APPM 5440: Solutions to Final Exam Review Problems: 6-10

6. Let $(\mathbb{X}, \tau_X)$ homeomorphic to $(\mathbb{Y}, \tau_Y)$ and $(\mathbb{Y}, \tau_Y)$ homeomorphic to $(\mathbb{Z}, \tau_Z)$ implies there exist homeomorphisms $f : \mathbb{X} \to \mathbb{Y}$ and $g : \mathbb{Y} \to \mathbb{Z}$.

   Claim: The composition gf is a homeomorphism from $\mathbb{X}$ to $\mathbb{Z}$.

   Proof of Claim:

   - gf is one-to-one: Suppose $g(f(x_1)) = g(f(x_2))$. Since g is a homeomorphism, g is one-to-one and so this implies that $f(x_1) = f(x_2)$. Now since f is a homeomorphism, f is one-to-one, which implies that $x_1 = x_2$. Hence, gf is one-to-one.
   - gf is onto: Let $z \in Z$. Since g is a homeomorphism, g is onto, so there exists a $y \in \mathbb{Y}$ such that $g(y) = z$. Since f is a homeomorphism, f is onto and so there exists an $x \in \mathbb{X}$ such that $f(x) = y$. Thus, $g(f(x)) = g(y) = z$, so gf is onto.
   - gf is continuous: Take any $U \in \tau_Z$. Since g is a homeomorphism, g is continuous and hence $g^{-1}(U) \in \tau_Y$. Since f is a homeomorphism and $g^{-1}(U) \in \tau_Y$, $f^{-1}(g^{-1}(U)) \in \tau_X$. But, $(gf)^{-1}(U) = f^{-1}(g^{-1}(U)) \in \tau_X$, so gf is continuous.
   - $(gf)^{-1}$ is continuous: Take any $U \in \tau_X$. Since f is a homeomorphism, f^{-1} is continuous and so $f(U)$ which is the inverse image of $f^{-1}(U)$ is in τ_Y. Similarly, since g is a homeomorphism, g^{-1} is continuous and so $g(f(U))$ is the inverse image of $g^{-1}(f(U))$ is in τ_Z. But, the inverse image of U under $(gf)^{-1}$ is $g(f(U))$ which is in τ_Z, so $(gf)^{-1}$ is continuous.

   Hence, gf is a homeomorphism.

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7. Let $\mathbb{X}$ be a finite dimensional space and let $T : \mathbb{X} \to \mathbb{Y}$ be a linear operator. We want to show that there exists some $M > 0$ such that $||Tx|| \leq M||x||$ for all $x \in \mathbb{X}$.

   Let $\{b_1, b_2, \ldots, b_k\}$ be a basis for $\mathbb{X}$.

   Take any $x \in \mathbb{X}$. Then x can be written as $x = \sum_{i=1}^k \alpha_i b_i$ for some $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$.

   Then
$$\begin{aligned} ||Tx|| = \left|\left| T\left(\sum_{i=1}^k \alpha_i b_i\right)\right|\right| &= \left|\left|\sum_{i=1}^k \alpha_i T b_i\right|\right| \leq \sum_{i=1}^k |\alpha_i|\, ||Tb_i|| \\ &\leq \left(\max_{1\leq i\leq k} ||Tb_i||\right) \sum_{i=1}^k |\alpha_i| \end{aligned}$$

   Recall that, for any finite dimensional vector space, there are constants $c, C > 0$ such that
$$c\sum_{i=1}^k |\alpha_i| \leq ||x|| \leq C\sum_{i=1}^k |\alpha_i|.$$

   So,
$$||Tx|| \leq \left(\max_{1\leq i\leq k} ||Tb_i||\right) \sum_{i=1}^k |\alpha_i| \leq \left(\max_{1\leq i\leq k} ||Tb_i||\right) \frac{1}{c}||x||.$$

So, define

$$M = \left(\max_{1\le i\le k} ||Tb_i||\right)\frac{1}{c}$$

and we have $||Tx|| \le M||x||$ for all $x \in \mathbb{X}$, as desired.

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8. $\boxed{\Rightarrow}$ Suppose that $\exists\ c > 0$ such that $||Tx|| \ge c||x||$.

Let (y_n) be a convergent sequence in $range(T)$ with $y_n \to y \in \mathbb{Y}$. We want to show that $y \in range(T)$.

For each n, $y_n \in range(T) \Rightarrow \exists\ x_n \in \mathbb{X}$ such that $y_n = Tx_n$.

(y_n) convergent $\Rightarrow$ (y_n) Cauchy $\Rightarrow$ (x_n) Cauchy since

$$||x_n - x_m|| \le \frac{1}{c}||T(x_n - x_m)|| = \frac{1}{c}||y_n - y_m||$$

(x_n) Cauchy in $\mathbb{X}$ and $\mathbb{X}$ complete $\Rightarrow x_n \to x \in \mathbb{X}$.

T bounded $\Rightarrow$ T continuous $\Rightarrow$

$$Tx = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} y_n = y$$

which implies that $y \in range(T)$. $\checkmark$

$\boxed{\Leftarrow}$ Suppose that $range(T)$ is closed.

Claim: A closed subspace of a Banach space is Banach.

Proof of Claim: Let $\mathbb{X}$ be Banach and let C be a closed subset of $\mathbb{X}$. Let (x_n) be a Cauchy sequence in C. Then (x_n) is a Cauchy sequence in $\mathbb{X}$ and since $\mathbb{X}$ is Banach, $x_n \to x \in \mathbb{X}$. Since C is closed, every sequence in C that has a limit in $\mathbb{X}$ has this limit in C. Thus, our arbitrary Cauchy sequence in C converges to a limit in C and therefore C is Banach.

So, $range(T)$ closed and $range(T) \subseteq \mathbb{Y}$ which is Banach $\Rightarrow$ $range(T)$ is Banach.

T is bounded is a one-to-one and onto the set $range(T)$. Since $\mathbb{X}$ and $range(T)$ are Banach, we can apply the Open Mapping Theorem to say that $T^{-1} : range(Y) \to \mathbb{X}$ is bounded.

Hence, $\exists\ M > 0$ such that $||T^{-1}y|| \le M||y||\ \forall\ y \in range(T)$.

Thus, for any $x \in \mathbb{X}$, let $y = Tx$ which is obviously in $range(T)$. So,

$$||T^{-1}Tx|| \le M||Tx||$$

which implies

$$x \le M||Tx||.$$

Take $c = 1/M$. Then we have that $||Tx|| \ge c||x||$ as desired.

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9. First of all note that $||T_n|| \to ||T||$ is a convergence of real numbers!

So

$$|\,||T_n|| - ||T||\,| \quad = \quad |\,||T_n - 0|| - ||T - 0||\,|$$

where 0 is the zero element (an operator) of the linear space $B(\mathbb{X}, \mathbb{Y})$.

If we let $d(S,T)$ be the metric induced by the operator norm: $d(S,T) := ||S - T||$, then we have So

$$\begin{aligned} |\,||T_n|| - ||T||\,| &= |\,||T_n - 0|| - ||T - 0||\,| \\ &= |\,d(T_n, 0) - d(T, 0)\,| \\ &\leq d(T_n, T) = |\,d(T_n, T)\,| \\ &= ||T_n - T|| \end{aligned}$$

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10. (a) Let $x \in \mathbb{X}$ be finxed and non-zero. Consider the subspace $\mathbb{Y}$ defined as all scalar multiples of x:

$$\mathbb{Y} = \{\alpha x : \alpha \in \mathbb{R}\}.$$

Note that this is a linear subspace of $\mathbb{X}$. (i.e. It contains 0, and is closed under addition and scalar multiplication.)

Define $\psi : \mathbb{Y} \to \mathbb{R}$ as follows. For each $y \in \mathbb{Y}$, y can bew written as $y = \alpha x$ for some $\alpha \in \mathbb{R}$. Define $\psi(y) = \psi(\alpha x) = \alpha\,||x||$.

Note that this is a linear map since

$$\begin{aligned} \psi(a_1 y_1 + a_2 y_2) &= \psi(a_1 \alpha_1 x + a_2 \alpha_2 x) \\ &= \psi((a_1 \alpha_1 + a_2 \alpha_2) x) \\ &= (a_1 \alpha_1 + a_2 \alpha_2)\,||x|| \\ &= a_1 \alpha_1\,||x|| + a_2 \alpha_2\,||x|| \\ &= a_1 \psi(y_1) + a_2 \psi(y_2). \end{aligned}$$

Furthermore, ψ is bounded since, for $y = \alpha x$

$$|\psi(y)| = |\alpha|\,||x|| = ||\alpha x|| = ||y||.$$

The operator norm is

$$||\psi|| = \sup_{||y|| \neq 0} \frac{|\psi(y)|}{||y||} = \sup_{\alpha \neq 0} \frac{\alpha||x||}{||\alpha x||} == \sup_{\alpha \neq 0} \frac{\alpha||x||}{|\alpha|\,||x||} = \sup_{\alpha \neq 0} \frac{\alpha}{|\alpha|} = 1.$$

By the Hahn-Banach Theorem, there exists a $\phi : \mathbb{X} \to \mathbb{R}$ such that $\phi(y) = \psi(y)$ for all $y \in \mathbb{Y}$, (That is, $\phi(\alpha x) = \alpha||x||$ for all $\alpha \in \mathbb{R}$.), and $||\phi|| = ||\psi|| = 1$.

Also, since the fixed x is in $\mathbb{X}$ (since $x = 1 \cdot x$), we have that $\phi(x) = \psi(x) = 1 \cdot ||x|| = ||x||$, as desired.

(b) Suppose that $x, y \in \mathbb{X}$ are such that $\phi(x) = \phi(y)$ for all $\phi \in \mathbb{X}^*$.
Suppose further that $x \neq y$. We will show that this results in a contradiction.
Let $z := x - y$ Then $z \neq 0$..
From part (a), there is a bounded linear functional $\phi \in \mathbb{X}^*$ such that $||\phi|| = 1$ and $\phi(z) = ||z||$.
So,

$$\phi(x) - \phi(y) = \phi(x - y) = \phi(z) = ||z|| \neq 0.$$

This contradicts the fact that $\phi(x) = \phi(y)$. Thus, we must have that $x = y$.