{1}{2}} \right) = \hat{E}_{xh} \left(x, y_{N_y+\frac{1}{2}} \right) = E_{xh} \left(x, y_{\frac{1}{2}}^+ \right), \tag{3.22}$$

$$\hat{E}_{yh} \left(x_{\frac{1}{2}}, y \right) = \hat{E}_{yh} \left(x_{N_x+\frac{1}{2}}, y \right) = E_{yh} \left(x_{\frac{1}{2}}^+, y \right), \tag{3.23}$$

$$\hat{H}_{zh} \left(x, y_{\frac{1}{2}} \right) = H_{zh} \left(x, y_{N_y+\frac{1}{2}}^- \right), \tag{3.24}$$

$$\hat{H}_{zh} \left(x_{\frac{1}{2}}, y \right) = H_{zh} \left(x_{N_x+\frac{1}{2}}^-, y \right) \tag{3.25}$$

while keeping the other parts of the scheme unchanged. It is straightforward to get (3.16) with $c_0 = 0$ for our numerical solutions with the periodic flux (3.22)–(3.25) by adjusting (simplifying) the proof of Theorem 3.1. Note that in this case the numerical solutions satisfy the same energy equality as the exact solutions (3.15). It is also painless to modify (simplify) the proof of Theorem 3.2 to obtain an optimal error estimate for the periodic flux scheme. This error estimate is verified

numerically in Example 1. To conclude, our scheme is both optimal and non-dissipative with $\Gamma_m = \Gamma_e = 0$ and for periodic boundary conditions.

3.2. The error analysis

We denote the errors between the exact solutions $(E_x, E_y, H_z, J_x, J_y, K_z)$ of (2.7)–(2.12) and the corresponding numerical solutions $(E_{xh}, E_{yh}, H_{zh}, J_{xh}, J_{yh}, K_{zh})$ of the semi-discrete scheme (3.1)–(3.6) by

$$\begin{aligned} \mathcal{E}_x &= E_x - E_{xh}, & \mathcal{E}_y &= E_y - E_{yh}, & \mathcal{H}_z &= H_z - H_{zh}, & \mathcal{J}_x &= J_x - J_{xh}, & \mathcal{J}_y &= J_y - J_{yh}, \\ \mathcal{K}_z &= K_z - K_{zh}. \end{aligned} \tag{3.26}$$

Subtracting (3.1)–(3.6) from the weak formulation of (2.7)–(2.12) and assuming that the exact solutions are continuous in the domain Ω , we can obtain the error equations:

$$\epsilon_0 \int_{K_{i,j}} \frac{\partial \mathcal{E}_x}{\partial t} \phi - \int_{I_i} \left((\hat{\mathcal{H}}_z \phi^-)_{x,j+\frac{1}{2}} - (\hat{\mathcal{H}}_z \phi^+)_{x,j-\frac{1}{2}} \right) dx + \int_{K_{i,j}} \mathcal{H}_z \frac{\partial \phi}{\partial y} + \int_{K_{i,j}} \mathcal{J}_x \phi = 0, \tag{3.27}$$

$$\epsilon_0 \int_{K_{i,j}} \frac{\partial \mathcal{E}_y}{\partial t} \psi + \int_{J_j} \left((\hat{\mathcal{H}}_z \psi^-)_{i+\frac{1}{2},y} - (\hat{\mathcal{H}}_z \psi^+)_{i-\frac{1}{2},y} \right) dy - \int_{K_{i,j}} \mathcal{H}_z \frac{\partial \psi}{\partial x} + \int_{K_{i,j}} \mathcal{J}_y \psi = 0, \tag{3.28}$$

$$\begin{aligned} \mu_0 \int_{K_{i,j}} \frac{\partial \mathcal{H}_z}{\partial t} \chi + \int_{J_j} \left((\hat{\mathcal{E}}_y \chi^-)_{i+\frac{1}{2},y} - (\hat{\mathcal{E}}_y \chi^+)_{i-\frac{1}{2},y} \right) dy - \int_{K_{i,j}} \mathcal{E}_y \frac{\partial \chi}{\partial x} \\ - \int_{I_i} \left((\hat{\mathcal{E}}_x \chi^-)_{x,j+\frac{1}{2}} - (\hat{\mathcal{E}}_x \chi^+)_{x,j-\frac{1}{2}} \right) dx + \int_{K_{i,j}} \mathcal{E}_x \frac{\partial \chi}{\partial y} + \int_{K_{i,j}} \mathcal{K}_z \chi = 0, \end{aligned} \tag{3.29}$$

$$\frac{1}{\epsilon_0 \omega_{pe}^2} \int_{K_{i,j}} \frac{\partial \mathcal{J}_x}{\partial t} u_1 + \frac{\Gamma_e}{\epsilon_0 \omega_{pe}^2} \int_{K_{i,j}} \mathcal{J}_x u_1 - \int_{K_{i,j}} \mathcal{E}_x u_1 = 0, \tag{3.30}$$

$$\frac{1}{\epsilon_0 \omega_{pe}^2} \int_{K_{i,j}} \frac{\partial \mathcal{J}_y}{\partial t} u_2 + \frac{\Gamma_e}{\epsilon_0 \omega_{pe}^2} \int_{K_{i,j}} \mathcal{J}_y u_2 - \int_{K_{i,j}} \mathcal{E}_y u_2 = 0, \tag{3.31}$$

$$\frac{1}{\mu_0 \omega_{pm}^2} \int_{K_{i,j}} \frac{\partial \mathcal{K}_z}{\partial t} v + \frac{\Gamma_m}{\mu_0 \omega_{pm}^2} \int_{K_{i,j}} \mathcal{K}_z v - \int_{K_{i,j}} \mathcal{H}_z v = 0, \tag{3.32}$$

for all test functions $\phi, \psi, \chi, u_1, u_2, v \in V_h^k$.

To estimate those errors in (3.26), we need some projection operators often used in DG and LDG methods (e.g., [37,38]). Given a function $u \in H^1(I_i)$, the 1-D projections

$$P_x^\pm : H^1(I_i) \rightarrow \mathcal{P}_k(I_i)$$

are defined as the elements of the k th polynomial space $\mathcal{P}_k(I_i)$ that satisfy

$$\int_{I_i} (P_x^+ u - u) w \, dx = 0 \quad \forall w \in \mathcal{P}_{k-1}(I_i), \quad \text{and} \quad P_x^+ u \left(x_{i-\frac{1}{2}}^+ \right) = u \left(x_{i-\frac{1}{2}}^+ \right), \tag{3.33}$$

$$\int_{I_i} (P_x^- u - u) w \, dx = 0 \quad \forall w \in \mathcal{P}_{k-1}(I_i), \quad \text{and} \quad P_x^- u \left(x_{i+\frac{1}{2}}^- \right) = u \left(x_{i+\frac{1}{2}}^- \right). \tag{3.34}$$

The 1-D projections P_x^\pm in the y -direction can be defined similarly. It is easy to check that these 1-D projections are well-defined.

Furthermore, we denote the standard 1-D L^2 projections

$$P_x : H^1(I_i) \rightarrow \mathcal{P}_k(I_i), \quad P_y : H^1(J_j) \rightarrow \mathcal{P}_k(J_j).$$

For a 2-D rectangular element $K_{i,j} = I_i \times J_j$, we use projections that are tensor products of the 1-D projections. More specifically, we adopt the projection

$$\Pi_1 = P_x \otimes P_y^+ : H^2(K_{i,j}) \rightarrow Q_k(K_{i,j}),$$

which satisfies that [39]: For any $w \in H^2(K_{i,j})$ and any $v_h \in Q_k(K_{i,j})$,

$$\int_{K_{i,j}} \Pi_1 w(x, y) \frac{\partial v_h(x, y)}{\partial y} \, dx dy = \int_{K_{i,j}} w(x, y) \frac{\partial v_h(x, y)}{\partial y} \, dx dy, \tag{3.35}$$

$$\int_{I_i} \Pi_1 w \left(x, y_{j-\frac{1}{2}}^+ \right) v_h \left(x, y_{j-\frac{1}{2}}^+ \right) \, dx = \int_{I_i} w \left(x, y_{j-\frac{1}{2}}^+ \right) v_h \left(x, y_{j-\frac{1}{2}}^+ \right) \, dx; \tag{3.36}$$

the projection

$$\Pi_2 = P_x^+ \otimes P_y : H^2(K_{i,j}) \rightarrow Q_k(K_{i,j}),$$

which satisfies that: For any $w \in H^2(K_{i,j})$ and any $v_h \in Q_k(K_{i,j})$,

$$\int_{K_{i,j}} \Pi_2 w(x, y) \frac{\partial v_h(x, y)}{\partial x} dx dy = \int_{K_{i,j}} w(x, y) \frac{\partial v_h(x, y)}{\partial x} dx dy \tag{3.37}$$

$$\int_{J_j} \Pi_2 w \left(x_{i-\frac{1}{2}}^+, y \right) v_h \left(x_{i-\frac{1}{2}}^+, y \right) dy = \int_{J_j} w \left(x_{i-\frac{1}{2}}^+, y \right) v_h \left(x_{i-\frac{1}{2}}^+, y \right) dy; \tag{3.38}$$

the projection

$$\Pi_3 = P_x^- \otimes P_y^- : H^2(K_{i,j}) \rightarrow Q_k(K_{i,j}),$$

which satisfies that: For any $w \in H^2(K_{i,j})$ and any $v_h \in Q_{k-1}(K_{i,j})$,

$$\int_{K_{i,j}} \Pi_3 w(x, y) v_h(x, y) dx dy = \int_{K_{i,j}} w(x, y) v_h(x, y) dx dy, \tag{3.39}$$

$$\int_{I_i} \Pi_3 w \left(x, y_{j+\frac{1}{2}}^- \right) v_h \left(x, y_{j+\frac{1}{2}}^- \right) dx = \int_{I_i} w \left(x, y_{j+\frac{1}{2}}^- \right) v_h \left(x, y_{j+\frac{1}{2}}^- \right) dx, \tag{3.40}$$

$$\int_{I_i} \Pi_3 w \left(x_{i+\frac{1}{2}}^-, y \right) v_h \left(x_{i+\frac{1}{2}}^-, y \right) dy = \int_{I_i} w \left(x_{i+\frac{1}{2}}^-, y \right) v_h \left(x_{i+\frac{1}{2}}^-, y \right) dy, \tag{3.41}$$

$$\Pi_3 w \left(x_{i+\frac{1}{2}}^-, y_{j+\frac{1}{2}}^- \right) = w \left(x_{i+\frac{1}{2}}^-, y_{j+\frac{1}{2}}^- \right). \tag{3.42}$$

Note that we use H^2 for the point values to make sense by the Sobolev embedding $H^2 \subset C^0$ in two dimensional spaces. The usual two dimensional L^2 projection is denoted:

$$\Pi_4 = P_x \otimes P_y : L^2(K_{i,j}) \rightarrow Q_k(K_{i,j}).$$

It is easy to see that

Lemma 3.1. *There exists a unique polynomial $\Pi_1 w \in Q_k(K_{i,j})$ as defined above. The same applies to $\Pi_2 w, \Pi_3 w$ and $\Pi_4 w$.*

Lemma 3.2. *If $w(x, y)$ is a product of 1-D functions, i.e. $w(x, y) = f(x)g(y)$ where $f \in H^1(I_i), g \in H^1(J_j)$, then we have:*

$$\Pi_1 w(x, y) = P_x f(x) P_y^+ g(y), \quad \Pi_2 w(x, y) = P_x^+ f(x) P_y g(y),$$

$$\Pi_3 w(x, y) = P_x^- f(x) P_y^- g(y), \quad \Pi_4 w(x, y) = P_x f(x) P_y g(y).$$

This lemma explains why we write the 2-D projections as the tensor products of 1-D projections.

Moreover, it is known that these projections have the following property (e.g., [37,38]).

Lemma 3.3. *Let Π_i ($i = 1, 2, 3, 4$) be the projections defined above. Then for any $u \in H^{k+1}(\Omega)$ ($k \geq 1$),*

$$\|\Pi_i u - u\| \leq Ch^{k+1} \|u\|_{H^{k+1}(\Omega)}, \quad \forall i = 1, 2, 3, 4,$$

where the constant $C > 0$ is independent of h , the mesh size introduced in Section 2.

With the above preparation, we can prove the following optimal error estimate.

Theorem 3.2. *Let $(E_x, E_y, H_z, J_x, J_y, K_z)$ and $(E_{xh}, E_{yh}, H_{zh}, J_{xh}, J_{yh}, K_{zh})$ be the solutions of (2.7)–(2.12) and (3.1)–(3.6), respectively. The following optimal error estimate holds true:*

$$\begin{aligned} & (\epsilon_0 (\|E_x - E_{xh}\|^2 + \|E_y - E_{yh}\|^2) + \mu_0 \|H_z - H_{zh}\|^2) (t) \\ & + \left(\frac{1}{\epsilon_0 \omega_{pe}^2} (\|J_x - J_{xh}\|^2 + \|J_y - J_{yh}\|^2) + \frac{1}{\mu_0 \omega_{pm}^2} \|K_z - K_{zh}\|^2 \right) (t) \\ & \leq Ch^{2(k+1)} + (\epsilon_0 (\|E_x - E_{xh}\|^2 + \|E_y - E_{yh}\|^2) + \mu_0 \|H_z - H_{zh}\|^2) (0) \\ & + \left(\frac{1}{\epsilon_0 \omega_{pe}^2} (\|J_x - J_{xh}\|^2 + \|J_y - J_{yh}\|^2) + \frac{1}{\mu_0 \omega_{pm}^2} \|K_z - K_{zh}\|^2 \right) (0), \end{aligned}$$

where the constant $C > 0$ is independent of h , and $k \geq 1$ is the order of the basis function in V_h^k .

Proof. Using the defined projections, we can decompose the errors given in (3.26) as follows:

$$\begin{aligned} \mathcal{E}_x &= E_x - E_{xh} = (\Pi_1 E_x - E_{xh}) - (\Pi_1 E_x - E_x) := E_{x\xi} - E_{x\eta}, \\ \mathcal{E}_y &= E_y - E_{yh} = (\Pi_2 E_y - E_{yh}) - (\Pi_2 E_y - E_y) := E_{y\xi} - E_{y\eta}, \\ \mathcal{H}_z &= H_z - H_{zh} = (\Pi_3 H_z - H_{zh}) - (\Pi_3 H_z - H_z) := H_{z\xi} - H_{z\eta}, \\ \mathcal{J}_x &= J_x - J_{xh} = (\Pi_4 J_x - J_{xh}) - (\Pi_4 J_x - J_x) := J_{x\xi} - J_{x\eta}, \\ \mathcal{J}_y &= J_y - J_{yh} = (\Pi_4 J_y - J_{yh}) - (\Pi_4 J_y - J_y) := J_{y\xi} - J_{y\eta}, \\ \mathcal{K}_z &= K_z - K_{zh} = (\Pi_4 K_z - K_{zh}) - (\Pi_4 K_z - K_z) := K_{z\xi} - K_{z\eta}. \end{aligned}$$

If we denote the summation of (3.27)–(3.32) as:

$$B(\mathcal{E}_x, \phi, \mathcal{E}_y, \psi, \mathcal{H}_z, \chi, \mathcal{J}_x, u_1, \mathcal{J}_y, u_2, \mathcal{K}_z, v) = 0. \tag{3.43}$$

Substituting the error decompositions into (3.43), and choosing the test functions $\phi = E_{x\xi}, \psi = E_{y\xi}, \chi = H_{z\xi}, u_1 = J_{x\xi}, u_2 = J_{y\xi}, v = K_{z\xi}$, we obtain

$$\begin{aligned} &B(E_{x\xi}, E_{x\xi}, E_{y\xi}, E_{y\xi}, H_{z\xi}, H_{z\xi}, J_{x\xi}, J_{x\xi}, J_{y\xi}, J_{y\xi}, K_{z\xi}, K_{z\xi}) \\ &= B(E_{x\eta}, E_{x\xi}, E_{y\eta}, E_{y\xi}, H_{z\eta}, H_{z\xi}, J_{x\eta}, J_{x\xi}, J_{y\eta}, J_{y\xi}, K_{z\eta}, K_{z\xi}). \end{aligned} \tag{3.44}$$

If we sum (3.44) over $i = 1 \dots N_x, j = 1 \dots N_y$ and look at the left hand side (LHS), we could observe that it is exactly the same thing as in the stability proof. Hence we have:

$$\begin{aligned} \text{LHS} &= \frac{1}{2} \frac{d}{dt} \left(\epsilon_0 \|E_{x\xi}\|^2 + \epsilon_0 \|E_{y\xi}\|^2 + \mu_0 \|H_{z\xi}\|^2 + \frac{1}{\epsilon_0 \omega_{pe}^2} (\|J_{x\xi}\|^2 + \|J_{y\xi}\|^2) + \frac{1}{\mu_0 \omega_{pm}^2} \|K_{z\xi}\|^2 \right) \\ &+ \frac{\Gamma_e}{\epsilon_0 \omega_{pe}^2} (\|J_{x\xi}\|^2 + \|J_{y\xi}\|^2) + \frac{\Gamma_m}{\mu_0 \omega_{pm}^2} \|K_{z\xi}\|^2 \\ &+ \sum_{i=1}^{N_x} \int_{I_i} \left((\hat{H}_{z\xi} - H_{z\xi}^+) E_{x\xi}^+ \right) \left(x, y_{\frac{1}{2}} \right) + \sum_{j=1}^{N_y} \int_{J_j} \left((H_{z\xi}^+ - \hat{H}_{z\xi}) E_{y\xi}^+ \right) \left(x_{\frac{1}{2}}, y \right) \\ &= \text{GT} + \sum_{i=1}^{N_x} c_0 \int_{I_i} \left(E_{x\xi}^+ \left(x, y_{\frac{1}{2}} \right) \right)^2 + \sum_{j=1}^{N_y} c_0 \int_{J_j} \left(E_{y\xi}^+ \left(x_{\frac{1}{2}}, y \right) \right)^2. \end{aligned}$$

Note that those terms in the first two lines in the above equation are referred as GT (good terms) and the third line is computed using the boundary fluxes (3.12) and (3.14).

Let us now consider the terms on the right hand side (RHS). By the definition of $\Pi_l, l \in \{1, 2, 3, 4\}$ in (3.35)–(3.39) and (3.42), the property of the projections $\Pi_l u_t = (\Pi_l u)_t$, and the fact that $E_{x\xi}, E_{y\xi}, H_{z\xi}, J_{x\xi}, J_{y\xi}, K_{z\xi} \in Q_k(K_{i,j})$, we can have

$$\begin{aligned} \text{RHS} &= \int_{\Omega} \left(\epsilon_0 \frac{\partial E_{x\eta}}{\partial t} E_{x\xi} + \epsilon_0 \frac{\partial E_{y\eta}}{\partial t} E_{y\xi} + \mu_0 \frac{\partial H_{z\eta}}{\partial t} H_{z\xi} - E_{x\eta} J_{x\xi} - E_{y\eta} J_{y\xi} - H_{z\eta} K_{z\xi} \right) \\ &+ \sum_{j=1}^{N_y} \text{TEX}_j + \sum_{i=1}^{N_x} \text{TEY}_i, \end{aligned}$$

where we denote

$$\text{TEX}_j = \sum_{i=1}^{N_x} \left(- \int_{I_i} \left((\hat{H}_{z\eta} E_{x\xi}^-) \left(x, y_{j+\frac{1}{2}} \right) - (\hat{H}_{z\eta} E_{x\xi}^+) \left(x, y_{j-\frac{1}{2}} \right) \right) + \int_{K_{i,j}} H_{z\eta} \frac{\partial E_{x\xi}}{\partial y} \right), \tag{3.45}$$

$$\text{TEY}_i = \sum_{j=1}^{N_y} \left(\int_{J_j} \left((\hat{H}_{z\eta} E_{y\xi}^-) \left(x_{i+\frac{1}{2}}, y \right) - (\hat{H}_{z\eta} E_{y\xi}^+) \left(x_{i-\frac{1}{2}}, y \right) \right) - \int_{K_{i,j}} H_{z\eta} \frac{\partial E_{y\xi}}{\partial y} \right). \tag{3.46}$$

Using the super-convergence results given in Lemmas 3.4 and 3.5 (similar to [37] and proved at the end of this section) to estimate TEX (terms of E_x) and TEY (terms of E_y), we obtain:

$$\begin{aligned} \text{GT} &\leq Ch^{2k+2} + C \|E_{x\xi}\|^2 + C \|E_{y\xi}\|^2 \\ &+ \int_{\Omega} \left(\epsilon_0 \frac{\partial E_{x\eta}}{\partial t} E_{x\xi} + \epsilon_0 \frac{\partial E_{y\eta}}{\partial t} E_{y\xi} + \mu_0 \frac{\partial H_{z\eta}}{\partial t} H_{z\xi} - E_{x\eta} J_{x\xi} - E_{y\eta} J_{y\xi} - H_{z\eta} K_{z\xi} \right). \end{aligned} \tag{3.47}$$

We can first bound all the right hand side terms of (3.47) using the Cauchy–Schwarz inequality and Lemma 3.3, then use the Gronwall inequality and the triangle inequality to conclude the proof. \square

Remark 3.3. We would like to remark that the optimal error estimate $O(h^{k+1})$ in the L^2 norm

$$\begin{aligned} & (\epsilon_0(\|E_x - E_{xh}\|^2 + \|E_y - E_{yh}\|^2) + \mu_0\|H_z - H_{zh}\|^2)(t) \\ & + \left(\frac{1}{\epsilon_0\omega_{pe}^2}(\|J_x - J_{xh}\|^2 + \|J_y - J_{yh}\|^2) + \frac{1}{\mu_0\omega_{pm}^2}\|K_z - K_{zh}\|^2 \right)(t) \leq Ch^{2(k+1)} \end{aligned}$$

can be simply achieved if we choose $O(h^{k+1})$ initial approximations:

$$\begin{aligned} E_{xh}(0) &= \Pi_1 E_x(0), & E_{yh}(0) &= \Pi_2 E_y(0), & H_{zh}(0) &= \Pi_3 H_z(0), \\ J_{xh}(0) &= \Pi_4 J_x(0), & J_{yh}(0) &= \Pi_4 J_y(0), & K_{zh}(0) &= \Pi_4 J_z(0). \end{aligned}$$

It is also worth mentioning that we cannot choose all the initial values as the standard L^2 projections (Π_4) of exact solutions, which are easier to implement. The numerical order of convergence would fluctuate if we use Π_4 for all the six initial values. The reason may be that no numerical dissipation is included to dissipate the initial error, see also [32].

In the rest of this section, we will prove the superconvergence results used above.

Lemma 3.4. Let TEX_i and TEY_j be defined by (3.45)–(3.46). Then we have:

$$\sum_{j=2}^{N_y} TEX_j \leq Ch^{2k+2} + \|E_{x\xi}\|^2, \quad \sum_{i=2}^{N_x} TEY_i \leq Ch^{2k+2} + \|E_{y\xi}\|^2,$$

where the positive constant C is independent of the mesh size h .

Proof. First we define the individual summands in TEX_j ($j \geq 2$) as $L_i(H_z, E_{x\xi})$:

$$\begin{aligned} L_i(H_z, E_{x\xi}) &= \int_{K_{i,j}} (\Pi_3 H_z - H_z) \frac{\partial E_{x\xi}}{\partial y} - \int_{I_i} ((\Pi_3 H_z - H_z)^- E_{x\xi}^-) \left(x, y_{j+\frac{1}{2}}\right) \\ &+ \int_{I_i} ((\Pi_3 H_z - H_z)^- E_{x\xi}^+) \left(x, y_{j-\frac{1}{2}}\right). \end{aligned} \tag{3.48}$$

Using the fact that $\Pi_3 H_z(x, y_{j-\frac{1}{2}}^-) = P_x^-(H_z(x, y_{j-\frac{1}{2}}))$, we can follow the same argument in [37] to prove:

$$L_i(H_z, E_{x\xi}) = 0, \quad \forall H_z \in \mathcal{P}_{k+1}(K_{i,j}), \quad i = 1, \dots, N_x, \quad j = 2, \dots, N_y \tag{3.49}$$

where \mathcal{P}_{k+1} is the standard notation for polynomials of degree not greater than $k + 1$, i.e. $\mathcal{P}_{k+1} = \{x^i \cdot y^j | i \geq 0, j \geq 0, i + j \leq k + 1\}$.

By the standard scaling argument [40] and Bramble–Hilbert lemma, we get:

$$\begin{aligned} L_i(H_z, E_{x\xi}) &= \inf_{p \in \mathcal{P}^{k+1}(K_{i,j})} L_i(H_z + p, E_{x\xi}) \\ &\leq Ch^{k+1} |H_z|_{H^{k+2}(K_{i,j})} \|E_{x\xi}\|_{L^2(K_{i,j})} \\ &\leq Ch^{2k+2} |H_z|_{H^{k+2}(K_{i,j})}^2 + \|E_{x\xi}\|_{L^2(K_{i,j})}^2, \end{aligned}$$

where $|\cdot|_{H^{k+2}}$ denotes the standard Sobolev semi-norm. We can now proceed to the proof of (3.48) from the above estimate

$$\sum_{j=2}^{N_y} TEX_j = \sum_{j=2}^{N_y} \sum_{i=1}^{N_x} L_i(H_z, E_{x\xi}) \leq Ch^{2k+2} |H_z|_{H^{k+2}(\Omega)}^2 + \|E_{x\xi}\|_{L^2(\Omega)}^2. \tag{3.50}$$

The proof of TEY is exactly the same as that of TEX by exchanging the role of x and y . \square

Lemma 3.5. Let TEX_i and TEY_j be defined by (3.45)–(3.46), and c_0 independent of mesh size h . Then we have:

$$TEX_1 - \sum_{i=1}^{N_x} c_0 \int_{I_i} \left(E_{x\xi}^+ \left(x, y_{\frac{1}{2}}\right)\right)^2 \leq Ch^{2k+2} + \|E_{x\xi}\|^2, \tag{3.51}$$

$$TEY_1 - \sum_{j=1}^{N_y} c_0 \int_{J_j} \left(E_{y\xi}^+ \left(x_{\frac{1}{2}}, y\right)\right)^2 \leq Ch^{2k+2} + \|E_{y\xi}\|^2, \tag{3.52}$$

where the positive constant C is independent of the mesh size h .

Proof. We define the bilinear form:

$$L_A(H_z, E_{x\xi}) = \int_{K_{i,1}} H_{z\eta} \frac{\partial E_{x\xi}}{\partial y} - \int_{I_i} (H_{z\eta}^- E_{x\xi}^-) \left(x, y_{\frac{3}{2}}\right) + \int_{I_i} (P_x^-(H_z(x, c)) - H_z(x, c)) E_{x\xi}^+(x, c). \tag{3.53}$$

Comparing (3.53) with (3.48), we see that L_A satisfies the same polynomial preserving properties. Hence we have

$$\sum_{i=1}^{N_x} L_A(H_z, E_{x\xi}) \leq Ch^{2k+2} |H_z|_{H^{k+2}(\Omega)}^2 + \|E_{x\xi}\|^2, \tag{3.54}$$

which leads to:

$$\begin{aligned} \text{TEX}_1 &- \sum_{i=1}^{N_x} c_0 \int_{I_i} \left(E_{x\xi}^+(x, y_{\frac{1}{2}})\right)^2 \\ &= \sum_{i=1}^{N_x} L_A(H_z, E_{x\xi}) + \sum_{i=1}^{N_x} \left(- \int_{I_i} (P_x^-(H_z(x, c)) - H_z(x, c)) E_{x\xi}^+(x, c) + \int_{I_i} (H_{z\eta}^+ E_{x\xi}^+)(x, c)\right) \\ &\quad + c_0 \int_a^b (E_{x\eta}^+ E_{x\xi}^+)(x, c) - c_0 \int_a^b (E_{x\xi}^+(x, c))^2 \\ &\leq Ch^{2k+2} |H_z|_{H^{k+2}(\Omega)}^2 + \|E_{x\xi}\|^2 + C \int_a^b (P_x^-(H_z(x, c)) - H_z(x, c))^2 + \int_a^b H_{z\eta}^+(x, c)^2 \\ &\leq Ch^{2k+2} |H_z|_{H^{k+2}(\Omega)}^2 + \|E_{x\xi}\|^2 + Ch^{2k+2} |H_z|_{H^{k+1}(\Gamma)}^2 \end{aligned}$$

where we denote $\Gamma = \{(x, c) : x \in (a, b)\}$. In deriving the first inequality, we used the definition of Π_1 to obtain

$$\int_{I_i} (E_{x\eta}^+ E_{x\xi}^+)(x, c) dx = 0.$$

In the last step we used the approximation properties of the projections.

The proof of (3.52) follows the same line as (3.51). \square

Remark 3.4. If $c_0 = 0$, we get a straightforward PEC boundary condition without the jump terms in (3.12) and (3.14). In this case, we can only control the term $\sum_{i=1}^{N_x} \int_{I_i} (H_{z\eta}^+ E_{x\xi}^+)(x, c)$ as follows:

$$\begin{aligned} \sum_{i=1}^{N_x} \int_{I_i} (H_{z\eta}^+ E_{x\xi}^+)(x, c) &\leq h^{-1} \int_a^b (H_{z\eta}^+)^2(x, c) + h \int_a^b (E_{x\xi}^+)^2(x, c) \\ &\leq C \cdot h^{2k+1} + \|E_{x\xi}\|^2 \quad (\text{by inverse inequality}). \end{aligned}$$

Therefore, we lose half an order. It is verified numerically that for basis function of odd orders of polynomial degree, like Q_1 and Q_3 , we do observe this sub-optimal order of convergence. See Tables 5.3 and 5.5.

4. The fully-discrete DG method

We consider the following leap-frog DG scheme: For any $n \geq 0$, find $E_{xh}^{n+1}, E_{yh}^{n+1}, H_{zh}^{n+\frac{3}{2}}, J_{xh}^{n+\frac{3}{2}}, J_{yh}^{n+\frac{3}{2}}, K_{zh}^{n+2} \in V_h^k$ such that

$$\epsilon_0 \int_{K_{i,j}} \frac{E_{xh}^{n+1} - E_{xh}^n}{\tau} \phi - \int_{I_i} \left(\left(\hat{H}_{zh}^{n+\frac{1}{2}} \phi^- \right)_{x,j+\frac{1}{2}} - \left(\hat{H}_{zh}^{n+\frac{1}{2}} \phi^+ \right)_{x,j-\frac{1}{2}} \right) dx + \int_{K_{i,j}} H_{zh}^{n+\frac{1}{2}} \frac{\partial \phi}{\partial y} + \int_{K_{i,j}} J_{xh}^{n+\frac{1}{2}} \phi = 0, \tag{4.1}$$

$$\epsilon_0 \int_{K_{i,j}} \frac{E_{yh}^{n+1} - E_{yh}^n}{\tau} \psi + \int_{J_j} \left(\left(\hat{H}_{zh}^{n+\frac{1}{2}} \psi^- \right)_{i+\frac{1}{2},y} - \left(\hat{H}_{zh}^{n+\frac{1}{2}} \psi^+ \right)_{i-\frac{1}{2},y} \right) dy - \int_{K_{i,j}} H_{zh}^{n+\frac{1}{2}} \frac{\partial \psi}{\partial x} + \int_{K_{i,j}} J_{yh}^{n+\frac{1}{2}} \psi = 0, \tag{4.2}$$

$$\begin{aligned} \mu_0 \int_{K_{i,j}} \frac{H_{zh}^{n+\frac{3}{2}} - H_{zh}^{n+\frac{1}{2}}}{\tau} \chi + \int_{J_j} \left((\hat{E}_{yh}^{n+1} \chi^-)_{i+\frac{1}{2},y} - (\hat{E}_{yh}^{n+1} \chi^+)_{i-\frac{1}{2},y} \right) dy - \int_{K_{i,j}} E_{yh}^{n+1} \frac{\partial \chi}{\partial x} \\ - \int_{I_i} \left((\hat{E}_{xh}^{n+1} \chi^-)_{x,j+\frac{1}{2}} - (\hat{E}_{xh}^{n+1} \chi^+)_{x,j-\frac{1}{2}} \right) dx + \int_{K_{i,j}} E_{xh}^{n+1} \frac{\partial \chi}{\partial y} + \int_{K_{i,j}} K_{zh}^{n+1} \chi = 0, \end{aligned} \tag{4.3}$$

$$\frac{1}{\epsilon_0 \omega_{pe}^2} \int_{K_{i,j}} \frac{J_{xh}^{n+\frac{3}{2}} - J_{xh}^{n+\frac{1}{2}}}{\tau} u_1 + \frac{\Gamma_e}{\epsilon_0 \omega_{pe}^2} \int_{K_{i,j}} \frac{J_{xh}^{n+\frac{3}{2}} + J_{xh}^{n+\frac{1}{2}}}{2} u_1 - \int_{K_{i,j}} E_{xh}^{n+1} u_1 = 0, \tag{4.4}$$

$$\frac{1}{\epsilon_0 \omega_{pe}^2} \int_{K_{i,j}} \frac{J_{yh}^{n+\frac{3}{2}} - J_{yh}^{n+\frac{1}{2}}}{\tau} u_2 + \frac{\Gamma_e}{\epsilon_0 \omega_{pe}^2} \int_{K_{i,j}} \frac{J_{yh}^{n+\frac{3}{2}} + J_{yh}^{n+\frac{1}{2}}}{2} u_2 - \int_{K_{i,j}} E_{yh}^{n+1} u_2 = 0, \tag{4.5}$$

$$\frac{1}{\mu_0 \omega_{pm}^2} \int_{K_{i,j}} \frac{K_{zh}^{n+2} - K_{zh}^{n+1}}{\tau} v + \frac{\Gamma_m}{\mu_0 \omega_{pm}^2} \int_{K_{i,j}} \frac{K_{zh}^{n+2} + K_{zh}^{n+1}}{2} v - \int_{K_{i,j}} H_{zh}^{n+\frac{3}{2}} v = 0, \tag{4.6}$$

for all test functions $\phi, \psi, \chi, u_1, u_2, v \in V_h^k$. With the following fluxes:

$$\hat{E}_{xh}^{n+1} \left(x, y_{j+\frac{1}{2}} \right) = E_{xh}^{n+1,+} \left(x, y_{j+\frac{1}{2}} \right) \quad \forall j = 1, \dots, N_y - 1, \tag{4.7}$$

$$\hat{E}_{xh}^{n+1} \left(x, y_{\frac{1}{2}} \right) = \hat{E}_{xh}^{n+1} \left(x, y_{N_y+\frac{1}{2}} \right) = 0, \tag{4.8}$$

$$\hat{E}_{yh}^{n+1} \left(x_{i+\frac{1}{2}}, y \right) = E_{yh}^{n+1,+} \left(x_{i+\frac{1}{2}}, y \right) \quad \forall i = 1, \dots, N_x - 1, \tag{4.9}$$

$$\hat{E}_{yh}^{n+1} \left(x_{\frac{1}{2}}, y \right) = \hat{E}_{yh}^{n+1} \left(x_{N_x+\frac{1}{2}}, y \right) = 0, \tag{4.10}$$

$$\hat{H}_{zh}^{n+\frac{1}{2}} \left(x, y_{j+\frac{1}{2}} \right) = H_{zh}^{n+\frac{1}{2},-} \left(x, y_{j+\frac{1}{2}} \right) \quad \forall j = 1, \dots, N_y, \tag{4.11}$$

$$\hat{H}_{zh}^{n+\frac{1}{2}} \left(x, y_{\frac{1}{2}} \right) = H_{zh}^{n+\frac{1}{2},+} \left(x, y_{\frac{1}{2}} \right) + \frac{C_0}{2} \left(E_{xh}^{n+1} \left(x, y_{\frac{1}{2}}^+ \right) + E_{xh}^n \left(x, y_{\frac{1}{2}}^+ \right) \right), \tag{4.12}$$

$$\hat{H}_{zh}^{n+\frac{1}{2}} \left(x_{i+\frac{1}{2}}, y \right) = H_{zh}^{n+\frac{1}{2},-} \left(x_{i+\frac{1}{2}}, y \right) \quad \forall i = 1, \dots, N_x, \tag{4.13}$$

$$\hat{H}_{zh}^{n+\frac{1}{2}} \left(x_{\frac{1}{2}}, y \right) = H_{zh}^{n+\frac{1}{2},+} \left(x_{\frac{1}{2}}, y \right) - \frac{C_0}{2} \left(E_{yh}^{n+1} \left(x_{\frac{1}{2}}^+, y \right) + E_{yh}^n \left(x_{\frac{1}{2}}^+, y \right) \right). \tag{4.14}$$

Note that by enforcing the PEC boundary conditions (2.14), $E_{xh}^{n+1}(x, y_{\frac{1}{2}}^+) = E_{xh}^{n+1}(x, y_{\frac{1}{2}}^+) - E_{xh}^{n+1}(x, y_{\frac{1}{2}}^-) = \llbracket E_{xh}^{n+1}(x, y_{\frac{1}{2}}) \rrbracket$ in (4.12), and the same for other artificial viscosity in (4.12) and (4.14).

To prove the stability for our fully-discrete scheme, we need some lemmas. Let us denote boundary integral terms

$$\begin{aligned} \text{sum}_{lh} := & - \int_{l_i} \left(\left(\hat{H}_{zh}^{n+\frac{1}{2}} (E_{xh}^{n+1} + E_{xh}^n)^- \right)_{x,j+\frac{1}{2}} - \left(\hat{H}_{zh}^{n+\frac{1}{2}} (E_{xh}^{n+1} + E_{xh}^n)^+ \right)_{x,j-\frac{1}{2}} \right) dx \\ & - \int_{l_i} \left(\left(\hat{E}_{xh}^{n+1} \left(H_{zh}^{n+\frac{3}{2}} + H_{zh}^{n+\frac{1}{2}} \right)^- \right)_{x,j+\frac{1}{2}} - \left(\hat{E}_{xh}^{n+1} \left(H_{zh}^{n+\frac{3}{2}} + H_{zh}^{n+\frac{1}{2}} \right)^+ \right)_{x,j-\frac{1}{2}} \right) dx \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} \text{sum}_{jh} := & \int_{j_j} \left(\left(\hat{H}_{zh}^{n+\frac{1}{2}} (E_{yh}^{n+1} + E_{yh}^n)^- \right)_{i+\frac{1}{2},y} - \left(\hat{H}_{zh}^{n+\frac{1}{2}} (E_{yh}^{n+1} + E_{yh}^n)^+ \right)_{i-\frac{1}{2},y} \right) dy \\ & + \int_{j_j} \left(\left(\hat{E}_{yh}^{n+1} \left(H_{zh}^{n+\frac{3}{2}} + H_{zh}^{n+\frac{1}{2}} \right)^- \right)_{i+\frac{1}{2},y} - \left(\hat{E}_{yh}^{n+1} \left(H_{zh}^{n+\frac{3}{2}} + H_{zh}^{n+\frac{1}{2}} \right)^+ \right)_{i-\frac{1}{2},y} \right) dy. \end{aligned} \tag{4.16}$$

Lemma 4.1. With the flux choices of (4.7)–(4.14) for any $m \geq 1$ we have

$$\begin{aligned} & \sum_{n=0}^m \sum_{i,j} \int_{K_{i,j}} \left\{ H_{zh}^{n+\frac{1}{2}} \frac{\partial}{\partial y} (E_{xh}^{n+1} + E_{xh}^n) + E_{xh}^{n+1} \frac{\partial}{\partial y} \left(H_{zh}^{n+\frac{3}{2}} + H_{zh}^{n+\frac{1}{2}} \right) \right\} + \sum_{n=0}^m \sum_{i,j} \text{sum}_{lh} \\ & = \sum_{j=1}^{N_y-1} \left(\int_a^b \left(E_{xh}^{m+1,+} \left[\left[H_{zh}^{m+\frac{3}{2}} \right] \right]_{x,j+\frac{1}{2}} - \int_a^b \left(E_{xh}^{0,+} \left[\left[H_{zh}^{\frac{1}{2}} \right] \right]_{x,j+\frac{1}{2}} \right) \right) \right. \\ & \quad \left. + \sum_{i,j} \int_{K_{i,j}} \left\{ -E_{xh}^0 \frac{\partial}{\partial y} H_{zh}^{\frac{1}{2}} + E_{xh}^{m+1} \frac{\partial}{\partial y} H_{zh}^{m+\frac{3}{2}} \right\} + \frac{C_0}{2} \sum_{n=0}^m \int_a^b (E_{xh}^{n+1,+} + E_{xh}^{n,+})_{x,\frac{1}{2}}^2 \right) \end{aligned} \tag{4.17}$$

where for simplicity we denote $\sum_{i,j} := \sum_{1 \leq i \leq N_x} \sum_{1 \leq j \leq N_y}$.

Proof. Using integration by parts, we see that

$$\begin{aligned} & \sum_{n=0}^m \sum_{i,j} \int_{K_{i,j}} \left\{ H_{zh}^{n+\frac{1}{2}} \frac{\partial}{\partial y} (E_{xh}^{n+1} + E_{xh}^n) + E_{xh}^{n+1} \frac{\partial}{\partial y} \left(H_{zh}^{n+\frac{3}{2}} + H_{zh}^{n+\frac{1}{2}} \right) \right\} dx dy \\ &= \sum_{n=0}^m \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \int_{I_i} \left(E_{xh}^{n+1,-} H_{zh}^{n+\frac{1}{2},-} \right)_{x,j+\frac{1}{2}} - \sum_{n=0}^m \sum_{i=1}^{N_x} \sum_{j=0}^{N_y-1} \int_{I_i} \left(E_{xh}^{n+1,+} H_{zh}^{n+\frac{1}{2},+} \right)_{x,j+\frac{1}{2}} \\ &+ \sum_{n=0}^m \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \int_{I_i} \left(E_{xh}^{n,-} H_{zh}^{n+\frac{1}{2},-} \right)_{x,j+\frac{1}{2}} - \sum_{n=0}^m \sum_{i=1}^{N_x} \sum_{j=0}^{N_y-1} \int_{I_i} \left(E_{xh}^{n,+} H_{zh}^{n+\frac{1}{2},+} \right)_{x,j+\frac{1}{2}} \\ &+ \sum_{i,j} \int_{K_{i,j}} \left\{ -E_{xh}^0 \frac{\partial}{\partial y} H_{zh}^{\frac{1}{2}} + E_{xh}^{m+1} \frac{\partial}{\partial y} H_{zh}^{m+\frac{3}{2}} \right\}. \end{aligned} \tag{4.18}$$

Combining (4.18) and the definition of sum_{lh} in (4.15), we get the equality (4.17). \square

Using the similar technique, we can also prove the following lemma.

Lemma 4.2. With the flux choices of (4.7)–(4.14) for any $m \geq 1$ we have

$$\begin{aligned} & - \sum_{n=0}^m \sum_{i,j} \int_{K_{i,j}} \left\{ H_{zh}^{n+\frac{1}{2}} \frac{\partial}{\partial x} (E_{yh}^{n+1} + E_{yh}^n) + E_{yh}^{n+1} \frac{\partial}{\partial x} \left(H_{zh}^{n+\frac{3}{2}} + H_{zh}^{n+\frac{1}{2}} \right) \right\} + \sum_{n=0}^m \sum_{i,j} sum_{lh} \\ &= \sum_{i=1}^{N_x-1} \left(- \int_c^d \left(E_{yh}^{m+1,+} \left[\left[H_{zh}^{m+\frac{3}{2}} \right] \right]_{i+\frac{1}{2},y} \right) + \int_c^d \left(E_{yh}^{0,+} \left[\left[H_{zh}^{\frac{1}{2}} \right] \right]_{i+\frac{1}{2},y} \right) \right) \\ &+ \sum_{i,j} \int_{K_{i,j}} \left\{ E_{yh}^0 \frac{\partial}{\partial x} H_{zh}^{\frac{1}{2}} - E_{yh}^{m+1} \frac{\partial}{\partial x} H_{zh}^{m+\frac{3}{2}} \right\} + \frac{c_0}{2} \sum_{n=0}^m \int_c^d (E_{yh}^{n+1,+} + E_{yh}^{n,+})_{\frac{1}{2},y}^2. \end{aligned} \tag{4.19}$$

With the above preparation, we can now prove the following stability. To shorten the notation, we introduce the vector L^2 norm $\|E_h\|^2 = \|E_{xh}\|^2 + \|E_{yh}\|^2$ for vector $E_h = (E_{xh}, E_{yh})$. Similar notation will be used for $\|J_h\|^2$ for vector $J_h = (J_{xh}, J_{yh})$.

Theorem 4.1. Denote $C_v = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$ for the speed of light, and C_{inv} for the positive constant appearing in (4.25). Under the assumption

$$\tau \leq \min \left(\frac{1}{2\omega_{pe}}, \frac{1}{2\omega_{pm}}, \frac{h}{2C_{inv}C_v} \right), \tag{4.20}$$

for any $m \geq 1$ we have

$$\begin{aligned} & \epsilon_0 \|E_h^{m+1}\|^2 + \mu_0 \left\| H_{zh}^{m+\frac{3}{2}} \right\|^2 + \frac{1}{\epsilon_0 \omega_{pe}^2} \left\| J_h^{m+\frac{3}{2}} \right\|^2 + \frac{1}{\mu_0 \omega_{pm}^2} \|K_{zh}^{m+2}\|^2 \\ & \leq C \left(\epsilon_0 \|E_h^0\|^2 + \mu_0 \left\| H_{zh}^{\frac{1}{2}} \right\|^2 + \frac{1}{\epsilon_0 \omega_{pe}^2} \left\| J_h^{\frac{1}{2}} \right\|^2 + \frac{1}{\mu_0 \omega_{pm}^2} \|K_{zh}^1\|^2 \right), \end{aligned} \tag{4.21}$$

where the constant $C > 1$ is independent of the mesh size h and the time step size τ .

Proof. Choosing $\phi = \tau(E_{xh}^{n+1} + E_{xh}^n)$, $\psi = \tau(E_{yh}^{n+1} + E_{yh}^n)$, $\chi = \tau(H_{zh}^{n+\frac{3}{2}} + H_{zh}^{n+\frac{1}{2}})$, $u_1 = \tau(J_{xh}^{n+\frac{3}{2}} + J_{xh}^{n+\frac{1}{2}})$, $u_2 = \tau(J_{yh}^{n+\frac{3}{2}} + J_{yh}^{n+\frac{1}{2}})$, $v = \tau(K_{zh}^{n+2} + K_{zh}^{n+1})$ in (4.1)–(4.6), summing them up over $1 \leq i \leq N_x$, $1 \leq j \leq N_y$, time levels $0 \leq n \leq m$ for any $m \geq 1$, and using Lemmas 4.1–4.2, we have

$$\begin{aligned} & \epsilon_0 (\|E_h^{m+1}\|^2 - \|E_h^0\|^2) + \mu_0 \left(\left\| H_{zh}^{m+\frac{3}{2}} \right\|^2 - \left\| H_{zh}^{\frac{1}{2}} \right\|^2 \right) \\ & + \frac{1}{\epsilon_0 \omega_{pe}^2} \left(\left\| J_h^{m+\frac{3}{2}} \right\|^2 - \left\| J_h^{\frac{1}{2}} \right\|^2 \right) + \frac{1}{\mu_0 \omega_{pm}^2} (\|K_{zh}^{m+2}\|^2 - \|K_{zh}^1\|^2) \\ & \leq - \sum_{0 \leq n \leq m} \int_{\Omega} J_h^{n+\frac{1}{2}} \cdot \tau (E_h^{n+1} + E_h^n) - \sum_{0 \leq n \leq m} \int_{\Omega} K_{zh}^{n+1} \cdot \tau \left(H_{zh}^{n+\frac{3}{2}} + H_{zh}^{n+\frac{1}{2}} \right) \end{aligned}$$

$$\begin{aligned}
 & + \sum_{0 \leq n \leq m} \int_{\Omega} \mathbf{E}_h^{n+1} \cdot \tau \left(\mathbf{J}_h^{n+\frac{3}{2}} + \mathbf{J}_h^{n+\frac{1}{2}} \right) + \sum_{0 \leq n \leq m} \int_{\Omega} H_{zh}^{n+\frac{3}{2}} \cdot \tau (K_{zh}^{n+2} + K_{zh}^{n+1}) \\
 & + B^y \left(E_{xh}^0, H_{zh}^{\frac{1}{2}} \right) - B^y \left(E_{xh}^{m+1}, H_{zh}^{m+\frac{3}{2}} \right) - B^x \left(E_{yh}^0, H_{zh}^{\frac{1}{2}} \right) + B^x \left(E_{yh}^{m+1}, H_{zh}^{m+\frac{3}{2}} \right),
 \end{aligned} \tag{4.22}$$

where we introduced the bilinear forms (cf. [41, (4.1)])

$$\begin{aligned}
 B^y(u, v) &= \tau \left(\sum_{i,j} \int_{K_{i,j}} u \partial_y v + \sum_{1 \leq j \leq N_y - 1} \int_a^b u_{j+\frac{1}{2}}^+ \llbracket v \rrbracket_{j+\frac{1}{2}} dx \right), \\
 B^x(u, v) &= \tau \left(\sum_{i,j} \int_{K_{i,j}} u \partial_x v + \sum_{1 \leq i \leq N_x - 1} \int_c^d u_{i+\frac{1}{2}}^+ \llbracket v \rrbracket_{i+\frac{1}{2}} dy \right).
 \end{aligned}$$

Note that the sum of the first and third terms of (4.22) can be estimated as follows:

$$\begin{aligned}
 S_1 + S_3 &= -\tau \sum_{0 \leq n \leq m} \int_{\Omega} \left(\mathbf{J}_h^{n+\frac{1}{2}} \cdot \mathbf{E}_h^{n+1} - \mathbf{J}_h^{n+\frac{3}{2}} \cdot \mathbf{E}_h^{n+1} \right) = -\tau \int_{\Omega} \left(\mathbf{J}_h^{\frac{1}{2}} \cdot \mathbf{E}_h^0 - \mathbf{J}_h^{m+\frac{3}{2}} \cdot \mathbf{E}_h^{m+1} \right) \\
 &\leq \frac{\tau \omega_{pe}}{2} \left(\frac{1}{\epsilon_0 \omega_{pe}^2} \left\| \mathbf{J}_h^{\frac{1}{2}} \right\|^2 + \epsilon_0 \left\| \mathbf{E}_h^0 \right\|^2 \right) + \frac{\tau \omega_{pe}}{2} \left(\frac{1}{\epsilon_0 \omega_{pe}^2} \left\| \mathbf{J}_h^{m+\frac{3}{2}} \right\|^2 + \epsilon_0 \left\| \mathbf{E}_h^{m+1} \right\|^2 \right),
 \end{aligned} \tag{4.23}$$

where in the last step we used the Cauchy–Schwarz inequality.

Similarly, we can bound the sum of the second and fourth terms of (4.22) as follows:

$$S_2 + S_4 \leq \frac{\tau \omega_{pm}}{2} \left(\frac{1}{\mu_0 \omega_{pm}^2} \left\| K_{zh}^1 \right\|^2 + \mu_0 \left\| H_{zh}^{\frac{1}{2}} \right\|^2 \right) + \frac{\tau \omega_{pm}}{2} \left(\frac{1}{\mu_0 \omega_{pm}^2} \left\| K_{zh}^{m+2} \right\|^2 + \mu_0 \left\| H_{zh}^{m+\frac{3}{2}} \right\|^2 \right). \tag{4.24}$$

By using the Cauchy–Schwarz inequality and the inverse estimate, we have (cf. [41, Lemma 4.1]):

$$\begin{aligned}
 B^y \left(E_{xh}^{m+1}, H_{zh}^{m+\frac{3}{2}} \right) &\leq \tau \cdot C_{inv} h^{-1} \left\| E_{xh}^{m+1} \right\| \cdot \left\| H_{zh}^{m+\frac{3}{2}} \right\| = \tau \cdot C_{inv} C_v h^{-1} \sqrt{\epsilon_0} \left\| E_{xh}^{m+1} \right\| \sqrt{\mu_0} \left\| H_{zh}^{m+\frac{3}{2}} \right\| \\
 &\leq \tau \cdot \frac{C_{inv} C_v}{2h} \left(\epsilon_0 \left\| E_{xh}^{m+1} \right\|^2 + \mu_0 \left\| H_{zh}^{m+\frac{3}{2}} \right\|^2 \right),
 \end{aligned} \tag{4.25}$$

where the positive constant C_{inv} is independent of τ and h .

Similarly, we can obtain

$$B^y \left(E_{xh}^0, H_{zh}^{\frac{1}{2}} \right) \leq \tau \cdot \frac{C_{inv} C_v}{2h} \left(\epsilon_0 \left\| E_{xh}^0 \right\|^2 + \mu_0 \left\| H_{zh}^{\frac{1}{2}} \right\|^2 \right), \tag{4.26}$$

$$B^x \left(E_{yh}^{m+1}, H_{zh}^{m+\frac{3}{2}} \right) \leq \tau \cdot \frac{C_{inv} C_v}{2h} \left(\epsilon_0 \left\| E_{yh}^{m+1} \right\|^2 + \mu_0 \left\| H_{zh}^{m+\frac{3}{2}} \right\|^2 \right), \tag{4.27}$$

$$B^x \left(E_{yh}^0, H_{zh}^{\frac{1}{2}} \right) \leq \tau \cdot \frac{C_{inv} C_v}{2h} \left(\epsilon_0 \left\| E_{yh}^0 \right\|^2 + \mu_0 \left\| H_{zh}^{\frac{1}{2}} \right\|^2 \right). \tag{4.28}$$

The proof is completed by substituting the above estimates (4.23)–(4.28) into (4.22) and using the assumptions $\frac{\tau \omega_{pm}}{2}, \frac{\tau \omega_{pe}}{2}, \frac{\tau C_{inv} C_v}{2h} \leq \frac{1}{4}$. \square

Remark 4.1. We would like to remark that under the same assumption as Theorem 4.1 coupled with the assumptions on the initial value:

$$\epsilon_0 \left\| \mathbf{E}^0 - \mathbf{E}_h^0 \right\|^2 + \mu_0 \left\| H_z^{\frac{1}{2}} - H_{zh}^{\frac{1}{2}} \right\|^2 + \frac{1}{\epsilon_0 \omega_{pe}^2} \left\| \mathbf{J}^{\frac{1}{2}} - \mathbf{J}_h^{\frac{1}{2}} \right\|^2 + \frac{1}{\mu_0 \omega_{pm}^2} \left\| K_z^1 - K_{zh}^1 \right\|^2 \leq Ch^{2(k+1)},$$

we can prove the following optimal error estimate: for any $m \geq 1$

$$\begin{aligned} \epsilon_0 \| \mathbf{E}^{m+1} - \mathbf{E}_h^{m+1} \|^2 + \mu_0 \left\| H_z^{m+\frac{3}{2}} - H_{zh}^{m+\frac{3}{2}} \right\|^2 \\ + \frac{1}{\epsilon_0 \omega_{pe}^2} \left\| \mathbf{J}^{m+\frac{3}{2}} - \mathbf{J}_h^{m+\frac{3}{2}} \right\|^2 + \frac{1}{\mu_0 \omega_{pm}^2} \| K_z^{m+2} - K_{zh}^{m+2} \|^2 \leq C (h^{(k+1)} + \tau^2)^2 \end{aligned}$$

where $\mathbf{J} = (J_x, J_y)$, and $\mathbf{E} = (E_x, E_y)$. The proof can be carried out by following the similar idea to the proofs of [Theorems 3.2](#) and [4.1](#) coupled with time discretization estimates (cf. [[10](#), Ch.3]). Considering that the proof is very lengthy and technical, we skip it.

5. Numerical results

To validate our theoretical analysis, we solve the model problem [\(2.7\)–\(2.12\)](#) with imposed sources f_x, f_y and g on $\Omega = [0, 1]^2$:

$$\frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y} - J_x + f_x \tag{5.1}$$

$$\frac{\partial E_y}{\partial t} = -\frac{\partial H_z}{\partial x} - J_y + f_y \tag{5.2}$$

$$\frac{\partial H_z}{\partial t} = -\frac{\partial E_y}{\partial x} + \frac{\partial E_x}{\partial y} - K_z + g \tag{5.3}$$

$$\frac{1}{\omega^2 \pi^2} \frac{\partial J_x}{\partial t} = -\frac{2}{\omega \pi} J_x + E_x \tag{5.4}$$

$$\frac{1}{\omega^2 \pi^2} \frac{\partial J_y}{\partial t} = -\frac{2}{\omega \pi} J_y + E_y \tag{5.5}$$

$$\frac{1}{\omega^2 \pi^2} \frac{\partial K_z}{\partial t} = -\frac{2}{\omega \pi} K_z + H_z, \tag{5.6}$$

which has the exact solutions:

$$E_x(x, y, t) = \cos(\omega \pi x) \sin(\omega \pi y) e^{-\omega \pi t}, \quad E_y(x, y, t) = -\sin(\omega \pi x) \cos(\omega \pi y) e^{-\omega \pi t}, \tag{5.7}$$

$$H_z(x, y, t) = \cos(\omega \pi x) \cos(\omega \pi y) e^{-\omega \pi t}, \tag{5.8}$$

$$J_x(x, y, t) = \omega \pi E_x(x, y, t), \quad J_y(x, y, t) = \omega \pi E_y(x, y, t), \quad K(x, y, t) = \omega \pi H(x, y, t), \tag{5.9}$$

and the source terms: $f_x = J_x$, $f_y = J_y$, $g = -2\omega \pi H$.

With the added source terms, we only need to modify the original scheme [\(4.1\)–\(4.6\)](#) by adding

$$\int_{K_{i,j}} f_x^{n+\frac{1}{2}} \phi, \quad \int_{K_{i,j}} f_y^{n+\frac{1}{2}} \psi, \quad \int_{K_{i,j}} g^{n+1} \chi$$

to the right hand sides of [\(4.1\)–\(4.3\)](#), respectively. For the initial values at time $t = 0$, we set

$$E_{xh}(\cdot, 0) = \Pi_1 E_x(\cdot, 0), \quad E_{yh}(\cdot, 0) = \Pi_2 E_y(\cdot, 0), \quad H_{zh}(\cdot, 0) = \Pi_3 H_z(\cdot, 0),$$

and

$$J_{xh}(\cdot, 0) = \Pi_4 J_x(\cdot, 0), \quad J_{yh}(\cdot, 0) = \Pi_4 J_y(\cdot, 0), \quad K_{zh}(\cdot, 0) = \Pi_4 K_z(\cdot, 0).$$

The first step values $H_{zh}^{\frac{1}{2}}, J_{xh}^{\frac{1}{2}}, J_{yh}^{\frac{1}{2}}, K_{zh}^1$ are calculated by using the third order Runge–Kutta scheme, see e.g. [[16](#)].

Example 1: periodic boundary conditions (BCs)

Since the exact solutions are all periodic, we can simply enforce the periodic boundary conditions in our DG scheme. To be more precise, we can replace the flux in [\(3.8\)](#), [\(3.10\)](#), [\(3.12\)](#), and [\(3.14\)](#), by [\(3.22\)–\(3.25\)](#).

We present the numerical results in [Tables 5.1](#) and [5.2](#), which are obtained with bilinear (denoted as Q_1) and biquadratic (denoted as Q_2) basis functions, respectively. We calculate the L^2 errors on a series of uniformly refined rectangular meshes with N partitions in both x - and y -directions, with the time steps (Δt) shown on the header of each table. [Tables 5.1](#) and [5.2](#) clearly show that the L^2 errors for all variables are $O(h^{k+1})$ for Q_k , $k = 1, 2$, where the mesh size $h = \frac{1}{N}$. We take final time $T = 0.1$ for all our simulations in this section.

Table 5.1
Periodic BCs, Q_1 basis function, $T = 0.1$, $\Delta t = 0.2h$.

N	Error of E_x L^2 error	Order	Error of E_y L^2 error	Order	Error of H_z L^2 error	Order
10	8.51e-03	–	8.51e-03	– 10	9.57e-03	–
20	2.08e-03	2.03	2.08e-03	2.03	2.40e-03	1.99
40	5.17e-04	2.01	5.17e-04	2.01	6.01e-04	2.00
80	1.29e-04	2.00	1.29e-04	2.00	1.50e-04	2.00
160	3.21e-05	2.01	3.21e-05	2.01	3.72e-05	2.01
320	7.99e-06	2.01	7.99e-06	2.01	9.26e-06	2.01
N	Error of J_x L^2 error	Order	Error of J_y L^2 error	Order	Error of K_z L^2 error	Order
10	3.56e-02	–	3.56e-02	–	3.93e-02	–
20	9.10e-03	1.97	9.10e-03	1.97	1.02e-02	1.94
40	2.30e-03	1.98	2.30e-03	1.98	2.60e-03	1.97
80	5.79e-04	1.99	5.79e-04	1.99	6.57e-04	1.99
160	1.45e-04	2.00	1.45e-04	2.00	1.65e-04	1.99
320	3.64e-05	2.00	3.64e-05	2.00	4.14e-05	2.00

Table 5.2
Periodic BCs, Q_2 basis function, $T = 0.1$, $\Delta t = 0.6h^{\frac{3}{2}}$.

N	Error of E_x L^2 error	Order	Error of E_y L^2 error	Order	Error of H_z L^2 error	Order
10	4.27e-04	–	4.27e-04	–	5.40e-04	–
20	5.40e-05	2.98	5.40e-05	2.98	7.66e-05	2.82
40	6.87e-06	2.98	6.87e-06	2.98	9.79e-06	2.97
80	8.54e-07	3.01	8.54e-07	3.01	1.23e-06	2.99
160	1.08e-07	2.99	1.08e-07	2.99	1.56e-07	2.98
320	1.35e-08	3.00	1.35e-08	3.00	1.95e-08	3.00
N	Error of J_x L^2 error	Order	Error of J_y L^2 error	Order	Error of K_z L^2 error	Order
10	2.36e-03	–	2.36e-03	–	2.37e-03	–
20	3.41e-04	2.79	3.41e-04	2.79	3.44e-04	2.79
40	4.27e-05	3.00	4.27e-05	3.00	4.39e-05	2.97
80	5.44e-06	2.97	5.44e-06	2.97	5.60e-06	2.97
160	6.84e-07	2.99	6.84e-07	2.99	7.05e-07	2.99
320	8.56e-08	3.00	8.56e-08	3.00	8.83e-08	3.00

Table 5.3
Straightforward PEC BCs ($c_0 = 0$), Q_1 basis function, $T = 0.1$, $\Delta t = 0.2h$.

N	Error of E_x L^2 error	Order	Error of E_y L^2 error	Order	Error of H_z L^2 error	Order
10	2.02e-02	–	2.02e-02	–	1.21e-02	–
20	6.69e-03	1.60	6.69e-03	1.60	2.80e-03	2.12
40	2.30e-03	1.54	2.30e-03	1.54	6.70e-04	2.06
80	8.01e-04	1.52	8.01e-04	1.52	1.58e-04	2.08
160	2.81e-04	1.51	2.81e-04	1.51	3.83e-05	2.05
320	9.91e-05	1.50	9.91e-05	1.50	9.40e-06	2.03
N	Error of J_x L^2 error	Order	Error of J_y L^2 error	Order	Error of K_z L^2 error	Order
10	4.80e-02	–	4.80e-02	–	3.99e-02	–
20	1.37e-02	1.81	1.37e-02	1.81	1.05e-02	1.93
40	4.17e-03	1.72	4.17e-03	1.72	2.64e-03	1.99
80	1.34e-03	1.64	1.34e-03	1.64	6.61e-04	2.00
160	4.47e-04	1.58	4.47e-04	1.58	1.66e-04	2.00
320	1.53e-04	1.54	1.53e-04	1.54	4.15e-05	2.00

Example 2: PEC BCs with pure alternating fluxes

To see the important role of those jump terms in the fluxes $\hat{H}_{zh}(x, y_{\frac{1}{2}})$ and $\hat{H}_{zh}(x_{\frac{1}{2}}, y)$ in (3.12) and (3.14), we let $c_0 = 0$ and run the numerical tests. The numerical errors obtained in this case are presented in Tables 5.3–5.6. It seems that the accuracy becomes sub-optimal for odd orders of polynomial degree, but stays optimal for even orders.

Table 5.4

Straightforward PEC BCs ($c_0 = 0$), Q_2 basis function, $T = 0.1$, $\Delta t = 0.6h^{\frac{3}{2}}$.

N	Error of E_x L^2 error	Order	Error of E_y L^2 error	Order	Error of H_z L^2 error	Order
10	5.42e-04	–	5.42e-04	–	6.08e-04	–
20	5.94e-05	3.19	5.94e-05	3.19	7.72e-05	2.98
40	7.17e-06	3.05	7.17e-06	3.05	9.80e-06	2.98
80	8.75e-07	3.03	8.75e-07	3.03	1.23e-06	2.99
160	1.09e-07	3.01	1.09e-07	3.01	1.56e-07	2.98
320	1.35e-08	3.01	1.35e-08	3.01	1.95e-08	3.00
N	Error of J_x L^2 error	Order	Error of J_y L^2 error	Order	Error of K_z L^2 error	Order
10	2.38e-03	–	2.38e-03	–	2.39e-03	–
20	3.42e-04	2.80	3.42e-04	2.80	3.44e-04	2.79
40	4.27e-05	3.00	4.27e-05	3.00	4.39e-05	2.97
80	5.44e-06	2.97	5.44e-06	2.97	5.60e-06	2.97
160	6.84e-07	2.99	6.84e-07	2.99	7.05e-07	2.99
320	8.56e-08	3.00	8.56e-08	3.00	8.83e-08	3.00

Table 5.5

Straightforward PEC BCs ($c_0 = 0$), Q_3 basis function, $T = 0.1$, $\Delta t = 0.6h^2$.

N	Error of E_x L^2 error	Order	Error of E_y L^2 error	Order	Error of H_z L^2 error	Order
10	7.81e-05	–	7.81e-05	–	4.18e-05	–
20	7.02e-06	3.48	7.02e-06	3.48	2.84e-06	3.88
40	6.10e-07	3.52	6.10e-07	3.52	1.73e-07	4.04
80	5.40e-08	3.50	5.40e-08	3.50	1.09e-08	3.99
160	4.75e-09	3.51	4.75e-09	3.51	6.76e-10	4.02
N	Error of J_x L^2 error	Order	Error of J_y L^2 error	Order	Error of K_z L^2 error	Order
10	2.35e-04	–	2.35e-04	–	1.90e-04	–
20	1.68e-05	3.80	1.68e-05	3.80	1.26e-05	3.91
40	1.24e-06	3.76	1.24e-06	3.76	8.02e-07	3.98
80	9.62e-08	3.69	9.62e-08	3.69	5.03e-08	3.99
160	7.84e-09	3.62	7.84e-09	3.62	3.15e-09	4.00

Table 5.6

Straightforward PEC BCs ($c_0 = 0$), Q_4 basis function, $T = 0.1$, $\Delta t = 0.8h^{\frac{5}{2}}$.

N	Error of E_x L^2 error	Order	Error of E_y L^2 error	Order	Error of H_z L^2 error	Order
10	3.20e-06	–	3.20e-06	–	7.08e-06	–
20	1.00e-07	4.99	1.00e-07	4.99	2.28e-07	4.96
40	3.12e-09	5.01	3.12e-09	5.01	7.15e-09	4.99
80	9.71e-11	5.01	9.71e-11	5.01	2.24e-10	5.00
N	Error of J_x L^2 error	Order	Error of J_y L^2 error	Order	Error of K_z L^2 error	Order
10	3.52e-05	–	3.52e-05	–	3.09e-05	–
20	1.13e-06	4.96	1.13e-06	4.96	1.00e-06	4.94
40	3.55e-08	4.99	3.55e-08	4.99	3.16e-08	4.99
80	1.11e-09	5.00	1.11e-09	5.00	9.87e-10	5.00

Example 3: PEC BCs with the fluxes used in the proof

Here we solve the same problem as Example 2 using the alternating fluxes modified on the PEC boundaries as defined in (3.7)–(3.14) with $c_0 = \frac{1}{2}$. As expected in Remark 3.3, results obtained in Tables 5.7–5.10 clearly show that the optimal convergence rates are obtained for all variables.

Table 5.7

Modified PEC BCs ($c_0 = \frac{1}{2}$), Q_1 basis function, $T = 0.1$, $\Delta t = 0.2h$.

N	Error of E_x L^2 error	Order	Error of E_y L^2 error	Order	Error of H_z L^2 error	Order
10	1.19e-02	–	1.19e-02	–	1.01e-02	–
20	2.81e-03	2.08	2.81e-03	2.08	2.54e-03	1.99
40	6.08e-04	2.21	6.08e-04	2.21	6.16e-04	2.04
80	1.40e-04	2.12	1.40e-04	2.12	1.52e-04	2.02
160	3.35e-05	2.06	3.35e-05	2.06	3.74e-05	2.02
320	8.16e-06	2.04	8.16e-06	2.04	9.29e-06	2.01
N	Error of J_x L^2 error	Order	Error of J_y L^2 error	Order	Error of K_z L^2 error	Order
10	3.98e-02	–	3.98e-02	–	3.92e-02	–
20	9.93e-03	2.00	9.93e-03	2.00	1.02e-02	1.94
40	2.44e-03	2.03	2.44e-03	2.03	2.60e-03	1.97
80	5.97e-04	2.03	5.97e-04	2.03	6.57e-04	1.98
160	1.48e-04	2.02	1.48e-04	2.02	1.65e-04	1.99
320	3.67e-05	2.01	3.67e-05	2.01	4.14e-05	2.00

Table 5.8

Modified PEC BCs ($c_0 = \frac{1}{2}$), Q_2 basis function, $T = 0.1$, $\Delta t = 0.6h^{\frac{3}{2}}$.

N	Error of E_x L^2 error	Order	Error of E_y L^2 error	Order	Error of H_z L^2 error	Order
10	4.54e-04	–	4.54e-04	–	5.89e-04	–
20	5.36e-05	3.08	5.36e-05	3.08	7.68e-05	2.94
40	6.84e-06	2.97	6.84e-06	2.97	9.79e-06	2.97
80	8.54e-07	3.00	8.54e-07	3.00	1.23e-06	2.99
160	1.08e-07	2.99	1.08e-07	2.99	1.56e-07	2.98
320	1.35e-08	3.00	1.35e-08	3.00	1.95e-08	3.00
N	Error of J_x L^2 error	Order	Error of J_y L^2 error	Order	Error of K_z L^2 error	Order
10	2.35e-03	–	2.35e-03	–	2.38e-03	–
20	3.41e-04	2.79	3.41e-04	2.79	3.44e-04	2.79
40	4.27e-05	3.00	4.27e-05	3.00	4.39e-05	2.97
80	5.44e-06	2.97	5.44e-06	2.97	5.60e-06	2.97
160	6.84e-07	2.99	6.84e-07	2.99	7.05e-07	2.99
320	8.56e-08	3.00	8.56e-08	3.00	8.83e-08	3.00

Table 5.9

Modified PEC BCs ($c_0 = \frac{1}{2}$), Q_3 basis function, $T = 0.1$, $\Delta t = 0.6h^2$.

N	Error of E_x L^2 error	Order	Error of E_y L^2 error	Order	Error of H_z L^2 error	Order
10	3.00e-05	–	3.00e-05	–	4.21e-05	–
20	1.70e-06	4.14	1.70e-06	4.14	2.73e-06	3.95
40	1.00e-07	4.09	1.00e-07	4.09	1.72e-07	3.99
80	6.07e-09	4.04	6.07e-09	4.04	1.08e-08	3.99
160	3.74e-10	4.02	3.74e-10	4.02	6.75e-10	4.00
N	Error of J_x L^2 error	Order	Error of J_y L^2 error	Order	Error of K_z L^2 error	Order
10	2.08e-04	–	2.08e-04	–	1.90e-04	–
20	1.35e-05	3.95	1.35e-05	3.95	1.26e-05	3.91
40	8.46e-07	3.99	8.46e-07	3.99	8.03e-07	3.98
80	5.29e-08	4.00	5.29e-08	4.00	5.04e-08	3.99
160	3.30e-09	4.00	3.30e-09	4.00	3.15e-09	4.00

Example 4: PEC BCs with the fluxes used in the proof and varying constant

In Lemma 3.5, the constant c_0 is independent of h . As we can see in the proof of that lemma, the purpose of the c_0 term is to control a less accurate error term, hence we could make c_0 larger without affecting its controlling power. Therefore, we can relax the condition of Lemma 3.5 to be $c_0 = \Omega_h(1)$, meaning that c_0 is asymptotically greater than a constant in terms of

Table 5.10

Modified PEC BCs ($c_0 = \frac{1}{2}$), Q_4 basis function, $T = 0.1$, $\Delta t = 0.8h^{\frac{5}{2}}$.

N	Error of E_x L^2 error	Order	Error of E_y L^2 error	Order	Error of H_z L^2 error	Order
10	3.11e-06	-	3.11e-06	-	7.08e-06	-
20	9.89e-08	4.98	9.89e-08	4.98	2.28e-07	4.96
40	3.10e-09	5.00	3.10e-09	5.00	7.15e-09	4.99
80	9.68e-11	5.00	9.68e-11	5.00	2.24e-10	5.00
N	Error of J_x L^2 error	Order	Error of J_y L^2 error	Order	Error of K_z L^2 error	Order
10	3.52e-05	-	3.52e-05	-	3.09e-05	-
20	1.13e-06	4.96	1.13e-06	4.96	1.00e-06	4.94
40	3.55e-08	4.99	3.55e-08	4.99	3.16e-08	4.99
80	1.11e-09	5.00	1.11e-09	5.00	9.87e-10	5.00

Table 5.11

Modified PEC BCs ($c_0 = \frac{1}{h}$), Q_1 basis function, $T = 0.1$, $\Delta t = 0.2h$.

N	Error of E_x L^2 error	Order	Error of E_y L^2 error	Order	Error of H_z L^2 error	Order
10	8.58e-03	-	8.58e-03	-	9.21e-03	-
20	2.09e-03	2.04	2.09e-03	2.04	2.40e-03	1.94
40	5.19e-04	2.01	5.19e-04	2.01	6.02e-04	1.99
80	1.29e-04	2.00	1.29e-04	2.00	1.50e-04	2.00
160	3.22e-05	2.01	3.22e-05	2.01	3.72e-05	2.01
320	8.02e-06	2.00	8.02e-06	2.00	9.32e-06	2.00
N	Error of J_x L^2 error	Order	Error of J_y L^2 error	Order	Error of K_z L^2 error	Order
10	3.55e-02	-	3.55e-02	-	3.86e-02	-
20	9.09e-03	1.96	9.09e-03	1.96	1.02e-02	1.93
40	2.30e-03	1.98	2.30e-03	1.98	2.60e-03	1.97
80	5.79e-04	1.99	5.79e-04	1.99	6.57e-04	1.98
160	1.45e-04	2.00	1.45e-04	2.00	1.65e-04	1.99
320	3.64e-05	2.00	3.64e-05	2.00	4.14e-05	2.00

Table 5.12

Modified PEC BCs ($c_0 = \frac{1}{h}$), Q_2 basis function, $T = 0.1$, $\Delta t = 0.6h^{\frac{3}{2}}$.

N	Error of E_x L^2 error	Order	Error of E_y L^2 error	Order	Error of H_z L^2 error	Order
10	4.40e-04	-	4.40e-04	-	5.63e-04	-
20	5.41e-05	3.02	5.41e-05	3.02	7.66e-05	2.88
40	6.87e-06	2.98	6.87e-06	2.98	9.79e-06	2.97
80	8.55e-07	3.01	8.55e-07	3.01	1.23e-06	2.99
160	1.08e-07	2.99	1.08e-07	2.99	1.56e-07	2.98
320	1.35e-08	3.00	1.35e-08	3.00	1.95e-08	3.00
N	Error of J_x L^2 error	Order	Error of J_y L^2 error	Order	Error of K_z L^2 error	Order
10	2.36e-03	-	2.36e-03	-	2.38e-03	-
20	3.41e-04	2.79	3.41e-04	2.79	3.44e-04	2.79
40	4.27e-05	3.00	4.27e-05	3.00	4.39e-05	2.97
80	5.44e-06	2.97	5.44e-06	2.97	5.60e-06	2.97
160	6.84e-07	2.99	6.84e-07	2.99	7.05e-07	2.99
320	8.56e-08	3.00	8.56e-08	3.00	8.83e-08	3.00

h . $c_0 = \frac{1}{h}$ would be such an example. However, choosing $c_0 = o_h(1)$, i.e. c_0 being asymptotically less than a constant in terms of h , would eliminate the controlling power of the c_0 term. The proof of Lemma 3.5 would not work with an asymptotically small c_0 , such as $c_0 = h$ as an example. Tables 5.11–5.13 manifest that the convergence rates are still optimal when $c_0 = \frac{1}{h}$. Tables 5.14–5.16 show suboptimal convergence rates for $c_0 = h$, which are the same as those with $c_0 = 0$ in Tables 5.3–5.5. We conclude that these results are consistent with our analysis.

Table 5.13

Modified PEC BCs ($c_0 = \frac{1}{h}$), Q_3 basis function, $T = 0.1$, $\Delta t = 0.6h^2$.

N	Error of E_x L^2 error	Order	Error of E_y L^2 error	Order	Error of H_z L^2 error	Order
10	2.36e-05	–	2.36e-05	–	4.19e-05	–
20	1.50e-06	3.98	1.50e-06	3.98	2.72e-06	3.94
40	9.41e-08	3.99	9.41e-08	3.99	1.72e-07	3.98
80	5.89e-09	4.00	5.89e-09	4.00	1.08e-08	3.99
N	Error of J_x L^2 error	Order	Error of J_y L^2 error	Order	Error of K_z L^2 error	Order
10	2.04e-04	–	2.04e-04	–	1.91e-04	–
20	1.33e-05	3.94	1.33e-05	3.94	1.27e-05	3.91
40	8.41e-07	3.98	8.41e-07	3.98	8.03e-07	3.98
80	5.27e-08	4.00	5.27e-08	4.00	5.04e-08	3.99

Table 5.14

Modified PEC BCs ($c_0 = h$), Q_1 basis function, $T = 0.1$, $\Delta t = 0.2h$.

N	Error of E_x L^2 error	Order	Error of E_y L^2 error	Order	Error of H_z L^2 error	Order
10	1.75e-02	–	1.75e-02	–	1.15e-02	–
20	5.78e-03	1.60	5.78e-03	1.60	2.70e-03	2.08
40	1.97e-03	1.55	1.97e-03	1.55	6.53e-04	2.05
80	6.85e-04	1.53	6.85e-04	1.53	1.56e-04	2.06
160	2.40e-04	1.51	2.40e-04	1.51	3.80e-05	2.04
320	8.44e-05	1.51	8.44e-05	1.51	9.38e-06	2.02
N	Error of J_x L^2 error	Order	Error of J_y L^2 error	Order	Error of K_z L^2 error	Order
10	1.75e-02	–	1.75e-02	–	1.15e-02	–
20	5.78e-03	1.60	5.78e-03	1.60	2.70e-03	2.08
40	1.97e-03	1.55	1.97e-03	1.55	6.53e-04	2.05
80	6.85e-04	1.53	6.85e-04	1.53	1.56e-04	2.06
160	2.40e-04	1.51	2.40e-04	1.51	3.80e-05	2.04
320	8.44e-05	1.51	8.44e-05	1.51	9.38e-06	2.02

Table 5.15

Modified PEC BCs ($c_0 = h$), Q_2 basis function, $T = 0.1$, $\Delta t = 0.6h^{\frac{3}{2}}$.

N	Error of E_x L^2 error	Order	Error of E_y L^2 error	Order	Error of H_z L^2 error	Order
10	5.01e-04	–	5.01e-04	–	6.05e-04	–
20	5.65e-05	3.15	5.65e-05	3.15	7.72e-05	2.97
40	6.99e-06	3.01	6.99e-06	3.01	9.80e-06	2.98
80	8.63e-07	3.02	8.63e-07	3.02	1.23e-06	2.99
160	1.08e-07	3.00	1.08e-07	3.00	1.56e-07	2.98
N	Error of J_x L^2 error	Order	Error of J_y L^2 error	Order	Error of K_z L^2 error	Order
10	2.37e-03	–	2.37e-03	–	2.38e-03	–
20	3.41e-04	2.79	3.41e-04	2.79	3.44e-04	2.79
40	4.27e-05	3.00	4.27e-05	3.00	4.39e-05	2.97
80	5.44e-06	2.97	5.44e-06	2.97	5.60e-06	2.97
160	6.84e-07	2.99	6.84e-07	2.99	7.05e-07	2.99

6. Conclusions

In this paper, we have developed and analyzed a non-dissipative DG method for solving the time domain Maxwell's equations when metamaterials are involved. Stability and optimal error estimates are proved for the semi-discrete scheme. The fully discrete DG scheme with leap-frog time discretization has also been investigated, and its discrete stability has been proved. Numerical results have demonstrated the effectiveness of this DG scheme.

In the future we plan to extend the same idea to develop and analyze the DG scheme for metamaterial Maxwell's equations on triangular elements. Other time discretizations such as explicit and implicit-explicit (IMEX) Runge-Kutta methods will also be considered.

Table 5.16
Modified PEC BCs ($c_0 = h$), Q_3 basis function, $T = 0.1$, $\Delta t = 0.6h^2$.

N	Error of E_x L^2 error	Order	Error of E_y L^2 error	Order	Error of H_z L^2 error	Order
10	5.69e−05	–	5.69e−05	–	4.18e−05	–
20	4.98e−06	3.51	4.98e−06	3.51	2.75e−06	3.92
40	4.29e−07	3.54	4.29e−07	3.54	1.72e−07	4.00
80	3.77e−08	3.51	3.77e−08	3.51	1.08e−08	3.99
N	Error of J_x L^2 error	Order	Error of J_y L^2 error	Order	Error of K_z L^2 error	Order
10	2.22e−04	–	2.22e−04	–	1.90e−04	–
20	1.55e−05	3.84	1.55e−05	3.84	1.26e−05	3.91
40	1.10e−06	3.82	1.10e−06	3.82	8.02e−07	3.98
80	8.16e−08	3.75	8.16e−08	3.75	5.03e−08	3.99

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