Solutions Manual

to problems in

Failure, Fracture, Fatigue

An Introduction

by

Tore Dahlberg **Anders Ekberg**

Studentlitteratur, Lund 2002, ISBN 91-44-02096-1.

This manual contains solutions to problems in the textbook

Failure, Fracture, Fatigue - An Introduction
Studentlitteratur, Lund 2002, ISBN 91-44-02096-1

At present, solutions to all problems given in Chapters 1 to 6 and Chapters 8 and 9 are available in this document (Chapter 7 does not contain any problems and solutions to the problems in Chapter 10 will be available later).

It is our hope that these solutions will be to some help for teachers and students when studying the topics covered in the textbook.

Samples of examinations are available (2010) on the home page of Solid Mechanics, IEI, at Linköping University, Sweden, address:
www.solid.iei.liu.se
under the heading "Examinations" and course "Damage mechanics and life analysis TMHL61".

Linköping and Gothenburg in February 2010

The authors

Address to Tore Dahlberg after May 2010:
Åsmark 6, 563 92 Gränna, Sweden

Solutions to problems in
T Dahlberg and A Ekberg: Failure, Fracture, Fatigue - An Introduction.
Studentlitteratur, Lund 2002, ISBN 91-44-02096-1.

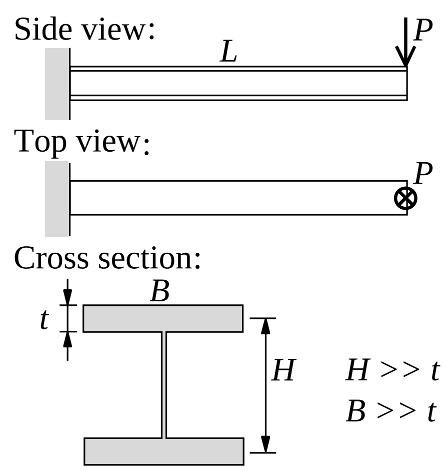
Chapter 1

Introduction, failure mechanisms

Problems with solutions

Elastic deformations

1/1.

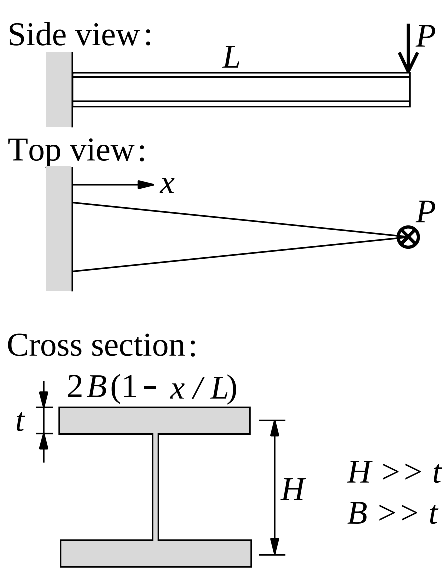


A cantilever beam, length L, carries a force P at its free end. The beam cross section is an ideal I profile, *i.e.*, the area of the flanges is width B by height t each ($B >> t$), and the area of the web can be neglected. The height is H, where $H >> t$, see figure.

When the beam is loaded by the force P, the deflection δ_1 becomes

$$\delta_1 = \frac{P\,L^3}{3\,EI} \quad \text{where} \quad I = 2\,Bt\left(\frac{H}{2}\right)^2 = \frac{Bt\,H^2}{2}$$

Thus, $\delta_1 = 2P\,L^3/3E\,BtH^2 = 0.67P\,L^3/E\,BtH^2$.



It has been found that this deflection is too large. Therefore, without increasing the weight (the mass) of the beam, another form of the cantilever beam cross section is tried. Material is “moved” from the flanges at the free end and “placed” at the flanges at the fixed end to give a beam cross section (still an ideal I profile) with a width varying with the coordinate x, see figure. The height of the profile is still H, but the width of the flanges is now $B(x) = 2B(1-x/L)$.

What deflection δ_2 will now be obtained when the load is applied at the free end?

Solution:

Use the differential equation

$$-EI(x)\,w''(x) = M(x) \tag{a}$$

to determine the deflection $w(x)$ of the beam. The bending moment $M(x)$ in the beam becomes $M(x) = -PL\,(1 - x/L)$, and the second moment of area becomes

$$I = 2 \times 2B\left(1 - \frac{x}{L}\right) t \left(\frac{H}{2}\right)^2 = Bt\,H^2\left(1 - \frac{x}{L}\right) \tag{b}$$

The direction of the moment $M(x)$ has been selected to give a positive deflection $w(x)$ downwards.

Entering $M(x)$ and $I(x)$ in the differential equation gives

$$w''(x) = \frac{-M(x)}{EI(x)} = \frac{PL\,(1 - x/L)}{E\;Bt\,H^2(1 - x/L)} = \frac{PL}{E\;Bt\,H^2} \tag{c}$$

Integration gives

$$w'(x) = \frac{PL}{E\;Bt\,H^2}\,x + C_1 \tag{d}$$

and

$$w(x) = \frac{PL}{E\;Bt\,H^2}\,\frac{x^2}{2} + C_1\,x + C_2 \tag{e}$$

Boundary conditions at $x = 0$ give (notation BC for boundary condition)

BC 1: $\quad w(0) = 0 \quad \text{gives} \quad C_2 = 0 \qquad$ (f)

BC 2: $\quad w'(0) = 0 \quad \text{gives} \quad C_1 = 0 \qquad$ (g)

The deflection $w(x)$ then becomes

$$w(x) = \frac{PL}{E\;Bt\,H^2}\,\frac{x^2}{2} \tag{h}$$

and at the free end (at the force P) one obtains

$$\delta_2 = w(L) = \frac{PL}{E\;Bt\,H^2}\,\frac{L^2}{2} = \frac{PL^3}{2E\;Bt\,H^2} \tag{i}$$

It is concluded that the rearrangement of the width of the flanges makes the deflection at the free end to decrease from $0.67\,PL^3/EBtH^2$ to $0.5\,PL^3/EBtH^2$, *i.e.*, the deflection decreases by 25 per cent.

Answer: The rearrangement of the width of the flanges causes the deflection at the free end to decrease from $0.67\,PL^3/EBtH^2$ to $0.5\,PL^3/EBtH^2$.

Yielding

1/2.

The stress components in a part of a structure have been calculated to σ_x = 120 MPa, σ_y = 80 MPa, and τ_{xy} = 60 MPa (all other stress components are zero). For this structure a safety factor s = 1.5 with respect to yielding is required. What yield strength σ_Y should the material have to fulfil this condition? Investigate both the Tresca and the von Mises yield criteria.

Solution:

First, determine the principal stresses in the material. In the *xy*-plane one obtains

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 100 \pm 63.2 \text{ (MPa)} \qquad \text{(a)}$$

and do not forget the third principal stress: σ_3 = 0.

Thus, the largest principal stress is σ_1 = 163.2 MPa and the smallest principal stress is σ_3 = 0. The effective stress according to Tresca then is $\sigma_e{}^T = \sigma_1 - \sigma_3$ = 163.2 MPa.

Using safety factor s = 1.5 on the yield limit gives

$$163.2 = \frac{\sigma_Y}{1.5} \quad \text{which gives} \quad \sigma_Y = 245 \text{ MPa} \qquad \text{(a)}$$

The von Mises equivalent stress is

$$\sigma_e^{vM} = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2} = 148.3 \text{ MPa} \qquad \text{(b)}$$

Again, using safety factor s = 1.5 on the yield limit gives

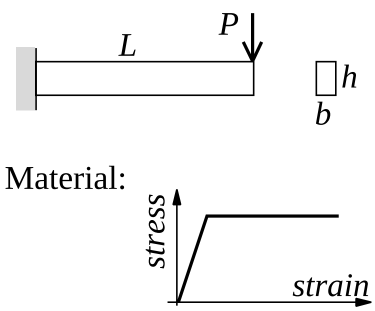
$$148.3 = \frac{\sigma_Y}{1.5} \quad \text{which gives} \quad \sigma_Y = 222 \text{ MPa} \qquad \text{(c)}$$

Thus, according to the more conservative criterion (the Tresca criterion) the yield strength of the material should be σ_Y = 245 MPa.

Answer: The yield strength of the material should be, according to the Tresca criterion, σ_Y = 245 MPa (and the von Mises criterion gives σ_Y = 222 MPa).

Yielding and plastic collapse

1/3.



A cantilever beam of length L and with a rectangular cross section (base b, height h) is loaded by the force P at the free end.

(a) Determine the load $P = P_{\text{elast}}$ the beam may be loaded with without inducing any plastic deformation in the beam (*i.e.*, P_{elast} is the maximum load not giving any yielding in the beam).

(b) Determine the maximum force $P_{\max}$ the beam may be loaded with. Assume that the material is linearly elastic, ideally plastic with yield limit σ_Y.

(c) Finally, determine the ratio of the load $P_{\max}$ to load P_{elast}.

Solution:

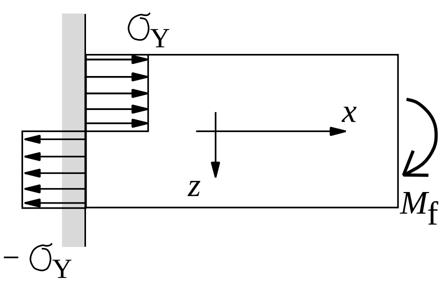
(a) To start with, the elastic solution giving P_{elast} will be determined. The stress in the beam is obtained as

$$\sigma = \frac{M_b}{W_b} \tag{a}$$

where $M_b = P_{\text{elast}} L$, and W_b is the section modulus: $W_b = bh^2/6$.

When $\sigma = \sigma_Y$ one obtains

$$P_{\text{elast}} = \frac{M_b}{L} = \frac{\sigma_Y\, b\, h^2}{6L} \tag{b}$$



(b) The maximum force that may be applied to the beam will produce a plastic hinge at the support of the beam. The bending moment needed to produce the hinge is

$$M_f = \int \sigma z\, dA = \int_{-h/2}^{0} -\sigma_Y z\, b\, dz + \int_{0}^{h/2} \sigma_Y z\, b\, dz = \frac{\sigma_Y\, b\, h^2}{4} \tag{c}$$

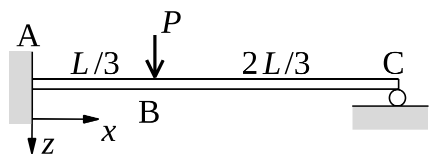
Thus $P_{\max} = M_f / L = \sigma_Y bh^2/4L$.

(c) The ratio $P_{\max}/P_{\text{elast}}$ becomes $P_{\max}/P_{\text{elast}} = 3/2$.

Answer: (a) $P_{\text{elast}} = \sigma_Y\, bh^2 / 6L$, (b) $P_{\max} = \sigma_Y\, bh^2 / 4L$, and (c) $P_{\max}/P_{\text{elast}} = 3/2$.

Plastic collapse

1/4.



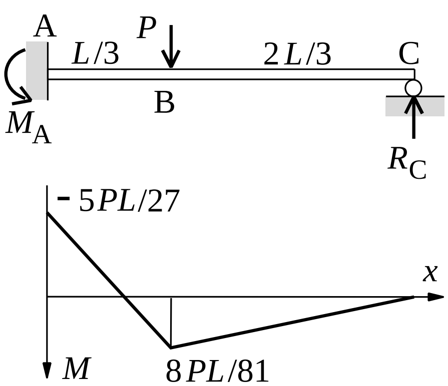
The beam AC is rigidly supported (clamped) at end A and simply supported at end C. The beam has length L and a rectangular cross section with base b and height h, and it is loaded at $x = L/3$ by a force P.

(a) Determine the force $P = P_0$ giving one plastic hinge on the beam (to give a plastic hinge, a cross section must be fully plastic).

(b) Determine the force $P = P_\mathrm{f}$ giving collapse of the structure. (For collapse, two plastic hinges are needed. Here they will appear at A and B.)

The material is linear elastic, ideally plastic with yield limit σ_Y.

Solution:



First determine the distribution of the bending moment M along the beam when the beam is fully elastic. Handbook formulae give, for an elastic beam, the moment M_A at support A as $M_\mathrm{A} = 5PL/27$. The reaction force at support C becomes $R_\mathrm{C} = P/3 - 5P/27 = 4P/27$. The bending moment M_B at B becomes $M_\mathrm{B} = R_\mathrm{C}\, 2L/3 = 8PL/81$. Note that M_A and M_B have different directions, see figure.

According to the conventions used here the moment M_A is negative, but in most cases only the numerical value of the moment (and not the direction) is of interest. Therefore, only numerical values of the moments will be discussed below. Note also that, numerically, $M_\mathrm{A} > M_\mathrm{B}$, implying that plastic yielding will start at point A.

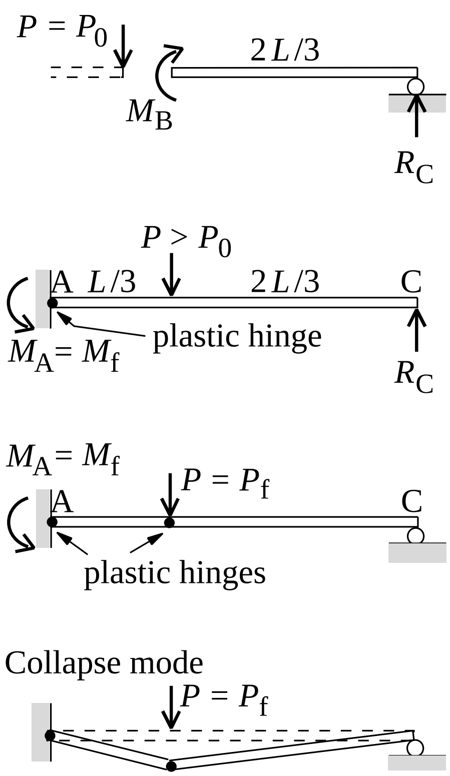
Next, determine the bending moment $M = M_\mathrm{f}$ needed to create a plastic hinge on the beam. When a plastic hinge appears, half the cross-sectional area A ($A = b \times h$) is yielding in tension and half of the area A is yielding in compression. This gives (see Problem 1/3)

$$M_\mathrm{f} = \int_A \sigma_x\, z\, \mathrm{d}A = 2\int_0^{h/2} \sigma_\mathrm{Y}\, z\, b\mathrm{d}z = 2\,\sigma_\mathrm{Y}\, b\, \frac{1}{2}\left(\frac{h}{2}\right)^2 = \sigma_\mathrm{Y}\,\frac{bh^2}{4} \qquad \text{(a)}$$

(a) Question (a) can now be answered. The first plastic hinge will appear in the beam when the largest moment (here M_A) reaches the value M_f. (It is assumed that $M_A > M_B$ is valid also during yielding at A.) Thus, $M_A = M_f$ gives (numerically)

$$M_f = \sigma_Y \frac{bh^2}{4} = \frac{5PL}{27} \quad \text{which gives} \quad P = P_0 = \frac{27}{20}\frac{bh^2}{L}\sigma_Y \qquad \text{(b,c)}$$

Thus, at load $P = P_0 = 1.35\,\sigma_Y\, bh^2/L$ one plastic hinge has developed (at A) in the beam.

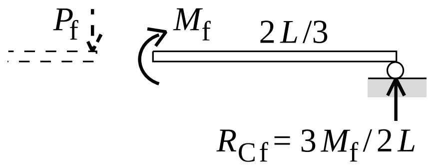


During the loading of the beam, after yielding has started at point A, the load distribution in the structure may change. This implies that the ratio M_A/M_B of the bending moments may change during yielding at A. Therefore, confirm that M_A is larger than M_B also when the hinge appears at A, *i.e.*, confirm that $M_A = M_f > M_B$ when $P = P_0$.

When $M_A = M_f$ and $P = P_0$, the support reaction R_C becomes $R_C = P_0/3 - M_f/L$, and the moment M_B at B becomes $M_B = 2R_C L/3 = 2\sigma_Y bh^2/15$. This moment is less that M_f, which implies that the beam is still elastic at B when the hinge appears at A, *i.e.*, at least part of the cross section at B is elastic, because the moment at M_B is not large enough to create a plastic hinge there.

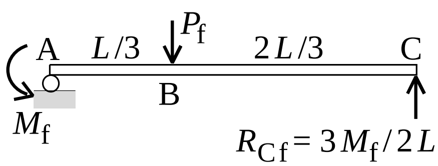
(b) The force P can be further increased, beyond P_0, because the part AC of the beam is still elastic. When $M_A = M_f$ the beam AC acts as a simply supported beam, supported at A and C, and loaded with a moment M_f at A and a force P ($> P_0$) at B. Collapse of the beam will appear when the bending moment at B reaches M_f, *i.e.*, when a plastic hinge is obtained also at B.

When the load $P > P_0$ has increased so much that it produces a hinge also at B, thus giving $M_B = M_f$, the structure will collapse. The force $P = P_f$ that gives a hinge also at B can be determined in the following way: the bending moment at B is $M_B = M_f = R_{Cf}(2L/3)$, which gives $R_C = R_{Cf} = M_B/(2L/3) = 3M_f/2L$, see figure below. The loading of the full beam AC then is $M_A = M_f$ at A, $P = P_f$ at B, and $R_C = R_{Cf} = 3M_f/2L$ at C, see figure. The force $P = P_f$ is still unknown.



Moment equilibrium with respect to point A gives an equation containing the unknown force $P = P_f$. One obtains

$$M_A = M_f = P_f \frac{L}{3} - R_{Cf} L = P_f \frac{L}{3} - \frac{3M_f}{2} \qquad \text{(d)}$$



from which is solved

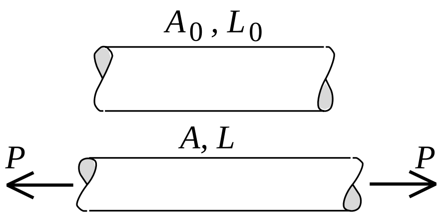
$$P_f = \frac{5}{2} M_f \frac{3}{L} = \frac{15}{8} \sigma_Y \frac{bh^2}{L} \qquad \text{(e)}$$

Thus, after the first hinge has appeared at load $P = P_0 = 1.35\,\sigma_Y bh^2/L$, giving one plastic hinge at A, the force P may be increased to $P = P_f = 1.875\,\sigma_Y bh^2/L$, at which load two plastic hinges are obtained, and the structure will collapse.

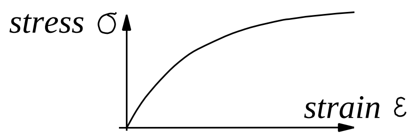
Answer: $P_0 = 1.35\,\sigma_Y bh^2/L$ gives one plastic hinge at A, and $P_f = 1.875\,\sigma_Y bh^2/L$ gives plastic collapse (two hinges).

Plastic instability

1/5.



Material:



Determine the maximum tensional force P_{max} the bar in the figure (see also Section 1.4.4) may be subjected to. Assume that the material is hardening due to plastic deformation and use the stress-strain relationship

$$\sigma = \sigma_0 \cdot \varepsilon^m$$

Use *true stress* and *natural (logarithmic) strain*. The true stress is defined as

$$\sigma = \frac{P}{A}$$

where A is the (contracted) cross-sectional area obtained after loading, *i.e.* $A < A_0$ when $P > 0$, and A depends on P.

The logarithmic strain is defined as

$$\varepsilon = \ln \frac{L}{L_0}$$

where L is the length of the bar when loaded, *i.e.* $L > L_0$ when $P > 0$, and L depends on P.

Also, remember that plastic deformation will take place with *no change of volume*.

Solution:

As in the example on plastic instability, see Section 1.4.4 in the textbook, one has

equilibrium:
$$P = \sigma \cdot A \tag{a}$$

the volume is constant:
$$V = A_0 \cdot L_0 = A \cdot L \tag{b}$$

large strain:
$$\varepsilon = \ln \frac{L}{L_0} \tag{c}$$

Differentiate (a), (b) and (c). It gives

$$\mathrm{d}P = \sigma \cdot \mathrm{d}A + A \cdot \mathrm{d}\sigma \tag{d}$$

$$\mathrm{d}V = A \cdot \mathrm{d}L + L \cdot \mathrm{d}A = 0 \tag{e}$$

$$\mathrm{d}\varepsilon = \frac{1}{L/L_0} \cdot \frac{\mathrm{d}L}{L_0} = \frac{\mathrm{d}L}{L} \tag{f}$$

Plastic instability occurs when the load cannot be increased any more, *i.e.*

when dP = 0. It gives

$$\sigma \cdot \mathrm{d}A + A \cdot \mathrm{d}\sigma = 0 \tag{g}$$

$$A \cdot \mathrm{d}L + L \cdot \mathrm{d}A = 0 \tag{h}$$

$$\mathrm{d}L = L \; \mathrm{d}\varepsilon \tag{i}$$

Eliminate A and L from (g), (h) and (i). It gives

$$\sigma = -\frac{A}{\mathrm{d}A}\,\mathrm{d}\sigma = \frac{L}{\mathrm{d}L}\,\mathrm{d}\sigma = \frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon}\,; \quad \text{thus} \quad \sigma = \frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon} \tag{j}$$

Introduce the material relationship $\sigma = \sigma_0 \cdot \varepsilon^m$ into (j). It gives

$$\sigma_0 \cdot \varepsilon^m = \sigma_0 \cdot m \cdot \varepsilon^{m-1} \quad \text{giving} \quad \varepsilon = m \tag{k}$$

Thus, the loading force P reaches its maximum value $P = P_{\max}$ when $\varepsilon = m$. Then

$$P_{\max} = \sigma \cdot A = \sigma_0\, m^m \cdot A = \sigma_0\, m^m \cdot A_0 \frac{L_0}{L} \tag{l}$$

The ratio L_0/L is obtained from equation (c). By use of $\varepsilon = m$, one obtains

$$\varepsilon = m = \ln\frac{L}{L_0} \quad \text{giving} \quad \mathrm{e}^m = \frac{L}{L_0}, \quad \text{or,} \quad \frac{L_0}{L} = e^{-m} \tag{m}$$

Finally, entering (m) into (l) gives

$$P_{\max} = \sigma_0\, m^m \cdot A_0\, e^{-m} \tag{n}$$

Alternative solution:

The loading force P may be written, by use of (a) and (c),

$$P = \sigma \cdot A = \sigma_0\, \varepsilon^m \cdot A_0 \frac{L_0}{L} = \sigma_0\, \varepsilon^m \cdot A_0\, \mathrm{e}^{-\varepsilon} \tag{o}$$

Determine the maximum value of P. One has

$$\frac{\mathrm{d}P}{\mathrm{d}\varepsilon} = \sigma_0 A_0\, \{\varepsilon^m \cdot (-1)\,\mathrm{e}^{-\varepsilon} + m\varepsilon^{m-1}\,\mathrm{e}^{-\varepsilon}\} = 0 \tag{p}$$

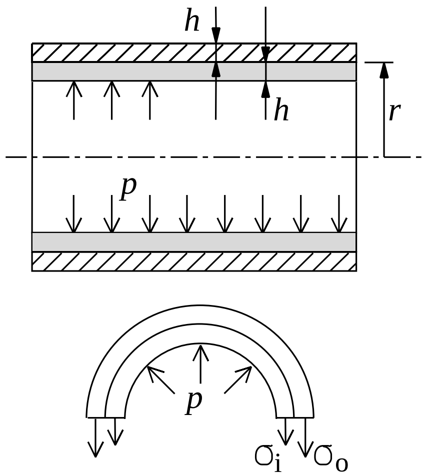
from which $\varepsilon = m$ is obtained. Entering $\varepsilon = m$ into (o) gives

$$P = P_{\max} = \sigma_0\, m^m \cdot A_0\, \mathrm{e}^{-m} \tag{q}$$

(Verify that a maximum value, and not a minimum value, has been determined.)

Answer: Plastic instability will occur when $P_{\max} = \sigma_0\, m^m\, \mathrm{e}^{-m}\, A_0$.

1/6.



A tube subjected to an internal pressure p is made of two circular cylinders of different materials. The outer cylinder has inner radius r, wall thickness h ($h \ll r$), and is made of a Hooke material with modulus of elasticity E_o = 207 GPa (*i.e.* linear elastic material with constitutive equation $\varepsilon = \sigma/E_o$).

The inner cylinder has outer radius r, wall thickness h, and is made of a Norton material. Thus, before the loading has been applied the inner cylinder fits exactly in the outer one.

The material equation of the inner cylinder reads

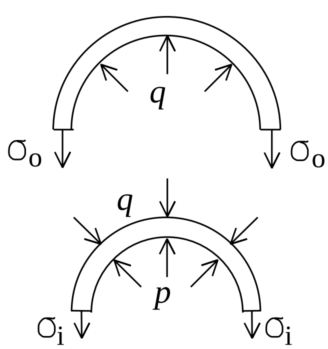
$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E_i} + \left(\frac{|\sigma|}{\sigma_n}\right)^n \frac{\operatorname{sgn}\sigma}{t^*}$$

where E_i is the elastic modulus of the material of the inner cylinder (E_i = 69 GPa), σ_n and n are material parameters ($n > 1$), $\operatorname{sgn}\sigma$ means "sign of σ" and is +1 if σ is positive and -1 if σ is negative, and t^* is a reference time.

(a) Calculate the stresses in the two cylinders immediately after the pressure p has been applied to the inner cylinder (no longitudinal stresses will appear, only circumferential).

(b) Investigate how the stresses will change with time in the two cylinders.

Solution:



(a) When, at time $t = 0$, the pressure p is applied to the inner cylinder, a contact pressure q will arise between the inner and the outer cylinder. The circumferential stress σ_i in the inner cylinder becomes $\sigma_i = (p - q)\,r/h$ and in the outer cylinder the circumferential stress σ_o becomes $\sigma_o = qr/h$ (index i for the inner cylinder and index o for the outer). The contact pressure q is unknown, see figure.

The contact pressure q is obtained from the deformation condition that the two cylinders have the same radial expansion u. Thus, $u_i = u_o$. For circular symmetry the circumferential strain ε equals $\varepsilon = u/r$ (see equation (1.18b) in Section 1.2.6 in the textbook) where u is the radial displacement (radial expansion) of the thin-walled cylinder. One obtains

$$u_i = r\,\varepsilon_i = r\,\varepsilon_o = u_o \qquad \text{(a)}$$

This gives, at time $t = 0$,

$$\frac{\sigma_i}{E_i} = \frac{\sigma_o}{E_o} \qquad \text{(b)}$$

Entering $\sigma_i = (p - q)\,r/h$ and $\sigma_o = qr/h$ in (b) gives

$$p - q = q\,\frac{E_i}{E_o} \qquad \text{(c)}$$

Using the numerical values of E_i and E_o one obtains $q = 3p/4$ (at $t = 0$).

Thus, at time $t = 0$, immediately after the pressure p has been applied, the circumferential stress in the inner cylinder is $\sigma_i = (p - 3p/4)\,r/h = pr/4h$, and in the outer cylinder the circumferential stress is $\sigma_o = qr/h = 3pr/4h$.

It is seen that when the pressure is applied (at $t = 0$), the outer cylinder carries most of the load ($q > p - q$) and $\sigma_o > \sigma_i$). This follows from the fact that the outer cylinder is stiffer than the inner one ($E_o > E_i$). Because of the lower modulus of elasticity of the inner cylinder, it is “easier” for the inner cylinder to expand, implying that the outer cylinder (which is less prone to expansion) has to carry most of the load. Due to creep, this stress distribution will be changed as time goes on.

(b) How will the circumferential stresses σ_i and σ_o develop with time?
The deformation condition $u_i = u_o$ is valid also at times $t > 0$. It gives

$$\dot{u}_i = \dot{u}_o \quad \text{giving} \quad \frac{\dot{\sigma}_i}{E_i} + k\sigma_i^n = \frac{\dot{\sigma}_o}{E_o} \qquad \text{(d,e)}$$

where k is a constant replacing $1/\tau\sigma_n^{\,n}$ in the material equation (thus, $k = 1/\tau\sigma_n^{\,n}$), and $\sigma_i > 0$ gives $\operatorname{sgn}\sigma_i = 1$. The dot ($\dot{}$) indicates differentiation with respect to time t.

The contact pressure q between the inner and outer cylinder will now depend on time t, thus $q = q(t)$. The initial condition on q, at time $t = 0$, was determined above, namely $q(0) = 3p/4$.

Entering $\sigma_i = (p - q)\, r/h$ and $\sigma_o = qr/h$ into (e) gives

$$(\dot{p} - \dot{q})\frac{r}{h\,E_i} + k\left\{(p - q)\frac{r}{h}\right\}^n = \dot{q}\,\frac{r}{h\,E_o} \qquad \text{(f)}$$

Here $\dot{p} = 0$, because p is the constant pressure in the cylinder. Rearranging the expression (f) gives

$$(\dot{q})\frac{r}{h}\left\{\frac{1}{E_i} + \frac{1}{E_o}\right\} + k\left\{q\,\frac{r}{h}\right\}^n = k\left\{\frac{p\,r}{h}\right\}^n \qquad \text{(g)}$$

or

$$\dot{q} + C_1\, q^n = C_2 \qquad \text{(h)}$$

where

$$C_1 = \frac{k\,\{\, r/h\,\}^n}{r\,\{\,1/E_i + 1/E_o\,\}/h} \quad \text{and} \quad C_2 = \frac{k\{\,p\,r/h\,\}^n}{r\,\{\,1/E_i + 1/E_o\,\}/h}$$

One notices that $C_2/C_1 = p^n$; a relation that will be used below.

The homogeneous part of equation (h) reads

$$\dot{q} + C_1\, q^n = 0 \qquad \text{(i)}$$

This equation may be separated into one part containing q only and one part containing the time t. The solution to (i) becomes, for $n > 1$,

$$q = q_{\text{hom}} = \{(-n+1)\,(-C_1\,t + C_0)\}^{1/(-n+1)} \qquad \text{(j)}$$

where C_0 is an integration constant to be determined from a boundary (initial) condition.

The particular solution to (h) becomes

$$q = q_{\text{part}} = \left\{\frac{C_2}{C_1}\right\}^{1/n} = p \qquad \text{(k)}$$

and the complete solution is

$$q = q_{\text{hom}} + q_{\text{part}} = \{\,(-n+1)\,(-C_1\,t + C_0)\,\}^{1/(-n+1)} + p \qquad \text{(l)}$$

The boundary (initial) condition $q(0) = 3p/4$ gives the constant C_0. One obtains

$$q(0) = \{(-n+1)\,C_0\}^{1/(-n+1)} + p = \frac{3p}{4} \qquad \text{(m)}$$

giving

$$C_0 = \frac{1}{-n+1}\left(\frac{3p}{4} - p\right)^{(-n+1)} = \frac{1}{-n+1}\left(\frac{-p}{4}\right)^{(-n+1)} \qquad \text{(n)}$$

Thus,

$$q = \left\{ (-n+1)\left(-C_1\, t + \frac{1}{-n+1}\left(\frac{-p}{4}\right)^{(-n+1)} \right) \right\}^{1/(-n+1)} + p$$

$$= \left\{ (n-1)\, C_1\, t + \left(\frac{-p}{4}\right)^{(-n+1)} \right\}^{1/(-n+1)} + p \qquad \text{(o)}$$

Finally, one obtains

$$\sigma_{\rm i}(t) = (p-q)\,\frac{r}{h} = -\frac{r}{h}\left\{ (n-1)\, C_1\, t + \left(\frac{-p}{4}\right)^{(-n+1)} \right\}^{1/(-n+1)} \qquad \text{(p)}$$

and

$$\sigma_{\rm o}(t) = \frac{qr}{h} = \frac{r}{h}\left\{ (n-1)\, C_1\, t + \left(\frac{-p}{4}\right)^{(-n+1)} \right\}^{1/(-n+1)} + \frac{r}{h}\,p \qquad \text{(q)}$$

One notices that for t = 0 the initial stresses $\sigma_{\rm i} = pr/4h$ and $\sigma_{\rm o} = 3pr/4h$ are obtained (as one should). For large values of time t, the term containing the time t will dominate over the second term in the factor enclosed by the curly brackets in (p) and (q). The factor in the curly brackets will then be proportional to $t^{-1/(n-1)}$, which tends to zero for large values of time t. This implies that after a long time (for large values of time t) the expression giving $\sigma_{\rm i}$ in (p), and the corresponding term in (q), will tend to zero, and the stresses tend to $\sigma_{\rm i} = 0$ and $\sigma_{\rm o} = pr/h$, respectively. Thus, the creep in the inner cylinder force the outer cylinder to carry a larger part of the load. After some time the inner cylinder does not carry any load at all (or it carries a very small part of the load). The outer cylinder will then carry the main part of the load. The function of the inner cylinder could then be, for example, to protect the outer cylinder from chemical reactions, or something else.

Answer: (a) At time t = 0, the stress in the inner cylinder is $\sigma_{\rm i} = pr/4h$, and in the outer cylinder the stress is $\sigma_{\rm o} = 3pr/4h$.

(b)

$$\sigma_{\rm i}(t) = (p-q)\,\frac{r}{h} = -\frac{r}{h}\left\{ (n-1)\, C_1\, t + \left(\frac{-p}{4}\right)^{(-n+1)} \right\}^{1/(-n+1)}$$

and

$$\sigma_{\rm o}(t) = \frac{qr}{h} = \frac{r}{h}\left\{ (n-1)\, C_1\, t + \left(\frac{-p}{4}\right)^{(-n+1)} \right\}^{1/(-n+1)} + \frac{r}{h}\,p$$

(where $\sigma_{\rm i}$ tends to zero and $\sigma_{\rm o}$ tends to pr/h when time t becomes large).

Fracture, maximum normal stress

1/7.

The stress components in a part of a structure have been calculated to σ_{xx} = 120 MPa, σ_{yy} = 80 MPa, and τ_{xy} = 60 MPa (all other stress components are zero). Using the maximum normal stress criterion, investigate material failure. The material is brittle and it has the ultimate strength σ_{Ut} = 150 MPa in tension and the ultimate strength σ_{Uc} = 200 MPa in compression.

Solution:

First, determine the principal stresses in the material (and don't forget the third principal stress). One obtains, in the *xy*-plane,

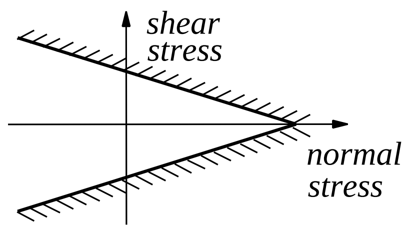
$$\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} = 100 \pm 63.2 \text{ (MPa)} \qquad \text{(a)}$$

Thus, the largest principal stress is σ_1 = 163.2 MPa and the smallest principal stress is σ_3 = 0 (= σ_{zz}). Failure is expected, since in tension the ultimate strength of the material is σ_{Ut} = 150 MPa only.

Answer: Yes, failure is expected because the maximum principal stress in the material is σ_1 = 163 MPa, whereas in tension the strength of the material is σ_{Ut} = 150 MPa, only.

Fracture, Mohr's failure criterion

1/8.



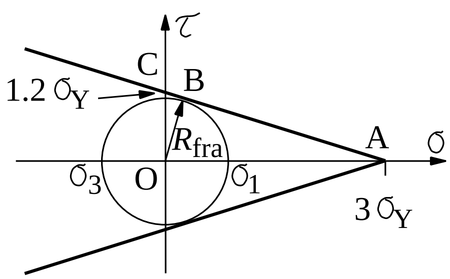
The Mohr criterion for brittle fracture is, for the material studied here, formulated as (see figure)

$$\pm\tau + 0.4\,\sigma = 1.2\sigma_Y$$

where σ_Y is the yield limit of the material. A test specimen of this material is subjected to a torque, which is increased until fracture occurs.

Will there be any plastic deformation of the specimen prior to the fracture? Use the Tresca yield criterion.

Solution:



Pure shear gives Mohr's circle as shown in the figure (centre of the circle at the origine O). Determine the shear stress giving fracture, *i.e.* determine the radius of the circle. Triangles OAB and OAC give

$$\frac{R_{fra}}{3\sigma_Y} = \frac{1.2\sigma_Y}{\sqrt{1.2^2 + 3^2}\,\sigma_Y}$$

which gives R_{fra} = 1.1142 σ_Y

Thus, at *fracture* one has $\tau_{fra} = R_{fra}$ = 1.1142 σ_Y.

At which shear stress $\tau = \tau_{yield}$ will yielding occur?
Yielding will occur when $\sigma_e^{Tresca} = \sigma_Y = \sigma_1 - \sigma_3 = \tau_{yield} - (-\tau_{yield}) = 2\tau_{yield}$.
Thus, at *yielding* one has (according to Tresca) $\tau_{yield} = \sigma_Y/2$.

It is concluded that yielding will occur before fracture (yielding when τ = $0.5\sigma_Y$ and fracture when $\tau = 1.1142\sigma_Y$). This means that the material must be deformation hardening so that shear stress $\tau = 1.1142\sigma_Y$ can be reached. When that stress is reached the fracture will occur.

Answer: Plastic deformation will occur because $\tau = \tau_{fra} = 1.114\sigma_Y$ at fracture whereas $\tau = \tau_{yield} = 0.5\sigma_Y$ at yielding.

Fracture, Mohr's failure criterion

1/9.

For a brittle material the ultimate strength was determined to be 100 MPa in tension and 400 MPa in compression. Within these limits the material is linearly elastic. Assume that the Mohr criterion for brittle fracture is given by straight lines in the shear stress *versus* normal stress diagram (as in Problem 1/8). The volume of the test specimens used to determine the ultimate strengths was 10^4 mm^3.

This material is used in a circular thin-walled cylinder of length L = 1 m, diameter d = 200 mm, and wall thickness h = 5 mm. The cylinder is loaded with a torque T. With respect to brittle fracture, determine the maximum allowable torque $T_{\max}$ the cylinder may be subjected to. The size effect according to Weibull should be taken into account.

According to Weibull the ultimate strength of a large volume of material is less than the ultimate strength of a small volume of the same material. (One reason for this could be that there is a larger probability of finding a flaw in a larger volume than in a smaller volume. The brittle fracture may start at the flaw.) Assume that the ultimate strength of a material of volume V_0 is found to be σ_0. According to Weibull, the ultimate strength of the same material, but of volume V, would be

$$\sigma = \sigma_0 \left(\frac{V_0}{V} \right)^{1/m}$$

For the material above, use factor m = 6 in the Weibull relationship.

Solution:

The ultimate strength in tension, σ_{Ut} = 100 MPa, and in compression, σ_{Uc} = 400 MPa, give the fracture limit curve according to the figure below.

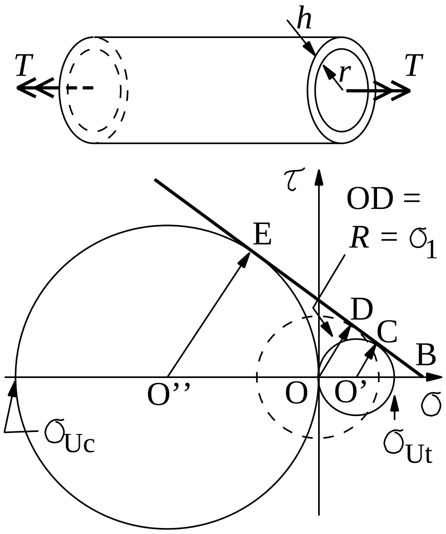
The shear stress in the circular thin-walled cylinder is

$$\tau = \frac{T}{W_{\text{v}}} = \frac{T}{2\pi r^2 h} \tag{a}$$

Denote this stress τ_{xy}. Thus, $\tau_{xy} = \tau = T/2\pi r^2 h$. The principal stresses (at plane stress, $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$) are given by

$$\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2} \right)^2 + \tau_{xy}^2} \tag{b}$$

where, here, $\sigma_{xx} = 0$, $\sigma_{yy} = 0$, and $\tau_{xy} = \tau$.



The principal stresses thus are (re-numbered)

$$\sigma_1 = \tau, \quad \sigma_2 = 0 \quad \text{and} \quad \sigma_3 = -\tau \tag{c}$$

Mohr's circle, as given by the principal stresses σ_1 and σ_3, is entered into the diagram. The new circle will have the radius σ_1 (= OD in the figure). Fracture is expected when the circle reaches the fracture limit curve. The radius of the circle is obtained from triangles OBD and O'BC.

Determine first, however, the distance OB. From triangles O'BC and O''BE one obtains

$$\frac{50}{200} = \frac{\text{OB} - 50}{\text{OB} + 200} \tag{d}$$

giving OB = 133 MPa.

Triangles O'BC and OBD give (with OD = σ_1)

$$\frac{50}{\sigma_1} = \frac{\text{OB} - 50}{\text{OB}} \tag{e}$$

from which σ_1 = 80 MPa is solved. Thus $\tau = \sigma_1$ = 80 MPa *before* any reduction of the stress with respect to volume effects has been done.

(With no reduction due to volume effects, the allowable torque T would have been $T = \tau W_v$ = 25 kNm.)

Reduce the allowable stress with respect to the volume effect. Weibull gives

$$\frac{\sigma}{\sigma_0} = \left(\frac{V_0}{V}\right)^{1/m} \tag{f}$$

Thus,

$$\frac{\tau_{\text{reduced}}}{\tau} = \left(\frac{V_0}{V}\right)^{1/6}$$

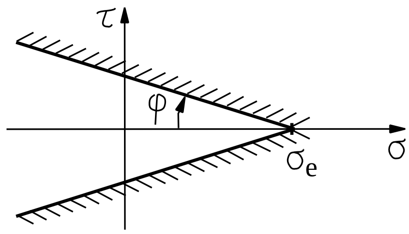
giving

$$\tau_{\text{reduced}} = \tau \left(\frac{10^4}{2\pi 100 \cdot 5 \cdot 1000}\right)^{1/6} = 0.3835\tau = 30.7 \text{ MPa} \tag{g}$$

The allowable torque T becomes $T_{\text{allow}} = \tau_{\text{reduced}} W_v$ = 9.6 kNm.

Answer: $T_{\max}$ = 9.6 kNm (and $T_{\max}$ would have been 25 kNm, approximately, if the size effect had been neglected).

Fracture, Mohr's failure criterion
1/10.

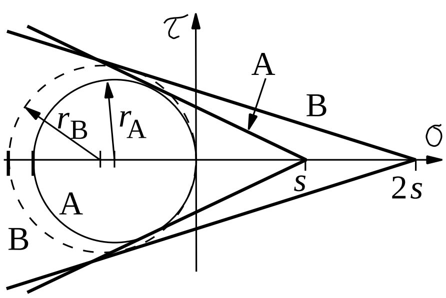


A circular bar made of a brittle material will be loaded in axial compression. Two different materials are available. Both materials have linear fracture limit curves, see figure, with the following data

Material	σ_e	tan φ
A	s	1/2
B	$2s$	1/3

Which one of the materials is the better if brittle fracture (in compression) of the bar is considered?

Solution:



The fracture limit of material A intersects the τ axis at $\tau_A = s\tan\varphi = s/2$. Let the radius of circle A be r_A. Similar triangles give

$$\frac{r_A}{r_A + s} = \frac{s/2}{s\sqrt{1^2 + (1/2)^2}} \qquad \text{(a)}$$

giving $r_A = 0.809s$. This gives $\sigma_{Ucomp}^{A} = 2r_A = 1.62s$.

The fracture limit of material B intersects the τ axis at $\tau_A = 2s\tan\varphi = 2s/3$. Let the radius of circle B be r_B. Similar triangles give

$$\frac{r_B}{r_B + 2s} = \frac{2s/3}{s\sqrt{2^2 + (2/3)^2}} \qquad \text{(b)}$$

giving $r_B = 0.925s$. This gives $\sigma_{Ucomp}^{B} = 2r_B = 1.85s$.

Material B has the largest fracture strength in (uni-axial) compression: $\sigma_{Ucomp}^{B} = 2r_B = 1.85s$. This material should be selected.

Answer: Material B should be selected because it has the largest ultimate strength in compression: $\sigma_{Ucomp}^{B} = 1.85s$ (whereas $\sigma_{Ucomp}^{A} = 1.62s$).

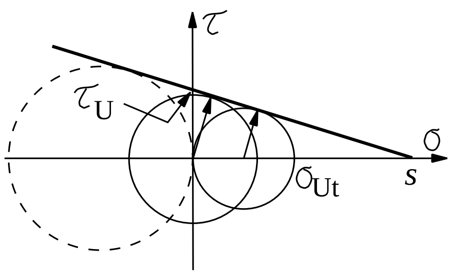
Fracture, Mohr's failure criterion

1/11.

Two materials, C and D, have linear fracture limit curves. The two materials have the same ultimate strength τ_U when loaded in shear. When loaded in tension, however, material C has a larger ultimate strength than material D, thus $\sigma_{Ut}^C > \sigma_{Ut}^D$.

What can be said about the two materials' ultimate strengths in compression?

Solution:



From the figure it can be seen that the larger the ultimate strength in tension σ_{Ut} is, the more horizontal will the fracture limit curve be, and the smaller will the circle to the left of the τ-axis be. Thus, as $\sigma_{Ut}^C > \sigma_{Ut}^D$ one will obtain $\sigma_{Ucomp}^C < \sigma_{Ucomp}^D$.

It can also be shown that (see below)

$$\frac{1}{\tau_U} = \frac{1}{\sigma_{Ut}} + \frac{1}{\sigma_{Ucomp}} \tag{a}$$

which gives the above result.

Similar triangles give

$$\frac{\tau_U}{s} = \frac{\sigma_{Ut}/2}{s - \sigma_{Ut}/2} \quad \text{giving} \quad \frac{1}{\tau_U} = \frac{1}{\sigma_{Ut}/2} - \frac{1}{s} \tag{b}$$

and

$$\frac{\tau_U}{s} = \frac{\sigma_{Ucomp}/2}{s + \sigma_{Ucomp}/2} \quad \text{giving} \quad \frac{1}{\tau_U} = \frac{1}{\sigma_{Ucomp}/2} + \frac{1}{s} \tag{c}$$

Summing (b) and (c) gives (a).

Answer: For the ultimate strengths in compression one has (numerically) $\sigma_{Ucomp}^C < \sigma_{Ucomp}^D$.

Fatigue

1/12.

A component of a machine is subjected to a repeated loading sequence. The loading sequence is repeated once a day. The loading is alternating, and one sequence contains the following loading amplitudes (*i.e.* stress amplitudes – the mean value of the stress is zero)

$$\sigma_a = 180,\ 160,\ 140 \text{ and } 100 \text{ MPa}$$

The number of loading cycles at each stress level is

$$n = 15,\ 20,\ 150, \text{ and } 500, \text{ respectively}$$

The Wöhler curve of the material is, in this stress range, given by the relation

$$\sigma_a = -50\log N + 400 \text{ MPa}$$

Determine the damage accumulation D due to one loading sequence, and then, determine how many loading sequences (how many days) the component might be used before fatigue failure is expected.

Solution: For the different stress levels, the expected fatigue life N is obtained from the Wöhler curve. One obtains

$$N = 10^{(400-\sigma_a)/50} \tag{a}$$

giving, respectively,

$$N = 25\,119,\ 63\,096,\ 158\,489, \text{ and } 1\,000\,000$$

The accumulated damage due to one loading sequence is, by use of the Palmgren-Miner damage accumulation rule,

$$D = \frac{15}{25\,119} + \frac{20}{63\,096} + \frac{150}{158\,489} + \frac{500}{1\,000\,000} = 0.00236 = \frac{1}{424} \tag{b}$$

Finally, the number of sequences N_s to expected fatigue failure is

$$N_s = 1/D \cong 424 \tag{c}$$

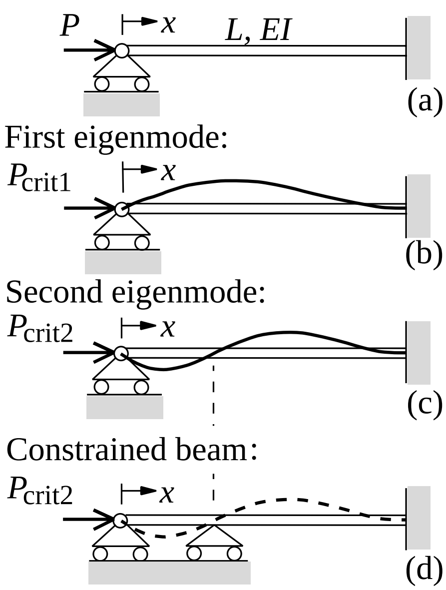
Thus, the component can be used in 424 days, approximately.

Answer: Damage D due to one sequence is

$$D = \frac{15}{25\,119} + \frac{20}{63\,096} + \frac{150}{158\,489} + \frac{500}{10^6} = 2.36 \cdot 10^{-3}$$

Expected number of sequences (days) N_s to fatigue failure is $N_s = 1/D \cong 424$.

1/13.



It is possible to increase the buckling load of a beam by introducing extra supports along the beam, see Section 1.4.9 (the example on instability). It can be shown that by use of one extra support the buckling load of a beam at most can be increase to the second eigenvalue of the original beam without the extra support, *i.e.* to the second critical load of the unconstrained beam. The extra support should then be placed at the node of the second eigenmode of the unconstrained beam (*i.e.* at the point of zero deflection of the second eigenmode).

If the extra support is placed at that node, then the beam is constrained to buckle according to the second eigenmode, and thereby the critical load $P = P_{\text{crit}}$ of the constrained beam (with the extra support) equals the second critical load of the unconstrained beam.

Determine where to place one extra support on the beam that is pinned (simply supported) at one end and fixed (rigidly supported) at the other end, as shown in figure (a), to obtain the largest possible critical load of the constrained beam shown in figure (d). In figure (b) the first eigenmode $w_1(x)$ of the pinned-fixed beam is sketched, and in figure (c) the second eigenmode $w_2(x)$ of the un-constrained beam (a) is sketched. Thus, determine the coordinate x where the mode $w_2(x)$ has zero deflection.

Solution:

The problem will be solved by use of the differential equation for an axially loaded beam. The differential equation reads

$$EI\, w^{\text{IV}}(x) + P\, w''(x) = 0 \qquad \text{(a)}$$

where $w(x)$ is the deflection of the beam, EI is the bending stiffness, and P is the force loading the beam in compression. The solution to this equation is

$$w(x) = C_1 + C_2\, px + C_3 \sin px + C_4 \cos px \qquad \text{(b)}$$

where $p = \sqrt{P/EI}$.

Boundary conditions give the constants C_1 to C_4. One has (BC for boundary condition)

BC 1: $w(0) = 0$ gives $C_1 + C_4 = 0$ (c)

BC 2: $M(0) = 0$ gives $-EIw''(0) = 0$ which gives

$$-C_3\, p^2 \sin p \cdot 0 - C_4\, p^2 \cos p \cdot 0 = 0 \quad \text{(d)}$$

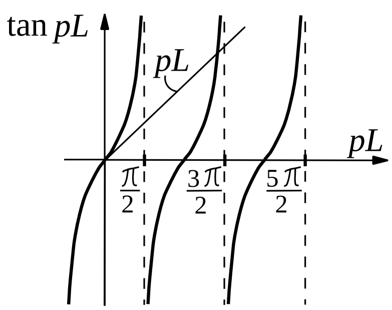
These two equations give $C_4 = 0$ and $C_1 = 0$, which is used in the following boundary conditions. At the beam end $x = L$ one has

BC 3: $w(L) = 0$ giving $C_2\, pL + C_3 \sin pL = 0$ (e)

BC 4: $w'(L) = 0$ giving $C_2\, p + C_3\, p \cos pL = 0$ (f)

To obtain a solution with at least one of C_2 and C_3 not equal to zero, one must have the determinant of the system of equations (e,f) equal to zero. It gives

$$\begin{vmatrix} pL & \sin pL \\ 1 & \cos pL \end{vmatrix} = 0 \quad \text{or} \quad \tan pL = pL \qquad \text{(g)}$$



The roots to equation (g) are determined numerically. From the figure it is concluded that the first root is close to $3\pi/2$, the second root very close to $5\pi/2$, and so on. One obtains

$$p_1 L = 4.4934 \qquad p_2 L = 7.7252 \approx 5\pi/2$$

$$p_3 L = 7\pi/2 \qquad \text{and so on.} \qquad \text{(h)}$$

The first root $p_1 L$ gives $p_1 L = \sqrt{P / EI} \cdot L = 4.4934$

which gives $$P = P_{\text{crit1}} = \frac{4.4934^2 EI}{L^2} = \frac{2.05\, \pi^2 EI}{L^2} \qquad \text{(i)}$$

The second root $p_2 L$ gives $p_2 L = \sqrt{P / EI} \cdot L = 7.7252$

which gives $$P = P_{\text{crit2}} = \frac{7.7252^2\, EI}{L^2} \qquad \text{(j)}$$

The eigenmodes, *i.e.* the forms of the deflection of the buckling beam, become (equation (f) gives $C_2 = -C_3 \cos pL$)

for $p_1 L$: $$w_1(x) = C_3\, \{ -p_1 x \cos p_1 L + \sin p_1 x \} \qquad \text{(k)}$$

for $p_2 L$: $$w_2(x) = C_3 \{ -p_2 x \cos p_2 L + \sin p_2 x \} \tag{l}$$

where C_3 is undetemined (and different in the different modes).

The second eigenmode becomes, by use of $p_2 L$ ($p_2 L$ in radians),

$$w_2(x) = C_3 \left\{ -7.7252 \frac{x}{L} \cos 7.7252 + \sin\left(7.7252 \frac{x}{L}\right) \right\} \tag{m}$$

Now the value of x giving a zero crossing of $w_2(x)$ can be determined. One obtains

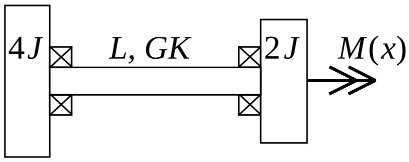
$$w_2(x) = C_3 \left\{ -7.7252 \frac{x}{L} \cos 7.7252 + \sin\left(7.7252 \frac{x}{L}\right) \right\} = 0 \tag{n}$$

which gives $x = 0.36\,L$.

Thus, if an extra support is placed at $x = 0.36\,L$, the critical load of the pinned-clamped beam increases from $P_{\text{crit1}} = 2.05\,\pi^2 EI / L^2 = 20.23 EI / L^2$ (no extra support) to $P_{\text{crit2}} = 7.7252^2 EI / L^2 = 59.68 EI / L^2$ for one optimally placed support.

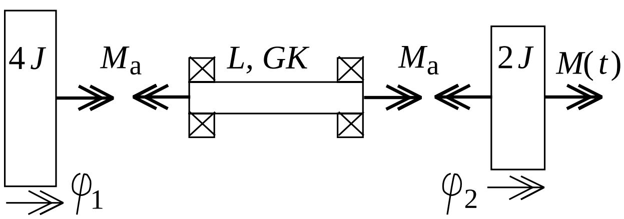
Answer: The support should be placed at $x = 0.36\,L$.

1/14.



A shaft carries two flywheels according to the figure. One of the flywheels is loaded with a torque $M(t) = M_0 \sin \Omega t$. Determine the maximum torque in the shaft if $\Omega = 0.75\omega_e$ and $\Omega = 0.90\omega_e$, where ω_e is the eigenfrequency of the structure. What happens if $\Omega = \omega_e$? The moments of inertia of the two flywheels are $4J$ and $2J$ respectively, and the inertia of the shaft can be neglected.

Solution:



Separate the flywheels from the axle and enter the torque M_a on the axle and on the flywheels. (The inertia of the axle, compared to the flywheels, is supposed to negligible here. Therefore the torque is the same at the two ends of the axle.)

Introduce the rotations φ_1 and φ_2 of the two flywheels. The directions of φ_1 and φ_2 are shown in the figure.

For the flywheel 1 the equation of motion becomes

$$4J\ddot{\phi}_1 = M_a \tag{a}$$

For the flywheel 2 the equation of motion becomes

$$2J\ddot{\phi}_2 = M(t) - M_a \tag{b}$$

The torsion of the axle is $\varphi_2 - \varphi_1$.
The moment-deformation relationship for the axle is

$$\phi_2 - \phi_1 = \frac{M_a L}{GK} \tag{c}$$

Eliminate φ_1 and φ_2 from the equations (a) to (c). That gives a differential equation in the unknown moment M_a. One obtains

$$\ddot{M}_a \frac{L}{GK} + \frac{M_a}{4J} + \frac{M_a}{2J} = \frac{M(t)}{2J} = \frac{M_0 \sin\Omega t}{2J} \tag{d}$$

Assume a particular solution to the differential equation (d) on the form

$$M_\mathrm{a} = A \sin \Omega t \tag{e}$$

Enter the assumption (e) into the differential equation (d). It gives an equation determining the unknown amplitude A. One obtains

$$\left(-\Omega^2 \frac{L}{GK} + \frac{3}{4J} \right) A = \frac{M_0}{2J} \tag{f}$$

from which is solved

$$A = \frac{M_0\, GK}{2JL\ (-\Omega^2\ + 3GK\ /\ 4JL\ )} \tag{g}$$

The (angular) eigenfrequency ω_e can now be determined. The eigenfrequency is the frequency $\Omega = \omega_\mathrm{e}$ giving that the amplitude A in (g) tends to infinity (which is equivalent to that the "system determinant" in (f) becomes zero). One obtains

$$\omega_\mathrm{e} = \sqrt{\frac{3GK}{4JL}} \tag{h}$$

The torque in the axle can now be calculated. One obtains, from (e) and (g)

$$M_\mathrm{a} = \frac{M_0\, GK}{2JL\ (-\Omega^2\ + 3GK\ /\ 4JL\ )} \sin\Omega t = \frac{2M_0}{3\,(1 - \Omega^2/\omega_\mathrm{e}^2)} \sin\Omega t \tag{i}$$

For $\Omega = 0.75\omega_\mathrm{e}$ one obtains

$$M_\mathrm{a} = \frac{32}{21} M_0 \sin\Omega t = 1.52\, M_0 \sin\Omega t \tag{j}$$

For $\Omega = 0.90\omega_\mathrm{e}$ one obtains

$$M_\mathrm{a} = \frac{200}{57} M_0 \sin\Omega t = 3.51\, M_0 \sin\Omega t \tag{k}$$

Finally, if $\Omega = \omega_\mathrm{e}$, the torque M_a in the axle tends to infinity.
It is concluded that an excitation frequency close to a resonance frequency of the structure might be very dangerous.

Answer: Normalised with respect to the load amplitude M_0, the amplitude M_a of the torque in the shaft is $M_\mathrm{a}\,/M_0 = M_\mathrm{shaft}\,/M_0 = 1.52$, 3.51 and ∞, respectively, for $\Omega = 0.75\omega_\mathrm{e}$, $\Omega = 0.90\omega_\mathrm{e}$, and $\Omega = \omega_\mathrm{e}$.

Empty page.

Solutions to problems in
T Dahlberg and A Ekberg: Failure, Fracture, Fatigue - An Introduction.
Studentlitteratur, Lund 2002, ISBN 91-44-02096-1.

Chapter 2

Stresses, stress concentration, stresses at crack tip, stress intensity factor, and fracture criteria

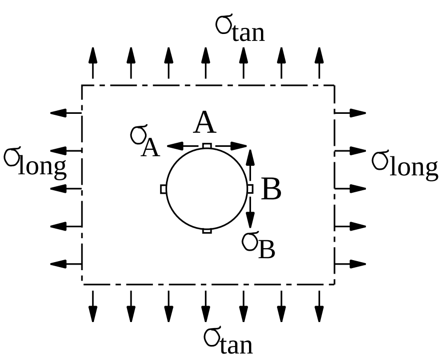
Problems with solutions

Stress concentration

2/1.

A thin-walled circular cylindrical pressure vessel has length L, radius a, and wall thickness h. For a tube connection a small circular hole is opened in the wall (and the tube is mounted). The vessel is then loaded with a pressure p. Determine the maximum stress at the hole due to the pressure if the influence of the tube connection is disregarded.

Solution:



The stresses in the pressure vessel wall are (tangentially and longitudinally)

$$\sigma_{tan} = p\,\frac{a}{h} \quad \text{and} \quad \sigma_{long} = p\,\frac{a}{2h} \tag{a,b}$$

Study the stress concentration at the hole. There are four critical points at the hole where high stress may occur: points A and B, see figure. Two points have stress σ_A and two points have stress σ_B. One obtains the stresses

$$\sigma_A = 3\,\sigma_{long} - \sigma_{tan} = 3\,p\,\frac{a}{2h} - p\,\frac{a}{h} = p\,\frac{a}{2h} \tag{c}$$

and

$$\sigma_B = 3\,\sigma_{tan} - \sigma_{long} = 3\,p\,\frac{a}{h} - p\,\frac{a}{2h} = \frac{5}{2}\,p\,\frac{a}{h} \tag{d}$$

The stress around the hole varies between these two values. The largest stress at the hole thus is $\sigma_{max} = \sigma_B = 5pa/2h$.

Answer: The largest stress at the hole is $\sigma_{max} = 5pa/2h$.

2/2.

A thin-walled circular cylindrical pressure vessel has a small circular hole in the wall. The hole will be covered with a square plate; the plate being screwed to the wall with four screws, one in each corner. By taking stress concentration into account, where do you want to place the screws in the pressure vessel wall?

(This is perhaps a good example to try to understand stress concentration, but it is not a good way to fasten the plate – you should *never* make a hole in a region where you already have stress concentration.)

Solution: See answer.

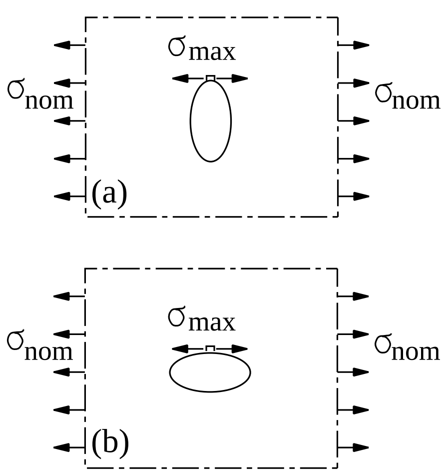
Answer: The screws should be mounted as far away from the stress concentration as possible; here it will be at 45° counted from the longitudinal axis of the pressure vessel.

2/3.

A large plate has a small elliptical hole in it. The ratio of the axes (major to minor axis) of the elliptical hole is 2 to 1. Determine the stress concentration factor K_t when the plate is loaded in parallel to the

(a) minor axis, (b) major axis of the ellipse.

Solution:



The maximum stress at an elliptical hole is given by

$$\sigma_{max} = \left(1 + 2\frac{a}{b}\right)\sigma_{nom} \qquad \text{(a)}$$

where a and b are the half-axes of the ellipse.

(a) If the load is parallel to the minor axis, see Figure (a), one obtains

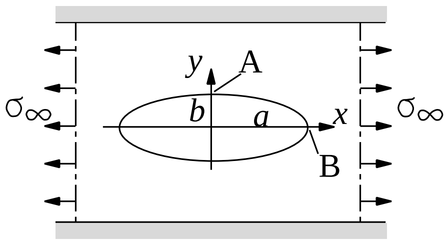
$$\sigma_{max} = \left(1 + 2\frac{2}{1}\right)\sigma_{nom} = 5\sigma_{nom} \qquad \text{(b)}$$

(b) If the load is parallel to the major axis, see Figure (b), one obtains

$$\sigma_{max} = \left(1 + 2\frac{1}{2}\right)\sigma_{nom} = 2\sigma_{nom} \qquad \text{(c)}$$

Answer: Stress concentration factor is K_t = 5 and 2, respectively.

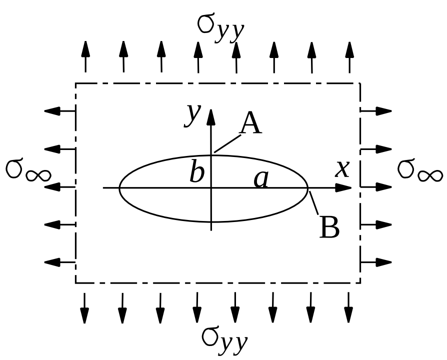
2/4.



A plate contains an elliptical hole as shown in the figure. The plate is supported at two opposite sides; the support being such that all motion in the *y*-direction is prevented whereas motion in the *x*-direction is possible. The plate is loaded with an uni-axial stress σ_∞ along the two other sides.

Under which condition (ratio b/a) will fracture start at point A and at point B, respectively? The material is linearly elastic (with Young's modulus E and Poisson's ratio ν) up to its brittle fracture at the ultimate strength σ_U.

Solution:



Due to the prevented contraction in the y direction (strain $\varepsilon_{yy} = 0$), stress σ_{yy} will appear in that direction. Let $\sigma_{xx} = \sigma_\infty$. Hooke's law gives

$$\varepsilon_{yy} = 0 = \frac{1}{E}\,[\sigma_{yy} - \nu\,(\sigma_{xx} + \sigma_{zz})] \qquad \text{(a)}$$

Knowing that $\sigma_{zz} = 0$, one obtains

$$\sigma_{yy} = \nu\,\sigma_{xx} \qquad \text{(b)}$$

Stresses σ_{xx} (= σ_∞) and $\sigma_{yy} = \nu\sigma_{xx}$ give rise to stresses at the elliptical hole. It is seen that if half-axis b is large ($b >> a$), then the largest stress at the hole will be at point A. Contrary, if $a >> b$, then the largest stress will appear at point B. Determine the ratio a/b that gives the same stress at point A as at point B. Using the elementary case for uni-axial stress:

$$\sigma_{\max} = \left(1 + 2\frac{a}{b}\right)\sigma_{\text{nom}} \quad \text{and} \quad \sigma_{\min} = -\,\sigma_{\text{nom}} \qquad \text{(c)}$$

one obtains ($\sigma_{yy} = \nu\sigma_{xx}$ is used)

$$\sigma_A = \left(1 + 2\frac{b}{a}\right)\sigma_{xx} - \nu\,\sigma_{xx} \qquad \text{(d)}$$

and

$$\sigma_B = \left(1 + 2\frac{a}{b}\right)\nu\,\sigma_{xx} - \sigma_{xx} \qquad \text{(e)}$$

Set $\sigma_A = \sigma_B$. It gives

$$\left(1 + 2\frac{b}{a}\right)\sigma_{xx} - \nu\,\sigma_{xx} = \left(1 + 2\frac{a}{b}\right)\nu\,\sigma_{xx} - \sigma_{xx} \qquad \text{(f)}$$

Simplifying (f) gives

$$1-\nu+\frac{b}{a}-\nu\frac{a}{b}=0 \tag{g}$$

Let $b = \beta a$. It gives

$$\beta^2+(1-\nu)\,\beta-\nu=0 \tag{h}$$

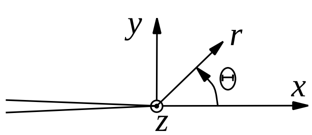
Knowing that $\beta > 0$, one obtains the solution $\beta = \nu$.

Thus, when $\beta = \nu$, i.e. when $b = \nu a$, one has $\sigma_A = \sigma_B$. If $b > \nu a$, then the stress at point A is the largest, whereas $b < \nu a$ gives that the stress at point B is the largest.

Answer: Ratio $b/a > \nu$ gives fracture (largest stress) at point A and $b/a < \nu$ gives fracture at point B.

Stresses at crack tip

2/5.



Calculate, and show in a graph, the distribution of the maximum shear stress close to a crack tip loaded in Mode I. The material is linearly elastic with Poisson's ratio $\nu = 0.3$.

The stress components close to the crack tip are

$$\sigma_{xx}=\frac{K_\mathrm{I}}{\sqrt{2\pi r}}\left[\cos\frac{\Theta}{2}\left(1-\sin\frac{\Theta}{2}\sin\frac{3\Theta}{2}\right)\right]$$

$$\sigma_{yy}=\frac{K_\mathrm{I}}{\sqrt{2\pi r}}\left[\cos\frac{\Theta}{2}\left(1+\sin\frac{\Theta}{2}\sin\frac{3\Theta}{2}\right)\right]$$

$$\tau_{xy}=\frac{K_\mathrm{I}}{\sqrt{2\pi r}}\left[\cos\frac{\Theta}{2}\sin\frac{\Theta}{2}\cos\frac{3\Theta}{2}\right]$$

$$\tau_{yz}=\tau_{zx}=0$$

and, at plane deformation, $\sigma_{zz}=\nu\,(\sigma_{xx}+\sigma_{yy})$

at plane stress $\sigma_{zz}=0$

Solution:

Determine the maximum shear stress by use of the Tresca criterion. The principal stresses are then needed. The maximum shear stress is

$$\tau_{\max} = \frac{1}{2}(\sigma^{\text{prs}}_{\max} - \sigma^{\text{prs}}_{\min}) \tag{a}$$

where superscript "prs" stands for "principal stress".

In this problem we have a plane state (either plane stress or plane deformation) in the *xy* plane. Therefore, the *z* direction is one principal direction and the stress component σ_{zz} is one of the principal stresses.

In case of plane stress one has the principal stress in the *z* direction

$$\sigma_{zz} = 0 \tag{b}$$

and in the case of plane deformation (plane strain) one obtains the principal stress

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) = 2\nu \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}} \cos\frac{\Theta}{2} \tag{c}$$

Next, determine the two other principal stresses, called σ_1 and σ_2. They are

$$\begin{aligned}
\sigma_{1,2} &= \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \\
&= \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}} \left\{ \cos\frac{\Theta}{2} \pm \sqrt{\left(\cos\frac{\Theta}{2}\sin\frac{\Theta}{2}\sin\frac{3\Theta}{2}\right)^2 + \cos^2\frac{\Theta}{2}\sin^2\frac{\Theta}{2}\cos^2\frac{3\Theta}{2}} \right\} \\
&= \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}} \left\{ \cos\frac{\Theta}{2} \pm \sqrt{\cos^2\frac{\Theta}{2}\sin^2\frac{\Theta}{2}\left(\sin^2\frac{3\Theta}{2} + \cos^2\frac{3\Theta}{2}\right)} \right\} \\
&= \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}} \cos\frac{\Theta}{2}\left(1 \pm \sin\frac{\Theta}{2}\right)
\end{aligned} \tag{d}$$

The principal stresses σ_1, σ_2, and σ_3 thus become, with $\nu = 0.3$,

$$\sigma_1 = \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}} \cos\frac{\Theta}{2}\left(1+\sin\frac{\Theta}{2}\right) \tag{e}$$

$$\sigma_2 = \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}} \cos\frac{\Theta}{2}\left(1-\sin\frac{\Theta}{2}\right) \tag{f}$$

$$\sigma_3 = 0 \qquad \text{if plane stress} \tag{g}$$

$$\sigma_3 = 0.6\,\frac{K_{\mathrm{I}}}{\sqrt{2\pi r}} \cos\frac{\Theta}{2} \quad \text{if plane deformation} \tag{h}$$

Plane stress

Now the plane stress situation will be investigated. The largest principal stress then is $\sigma_{\max}^{\mathrm{prs}} = \sigma_1$ and the smallest principal stress is $\sigma_{\min}^{\mathrm{prs}} = \sigma_3$ = 0. The maximum shear stress then becomes

$$\tau_{\max} = \frac{\sigma_1}{2} = \frac{1}{2}\,\frac{K_{\mathrm{I}}}{\sqrt{2\pi r}} \cos\frac{\Theta}{2}\left(1+\sin\frac{\Theta}{2}\right) \tag{i}$$

It is seen that $\tau_{\max}$ is a function of Θ. Find the value of Θ that gives the largest $\tau_{\max}$. By solving

$$\frac{\mathrm{d}\,\tau_{\max}}{\mathrm{d}\,\Theta} = 0 \tag{j}$$

one finds that $\tau_{\max}$ has its largest value when $\Theta = \pi/3$ = 60 degrees = 60$^{\mathrm{o}}$ (and the smallest value of $\tau_{\max}$ occurs when $\Theta = \pm\pi$ and it is then $\tau_{\max}$ = 0).

Thus, at plane stress the maximum shear stress will be found in the direction Θ = 60 degrees from the *x* axis. One has

$$\tau_{\max}(60^{\mathrm{o}}) = 0.65\,\frac{K_{\mathrm{I}}}{\sqrt{2\pi r}} \tag{k}$$

Plane deformation

At plane deformation one still has that stress σ_1 is the largest principal stress. But which one of σ_2 and σ_3 is the smallest? To investigate this, set σ_2 = σ_3, which gives

$$\text{if} \quad \Theta < 47^{\mathrm{o}} \quad \text{then} \quad \sigma_2 > \sigma_3$$

and

$$\text{if} \quad \Theta > 47^{\mathrm{o}} \quad \text{then} \quad \sigma_2 < \sigma_3 \tag{l}$$

Assume that the maximum value of τ_{max} will be obtained for $\Theta < 47^o$. Then τ_{max} will be given by the principal stresses σ_1 and σ_3. One obtains

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3)$$

$$= \frac{1}{2}\frac{K_I}{\sqrt{2\pi r}}\left(\cos\frac{\Theta}{2}\left(1+\sin\frac{\Theta}{2}\right) - 0.6\cos\frac{\Theta}{2}\right) \qquad \text{(m)}$$

Next, find the value of Θ that gives the largest value of τ_{max} in (m). By solving

$$\frac{d\,\tau_{max}}{d\,\Theta} = 0 \qquad \text{(k)}$$

one finds that τ_{max} has its largest value when Θ = 75.8 degrees. But this is *not* in agreement with the assumption made, namely that Θ should be smaller than 47^o. Thus, we have to repeat these calculations once again with a new assumption:

Assume that the maximum value of τ_{max} will be obtained for $\Theta > 47^o$. Then σ_2 is the smallest principal stress, and one obtains

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}\frac{K_I}{\sqrt{2\pi r}}\cos\frac{\Theta}{2}\cdot 2\cdot\sin\frac{\Theta}{2} \qquad \text{(n)}$$

Again, determine the value of Θ that gives the largest value of τ_{max}. By solving $d\,\tau_{max} / d\,\Theta = 0$ one finds that τ_{max} has its largest value when $\Theta = \pi/2$ = 90 degrees. This is now in agreement with the assumption that Θ should be larger than 47^o. It is then concluded that

$$\tau_{max}(90^o) = 0.5\frac{K_I}{\sqrt{2\pi r}} \qquad \text{(o)}$$

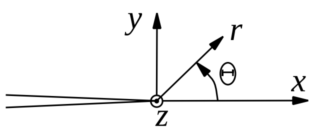
Answer: The maximum shear stress is

at plane stress: $\tau_{max}(60^o) = 0.65\frac{K_I}{\sqrt{2\pi r}}$

at plane deformation: $\tau_{max}(90^o) = 0.5\frac{K_I}{\sqrt{2\pi r}}$

Relations (i) and (n) give the graphs asked for (not shown here).

2/6.



The stress components in front of a crack tip loaded in Mode III are

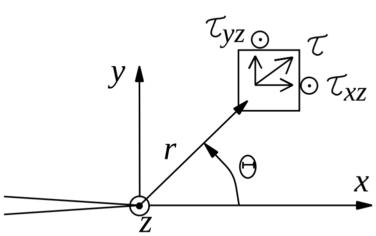
$$\tau_{xz} = \frac{-K_{\mathrm{III}}}{\sqrt{2\pi r}} \sin\frac{\Theta}{2}$$

$$\tau_{yz} = \frac{K_{\mathrm{III}}}{\sqrt{2\pi r}} \cos\frac{\Theta}{2}$$

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \tau_{xy} = 0$$

In which direction Θ will the shear stress have its maximum?

Solution:



The shear stress τ on a surface in the *xy*-plane is

$$\tau = \sqrt{\tau_{xz}^2 + \tau_{yz}^2} = \frac{K_{\mathrm{III}}}{\sqrt{2\pi r}} \sqrt{\sin^2\frac{\Theta}{2} + \cos^2\frac{\Theta}{2}}$$

$$= \frac{K_{\mathrm{III}}}{\sqrt{2\pi r}} = \tau_{\max} \qquad \text{(a)}$$

Thus, the shear stress τ (= $\tau_{\max}$) is the same in all directions (thus independent of the direction Θ).

Answer: The shear stress τ is the same in all directions.

Stress intensity factor

2/7.

Study the following crack geometries (a) to (f) and determine the characteristic lengths a of the cracks so that the probability of crack growth is the same for all cracks. All cracks are loaded in Mode I with a remote stress σ_0. The material is linear elastic.

(a) Central crack of length $2a$ in a large plate.

(b) Central crack of length $2a$ in a quadratical plate with side length $4a$.

(c) Edge crack of length a in a large plate.

(d) Two opposite edge cracks of length a in a rectangular plate with side length (width) $2W = 4a$ and height $h >> W$.

(e) Half-elliptical surface crack, a deep and $4a$ (= $2c$) long, in a thick plate.

(f) Elliptical crack with principal axes a and $2a$ embedded in a thick plate.

Solution:

Handbook formulae give stress intensity factors K_I (functions f_i refer to Appendix 3 in the textbook)

(a)
$$K_\mathrm{I} = \sigma_0 \sqrt{\pi a}\, f_1\left(\frac{a}{W}, \frac{h}{W}\right) \quad \text{where} \quad f_1 = 1.0 \quad \text{gives} \quad a = \frac{1}{\pi}\left(\frac{K_\mathrm{I}}{\sigma_0}\right)^2 = 0.318\left(\frac{K_\mathrm{I}}{\sigma_0}\right)^2$$

(b)
$$K_\mathrm{I} = \sigma_0 \sqrt{\pi a}\, f_1\left(\frac{a}{W}, \frac{h}{W}\right) \quad \text{where} \quad f_1 = 1.34 \quad \text{gives} \quad a = 0.177\left(\frac{K_\mathrm{I}}{\sigma_0}\right)^2$$

(c)
$$K_\mathrm{I} = \sigma_0 \sqrt{\pi a}\, f_5\left(\frac{a}{W}\right) \quad \text{where} \quad f_5 = 1.12 \quad \text{gives} \quad a = 0.254\left(\frac{K_\mathrm{I}}{\sigma_0}\right)^2$$

(d)
$$K_\mathrm{I} = \sigma_0 \sqrt{\pi a}\, f_4\left(\frac{a}{W}\right) \quad \text{where} \quad f_4 = 1.17 \quad \text{gives} \quad a = 0.233\left(\frac{K_\mathrm{I}}{\sigma_0}\right)^2$$

(e)
$$K_\mathrm{I} = \sigma_0 \sqrt{\pi a}\, f_7\left(\frac{a}{c}\right) \quad \text{where} \quad f_7 = 0.896 \quad \text{gives} \quad a = 0.397\left(\frac{K_\mathrm{I}}{\sigma_0}\right)^2$$

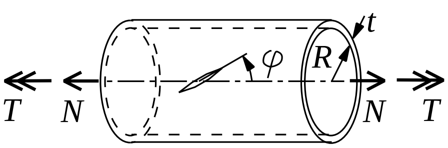
(f)
$$K_\mathrm{I} = \sigma_0 \sqrt{\pi a}\, f_8\left(\frac{a}{c}\right) \quad \text{where} \quad f_8 = 0.826 \quad \text{gives} \quad a = 0.467\left(\frac{K_\mathrm{I}}{\sigma_0}\right)^2$$

Answer: Ratio $a/(K_\mathrm{I}/\sigma_0)^2$ is

(a) 0.318; (b) 0.177; (c) 0.254; (d) 0.233; (e) 0.397; (f) 0.467.

Fracture modes

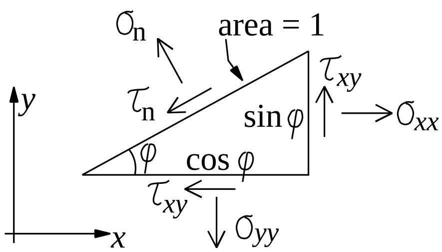
2/8.



A thin-walled circular cylinder contains a crack. The plane of the crack is inclined an angle ϕ (= φ) with respect to the longitudinal axis of the cylinder, see figure. The cylinder is loaded by a torque *T* and an axial tensile force *N*.

Determine a relationship between *T*, *N* and φ so that the crack is loaded in Mode I only, and in Mode II only. Assume that *R* >> *t* so that the curvature of the cylinder may be neglected when the stress intensity factor is determined.

Solution:



The axial tensile force *N* gives the normal stress σ_{xx} in the longitudinal direction, and the torque *T* gives shear stress τ_{xy}, where

$$\sigma_{xx} = \frac{N}{2\pi R t} \quad \text{and} \quad \tau_{xy} = \frac{-T}{2\pi R^2 t} \qquad \text{(a,b)}$$

The stress σ_{yy} is zero ($\sigma_{yy} = 0$).

Equilibrium equations in the directions of σ_n and τ_n give (in the equations below the area is written after the multiplication sign $\times$, *i.e.* the area equals 1 for the largest area and $\sin\phi$ and $\cos\phi$, respectively, for the two smaller areas)

$$\sigma_n \times 1 = \sigma_{xx} \sin\phi \times \sin\phi + 0 - \tau_{xy} \sin\phi \times \cos\phi - \tau_{xy} \cos\phi \times \sin\phi$$

$$= \sigma_{xx} \sin^2\phi - 2\tau_{xy} \cos\phi \sin\phi = \sin\phi \left[\frac{N}{2\pi R t} \sin\phi + \frac{2T}{2\pi R^2 t} \cos\phi \right] \qquad \text{(c)}$$

and

$$\tau_n \times 1 = \sigma_x \sin\phi \times \cos\phi - 0 + \tau_{xy} \sin\phi \times \sin\phi - \tau_{xy} \cos\phi \times \cos\phi$$

$$= \frac{N}{4\pi R t} \sin 2\phi + \frac{T}{2\pi R^2 t} \cos 2\phi \qquad \text{(d)}$$

The loading of the crack is in Mode I if $\sigma_n \neq 0$ and $\tau_n = 0$, which gives

$$\tau_n = 0 = \frac{N}{4\pi R t} \sin 2\phi + \frac{T}{2\pi R^2 t} \cos 2\phi \quad \text{giving} \quad \frac{T}{N} = -\frac{R}{2} \tan 2\phi \qquad \text{(e)}$$

Loading of the crack is in Mode II if $\sigma_n = 0$ and $\tau_n \neq 0$, which gives, either

$$\sigma_n = 0 = \sin\phi \quad \text{giving} \quad \phi = 0 \tag{f}$$

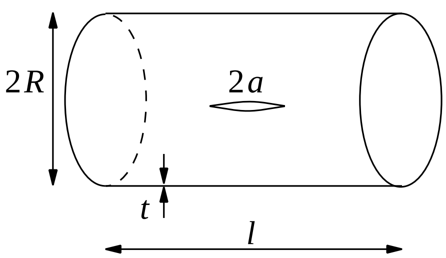
or

$$\sigma_n = 0 = \sin\phi\left[\frac{N}{2\pi R t}\sin\phi + \frac{2T}{2\pi R^2 t}\cos\phi\right] \quad \text{giving} \quad \frac{T}{N} = -\frac{R}{2}\tan\phi \tag{g}$$

Answer: Load ratio $T/N = (-R/2)\tan 2\phi$ gives Mode I loading of the crack, and ratio $T/N = (-R/2)\tan\phi$ gives Mode II loading.

Linear elastic fracture mechanics (LEFM)

2/9.



In a thin-walled pressure vessel there is a risk that a 300 mm (= 2a) long longitudinal through-thickness crack will develop almost immediately as the pressure vessel is loaded (due to a bad welding, for example). In an approximate estimation of the stress intensity factor the curvature of the pressure vessel wall was disregarded. The stress intensity factor K_I was then determined for a crack in a plane, infinitely large plate, and it was found that the stress intensity factor so determined was only half of the fracture toughness K_{Ic} of the material.

What would a more accurate analysis give? Could the pressure vessel be used without danger? The vessel is manufactured of steel, and it is loaded with a pressure p. The conditions for linear elastic fracture mechanics are assumed to be fulfilled.

Numerical data: pressure vessel radius is R = 0.4 m, overall length of pressure vessel is l = 5 m, crack length is 2a, where a = 0.15 m, and the wall thickness is t = 0.015 m.

Solution:

The stresses in the wall are

$$\sigma_{rr} = 0, \qquad \sigma_{\Theta\Theta} = \frac{pR}{t}, \qquad \sigma_{zz} = \frac{pR}{2t}, \quad \text{and all } \tau_{ij} = 0 \quad (i \neq j) \tag{a}$$

The requirements for linear elastic fracture mechanics (LEFM) to be valid are assumed to be fulfilled.

Case A. Study a crack in a large flat plate. The stress intensity factor then is

$$K_\mathrm{I} = \sigma_0 \sqrt{\pi a}\, f_1(0,1) = \sigma_0 \sqrt{\pi a} \cdot 1 = \frac{pR}{t}\sqrt{\pi a} \tag{b}$$

But $$K_\mathrm{I} = \frac{K_\mathrm{Ic}}{2} \quad \text{gives} \quad K_\mathrm{Ic} = 2\frac{pR}{t}\sqrt{\pi a} \tag{c}$$

The fracture toughness K_Ic of the material has now been determined.

Case B. Study a crack in a cylindrical wall. The stress intensity factor will now be obtained from a handbook (Case 14 in Appendix 3 of the textbook). The following conditions must be fulfilled:
$R >> t$, $a >> t$ and $L >> R$. They are all fulfilled.

Determine λ. One has

$$\lambda = \frac{a}{\sqrt{Rt}} = \frac{0.15}{\sqrt{0.4 \cdot 0.015}} = 1.936 \tag{d}$$

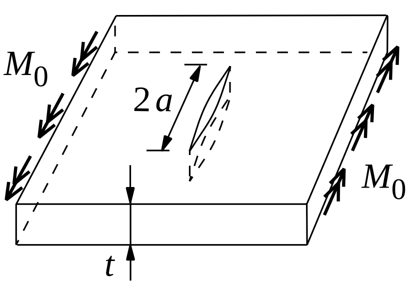
This gives $f_{14}(1.936) = 2.343$. Thus

$$K_\mathrm{I} = 2.343\frac{pR}{t}\sqrt{\pi a} \tag{e}$$

which is larger than the fracture toughness K_Ic and *fracture will be expected*!

Answer: Exact analysis (taking curvature into account) gives $K_\mathrm{I} \approx 1.17 K_\mathrm{Ic}$, implying that failure will occur.

2/10.



A through-thickness crack of length $2a$ has been found in a large plate. The plate is subjected to a bending moment M_0 (Nm/m) per unit length. Determine at which moment $M_{0\max}$ crack growth will occur.

Numerical data: a = 0.02 m (crack length $2a$), plate thickness t = 0.03 m, yield strength σ_Y = 1300 MPa, and fracture toughness K_Ic = 110 MN/m$^{3/2}$.

Solution:

Case 13 in the Appendix 3 of the textbook gives

$$K_I = \sigma_0 \sqrt{\pi a}\, f_{13}\left(\frac{a}{W}\right) \quad \text{where} \quad \sigma_0 = \frac{6M_0}{t^2} \quad \text{and} \quad M_0 \text{ in Nm/m} \tag{a}$$

Ratio $a / W = 0$ gives $f_{13} = 1$. Thus

$$K_I = \frac{6M_0}{t^2}\sqrt{\pi a} \tag{b}$$

Crack propagation is expected when $K_I = K_{Ic}$ if linear elastic fracture mechanics (LEFM) can be used. One has

$$2.5\left(\frac{K_{Ic}}{\sigma_Y}\right)^2 = 2.5\left(\frac{110}{1300}\right)^2 = 0.0179 \text{ m} \tag{c}$$

which is smaller than a, t and $W - a$. Thus LEFM can be used.

The fracture criterion $K_I = K_{Ic}$ gives

$$\frac{6M_0}{t^2}\sqrt{\pi a} = K_{Ic} \tag{d}$$

from which M_0 is solved. One obtains

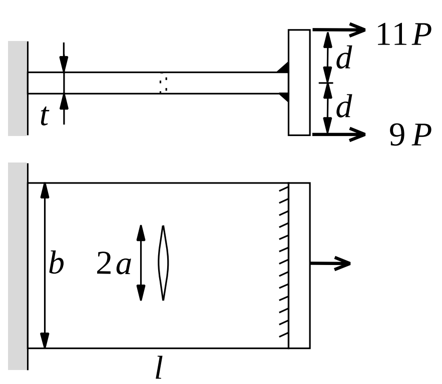
$$M_0 = \frac{K_{Ic}\, t^2}{6\sqrt{\pi a}} = \frac{110 \cdot 0.03^2}{6\sqrt{\pi\, 0.02}} = 0.0658 \ \frac{\text{MNm}}{\text{m}} \tag{e}$$

This value of the moment M_0 gives the stress σ in the plate. One obtains

$$\sigma = \frac{6M_0}{t^2} = 439 \text{ MPa} = 0.34\sigma_Y \tag{f}$$

Answer: Crack growth is expected when bending moment of the plate is M_{0max} = 66 kNm/m, giving maximum remote stress $\sigma = 0.34\,\sigma_Y$ in the plate.

2/11.



A cantilever beam contains a through-thickness crack. The beam is loaded by a system of forces (11P and 9P) according to the figure. Determine at which value of P failure may be expected. Numerical data: a = 0.02 m (crack length 2a), beam (plate) thickness t = 0.03 m, plate width b = 0.08 m, d = 0.06 m, l = 3.2 m, yield strength σ_Y = 600 MPa, and fracture toughness K_{Ic} = 50 MN/m$^{3/2}$.

Solution:

The cantilever beam (or plate) will be loaded in tension and in bending. The axial force becomes N = 20P loading the plate in tension, and the bending moment becomes M = 2Pd.

The stress intensity factor K_{I} is obtained by superposition of the two loading cases. One obtains (Cases 1 and 13 in Appendix 3 of the textbook)

$$K_{\mathrm{I}} = K_{\mathrm{I}}^{N} + K_{\mathrm{I}}^{M} = \frac{N}{bt}\sqrt{\pi a}\, f_1\left(\frac{a}{W}, \frac{h}{W}\right) + \frac{6M}{b\,t^2}\sqrt{\pi a}\, f_{13}\left(\frac{a}{W}\right)$$

$$= \frac{\sqrt{\pi a}}{bt}\left\{20P\ f_1\left(\frac{0.02}{0.04}, \frac{1.6}{0.04}\right) + \frac{12Pd}{t} f_{13}\left(\frac{0.02}{0.04}\right)\right\}$$

$$= \frac{P\sqrt{\pi a}}{bt}\left(20 \cdot 1.186 + 12\,\frac{d}{t}\,1.162\right)$$

$$= (23.72 + 27.89)\,\frac{P\sqrt{\pi a}}{bt} = 51.6\,\frac{P\sqrt{\pi a}}{bt} \qquad \text{(a)}$$

If the conditions for linear elastic fracture mechanics, LEFM, at plane strain are fulfilled, then failure will occur when

$$K_{\mathrm{I}} = K_{\mathrm{Ic}} \qquad \text{(b)}$$

Can LEFM be used?

$$2.5\left(\frac{K_{\mathrm{Ic}}}{\sigma_{\mathrm{Y}}}\right)^2 = 2.5\left(\frac{50}{600}\right)^2 = 0.0174\ \mathrm{m} \qquad \text{(c)}$$

This number (0.0174 m) is less than the crack (half-)length a, plate thickness t, and the distance $b/2 - a$ from the crack tip to the edge of the plate, so LEFM can used.

Thus,

$$K_{\mathrm{I}} = 51.6\,\frac{P\sqrt{\pi a}}{bt} = K_{\mathrm{Ic}} = 50\ \mathrm{MN/m^{3/2}} \qquad \text{(d)}$$

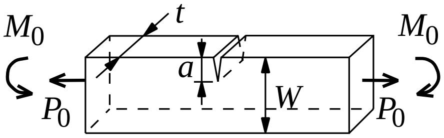
giving

$$P = P_{\mathrm{crit}} = \frac{K_{\mathrm{Ic}}\,bt}{51.6\sqrt{\pi a}} = \frac{50 \cdot 0.08 \cdot 0.03}{51.6\sqrt{\pi\, 0.02}} = 0.00928\ \mathrm{MN} = 9300\ \mathrm{N} \qquad \text{(e)}$$

Failure is expected at P = P_{crit} = 9.3 kN.

Answer: Failure is expected at critical load P_{crit} = 9.3 kN

2/12.



An edge crack has been discovered in a long beam, see figure. The fracture toughness K_{Ic} and the yield strength σ_Y of the material depend on temperature as given below.

(a) Over which temperature range may linear elastic fracture mechanics be used?

(b) For the upper limit of this temperature range, determine at which loadings P_0 and M_0 fracture may be expected.

Numerical data: crack length a = 0.1 m, beam height W = 0.5 m, beam width t = 0.2 m.

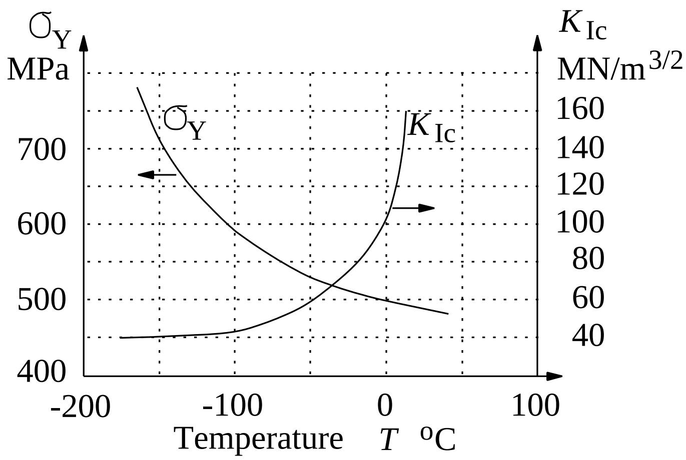


Figure. *Fracture toughness K_{Ic} and yield strength σ_Y as function of temperature T*

Solution:

(a) First, investigate if linear elastic fracture mechanics, LEFM, can be used. One has

$$2.5\left(\frac{K_{Ic}}{\sigma_Y}\right)^2 \le \text{minimum of } (a, t, W-a) = a$$

which gives
$$\left(\frac{K_{Ic}}{\sigma_Y}\right) \le \sqrt{\frac{a}{2.5}} = 0.2 \ \ (\sqrt{\text{m}}) \qquad \text{(a)}$$

Thus, numerically, one should have K_{Ic} < $0.2\sigma_Y$. This condition is fulfilled if temperature T is $T \le -3$°C, see diagram (at $T = -3$°C, approximately, one obtains K_{Ic} = 100 MN/m$^{3/2}$ and σ_Y = 500 MPa).

(b) Superposition of the two loading cases tension and bending gives (Case 5 and 6 in the Appendix 3 of the textbook)

$$K_{\mathrm{I}} = K_{\mathrm{I}}^{P} + K_{\mathrm{I}}^{M} = \frac{P_0}{bt}\sqrt{\pi a}\, f_5\left(\frac{a}{W}\right) + \frac{6M_0}{t\,W^2}\sqrt{\pi a}\, f_6\left(\frac{a}{W}\right)$$

$$= \frac{\sqrt{\pi a}}{tW}\left\{ P_0\, f_5\left(\frac{0.1}{0.5}\right) + \frac{6M_0}{W} f_6\left(\frac{0.1}{0.5}\right)\right\}$$

$$= \frac{\sqrt{\pi a}}{tW}(1.366P_0 + 12.902M_0) \qquad \text{(b)}$$

Fracture will occur when

$$K_{\mathrm{I}} = K_{\mathrm{Ic}} \qquad \text{(c)}$$

At temperature $T = -3^{\mathrm{o}}\mathrm{C}$ the fracture toughness K_{Ic} is K_{Ic} = 100 MN/m$^{3/2}$. Thus, at this temperature fracture will occur when

$$K_{\mathrm{I}} = \frac{\sqrt{\pi a}}{tW}(1.37P_0 + 12.90M_0) = K_{\mathrm{Ic}} \qquad \text{(d)}$$

giving

$$(1.37P_0 + 12.90M_0) = \frac{K_{\mathrm{Ic}}\, tW}{\sqrt{\pi a}} = \frac{100\cdot 0.2\cdot 0.5}{\sqrt{\pi\, 0.1}} = 17.84 \ \mathrm{MN} \qquad \text{(e)}$$

with P_0 in MN and M_0 in MNm.

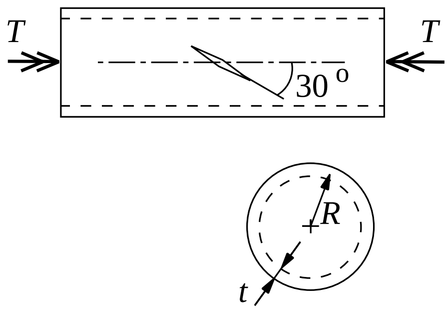
Thus, at fracture one has

$$P_{0\mathrm{c}} + 9.44M_{0\mathrm{c}} = 13.1 \ \mathrm{MN} \qquad \text{(f)}$$

where P_0 should be entered in MN and M_0 in MNm.

Answer: Temperature T should be $T \le -3^{\mathrm{o}}\mathrm{C}$. At temperature $T = -3^{\mathrm{o}}\mathrm{C}$ the yield strength is σ_{Y} = 500 MPa and the fracture toughness is K_{Ic} = 100 MN/m$^{3/2}$ giving the maximum load P_0 + 9.44M_0 = 13.1 MN, where P_0 should be entered in MN and M_0 in MNm.

2/13.



A thin-walled circular cylinder contains a crack forming an angle 30° to the longitudinal axis of the cylinder, see figure. Determine at which torque T crack propagation will be expected. In this case, the curvature of the wall may be neglected when the stress intensity factor is estimated (*i.e.*, the radius R is very large compared to crack length a and wall thickness t).

The equivalent stress intensity factor K_e is

$$K_e^2 = K_I^2 + K_{II}^2 + \frac{4}{\kappa + 1} K_{III}^2$$

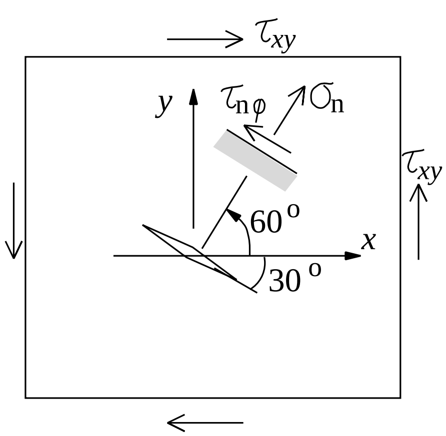
Numerical data: radius R = 0.2 m, crack (half-)length a and wall thickness t are a = t = 0.005 m, yield limit σ_Y = 1200 MPa, and fracture toughness K_{Ic} = 50 MN/m$^{3/2}$. (For a small crack in a large plate the stress intensity factor in Mode II is $K_{II} = \tau_{xy\infty}\sqrt{\pi a}\, g$, where (here) g = 1.)

Solution:

Can linear elastic fracture mechanics, LEFM, be used? One has

$$2.5\left(\frac{K_{Ic}}{\sigma_Y}\right)^2 = 2.5\left(\frac{50}{1200}\right)^2 = 0.00434 \text{ m} \tag{a}$$

Here a = t = 0.005 m > 0.00434 m (and $W - a$ is larger), so LEFM can be used.



Consider an element surrounding the crack, see figure. The stress on the element is

$$\tau_{xy} = \frac{T}{W_v} = \frac{T}{2\pi R^2 t} \tag{b}$$

where W_v is the section modulus in torsion of the cylinder.

The stress τ_{xy} gives a normal stress σ_n across the crack, loading the crack in Mode I, and a shear stress $\tau_{n\phi}$, loading the crack in Mode II.

The stresses σ_n and $\tau_{n\phi}$ become, with $\phi = 60^o$,

$$\sigma_n = \sigma_{xx} \cos 2\phi + \sigma_{yy} \sin^2 \phi + 2\tau_{xy} \cos\phi \sin\phi = \frac{\sqrt{3}}{2} \tau_{xy} \tag{c}$$

$$\tau_{n\phi} = -(\sigma_{xx} - \sigma_{yy}) \sin\phi \cos\phi + \tau_{xy} (\cos^2\phi - \sin^2\phi) = -\frac{1}{2} \tau_{xy} \tag{d}$$

In Mode I the stress intensity factor K_I becomes (for a small crack in a large plate the factor f_1 becomes $f_1 = 1$)

$$K_I = \sigma_n \sqrt{\pi a}\, f_1 = \frac{\sqrt{3}}{2} \tau_{xy} \sqrt{\pi a} \tag{e}$$

In Mode II the stress intensity factor K_{II} becomes (for a small crack in a large plate the factor g becomes $g = 1$)

$$K_{II} = \tau_{n\phi} \sqrt{\pi a}\, g = \frac{1}{2} \tau_{xy} \sqrt{\pi a} \tag{f}$$

The equivalent stress intensity factor K_e becomes

$$K_e^2 = K_I^2 + K_{II}^2 = \tau_{xy}^2 \pi a \left(\frac{3}{4} + \frac{1}{4} \right) = \tau_{xy}^2 \pi a \tag{g}$$

Fracture is expected when $K_e = K_{Ic}$ (if LEFM can be used). This gives

$$K_e = \tau_{xy} \sqrt{\pi a} = \frac{T}{2\pi R^2 t} \sqrt{\pi a} = K_{Ic} \tag{g}$$

Thus,

$$T = \frac{K_{Ic}\, 2\pi R^2 t}{\sqrt{\pi a}} = \frac{50 \cdot 2\pi\, 0.2^2 \cdot 0.005}{\sqrt{\pi \cdot 0.005}} = 0.501 \text{ MNm} \tag{h}$$

At torque T = 500 kNm crack propagation is expected. (The LEFM conditions were checked above.)

Answer: Crack propagation is expected when torque T = 500 kNm.

Solutions to problems in
T Dahlberg and A Ekberg: Failure, Fracture, Fatigue - An Introduction.
Studentlitteratur, Lund 2002, ISBN 91-44-02096-1.

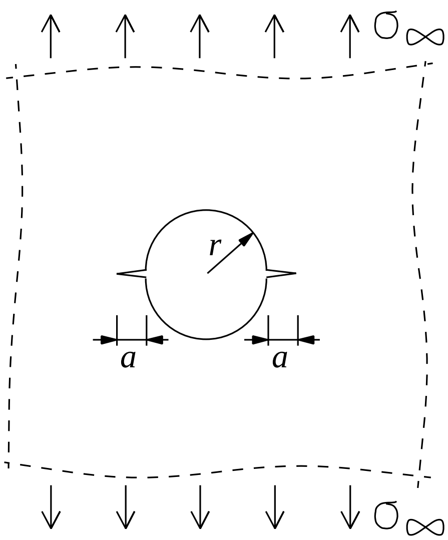
Chapter 3

Small plastic zone at crack tip

Problems with solutions

The Irwin correction

3/1.



At a circular hole (radius r = 10 mm) in a large plate two cracks (length a = 5 mm) have been discovered, see figure. Determine the maximum remote stress (the critical stress) the plate may be loaded with.

Data: plate thickness t = 20 mm, fracture toughness K_{Ic} = 40 MN/m$^{3/2}$, and yield limit σ_{Y} = 500 MPa.

Solution:

Can linear elastic fracture mechanics be used?

$$2.5\left(\frac{K_{\mathrm{Ic}}}{\sigma_{\mathrm{Y}}}\right)^2 = 2.5\left(\frac{40}{500}\right)^2 = 0.016 \text{ m} \tag{a}$$

The plate thickness t is large enough to ensure plane strain, but the crack length a is too short. Therefore, use the Irwin correction of the crack length. For plane strain (t is large enough), the correction at critical stress, *i.e.* when K_{I} = K_{Ic}, is

$$r_1 = \frac{1}{6\pi}\left(\frac{K_{\mathrm{Ic}}}{\sigma_{\mathrm{Y}}}\right)^2 = \frac{1}{6\pi}\left(\frac{40}{500}\right)^2 = 0.00034 \text{ m} \tag{b}$$

Correction of LEFM now gives

$$K_{\mathrm{I}} = \sigma_{\infty}\sqrt{\pi\,(a + r_1)}\cdot f_3\left(\frac{a + r_1}{r + a + r_1}\right) = \sigma_{\infty}\sqrt{\pi\,0.00534}\cdot 1.78$$

$$= K_{\mathrm{Ic}} = 40\cdot 10^6\ \mathrm{N/m}^{3/2} \tag{c}$$

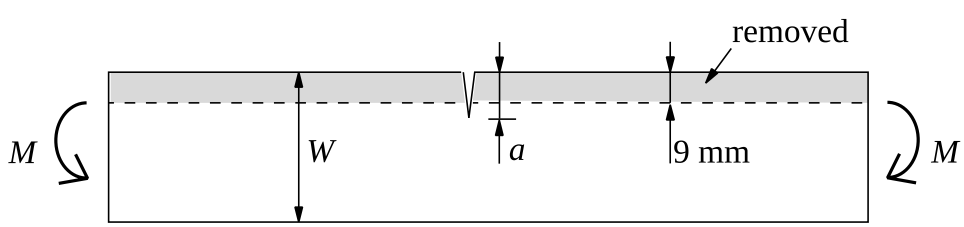
from which $\sigma_{\infty} = \sigma_{\mathrm{crit}} = 173$ MPa is solved.

Answer: The critical remote stress σ_{∞} is, approximately, $\sigma_{\infty} = \sigma_{\mathrm{crit}} = 173$ MPa.

3/2.

A beam with rectangular cross section, t = 20 mm and W = 40 mm, is loaded with a bending moment M as shown in the figure. A 12 mm deep crack (a = 12 mm) has appeared in the beam (see figure). In order to increase the load-carrying capacity of the cracked beam, the material around the crack is removed. It is decided that 9 mm of the beam height should be removed by grinding. After the grinding the beam height thus is 31 mm. The crack, however, is still there, and its depth is now 3 mm.

Material properties: K_{Ic} = 40 MN/m$^{3/2}$ and σ_{Y} = 900 MPa.



(a) How much has the ultimate bending moment been increased by the grinding? (Thus, calculate the ultimate load M of the beam before grinding and after the grinding.)

(b) The purpose of the grinding was perhaps to remove all the crack. What bending moment can be put on the beam if 12 mm were removed (*i.e.* even the crack tip has been removed)? In this case the yield limit of the material limits the loading.

Solution:

Can LEFM (linear elastic fracture mechanics) be used?

$$2.5\left(\frac{K_{\mathrm{Ic}}}{\sigma_{\mathrm{s}}}\right)^2 = 2.5\left(\frac{40}{900}\right)^2 = 4.94\cdot 10^{-3}\ \mathrm{m} = 4.94\ \mathrm{mm} \qquad \text{(a)}$$

This length, 4.94 mm, is less than t (= 20 mm), a (= 12 mm), and $W - a$ (= 28 mm), respectively. Thus, it is OK to use LEFM for the first crack length (12 mm) but not for the second (length 3 mm).

Determine the ultimate load before grinding, *i.e.* the ultimate bending moment for the beam with cross section 20 mm by 40 mm and crack depth 12 mm:
Case 6 in Appendix 3 of the textbook gives

$$K_{\mathrm{I}} = \frac{6M}{tW^2}\sqrt{\pi a}\, f_6\left(\frac{a}{W}\right) = \frac{6M}{0.020\cdot 0.040^2}\sqrt{\pi\, 0.012}\, f_6\left(\frac{0.012}{0.040}\right)$$

$$= K_{\mathrm{Ic}} = 40\cdot 10^6\ \mathrm{N/m^{3/2}} \qquad \text{(b)}$$

Diagram gives f_6 = 1.115, and one obtains the ultimate load $M = M_1$ = 985 Nm.

After the grinding the cross section becomes 20 mm by 31 mm, and crack depth is 3 mm. LEFM can not be used any longer. Use the Irwin correction of the crack length. One obtains for plane strain, when the stress intensity factor reaches K_{Ic},

$$r_1 = \frac{1}{6\pi}\left(\frac{K_{\mathrm{Ic}}}{\sigma_{\mathrm{Y}}}\right)^2 = 0.000105\ \mathrm{m} \qquad \text{(c)}$$

Case 6 gives

$$K_{\mathrm{Ieff}} = \frac{6M}{tW^2}\sqrt{\pi a_{\mathrm{eff}}}\, f_6\left(\frac{a_{\mathrm{eff}}}{W}\right) = \frac{6M}{0.020\cdot 0.031^2}\sqrt{\pi\, 0.003105}\, f_6\left(\frac{0.003105}{0.031}\right)$$

$$= K_{\mathrm{Ic}} = 40\cdot 10^6\cdot \mathrm{N/m^{3/2}} \qquad \text{(d)}$$

Diagram gives f_6 = 1.057, giving $M = M_2$ = 1227 Nm.

(b) If all the material surrounding the crack had been removed, *i.e.* if 12 mm of the beam height were taken away (this was perhaps the original purpose of the grinding operation), then fracture mechanics theory needs not be used at all. In that case, determine the bending moment the beam can be loaded with, if plastic deformation of the beam should be avoided. One obtains

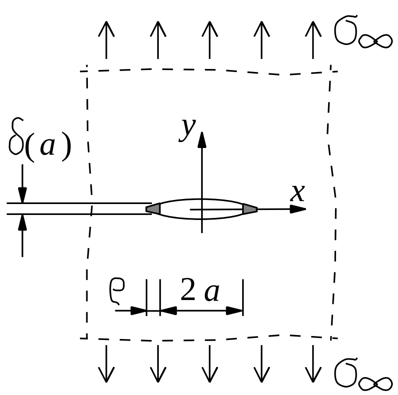
$$\sigma_{max} = \frac{M \cdot W/2}{I} = \frac{M \cdot 0.028/2}{0.020 \cdot 0.028^3/12} = \sigma_Y = 900 \text{ MPa} \qquad \text{(e)}$$

which gives M = 2352 Nm. (Conclusion: make sure that also the crack *tip* is removed, if you try to improve the load-carrying capacity of a structure by removing a crack.)

Answer: (a) The maximum bending moment M increases from 985 Nm to 1227 Nm, thus by 25 per cent. (b) The moment can be increased to M = 2352 Nm if the full crack (with crack tip included) is removed.

The Dugdale model

3/3.



A large steel plate of an elastic, ideally plastic material (σ_Y = 500 MPa) contains a through-thickness crack of total length 50 mm (= 2a). At a tension test of the plate the crack was found to start growing at the remote stress σ_∞ = 300 MPa.

(a) At which remote stress σ_∞ would a 150 mm long crack start to grow? Use the Dugdale model of crack tip opening displacement $\delta(a)$ as criterion for crack growth initiation (CTOD criterion). One has

$$\delta(a) = \frac{8\,a\,\sigma_Y}{\pi E} \ln\left\{\frac{1}{\cos\,(\pi\sigma_\infty/2\sigma_Y)}\right\}$$

(b) For the two cases 2a = 50 mm and 2a = 150 mm, calculate the size (length) ρ of the plastic zone at the crack tip when the stress is such that the crack starts to propagate.

Solution:

The first measurement, at crack length 2a = 50 mm, gives the critical value of the crack tip opening displacement (CTOD) $\delta(a)_{crit}$. One obtains

$$\delta(a)_{\rm crit} = \frac{8\,a_{\rm crit}\,\sigma_{\rm Y}}{\pi E} \ln\left\{\frac{1}{\cos\,(\pi\sigma_{\infty\,\rm crit}/2\sigma_{\rm Y})}\right\}$$

$$= \frac{8\cdot 0.025\,\sigma_{\rm Y}}{\pi E} \ln\left\{\frac{1}{\cos\,(\pi\cdot 300/2\cdot 500)}\right\} = 0.03382957\,\frac{\sigma_{\rm Y}}{E} \qquad \text{(a)}$$

(a) At which stress would a 150 mm long crack start to grow?
The CTOD criterion gives, with $\delta(a)_{\rm crit}$ just calculated in (a),

$$\delta(a)_{\rm crit} = 0.03382957\,\frac{\sigma_{\rm Y}}{E} = \frac{8\cdot 0.075\cdot\sigma_{\rm Y}}{\pi E} \ln\left\{\frac{1}{\cos\,(\pi\cdot\sigma_{\infty}/2\cdot 500)}\right\} \qquad \text{(b)}$$

Solving for σ_∞ gives σ_∞ = 184 MPa.

(b) The length of the plastic zone is obtained from

$$\rho = a\left(\frac{1}{\cos\,(\pi\,\sigma_{\infty}/2\,\sigma_{\rm Y})} - 1\right) \qquad \text{(c)}$$

For $2a$ = 50 mm, one obtains, with remote stress σ_∞ = 300 MPa,

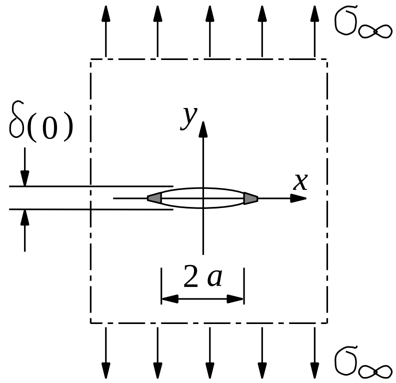
$$\rho = 0.025\left(\frac{1}{\cos\,(\pi\cdot 300/2\cdot 500)} - 1\right) = 0.0175325\ \text{m} = 17.5\ \text{mm} \qquad \text{(d)}$$

For $2a$ = 150 mm, one obtains, with remote stress σ_∞ = 184 MPa,

$$\rho = 0.075\left(\frac{1}{\cos\,(\pi\cdot 184/2\cdot 500)} - 1\right) = 0.0145492\ \text{m} = 14.5\ \text{mm} \qquad \text{(e)}$$

Answer: (a) At crack length 150 mm the critical stress is 184 MPa, (b) at crack length 50 mm the length of the plastic zone is 17.5 mm (for critical remote stress 300 MPa), and at crack length 150 mm the length of the plastic zone is 14.5 mm (for critical stress σ_∞ = 184 MPa).

3/4.



During a tension test of a large plate with a central crack the crack opening $\delta(0)$ was recorded as a function of the load. The recorded curve deviated from a straight line. The deviation from linearity may be explained partly by the appearance of a plastic zone at the crack tip and partly by crack growth during the loading.

(a) How large deviation from linearity will be expected due to the plastic zone at the crack tip? The remote stress σ_∞ is $\sigma_Y / 2$.

(b) The deviation from linearity of the recorded curve appeared to be 30 per cent. Calculate the crack propagation during the loading (assume that symmetry is maintained, *i.e.* the two crack tips move the same distance).

The material is linearly elastic, ideally plastic with yield strength σ_Y. Plane stress is at hand so that the Dugdale model may be used. The crack opening displacement (COD) is

$$\frac{\delta(0)}{a} = \frac{8\sigma_Y}{\pi E} \ln \frac{1 + \sin(\pi\sigma_\infty / 2\sigma_Y)}{\cos(\pi\sigma_\infty / 2\sigma_Y)}$$

and if $\sigma_\infty << \sigma_Y$ one obtains

$$\frac{\delta(0)}{a} = \frac{4K_I}{E\sqrt{\pi a}} \left\{ 1 + \frac{\pi^2}{24}\left(\frac{K_I}{\sigma_Y\sqrt{\pi a}}\right)^2 + \ldots.. \right\}$$

Solution:

(a) Using Dugdale's model, the crack opening displacement $\delta(0)$ becomes

$$\frac{\delta(0)}{a} = \frac{8\sigma_Y}{\pi E} \ln \frac{1 + \sin(\pi\sigma_\infty / 2\,\sigma_Y)}{\cos(\pi\sigma_\infty / 2\,\sigma_Y)}$$

$$= \frac{8 \cdot 2\sigma_\infty}{\pi E} \ln \frac{1 + \sin(\pi\sigma_\infty / 4\,\sigma_\infty)}{\cos(\pi\sigma_\infty / 4\sigma_\infty)} = 4.4888\,\frac{\sigma_\infty}{E} \qquad \text{(a)}$$

If $\sigma_\infty << \sigma_Y$ (*i.e.* if σ_Y is very large) one has

$$\frac{\delta(0)}{a} = \frac{4\,K_I}{E\,\sqrt{\pi a}}\left\{1+\frac{\pi^2}{24}\left(\frac{K_I}{\sigma_Y\,\sqrt{\pi\,a}}\right)^2 + \ldots..\right\} \qquad \text{(b)}$$

Using $K_I = \sigma_\infty\sqrt{\pi a}$ and $\sigma_\infty << \sigma_Y$ in (b), one obtains the linear elastic solution. (If σ_Y tends to infinity, no yielding will occur, and the solution obtained is for the fully elastic case.) This gives

$$\frac{\delta(0)}{a} = \frac{4\,\sigma_\infty\sqrt{\pi\,a}}{E\,\sqrt{\pi a}}\{1+0\} = 4\,\frac{\sigma_\infty}{E} \qquad \text{(c)}$$

The difference between the displacements in (a) and (c) comes from the plastic deformation at the crack tip. The deviation from linearity is

$$\frac{4.4888-4}{4} = 0.1222 = 12.2 \text{ per cent} \qquad \text{(d)}$$

(b) At the experiment, the deviation from linearity was 30 per cent. The plastic deformation at the crack tips gives 12.2 per cent deviation from linearity only, so the remaining deviation up to 30 per cent is explained by crack growth; *i.e.*, the crack has propagated during the loading of the structure.

Calculate the final crack length a_{final}. Using $\delta(0) = 1.3\,\delta_{\text{elastic}}$, where δ_{elastic} was calculated in (c), the Dugdale model gives

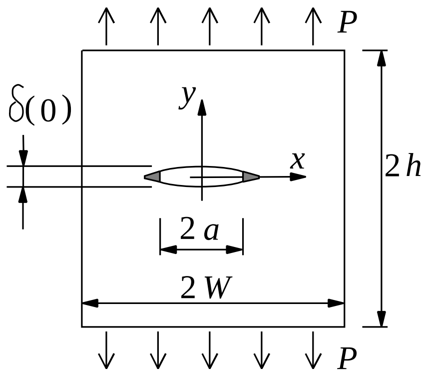
$$\delta(0) = 1.3\cdot 4\,\frac{\sigma_\infty\,a}{E} = \frac{8\cdot 2\sigma_\infty\,a_{\text{final}}}{\pi\,E}\ln\frac{1+\sin(\pi\,\sigma_\infty/4\,\sigma_\infty)}{\cos(\pi\,\sigma_\infty/4\,\sigma_\infty)} \qquad \text{(e)}$$

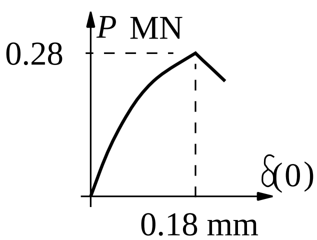
from which is solved

$$a_{\text{final}} = \frac{1.3\cdot 4}{4.4888}\,a = 1.158\,a \qquad \text{(f)}$$

Answer: (a) Deviation from linearity due to plastic zones is 12.2 per cent, and (b), crack has grown to final length $a_{\text{final}} = 1.158a$ during the loading, where a is the original crack length in the unloaded structure.

3/5.





During a tension test of a large plate with a central crack the crack opening δ(0) was recorded as a function of the load *P*. The recorded curve deviated from a straight line as given in the figure. Symmetric stable crack growth was observed during the loading of the plate. Calculate the crack growth during the loading.

The material is linearly elastic, ideally plastic with yield strength σ_Y. The plate is thin so that the stress state is plane (PS). Thus, the Dugdale model may be used.
Numerical data: crack length a_0 = 0.01 m, plate thickness t = 0.002 m, plate size $h = W$ = 0.2 m, yield limit σ_Y = 600 MPa, and modulus of elasticity E = 200 GPa.

Solution:
At fracture the remote tensile stress is

$$\sigma_{\infty c} = \frac{P_c}{2Wt} \tag{a}$$

The crack opening displacement (COD) at fracture is

$$\delta_c(0) = \frac{8\,\sigma_Y\,a_c}{\pi E}\ln\frac{1+\sin(\pi\sigma_\infty/2\sigma_Y)}{\cos(\pi\sigma_\infty/2\sigma_Y)} \tag{b}$$

which gives

$$a_c = \pi E\,\delta_c(0)\cdot\left\{8\,\sigma_Y \ln\frac{1+\sin(\pi P_c/4Wt\,\sigma_Y)}{\cos(\pi P_c/4Wt\,\sigma_Y)}\right\}^{-1} \tag{c}$$

According to the diagram, fracture occurred when P_c = 0.28 MN and $\delta_c(0) = 0.18\cdot10^{-3}$ m, which gives

$$a_c = 0.0218 \text{ m, giving } 2a_c = 0.0436 \text{ m} \tag{d}$$

During the loading of the plate, the crack has grown the amount

$$2\Delta a = 2a_c - 2a_0 = 0.0436 - 0.0200 \text{ m} = 0.0236 \text{ m} = 23.6 \text{ mm.} \tag{e}$$

Answer: The crack growth is $2\Delta a$ = 23.6 mm (thus, the crack has propagated 11.8 mm at each crack tip).

Solutions to problems in
T Dahlberg and A Ekberg: Failure, Fracture, Fatigue - An Introduction
Studentlitteratur, Lund 2002, ISBN 91-44-02096-1

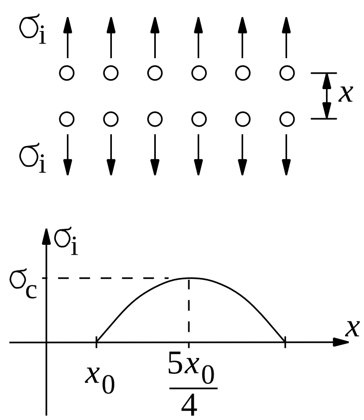
Chapter 4

Energy considerations

Problems with solutions

Surface energy, stresses at crack tip

4/1.



The surface energy w_s and the cohesive strength σ_c of a material may (approximately) be determined from the following simple model of the inter-atomic forces:
Assume that the force σ_i per unit area between atomic planes may be represented by the function

$$\sigma_i = \sigma_c \sin \alpha \pi \frac{x - x_0}{x_0}$$

where x_0 is the distance between the planes of the atoms in the unloaded state and x is the atomic separation when the material is loaded. The factor σ_c is a material parameter: the bond strength.
Also, assume that the fracture limit is obtained when the strain is 25 per cent, *i.e.* when $x = 5x_0/4$.

(a) Determine the parameters α, σ_c and w_s in terms of E and x_0, where E is the modulus of elasticity of the material.

(b) Compare the fracture criterion by Griffith (released energy equals surface energy) with a stress criterion for fracture of a material with a crack. Use a small through-thickness crack in a large plate and compare the remote stresses σ_∞ giving crack growth in the two cases.

For the stress criterion, assume that the mean stress over *n* planes of atoms in front of the crack tip is a measure of the loading of the material. Determine how far away in front of the crack tip (measured in number of atomic planes) the mean stress must be equal to the cohesive strength of the material in order to make the stress criterion equivalent to the energy criterion by Griffith. (The stress state may be considered as plane.)

Solution:

(a) At small displacements (*i.e.*, when x is close to x_0), Hooke' s law gives

$$\sigma_i = E\,\varepsilon = E\,\frac{x - x_0}{x_0} \tag{a}$$

which gives

$$\frac{d\,\sigma_i}{d\,x} = \frac{E}{x_0} \tag{b}$$

From the equation given in the problem formulation

$$\sigma_i = \sigma_c \sin \alpha\,\pi\,\frac{x - x_0}{x_0}$$

one obtains

$$\frac{d\,\sigma_i}{d\,x} = \sigma_c \frac{\alpha\,\pi}{x_0} \cos \alpha\,\pi\,\frac{x - x_0}{x_0} \tag{c}$$

Equations (c) and (b) give, for $x = x_0$,

$$\frac{d\sigma_i(x = x_0)}{dx} = \sigma_c \frac{\alpha\,\pi}{x_0} \cos 0 = \frac{E}{x_0} \qquad \text{giving} \qquad \sigma_c = \frac{E}{\alpha\,\pi} \tag{d}$$

From the problem formulation, see figure above, one has, when $x = 5x_0/4$,

$$\frac{d\,\sigma_i}{d\,x} = 0 \tag{e}$$

which gives

$$\frac{d\,\sigma_i(x = 5x_0/4)}{d\,x} = \sigma_c \frac{\alpha\,\pi}{x_0} \cos \alpha\,\pi\,\frac{5\,x_0/4 - x_0}{x_0} = 0 \tag{f}$$

Thus,

$$\cos \alpha\,\pi\,\frac{1}{4} = 0 \quad \text{which gives} \quad \alpha\,\pi\,\frac{1}{4} = \frac{\pi}{2} \quad \text{and} \quad \alpha = 2 \tag{g}$$

Enter $\alpha = 2$ into (d). It gives the cohesive stress (cohesive strength)

$$\sigma_c = \frac{E}{2\pi} \tag{h}$$

It is noted that this stress is very high: in the order of $E/10$. The ultimate strength σ_U of a very high-strength steel is in the order of $E/100$, which means that in practice, fracture will occur due to other mechanisms than separation of the atomic planes.

The surface energy may now be determined from the area below the curve $\sigma_i(x)$. One obtains

$$2w_s = \int_{x_0}^{3x_0/2} \sigma_i(x)\,dx = \int_{x_0}^{3x_0/2} \frac{E}{2\pi} \sin 2\pi \frac{x - x_0}{x_0}\,dx$$

$$= \frac{E}{2\pi}\left[-\frac{x_0}{2\pi} \cos 2\pi \frac{x - x_0}{x_0} \right]_{x_0}^{3x_0/2} = \frac{E\,x_0}{2\,\pi^2} \tag{i}$$

where the factor 2 in front of w_s is there because two new surfaces are created when the crack appears.

Thus

$$w_s = \frac{E\,x_0}{4\,\pi^2} \quad [\mathrm{Nm/m^2}] \tag{j}$$

(b) *The Griffith criterion*

According to Griffith, the energy that is released when a crack propagates should be equal to the energy needed to create the new crack surfaces. By use of the stress intensity factor K_I, this crack propagation criterion can be written, see equation (4.34) in the textbook,

$$\frac{\kappa + 1}{8\,\mu} K_I^2\, t \cdot \Delta a = 2w_s\, t \cdot \Delta a \tag{k}$$

where $\kappa = (3 - \nu)/(1 + \nu)$ at plane stress, and μ is the shear modulus of the material: $\mu = E/2(1 + \nu)$. The crack growth is Δa (the crack length a has increased by Δa) and the plate, in which the (through-thickness) crack is situated, has thickness t. Expression (k) is thus given for the new crack area $t\Delta a$ created. Solving (k) for K_I gives

$$K_I = \sqrt{\frac{2w_s\,8\mu}{\kappa + 1}} = \sqrt{\frac{2w_s\,8E}{2(1 + \nu)\left(\frac{3 - \nu}{1 + \nu} + 1\right)}} = \sqrt{2w_s\,E} \tag{l}$$

Thus, fracture will occur when

$$K_I = K_c, \quad \text{where} \quad K_c = \sqrt{2w_s\,E} \tag{m}$$

Now, determine the remote stress σ_∞^G (superscript G for Griffith) leading to failure. For a small through-thickness crack in a large plate, the stress intensity factor K_I is

$$K_I = \sigma_\infty^G \sqrt{\pi a} \tag{n}$$

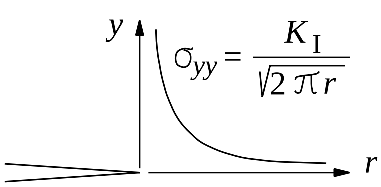
Crack propagation will occur (for plane stress conditions) when $K_I = K_c$, giving

$$K_I = \sigma_\infty^G \sqrt{\pi a} = K_c = \sqrt{2 w_s E} \tag{o}$$

Thus, according to the Griffith criterion the crack will propagate when the remote stress is

$$\sigma_\infty^G = \sqrt{\frac{2 w_s E}{\pi a}} = \sqrt{\frac{2E}{\pi a}\frac{E\, x_0}{4\pi^2}} = \frac{E}{\pi}\sqrt{\frac{x_0}{2\pi a}} \tag{p}$$

The stress criterion



We now turn to the stress criterion. The stress σ_{yy} in front of the crack tip is

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} = \frac{\sigma_\infty \sqrt{\pi a}}{\sqrt{2\pi r}} = \sigma_\infty \sqrt{\frac{a}{2r}} \tag{q}$$

where σ_∞ is the remote stress (far away from the crack). The coordinate r is used here in order to avoid confusion with the distance x, or x_0, between the atomic planes.

The mean value $\bar{\sigma}_{yy}$ of $\sigma_{yy} = \sigma_{yy}(r)$ over n atomic planes is

$$\bar{\sigma}_{yy} = \frac{1}{n\, x_0}\int_0^{n\, x_0} \sigma_{yy}(r)\, \mathrm{d}r = \frac{1}{n\, x_0}\int_0^{n\, x_0} \sigma_\infty \sqrt{\frac{a}{2r}}\, \mathrm{d}r$$

$$= \frac{1}{n\, x_0}\frac{\sigma_\infty \sqrt{a}}{\sqrt{2}}\left[\, 2\sqrt{r}\,\right]_0^{n\, x_0} = \sigma_\infty \sqrt{\frac{2a}{n\, x_0}} \tag{r}$$

At fracture the mean value $\bar{\sigma}_{yy}$ should reach the cohesive strength σ_c of the material. This gives

$$\bar{\sigma}_{yy} = \sigma_\infty \sqrt{\frac{2a}{n\, x_0}} = \sigma_c = \frac{E}{2\pi} \tag{s}$$

and

$$\sigma_\infty = \sigma_\infty^{\mathrm{S}} = \frac{E}{2\pi}\sqrt{\frac{n\,x_0}{2a}} \tag{t}$$

where superscript S stands for "stress criterion".

We have now determined the stress $\sigma_\infty^{\mathrm{G}}$ giving failure according to the Griffith criterion and the stress $\sigma_\infty^{\mathrm{S}}$ giving failure according to the stress criterion. In the case that these two criteria give failure at the same remote stress, one has $\sigma_\infty^{\mathrm{G}} = \sigma_\infty^{\mathrm{S}}$. This gives

$$\sigma_\infty^{\mathrm{G}} = \frac{E}{\pi}\sqrt{\frac{x_0}{2\pi a}} = \frac{E}{2\pi}\sqrt{\frac{n\,x_0}{2a}} = \sigma_\infty^{\mathrm{S}} \tag{u}$$

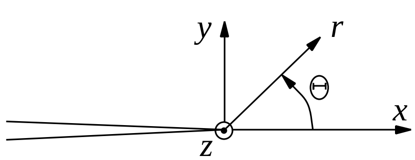
which gives n. Thus $n = 4/\pi = 1.27$.

In conclusion, if the mean stress $\bar{\sigma}_{yy}$ of $\sigma_{yy}(r)$ reaches the cohesive strength σ_{c} over $n = 4/\pi$ atomic planes, then the stress criterion (as formulated here) and the Griffith criterion will predict crack propagation at the same remote stress σ_∞. The material properties should then be as given in the equation in the problem: $\sigma_{\mathrm{i}} = \sigma_{\mathrm{c}} \sin \alpha\,\pi\,(x - x_0)/x_0$, with parameter values as calculated in problem (a).

Answer: (a) $\alpha = 2$, $\sigma_{\mathrm{c}} = E/2\pi$, $w_{\mathrm{s}} = E\,x_0/4\pi^2$, (b) $n = 4/\pi = 1.27$.

Tensor notations, strain energy

4/2.



Determine the elastic strain energy density in front of a crack tip loaded in mode III. In which direction Θ has the strain energy its maximum? The elastic strain energy density u is, for linearly elastic material,

$$u = \frac{\sigma_{ij}\,\varepsilon_{ij}}{2}$$

The stress components in front of a crack tip loaded in mode III are

$$\tau_{xz} = \frac{-K_{\mathrm{III}}}{\sqrt{2\,\pi\,r}}\sin\frac{\Theta}{2}\,; \qquad \tau_{yz} = \frac{K_{\mathrm{III}}}{\sqrt{2\,\pi\,r}}\cos\frac{\Theta}{2}$$

and

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \tau_{xy} = 0$$

Solution:

Use

$$u = \frac{\sigma_{ij}\,\varepsilon_{ij}}{2} \tag{a}$$

Calculate the strains. One has, for $i \neq j$,

$$\varepsilon_{ij} = \frac{1}{2\mu}\,\tau_{ij} \tag{b}$$

where μ is the shear modulus. Enter the stresses given in the problem, and the strains calculated in (b) into the expression (a). It gives

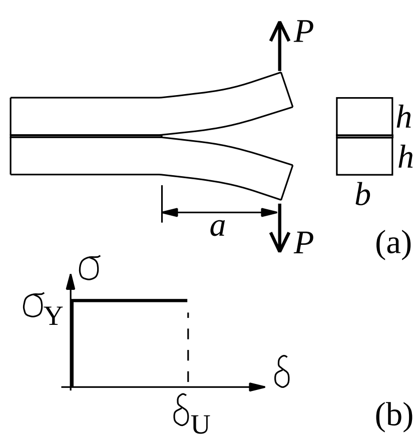
$$u = \frac{1}{2}\left\{\tau_{xz}\frac{\tau_{xz}}{2\mu} + \tau_{zx}\frac{\tau_{zx}}{2\mu} + \tau_{yz}\frac{\tau_{yz}}{2\mu} + \tau_{zy}\frac{\tau_{zy}}{2\mu}\right\} = \frac{1}{4\mu}\left(\frac{K_{\mathrm{III}}}{\sqrt{2\,\pi\,r}}\right)^2\left\{2\sin^2\frac{\Theta}{2} + 2\cos^2\frac{\Theta}{2}\right\}$$

$$= \frac{1}{2\mu}\left(\frac{K_{\mathrm{III}}}{\sqrt{2\,\pi\,r}}\right)^2 = \frac{1}{4\pi\,\mu\,r}K_{\mathrm{III}}^2 \tag{c}$$

Answer: The elastic strain energy u is $u = K_{\mathrm{III}}^2/4\pi\mu r$ (thus independent on Θ), μ is the shear modulus.

Energy balance

4/3.



Two rectangular bars (same cross section b by h) of linearly elastic material (modulus of elasticity E) are glued together along a portion of its length, see figure (a). The glue has σ-δ characteristics according to figure (b) (rigid, ideally plastic up to deformation δ_U and yield limit σ_Y). Determine the critical load $P = P_{crit}$. One has $a >> h$ and $a >> b$.

Solution:

Two solutions will be given.

(a) *Load control* gives the potential energy

$$\Pi = U - P\Delta \tag{a}$$

where

$$U = \int_{\mathrm{Vol}}\frac{\sigma_{ij}\,\varepsilon_{ij}}{2}\,\mathrm{d}V = \frac{P\Delta}{2} \tag{b}$$

which gives

$$\Pi = -P\Delta/2 \tag{c}$$

Assume that the two rectangular bars may be considered to be cantilever beams of length a. The displacement $\Delta/2$ of the cantilever beam end due to a force P then is

$$\frac{\Delta}{2} = \frac{P\,a^3}{3\,EI} \quad \text{giving} \quad \Delta = \frac{2\,P\,a^3}{3\,EI} \tag{d}$$

The potential energy of the structure becomes

$$\Pi = -\frac{P\Delta}{2} = -\frac{P^2\,a^3}{3\,EI} \tag{e}$$

The energy release due to a change of the crack area is

$$\frac{\mathrm{d}\,\Pi}{\mathrm{d}\,A} = \frac{\mathrm{d}\,\Pi}{\mathrm{d}\,(b\cdot a)} = \frac{\mathrm{d}}{\mathrm{d}\,(b\cdot a)}\left(\frac{-P^2\,(b\cdot a)^3}{3EI\,b^3}\right) = \frac{-P^2}{3\,EI}\,\frac{3(b\cdot a)^2}{b^3} = \frac{-P^2\,a^2}{EI\,b} \tag{f}$$

The work required to create the crack surface, *i.e.* the work required to overcome the glue strength, is

$$W_{\mathrm{s}} = A\cdot\sigma_{\mathrm{Y}}\,\delta_{\mathrm{U}} \;\; (= A\cdot 2w_{\mathrm{s}}) \tag{g}$$

which gives

$$\frac{\mathrm{d}\,W_{\mathrm{s}}}{\mathrm{d}\,A} = \sigma_{\mathrm{Y}}\,\delta_{\mathrm{U}} \tag{h}$$

Energy balance, namely

$$-\frac{\mathrm{d}\,\Pi}{\mathrm{d}\,A} = \frac{\mathrm{d}\,W_{\mathrm{s}}}{\mathrm{d}\,A}, \qquad \text{gives} \qquad \frac{P^2\,a^2}{EI\,b} = \sigma_{\mathrm{Y}}\,\delta_{\mathrm{U}} \tag{i}$$

that gives

$$P = P_{\mathrm{crit}} = \sqrt{\frac{EI\,b\,\sigma_{\mathrm{Y}}\,\delta_{\mathrm{U}}}{a^2}} = \sqrt{\frac{E\,b\,h^3\,b\,\sigma_{\mathrm{Y}}\,\delta_{\mathrm{U}}}{12\,a^2}} = \frac{bh}{a}\,\frac{\sqrt{E\,h\,\sigma_{\mathrm{Y}}\,\delta_{\mathrm{U}}}}{2\sqrt{3}} \tag{j}$$

(b) *Alternative solution*

From

$$\Pi = -\frac{P^2\,a^3}{3\,EI} \qquad \text{and} \qquad W_{\mathrm{s}} = b\cdot a\;\sigma_{\mathrm{Y}}\,\delta_{\mathrm{U}} \tag{k,l}$$

one obtaines, for a crack growth δa,

$$\delta\Pi = \frac{\partial\Pi}{\partial a}\,\delta a = -\frac{P^2\,3\,a^2}{3\,EI}\,\delta a \quad \text{and} \quad \delta W_{\mathrm{s}} = \frac{\partial W_{\mathrm{s}}}{\partial a}\,\delta a = b\;\sigma_{\mathrm{Y}}\,\delta_{\mathrm{U}}\,\delta a \tag{m,n}$$

Energy balance, namely

$$-\delta\Pi = \delta W_{\mathrm{s}}, \qquad \text{gives} \qquad \frac{P^2\,a^2}{EI} = b\;\sigma_{\mathrm{Y}}\,\delta_{\mathrm{U}} \tag{o}$$

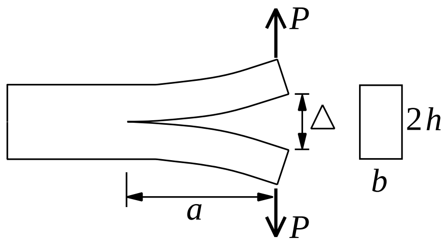
Thus

$$P = P_{\mathrm{crit}} = \sqrt{\frac{EI\,b\,\sigma_{\mathrm{Y}}\,\delta_{\mathrm{U}}}{a^2}} = \frac{bh}{a}\,\frac{\sqrt{E\,h\,\sigma_{\mathrm{Y}}\,\delta_{\mathrm{U}}}}{2\sqrt{3}} \tag{p}$$

Answer: Critical load is

$$P_{\mathrm{crit}} = \frac{b\,h}{2\,a\,\sqrt{3}}\sqrt{E\,\sigma_{\mathrm{Y}}\,\delta_{\mathrm{U}}\,h}$$

4/4.



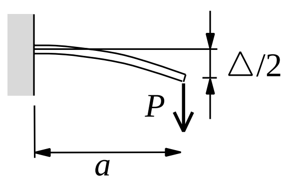
To determine the surface energy w_s of a brittle, linearly elastic material a test specimen, see figure, was manufactured from the material to be investigated. When loading the test specimen crack propagation was obtained when the force P was $P = P_{max}$. Determine the surface energy of the material. The modulus of elasticity is E, the cross section is b by $2h$, and the crack length is a where $a >> h$ and $a >> b$.

Solution:

Two solutions will be given.

(a) *Load control* gives the potential energy

$$\Pi = U - P\Delta = \frac{P\Delta}{2} - P\Delta = -\frac{P\Delta}{2} \qquad \text{(a)}$$



Assume that the rectangular bars may be approximated to two cantilever beams of length a. The displacement $\Delta/2$ of the beam end due to a force P is

$$\frac{\Delta}{2} = \frac{P\,a^3}{3\,EI} \quad \text{giving} \quad \Delta = \frac{2\,P\,a^3}{3\,EI} \qquad \text{(b)}$$

The potential energy of the structure is

$$\Pi = -\frac{P\Delta}{2} = -\frac{P^2\,(b\,a)^3}{3\,EI\,b^3} \qquad \text{(c)}$$

The energy release due to a change of the crack area $A = b\,a$ is

$$\frac{\mathrm{d}\,\Pi}{\mathrm{d}\,A} = \frac{\mathrm{d}\,\Pi}{\mathrm{d}\,(b\,a)} = -\frac{P^2}{3\,EI}\,\frac{3(b\,a)^2}{b^3} = -\frac{P^2\,a^2}{EI\;b} \qquad \text{(d)}$$

The work required to create the new surfaces is

$$W_s = 2w_s \cdot A \quad \text{which gives} \quad \frac{\mathrm{d}\,W_s}{\mathrm{d}\,A} = 2\,w_s \qquad \text{(e)}$$

Energy balance

$$(\,G =\,) -\frac{\mathrm{d}\,\Pi}{\mathrm{d}\,A} = \frac{\mathrm{d}\,W_s}{\mathrm{d}\,A} \qquad \text{gives} \qquad \frac{P^2\,a^2}{EI\;b} = 2w_s \qquad \text{(f)}$$

which gives

$$w_s = \frac{P^2\,a^2}{2\,b\;EI} = \frac{P^2\,a^2\;12}{2\,b\;E\;\,b h^3} = \frac{6\,P_{max}^2\,a^2}{E\;b^2\,h^3} \qquad \text{(g)}$$

Thus, the surface energy w_s is $w_s = 6P_{max}^{\,2}\, a^2 / E\, b^2\, h^3$.

(b) *Alternative solution*

From $$\Pi = -\frac{P^2 a^3}{3\,EI} \quad \text{and} \quad W_s = 2\,b \cdot a\; w_s \tag{h,i}$$

is obtained, for a crack growth δa,

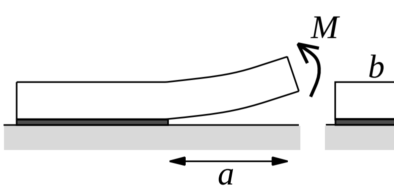
$$\delta\Pi = \frac{\partial\Pi}{\partial a}\,\delta a = -\frac{P^2\,3\,a^2}{3\,EI}\,\delta a \quad \text{and} \quad \delta W_s = \frac{\partial W_s}{\partial a}\,\delta a = 2\,b\;w_s\,\delta a \tag{j,k}$$

Energy balance, $-\delta\Pi = \delta W_s$, gives

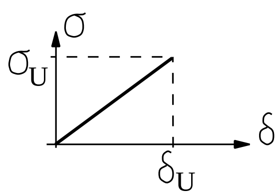
$$\frac{P^2 a^2}{EI} = 2\,b\;w_s \qquad \text{Thus} \quad w_s = \frac{P^2 a^2}{2\,b\,EI} \tag{l}$$

Answer: The surface energy w_s is $w_s = 6P_{max}^{\,2}\, a^2 / E\, b^2\, h^3$.

4/5.



(a)



(b)

A thin rectangular bar (cross section b by h) of linearly elastic material (Young's modulus E) is glued to a rigid foundation, see figure (a). The glue has a linear σ-δ characteristics according to figure (b) (rupture at stress σ_U when deformation is δ_U). The bar is loaded with a bending moment M. Determine the critical load M_{crit}. One has $a >> b > h$.

Solution:

Two solutions will be given.

(a) *Load control* gives the potential energy $\Pi = U - P\Delta$, which here gives

$$\Pi = U - M \cdot \alpha = \frac{M\alpha}{2} - M\alpha = -\frac{M\alpha}{2} \tag{a}$$

Assume that the free part of the rectangular bar may be approximated to a cantilever beam of length a. The rotation α of the beam end due to the bending moment M is

$$\alpha = \frac{M\;a}{EI} = \frac{M\;a\;12}{E\;bh^3} \tag{b}$$

The potential energy of the structure becomes (crack area A is $A = b{\cdot}a$)

$$\Pi = -\frac{M\alpha}{2} = -\frac{M^2\,a\;6}{E\;bh^3} = -\frac{M^2\,(ba)\;6}{E\;b^2h^3} \quad (\text{where} \quad A = ba) \tag{c}$$

The energy release due to a change of the crack area is

$$\frac{\mathrm{d}\,\Pi}{\mathrm{d}\,A} = -\frac{M^2\,6}{E\;b^2 h^3} \tag{d}$$

The work required to create the new surface is

$$W_\mathrm{s} = A \cdot \frac{1}{2}\,\sigma_\mathrm{U}\,\delta_\mathrm{U} \qquad \text{which gives} \qquad \frac{\mathrm{d}\,W_\mathrm{s}}{\mathrm{d}\,A} = \frac{1}{2}\,\sigma_\mathrm{U}\,\delta_\mathrm{U} \tag{e}$$

Energy balance,

$$-\frac{\mathrm{d}\,\Pi}{\mathrm{d}\,A} = \frac{\mathrm{d}\,W_\mathrm{s}}{\mathrm{d}\,A} \quad \text{gives} \quad \frac{M^2\,6}{E\;b^2 h^3} = \frac{1}{2}\,\sigma_\mathrm{U}\,\delta_\mathrm{U} \tag{f}$$

which gives

$$M^2 = M_\mathrm{crit}^2 = \frac{\sigma_\mathrm{U}\,\delta_\mathrm{U}}{12}\,E\;b^2 h^3 \quad \text{and} \quad M_\mathrm{crit} = \frac{bh}{2}\sqrt{\frac{\sigma_\mathrm{U}\,\delta_\mathrm{U}\,E\;h}{3}} \tag{g}$$

(b) *Alternative solution*

From
$$\Pi = -\frac{M^2\,6\,a}{E\;b\;h^3} \qquad \text{and} \qquad W_\mathrm{s} = \frac{1}{2}\,b\,a\;\sigma_\mathrm{U}\,\delta_\mathrm{U} \tag{h,i}$$

is obtained

$$\delta\Pi = \frac{\partial\Pi}{\partial a}\,\delta a = -\frac{6\,M^2}{E\;b\;h^3}\,\delta a \quad \text{and} \quad \delta W_\mathrm{s} = \frac{\partial W_\mathrm{s}}{\partial a}\,\delta a = \frac{1}{2}\,b\;\sigma_\mathrm{U}\,\delta_\mathrm{U}\,\delta a \tag{j,k}$$

Energy balance, $-\,\delta\Pi = \delta W_\mathrm{s}$, gives

$$\frac{6\,M^2}{E\;b\;h^3} = \frac{1}{2}\,b\;\sigma_\mathrm{U}\,\delta_\mathrm{U} \tag{l}$$

Thus

$$M^2 = M_\mathrm{crit}^2 = \frac{E\;\sigma_\mathrm{U}\,\delta_\mathrm{U}\,b^2 h^3}{12} \quad \left\{\text{dimension: } \frac{\mathrm{N}}{\mathrm{m}^2}\,\frac{\mathrm{N}}{\mathrm{m}^2}\,\mathrm{m}\,\mathrm{m}^2\,\mathrm{m}^3 = (\mathrm{Nm})^2\right\} \tag{m}$$

as obtained in (g).

Answer: Critical load M_crit is

$$M_\mathrm{crit} = \frac{b\;h}{2\sqrt{3}}\sqrt{E\;h\;\sigma_\mathrm{U}\,\delta_\mathrm{U}}$$

Solutions to problems in
T Dahlberg and A Ekberg: Failure, Fracture, Fatigue - An Introduction.
Studentlitteratur, Lund 2002, ISBN 91-44-02096-1.

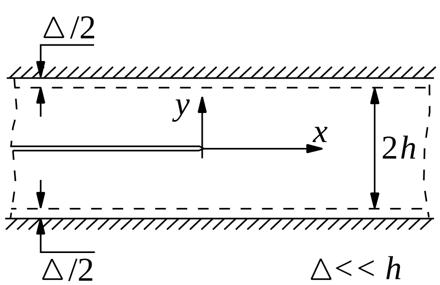
Chapter 5

Determination of stress intensity factor K_I

Problems with solutions

Stress intensity factor

5/1.



An infinitely long strip, height 2*h*, is split along half its length, see the figure (the strip thus has a semi-infinitely long crack in it). The upper and lower boundary of the strip have been given a prescribed displacement $\Delta/2$ each. Determine the stress intensity factor K_I for this configuration. Assume plane stress at the crack tip.

Solution:

The strategy to solve this problem is the following: the stress intensity factor K_I can be determined from the energy release rate *G*. To find *G*, an energy consideration is made. When the crack grows (say δa) the strip to the right of the crack tip becomes δa shorter, and the energy in that part (*i.e.* in a volume δa by 2*h* by strip thickness *t*) is released. This energy is used to determine *G*.

First the elastic strain energy density *u* stored in the strip far away from the crack tip is determined. To do this, we need to know the stresses and the strains in the strip.

The strains in the strip (far away to the right of the crack tip) are

$$\varepsilon_{yy} = \frac{\Delta}{2h}\,, \quad \varepsilon_{xx} = 0\,, \quad \text{and} \quad \varepsilon_{zz} = -\nu\,\varepsilon_{yy} \qquad \text{(a,b,c)}$$

The strain ε_{xx} is zero because no movements can occur in the *x* direction far away from the crack tip. Also, all shear strain components are zero because of symmetry. Hooke's law gives, with $\sigma_{zz} = 0$,

$$\varepsilon_{yy} = \frac{\Delta}{2h} = \frac{1}{E}\{\sigma_{yy} - \nu\sigma_{xx}\} \tag{d}$$

$$\varepsilon_{xx} = 0 = \frac{1}{E}\{\sigma_{xx} - \nu\sigma_{yy}\} \tag{e}$$

from which the stresses are solved:

$$\sigma_{yy} = \frac{E}{1-\nu^2}\frac{\Delta}{2h} \quad \text{and} \quad \sigma_{xx} = \nu\,\sigma_{yy} \tag{f,g}$$

The strain energy density u (Nm/m^3) becomes

$$u = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}(\sigma_{xx}\,\varepsilon_{xx} + \sigma_{yy}\,\varepsilon_{yy}) = \frac{1}{2}\frac{E}{1-\nu^2}\left(\frac{\Delta}{2h}\right)^2 \tag{h}$$

The energy δU in a section of the strip with length δa is (far away to the right of the crack tip)

$$\delta U = u \cdot \text{Volume} = u \cdot 2h\ \delta a\ t = \frac{1}{2}\frac{E}{1-\nu^2}\left(\frac{\Delta}{2h}\right)^2 \cdot 2h\ \delta a\ t \tag{i}$$

where t is the thickness of the strip.

The strain energy close to the crack tip is not calculated here, because close to the crack tip the stress field is "complicated", and, as will be seen, it is not necessary to calculate the strain energy in this region.

The energy release rate G is, at displacement control,

$$G = -\frac{\mathrm{d}\Pi}{\mathrm{d}A} = -\frac{\mathrm{d}U}{t\,\mathrm{d}a} = -\frac{1}{t}\frac{U(a+\mathrm{d}a) - U(a)}{\mathrm{d}a} = \frac{1}{t}\frac{U(a) - U(a+\mathrm{d}a)}{\mathrm{d}a} \tag{j}$$

Thus, the energy release when the crack grows a distance δa is $\delta U = U(a) - U(a+\delta a)$. The energy release rate G then becomes (at displacement control), by use of δU from (h),

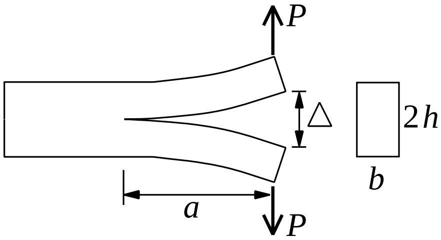
$$G = \frac{1}{t}\frac{\delta U}{\delta a} = \frac{1}{t}\frac{1}{2}\frac{E}{1-\nu^2}\left(\frac{\Delta}{2h}\right)^2 \cdot 2h\ t \tag{k}$$

Finally, K_I is obtained from $K_\mathrm{I} = \sqrt{E'G}$, ($E' = E$ at plane stress) giving

$$K_\mathrm{I} = \sqrt{EG} = \frac{E}{\sqrt{1-\nu^2}}\left(\frac{\Delta}{2h}\right) \cdot \sqrt{h} \tag{l}$$

Answer: The stress intensity factor is $K_\mathrm{I} = E\Delta / 2\sqrt{h(1-\nu^2)}$

5/2.



Use the compliance method to determine the energy release rate G and from that, determine the stress intensity factor K_{I} for a double cantilever beam (DCB) specimen. Crack length is a, thickness is b and specimen height is $2h$, where $a >> b$ and $a >> h$. The material is linear elastic with Young's modulus E. Investigate the two loading cases:

(a) Load control, with the loading force P prescribed.
(b) Displacement control, with displacement Δ prescribed.
Assume plane stress conditions.

Solution:

The force *versus* displacement relationship for a cantilever beam gives the displacement $\Delta/2$ of the beam end due to the loading force P as

$$\frac{\Delta}{2} = \frac{P\,a^3}{3\,EI} \quad \text{giving} \quad \Delta = \frac{2\,P\,a^3}{3\,EI} \tag{a}$$

The compliance C of the specimen is

$$C = \frac{\Delta}{P} = \frac{1}{P}\,\frac{2\,P\,a^3}{3\,EI} = \frac{2a^3}{3\,EI} \qquad \text{giving} \qquad \frac{\mathrm{d}C}{\mathrm{d}a} = \frac{2\,a^2}{EI} \tag{b}$$

(a) The energy release rate G due to a crack extension is, for prescribed load P,

$$G = \frac{P^2}{2\,b}\,\frac{\mathrm{d}C}{\mathrm{d}a} = \frac{P^2}{2\,b}\,\frac{2\,a^2}{EI} \tag{c}$$

In the case of *plane stress* at the crack tip, one obtains

$$G = \frac{K_{\mathrm{I}}^2}{E} \quad \text{giving} \quad K_{\mathrm{I}} = \sqrt{G \cdot E} = \sqrt{\frac{P^2\,a^2}{b\,I}} = \frac{P\,a}{b\,h}\,\frac{2\sqrt{3}}{\sqrt{h}} \tag{d}$$

It is seen that the stress intensity factor K_{I} increases as the crack grows, implying that unstable crack growth will be expected (at least if K_c does not depend on crack length a).

(b) The energy release rate G due to a crack extension is, for prescribed displacement Δ,

$$G = \frac{P^2}{2\,b}\,\frac{\mathrm{d}C}{\mathrm{d}a} = \frac{P^2}{2\,b}\,\frac{2\,a^2}{EI} \tag{e}$$

Using

$$P = \frac{3\,EI\,\Delta}{2\,a^3} \tag{f}$$

one obtains $$G = \frac{a^2}{b\,EI}\left(\frac{3\,EI\,\Delta}{2\,a^3}\right)^2 = \frac{9\,EI\,\Delta^2}{4\,b\,a^4} \qquad \text{(g)}$$

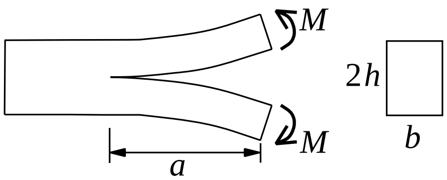
In the case of plane stress at the crack tip, one obtains

$$G = \frac{K_I^2}{E} \quad \text{giving} \quad K_I = \sqrt{G \cdot E} = \sqrt{\frac{9\,E^2\,I\,\Delta^2}{4\,b\,a^4}} = \frac{3\,E\,\Delta\,h\sqrt{h}}{\sqrt{12}\cdot 2\,a^2} = \frac{\sqrt{3}\,E\,\Delta\,h\sqrt{h}}{4\,a^2} \qquad \text{(h)}$$

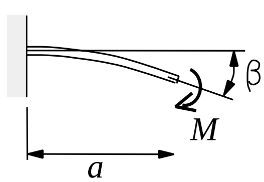
Here it is seen that the stress intensity factor K_I decreases as the crack grows, implying that stable crack growth will be expected (at least if K_c does not depend on crack length a).

Answer: Stress intensity factors are (a) $K_I = 2\sqrt{3}Pa / bh\sqrt{h}$, and (b) $K_I = \sqrt{3}E\Delta h\sqrt{h} / 4a^2$.

5/3.



Use the compliance method to determine the energy release rate G for a double cantilever beam (DCB) specimen. Crack length is a, thickness is b and specimen height is $2h$, where $a >> b$ and $a >> h$. The material is linear elastic with Young's modulus E. The specimen is loaded with bending moments M at the beam ends.



Solution:

The (generalised) force *versus* (generalised) displacement relationship (here this becomes moment *versus* rotation) of a cantilever beam gives the rotation angle β of the beam end due to the bending moment M. One obtains

$$\beta = \frac{M\,a}{EI} \quad \text{giving} \quad \alpha = 2\,\beta = \frac{2\,M\,a}{EI} \qquad \text{(a)}$$

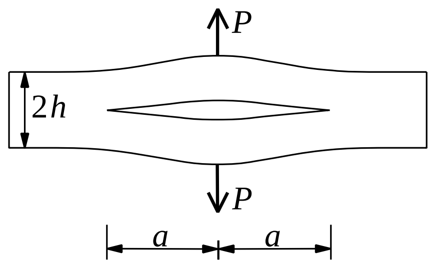
The compliance C of the specimen is $$C = \frac{\alpha}{M} = \frac{2\,a}{EI} \quad \text{giving} \quad \frac{\mathrm{d}C}{\mathrm{d}a} = \frac{2}{EI} \qquad \text{(b)}$$

The energy release rate G due to a crack extension is, for prescribed load M,

$$G = \frac{M^2}{2\,b}\frac{\mathrm{d}C}{\mathrm{d}a} = \frac{M^2}{2\,b}\frac{2}{EI} = \frac{12M^2}{E\,b^2\,h^3} \qquad \text{(c)}$$

Answer: The energy release rate G is $G = 12M^2 / Eb^2h^3$.

5/4.



Use the compliance method to determine the energy release rate G for a beam with a long central crack, see figure. The crack length is $2a$, beam thickness is b and beam height is $2h$, where $a >> b$ and $a >> h$. The material is linear elastic with modulus of elasticity E. The beam is loaded symmetrically with two opposite forces P at the beam centre. The crack is assumed to grow symmetrically.

Solution:

The force *versus* displacement relationship of a fixed-fixed beam (length $2a$) gives the displacement $\Delta / 2$ of the beam centre due to the loading force P. One obtains

$$\frac{\Delta}{2} = \frac{P\,(2a)^3}{3\,EI} \cdot \frac{1}{8} \cdot \frac{1}{8} \quad \text{giving} \quad \Delta = \frac{P\,a^3}{12\,EI} \tag{a}$$

The compliance C of the specimen is

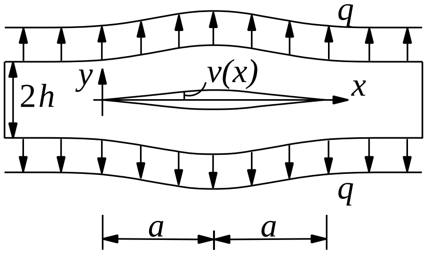
$$C = \frac{\Delta}{P} = \frac{a^3}{12\,EI} \quad \text{giving} \quad \frac{\mathrm{d}C}{\mathrm{d}a} = \frac{a^2}{4\,EI} \tag{b}$$

Note that, due to symmetry the crack is assumed to grow in both directions, which gives (in $\mathrm{d}\Pi/\mathrm{d}A$) that $\mathrm{d}A = \mathrm{d}(2ab) = 2b\mathrm{d}a$ (A is crack area) (or, it can be seen as one crack with width $2b$, *i.e.* two crack fronts of length b each). The energy release rate G due to a crack extension in both directions then is, for a prescribed force P,

$$G = \frac{P^2}{2 \cdot (2b)} \frac{\mathrm{d}C}{\mathrm{d}a} = \frac{P^2}{4\,b} \frac{a^2}{4\,EI} = \frac{3\,P^2\,a^2}{4\,E\,b^2\,h^3} \tag{c}$$

Answer: The energy release rate G is $G = 3P^2a^2 / 4Eb^2h^3$.

5/5.



Use the potential energy Π to determine the energy release rate G for a beam with a long central crack, see figure. The crack length is $2a$, beam thickness is b and beam height is $2h$, where $a >> b$ and $a >> h$. The material is linear elastic with modulus of elasticity E. The beam is loaded symmetrically with two opposite distributed loads q (q in N/m) at beam upper and lower surfaces.

The deflection of a clamped-clamped beam loaded with a distributed load q is, for $0 \le x \le 2a$,

$$v(x) = \frac{q}{24\,EI}\,(x^4 - 4ax^3 + 4a^2x^2)$$

Solution:

The energy release rate G for the beam is

$$G = -\frac{1}{2b}\frac{\mathrm{d}\Pi}{\mathrm{d}a} \qquad \text{(a)}$$

The crack front length (the extension of the crack tip in the z direction) is $2b$, because here there are two crack tips and each crack font has length b.

The potential energy is, see equation (4.4) in the textbook,

$$\Pi = U - W = \int_{\text{Vol}} u \,\mathrm{d}V - \int_{\text{boundary}} q \cdot v \,\mathrm{d}S$$

$$= \frac{1}{2}\int_{\text{boundary}} q \cdot v \,\mathrm{d}S - \int_{\text{boundary}} q \cdot v \,\mathrm{d}S = -\frac{1}{2}\int_{\text{boundary}} q \cdot v \,\mathrm{d}S \qquad \text{(b)}$$

where $U = W/2$ has been used (the material is linear elastic). Expression (b) gives

$$\Pi = -\frac{1}{2}\left\{\int_0^{2a} q \cdot v(x)\,\mathrm{d}x + \int_0^{2a} -q \cdot (-v(x))\,\mathrm{d}x\right\} \qquad \text{(c)}$$

where the first term refers to the upper surface of the beam (loaded upwards) and the second term refers to the lower surface (loaded downwards). Insert $v(x)$ into (c) and integrate. It gives

$$\Pi = -\int_0^{2a} q \cdot v(x)\,\mathrm{d}x = \frac{-q^2}{24EI}\left\{\frac{1}{5}(2a)^5 - \frac{4a}{4}(2a)^4 + 4a^2\frac{1}{3}(2a)^3\right\}$$

$$= -\frac{2}{45}\frac{q^2 a^5}{EI} \qquad \text{(d)}$$

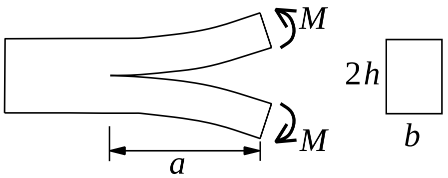
Finally the energy release rate G can be calculated:

$$G = -\frac{1}{2b}\frac{\mathrm{d}\Pi}{\mathrm{d}a} = -\frac{1}{2b}\frac{(-2\,q^2)}{45\,EI}\,5a^4 = \frac{1}{9}\frac{q^2\,a^4}{b\,EI} = \frac{4}{3}\frac{q^2\,a^4}{E\,b^2 h^3} \tag{e}$$

Answer: Energy release rate G is $G = 4q^2a^4/3Eb^2h^3$.

Reciprocity relation

5/6.



Determine the stress intensity factor for the structure in Problem 5/3 by use of the reciprocity relation. Use Problem 5/2 as the known Case (1).

Solution:

Use equation (5.10) in the textbook to determine the stress intensity factor. One has

$$K_{\mathrm{I}}^{(2)} = \frac{4\mu}{(\kappa+1)\,K_{\mathrm{I}}^{(1)}}\int_{\Gamma} T_i^{(2)}\frac{\partial u_i^{(1)}}{\partial a}\,\mathrm{d}\Gamma \tag{5.10}$$

The stress intensity factor in Problem 5/2 is known. Thus, the factor $K_{\mathrm{I}}^{(1)}$ to use in the right hand side of (5.10) is (from Problem 5/2)

$$K_{\mathrm{I}}^{(1)} = \frac{P\,a}{b\,h}\frac{2\sqrt{3}}{\sqrt{h}} \tag{a}$$

From loading case (1) (Problem 5/2) one needs to know also the (generalized) displacement at the point where the load is going to be applied in the second loading case. This means that one needs to know the rotation angle in Case (1) (due to load P) at the end of the cantilever beam (where the load M is applied in Case (2)). One obtains

$$\theta = \frac{Pa^2}{2EI} \tag{b}$$

Also, $\mathrm{d}\theta/\mathrm{d}a$ is needed. One obtains

$$\frac{\mathrm{d}\theta}{\mathrm{d}a} = \frac{2Pa}{2EI} = \frac{Pa}{EI} \tag{c}$$

In Problem 5/3 the load, *i.e.* the bending moment M, is applied in two discrete points. This means that the intergration in (5.10) along the perimeter can be discretized. Note that the integral in (5.10) has dimension N/m. In a general case one has $T_i^{(2)}$ (N/m^2) times $\mathrm{d}u_i^{(1)}/\mathrm{d}a$ (m/m) times $\mathrm{d}\Gamma$ (m), giving (N/m^2) $\cdot$ (m/m) $\cdot$ m = N/m. When discretizing the integral, $T_i^{(2)}\mathrm{d}\Gamma$ will have the dimension N/m. In case $T_i^{(2)}$ is a stress (N/m^2), $T_i^{(2)}\mathrm{d}\Gamma$ becomes force per unit width (width in the z direction) of the structure.

Here, when the (generalized) displacement is an angle θ, one obtains the dimension of dθ/da as 1/m. The expression $T_i^{(2)}$dΓ then must have dimension N (unit of force) to give N/m when multiplying $T_i^{(2)}$dΓ with dθ/da. This implies that the loading moment M (given in Nm in the problem) should be applied as moment per unit width (in the z direction) when the moment is entered into (5.10). Thus, the discretized version of (5.10) becomes, in this case,

$$K_{\rm I}^{(2)} = \frac{4\mu}{(\kappa+1)\,K_{\rm I}^{(1)}} \int_\Gamma T_i^{(2)} \frac{\partial u_i^{(1)}}{\partial a}\,\mathrm{d}\Gamma = \frac{4\mu}{(\kappa+1)\,K_{\rm I}^{(1)}} \cdot 2 \cdot \frac{M}{b} \cdot \frac{Pa}{EI} \tag{d}$$

Here, at the end of the expression one has dθ/da from (c), next to that the bending moment per unit width M/b, and in front of M/b the factor 2, which is there because there are two bending moments.

Using that $\mu = E/2(1+\nu)$ (the shear modulus), and from equatio (4.34) that $\kappa = (3-\nu)/(1+\nu)$ for plane stress conditions at the crack tip, one obtains

$$K_{\rm I}^{(2)} = \frac{4E}{2(1+\nu)} \frac{(1+\nu)}{4} \cdot \frac{bh\sqrt{h}}{Pa\;2\sqrt{3}} \cdot 2 \cdot \frac{M}{b} \cdot \frac{Pa}{EI} = \frac{2\sqrt{3}\,M}{bh\sqrt{h}} \tag{e}$$

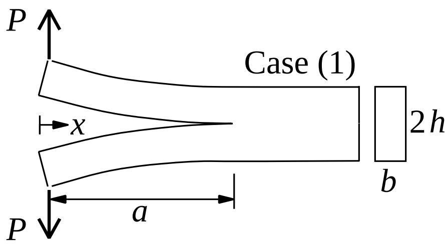
From Problem 5/3 is obtained

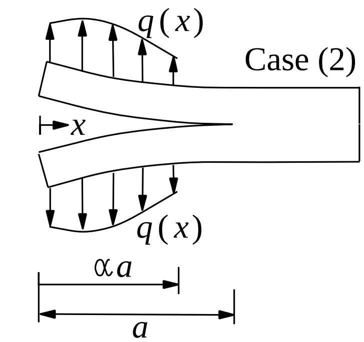
$$K_{\rm I}^{(1)} = \sqrt{EG} = \sqrt{\frac{E\;12M^2}{E\;b^2h^3}} = \frac{2\sqrt{3}M}{bh\sqrt{h}} \tag{f}$$

which is in agreement with (e).

Answer: The stress intensity factor is, *cf*. answer to Problem 5/3, $K_{\rm I} = \dfrac{2\sqrt{3}M}{bh\sqrt{h}}$

5/7.





Determine the stress intensity factor for the structure in the figure by use of the reciprocity relation. Use Problem 5/2 as the known Case (1).

Investigate some different loading cases of the structure (structure and loadings are symmetric):

(a) a general loading $q(x)$ (N/m) along part of the beams, *i.e.*, the loading is $q(x)$ for $0 < x < \alpha a$ (note that $x = 0$ at the free surface and $x = a$ at the crack tip),

(b) the loading $q(x)$ is constant, $q(x) = q_0$, along part of the beams, *i.e.*, the loading is q_0 for $0 < x < \alpha a$,

(c) what happens in Case (b) above if the factor α becomes small ($\alpha << 1$) but the force $q_0 \alpha a$ at the beam end is a constant, say $q_0 \alpha a = Q$?

Assume plane stress conditions, and compare with Problem 5/2.

Solution:

Use equation (5.10) in the textbook to determine the stress intensity factor. One has

$$K_\mathrm{I}^{(2)} = \frac{4\mu}{(\kappa + 1)\, K_\mathrm{I}^{(1)}} \int_\Gamma T_i^{(2)} \frac{\partial u_i^{(1)}}{\partial a}\, \mathrm{d}\Gamma \qquad (5.10)$$

The stress intensity factor in Problem 5/2 is known. Thus, the factor $K_\mathrm{I}^{(1)}$ to use in the right hand side of (5.10) is (from Problem 5/2)

$$K_\mathrm{I}^{(1)} = \frac{P\, a}{b\, h} \frac{2\sqrt{3}}{\sqrt{h}} \qquad \text{(a)}$$

From loading case (1) (Problem 5/2) one needs to know also the (generalized) displacement at the points where the load is going to be applied in the second loading case. This means that one needs to know the deflection in Case (1) (due to load P) of the cantilever beam where the load $q(x)$ is applied in Case (2). One obtains, with $u_i^{(1)}(x) = v(x)$,

$$v(x) = \frac{P\, a^3}{6EI}\left(2 - \frac{3x}{a} + \frac{x^3}{a^3}\right) \qquad \text{(b)}$$

Also, $\partial v / \partial a$ is needed. One obtains

$$\frac{\partial v(x)}{\partial a} = \frac{P}{6EI}\,(2 \cdot 3a^2 - 3x \cdot 2a + 0\,) \qquad \text{(c)}$$

Equation (5.10) now gives

$$K_I^{(2)} = \frac{4\mu}{(\kappa+1)\,K_{\mathrm{I}}^{(1)}}\,2\int_0^{\alpha a}\frac{q(x)}{b}\,\frac{P}{6EI}(6a^2 - 6xa)\,\mathrm{d}x \tag{d}$$

where $\mu = E/2(1+\nu)$ is the shear modulus, $\kappa = (3-\nu)/(1+\nu)$ (plane stress), and the factor 2 is there due to the two cantilever beams (integration along two beams). In (5.10) $T^{(2)}$ is stress. Dividing $q(x)$ (N/m) by the thickness b gives the stress on the boundary.

(b) Enter $q(x) = q_0$ into (d). It gives

$$K_I^{(2)} = \frac{4\mu}{(\kappa+1)\,K_{\mathrm{I}}^{(1)}}\,2\int_0^{\alpha a}\frac{q_0\,a}{b}\,\frac{P}{EI}(a-x)\,\mathrm{d}x$$

$$= \frac{4\mu}{(\kappa+1)\,K_{\mathrm{I}}^{(1)}}\,2\,\frac{q_0\,a}{b}\,\frac{P}{EI}\left(a\;\alpha a - \frac{1}{2}(\alpha a)^2\right) \tag{e}$$

Thus,

$$K_{\mathrm{I}}^{(2)} = \frac{8\mu}{(\kappa+1)\,K_{\mathrm{I}}^{(1)}}\,\frac{P\,a^2}{EI}\,\frac{q_0\,\alpha a}{b}\left(1-\frac{\alpha}{2}\right) \tag{f}$$

(c) Enter $q_0{\cdot}\alpha a = Q$ into (f), and then let α go to zero. It gives

$$K_{\mathrm{I}}^{(2)} = \frac{8\mu}{(\kappa+1)\,K_{\mathrm{I}}^{(1)}}\,\frac{P\,a^2}{EI}\,\frac{Q}{b} = \frac{8\mu\,b\,h\,\sqrt{h}}{(\kappa+1)\,P\,a\;2\sqrt{3}}\,\frac{P\,a^2}{EI}\,\frac{Q}{b}$$

$$= \frac{8E\,(1+\nu)\,b\,h\,\sqrt{h}}{2(1+\nu)\,4\;2\sqrt{3}}\,\frac{a\;12}{Ebh^3}\,\frac{Q}{b} = \frac{Qa\;2\sqrt{3}}{bh\sqrt{h}} \tag{g}$$

as obtained in Case (1) (at plane stress), see expression (a) above.

Answer: (a)

$$K_{\mathrm{I}}^{(2)} = \frac{4\mu}{(\kappa+1)\,K_{\mathrm{I}}^{(1)}}\,2\int_0^{\alpha a}\frac{q(x)}{b}\,\frac{Pa}{EI}\,(a-x)\,\mathrm{d}x \tag{d}$$

(b) Entering $q(x) = q_0$ into (d) gives

$$K_{\mathrm{I}}^{(2)} = \frac{8\mu}{(\kappa+1)\,K_{\mathrm{I}}^{(1)}}\,\frac{P\,a^2}{EI}\,\frac{q_0\,\alpha a}{b}\left(1-\frac{\alpha}{2}\right) \tag{f}$$

(c) Enter $Q = q_0{\cdot}\alpha a$ and $\alpha = 0$ into (f). It gives

$$K_{\mathrm{I}}^{(2)} = \frac{Qa\;2\sqrt{3}}{bh\sqrt{h}}$$

as in Case (1) (at plane stress), *cf.* expression (a).

Superposition

5/8.

Demonstrate (no calculations needed) why the two loading cases in the figures have the same stress intensity factor K_{I}, *cf.* (j) and (k) in Section 5.2.3 in the textbook. The plates are large. In Case (2) the stress σ is a compressive stress acting on the two crack surfaces (*i.e.*, σ is a pressure loading the crack surfaces).

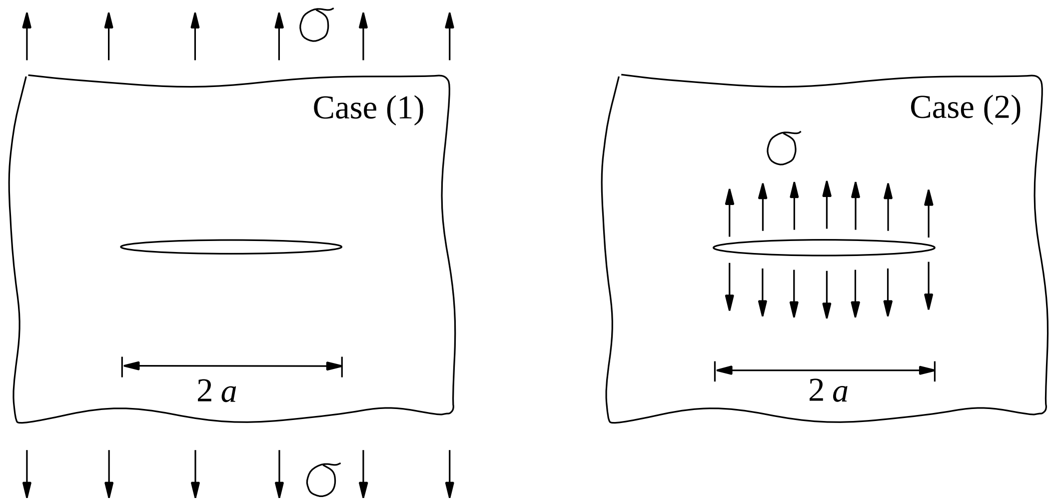


Figure *Loading case* (1) *with remote stress* σ *and loading case* (2) *with same stress* σ *acting on the crack surfaces*

Solution:

In Case (2), replace the compressive stress σ on the crack surfaces with a tension stress σ. Call this loading case for Case (2b). Superimpose loading case (2b) on Case (1). It gives the same stress distribution in the plate (in the plane of the crack) as in a plate with no crack at all, see figure below. This implies that the stress intensity factors in the two cases must cancel each other, and therefore they must be the same.

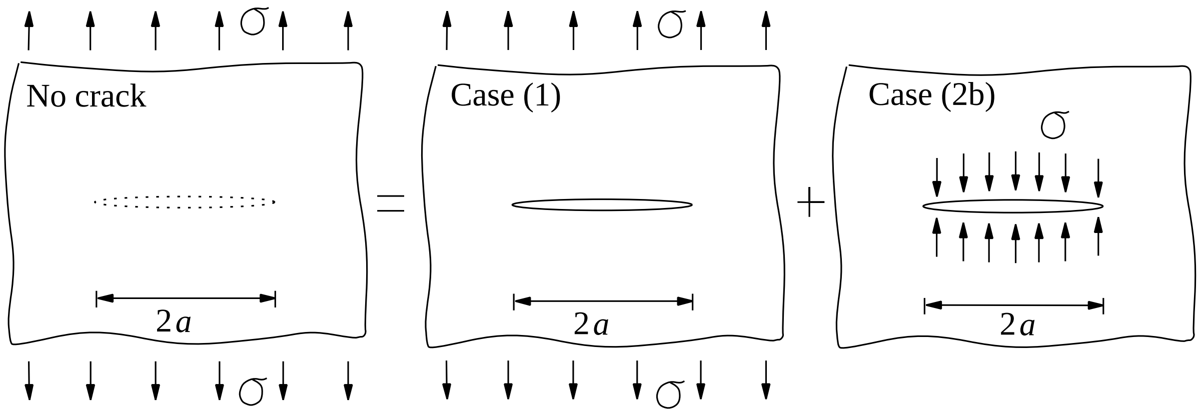


Figure *Loading case* (1) *with remote stress* σ *and loading case* (2b) *with tension stress* σ *acting on the crack surfaces*

Or, doing it in anther way: Start with a plate without crack, see the leftmost part of the figure. The stress in the left plate is uniaxial (value σ) and evenly distributed all over the plate. Create loading case (1) by cutting a crack in the first plate. It means that in the second plate the stress σ cannot be transferred from the upper crack surface to the lower crack surface (as was done in the first plate). This gives the loading case (1) with a certain stress intensity factor $K_I^{(1)}$.

Next, the stress σ that could not be transferred over the crack in loading case (1) is applied on the crack surfaces of another plate. This gives loading case (2b), see the figure. The loading case (2b) gives a stress intensity factor $K_I^{(2b)}$. Superimposing Cases (1) and (2b) gives the first plate with no crack. But the first plate has no stress singularity, so one must have

$$K_I^{(1)} + K_I^{(2b)} = 0 \quad \text{giving} \quad K_I^{(1)} = -K_I^{(2b)} \tag{a}$$

Finally, changing the sign of the stress σ in loading case (2b) gives back the original loading case (2), and one has

$$K_I^{(1)} = K_I^{(2)} \tag{b}$$

Answer: Replace the compressive stress σ in Case (2) by a tensile stress (of magnitude σ) on the crack surfaces. This gives the Case (2b). Superimpose Case (1) and the new Case (2b). It gives a plate with a homogeneous stress field throughout the plate, without stress concentration. Thus, the two stress intensity factors cancel each other, and it follows that Case (1) and Case (2) must have the same stress intensity factor.

Solutions to problems in
T Dahlberg and A Ekberg: Failure, Fracture, Fatigue - An Introduction.
Studentlitteratur, Lund 2002, ISBN 91-44-02096-1.

Chapter 6

Crack propagation under cyclic loading

Problems with solutions

Crack propagation at cyclic loading, Paris' law

6/1.

Crack-like flaws may develop in most materials. These flaws should be treated as if they were cracks. Therefore, assume that the flaws are penny-shaped (the crack looks like a coin with a sharp egde). Determine the largest flaw allowable with respect to fatigue crack propagation in the following materials. The loading is cyclic and the stress range is $0.5\sigma_Y$ (and σ_Y is the yield limit of the material).

(a) A low-strength steel with ΔK_{th} = 7 MN/m$^{3/2}$ and σ_Y = 300 MPa, and

(b) a high-strength steel with ΔK_{th} = 4 MN/m$^{3/2}$ and σ_Y = 1500 MPa.

Solution:

A penny-shaped crack in the interior of the material will give the stress intensity factor $K_I = \sigma_0 \sqrt{\pi a}\, f_8(a)$. Case 8 in Appendix 3 of the textbook gives (terms containing a/t can be neglected here so that f_8 = 0.637 is obtained)

$$K_I = 0.637\,\sigma_0 \sqrt{\pi a} \tag{a}$$

(a) No crack propagation is allowed. Then the stress intensity range ΔK_I must be less than (or equal to) the threshold value ΔK_{th}. Using ΔK_{th} = 7 MN/m$^{3/2}$ and $\Delta\sigma_0 = \sigma_Y / 2$ = 150 MPa, one obtains

$$\Delta K_I = 0.637 \cdot 150 \sqrt{\pi a} = 7 \text{ MN/m}^{3/2} \tag{b}$$

giving a = 1.7 mm.

(b) Similarly, using ΔK_{th} = 4 MN/m$^{3/2}$ and $\Delta\sigma_0 = \sigma_Y / 2$ = 750 MPa, one obtains

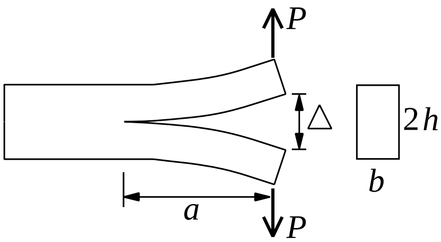
$$\Delta K_I = 0.637 \cdot 750 \sqrt{\pi a} = 4 \text{ MN/m}^{3/2} \tag{c}$$

giving a = 0.0223 mm.

It is concluded that the high-strength steel is much more sensitive to cracks than the low-strength steel. (One reason is that the high-strength steel is loaded at higher stress.)

Answer: Critical crack length is (a) a = 1.7 mm, and (b) a = 0.022 mm.

6/2.



A double cantilever beam (DCB) specimen is subjected to a pulsating loading. The crack length is a, thickness is b and specimen height is $2h$, where $a >> b$ and $a >> h$. The material is linear elastic with Young's modulus E. Use Paris' law

$$\frac{da}{dN} = C\ (\Delta K_I)^n$$

to determine the number of loading cycles required to make the crack grow from its initial length a_i to a final length a, if

(a) the load is prescribed; the load varies between 0 and P_{max},
(b) the displacement is prescribed; the displacement varies between 0 and v_{max} (in the figure one has $\Delta = 2v_{max}$).

Solution:
(a) The stress intensity range becomes ($\Delta p = P_{max}/b$ (N/m) is the load range)

$$\Delta K_I = \frac{2\sqrt{3}\,\Delta p}{h\sqrt{h}}\ a = \frac{2\sqrt{3}\,P_{max}}{b\ h\sqrt{h}}\ a \qquad \text{(a)}$$

Entering (a) into Paris' law gives

$$\frac{da}{dN} = C\ (\Delta K_I)^n = C\left(\frac{2\sqrt{3}\,P_{max}}{b\ h\sqrt{h}}\ a\right)^n \qquad \text{(b)}$$

Separating (b) gives

$$\frac{da}{a^n} = C\left(\frac{2\sqrt{3}\,P_{max}}{b\ h\sqrt{h}}\right)^n dN \qquad \text{(c)}$$

Integrating (c) and solving for N give

$$N = \frac{1}{C\,(1-n)}\left\{\frac{b\ h\sqrt{h}}{2\sqrt{3}\,P_{max}}\right\}^n (a^{1-n} - a_i^{1-n}) \qquad (n \neq 1) \qquad \text{(d)}$$

(b) The stress intensity range becomes (here $v_{max} = \Delta/2$ is the range of the displacement)

$$\Delta K_I = \frac{\sqrt{3}\, E\, h^2\, v_{max}}{2\sqrt{h}} \frac{1}{a^2} \qquad \text{(e)}$$

Entering (e) into Paris' law gives

$$\frac{da}{dN} = C\,(\Delta K_I)^n = C\left(\frac{\sqrt{3}\, E\, h^2\, v_{max}}{2\sqrt{h}} \frac{1}{a^2}\right)^n \qquad \text{(f)}$$

Separating (f) gives

$$\int_{a_i}^{a} a^{2n}\,da = \int_0^N C\left(\frac{\sqrt{3}\, E\, h^2\, v_{max}}{2\sqrt{h}}\right)^n dN \qquad \text{(g)}$$

Integrating (g) and solving for N give

$$N = \frac{1}{C\,(2n+1)}\left\{\frac{2\sqrt{h}}{\sqrt{3}\, E\, h^2\, v_{max}}\right\}^n (a^{2n+1} - a_i^{2n+1}) \qquad \text{(h)}$$

Answer: Number of cycles required is, respectively,

(a)
$$N = \frac{1}{C\,(1-n)}\left\{\frac{b\, h\sqrt{h}}{2\sqrt{3}\, P_{max}}\right\}^n (a^{1-n} - a_i^{1-n}) \qquad (n \neq 1)$$

(b)
$$N = \frac{1}{C\,(2n+1)}\left\{\frac{2\sqrt{h}}{\sqrt{3}\, E\, h^2\, v_{max}}\right\}^n (a^{2n+1} - a_i^{2n+1})$$

6/3.

A structure is subjected to a pulsating loading, *i.e.*. $\sigma_{min} = 0$. Embedded circular (penny-shaped) cracks are discovered in the structure. The crack diameters are $2a_0$, which is much smaller than the thickness of the material. The structure is made of a material that follows the crack propagation law

$$\frac{da}{dN} = C\,(\Delta K_I)^n$$

(a) Determine the largest (remote) stress range $\Delta\sigma_0$ the structure may be subjected to if crack propagation is not allowed.

(b) Determine the maximum stress range that may be allowed if the structure is loaded by N = 10 000 cycles. During the loading the crack is not allowed to grow so large that it becomes unstable. Therefore, choose safety factor s = 1.5.

Numerical data: crack length a_0 = 0.0005 m, σ_Y = 620 MPa, ΔK_{th} = 2 MN/m$^{3/2}$, K_{Ic} = 36 MN/m$^{3/2}$, n = 4.0 and $C = 1.12{\cdot}10^{-11}$ m^7/MN^4.

Solution:

(a) *No crack propagation*

If no crack propagation is allowed, the stress intensity range ΔK_I must be less than or equal to the threshold value ΔK_{th}. Assume that there exists a crack with the most unfavourable direction. The stress intensity factor K_I then is (Case 8 in Appendix 3 in the textbook, with $a << t$)

$$K_I = \sigma_0 \sqrt{\pi a}\, f_8\left(\frac{a}{c}, 0\right) \tag{a}$$

where $a = c$, which gives $f_8(1,0) = 0.637$. Thus

$$K_I = 0.637\, \sigma_0 \sqrt{\pi a} \tag{b}$$

The stress intensity range ΔK_I becomes ($\Delta\sigma_0$ is stress range)

$$\Delta K_I = 0.637\, \Delta\sigma_0 \sqrt{\pi a} \tag{c}$$

No crack propagation is expected if $\Delta K_I \leq \Delta K_{th}$, which gives

$$0.637\, \Delta\sigma_0 \sqrt{\pi a} \leq \Delta K_{th} \tag{d}$$

Solving for $\Delta\sigma_0$ gives

$$\Delta\sigma_0 \leq \frac{\Delta K_{th}}{0.637\sqrt{\pi a}} = \frac{2 \cdot 10^6}{0.637\sqrt{\pi \cdot 0.0005}} = 79 \cdot 10^6 \text{ N/m}^2 = 79 \text{ MPa} \tag{e}$$

Thus, if $\Delta\sigma_0 \leq 79$ MPa no crack growth is expected.

(b) *Crack propagation*

After $N = 10^4$ loading cycles the crack should give a stress intensity factor $K_{Imax} = K_{Ic} / s$. Determine the critical crack length a_c giving this stress intensity factor.

Assume that the crack stays circular (penny-shaped) during the propagation.

One then has

$$K_I = 0.637\, \sigma_0 \sqrt{\pi a} \tag{f}$$

Fracture will occur (if LEFM is valid) when

$$K_{Imax} = \frac{K_{Ic}}{s} \tag{g}$$

Equations (f) and (g) give

$$0.637\sigma_0\sqrt{\pi\, a_c} = \frac{K_{Ic}}{s} \quad \text{giving} \quad a_c = \frac{1}{\pi}\left(\frac{K_{Ic}}{0.637\, \sigma_0\, s}\right)^2 = \frac{1}{\pi}\left(\frac{36}{0.637\, \sigma_0 \cdot 1.5}\right)^2 = \frac{452}{\sigma_0^2} \tag{h}$$

where, here, σ_0 (in MPa) equals the stress range $\Delta\sigma_0$ (in MPa) because $\sigma_{min} = 0$ (load is pulsating).

Now the initial crack length a_0 and the final crack length a_c (expressed in the unknown stress range $\Delta\sigma_0$) are known. During N cycles the crack will grow from a_0 to a_c. The crack propagation law (Paris' law) gives

$$\frac{da}{dN} = C\ (\Delta K_I)^n = C(0.637\,\Delta\sigma_0\sqrt{\pi a})^n \qquad \text{(i)}$$

which gives

$$\int_{a_0}^{a_c} \frac{da}{(\sqrt{a}\ )^n} = \int_0^N C\ (0.637\,\Delta\sigma_0\sqrt{\pi}\ )^n\ dN \qquad \text{(j)}$$

Evaluation of the integrals gives

$$\frac{a_c^{1-n/2} - a_0^{1-n/2}}{1-n/2} = C\ (0.637\,\Delta\sigma_0\sqrt{\pi}\ )^n \cdot N \qquad \text{(k)}$$

Entering the values of a_0, a_c, C, n and N into (k) gives

$$\frac{(452/(\Delta\sigma_0)^2)^{-1} - (0.5\cdot 10^{-3}\ )^{-1}}{-1} = 1.12\cdot 10^{-11}\,(0.637\,\Delta\sigma_0\sqrt{\pi}\ )^4 \cdot 10^4 \qquad \text{(l)}$$

Solving for $\Delta\sigma_0$ gives $\Delta\sigma_0$ = 324 MPa = σ_0.

Finally, LEFM was used when a_c was determined. Is LEFM valid?
The stress σ_0 = 324 MPa (= σ_{max}) gives the critical crack length

$$a_c = \frac{452}{324^2} = 0.0043\text{ m} \qquad \text{(m)}$$

to be compared with

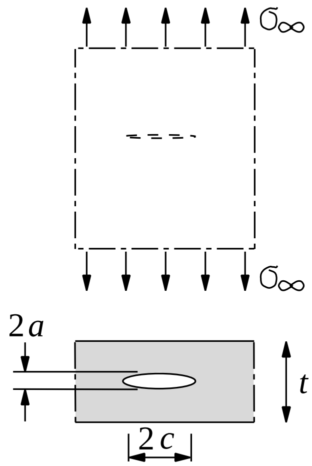
$$2.5\left(\frac{K_{Ic}}{s\cdot\sigma_Y}\right)^2 = 2.5\left(\frac{36}{1.5\cdot 620}\right)^2 = 0.0037\text{ m} \qquad \text{(n)}$$

Thus, a_c is larger than 0.0037 m so LEFM may be used (t and $W-a$ are assumed large enough).

Note that only the crack length a_c at fracture is of interest here. The fracture criterion K_{Imax} = K_{Ic}/s was used to determine a_c. Only then LEFM must be valid, not during the crack propagation phase.

Answer: The stress range should be less than (a) $\Delta\sigma_0$ = 79 MPa, (b) $\Delta\sigma_0$ = 324 MPa.

6/4.



A large plate of thickness t contains an embedded elliptical crack, see figure. Determine the cyclic life (the number of cycles to failure) of the plate if
(a) the load varies between 0 and σ_∞,
(b) the load varies between $0.5\sigma_\infty$ and σ_∞.
The influence of the mean value of the stress intensity factor is disregarded. The crack is assumed to keep its form during the crack propagation (*i.e.*, ratio a/c is constant during the crack propagation).

The material follows Paris' law for crack propagation. Use safety factor s = 1.4.
Numerical data: crack length a_0 = 0.001 m and c_0 = 0.002 m, thickness t = 0.10 m, yield limit σ_Y = 1200 MPa, threshold value $\Delta K_{th} < 6$ MN/m$^{3/2}$, fracture toughness K_{Ic} = 60 MN/m$^{3/2}$, and crack propagation parameters n = 4.0 and C = $5\cdot10^{-13}$ m^7/MN^4, σ_∞ = 400 MPa.

Solution:
For an embedded elliptical crack the stress intensity factor is (Case 8 in Appendix 3; $a << t$ implies that terms containing a/t may be neglected)

$$K_I = \sigma_\infty \sqrt{\pi a}\, f_8\left(\frac{a}{c}, 0\right) = \sigma_\infty \sqrt{\pi a}\, f_8\left(\frac{1}{2}, 0\right) = 0.826\, \sigma_\infty \sqrt{\pi a} \qquad \text{(a)}$$

The stress intensity range ΔK_I becomes

$$\Delta K_I = 0.826\, \Delta\sigma_\infty \sqrt{\pi a} \qquad \text{(b)}$$

The initial crack length a_0 = 0.001 m gives the stress intensity range
Case (a):

$$\Delta K_I = 0.826\, \Delta\sigma_\infty \sqrt{\pi a} = 0.826 \cdot 400 \sqrt{\pi\, 0.001} = 18.5 \text{ MN/m}^{3/2} > \Delta K_{th} \qquad \text{(c)}$$

Case (b):

$$\Delta K_I = 0.826\, \Delta\sigma_\infty \sqrt{\pi a} = 0.826 \cdot 200 \sqrt{\pi\, 0.001} = 9.26 \text{ MN/m}^{3/2} > \Delta K_{th} \qquad \text{(d)}$$

Thus, the crack will grow in both cases.

Next, determine the critical crack length a_c. Failure will occur (if linear elastic fracture mechanics theory, LEFM, may by used) when

$$K_{\mathrm{Imax}} = \frac{K_{\mathrm{Ic}}}{s} \quad \text{giving} \quad 0.826 \cdot 400 \sqrt{\pi\, a_c} = \frac{60}{1.4} \tag{e}$$

which gives a_c = 0.0054 m at failure (or, to be exact, the crack will start to grow rapidly at this crack length and the failure will come after a few more cycles).

Check the conditions for LEFM. One has

$$2.5 \left(\frac{K_{\mathrm{Ic}}}{s\ \sigma_{\mathrm{Y}}} \right)^2 = 2.5 \left(\frac{60}{1.4 \cdot 1200} \right)^2 = 0.0032 \text{ m} \tag{f}$$

Thus, a_c > 0.0032 m and LEFM can be used (also t and $W - a$ are large enough).

Paris' law now gives

$$\frac{\mathrm{d}a}{\mathrm{d}N} = C\ (\Delta K_{\mathrm{I}})^n = C\ (0.826\, \Delta\sigma_0 \sqrt{\pi a}\ )^n \tag{g}$$

Separating and integrating give

$$\int_{a_0}^{a_{\mathrm{cr}}} \frac{1}{(\sqrt{a}\ )^n} \mathrm{d}a = \int_0^N C\ (0.826\, \Delta\sigma_0 \sqrt{\pi}\ )^n\ \mathrm{d}N \tag{h}$$

Thus

$$N = \frac{a_{\mathrm{cr}}^{1-n/2} - a_0^{1-n/2}}{(1 - n/2)\ C\ (0.826\, \Delta\sigma_0 \sqrt{\pi}\ )^n} \tag{i}$$

Case (a) gives:

$$N^{(\mathrm{a})} = \frac{0.0054^{-1} - 0.001^{-1}}{(-1)\ 5 \cdot 10^{-13}\ (0.826 \cdot 400 \sqrt{\pi}\ )^4} = 13\,855 \text{ cycles} \tag{j}$$

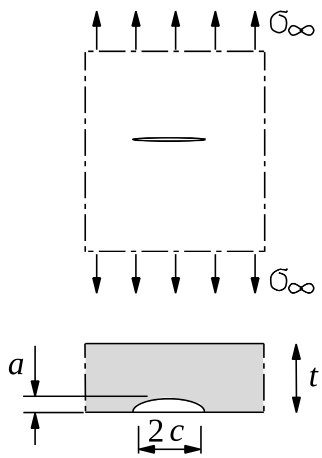
Case (b) gives:

$$N^{(\mathrm{b})} = \frac{0.0054^{-1} - 0.001^{-1}}{(-1)\ 5 \cdot 10^{-13}\ (0.826 \cdot 200 \sqrt{\pi}\ )^4} = 2^4\, N^{(\mathrm{a})} = 221\,680 \text{ cycles} \tag{k}$$

Thus $N^{(b)} = 2^4\ N^{(a)}$ = 221 600 cycles.

Answer: Cyclic life to failure is, approximately, (a) N = 13 850 cycles, and (b) N = 221 600 cycles (*cf.* Problem 6/9).

6/5.



A large plate of thickness t contains an elliptical surface crack, see figure. The load is cyclic and it varies between 0 and σ_∞.

(a) Determine the relationship between the remote stress σ_∞ and the number of loading cycles N for crack propagation to depths a = 5, 10, 15 and 20 mm.

(b) Determine also the stress level that may give unstable crack growth at the different crack depths. The crack is assumed to keep its form during the crack propagation. The material follows Paris' crack propagation law. Use safety factor s = 1.5.

Numerical data: crack length a_0 = 0.002 m, c_0 = 0.004 m, t = 0.10 m, σ_Y = 1200 MPa, ΔK_{th} = 6 MN/m$^{3/2}$, K_{Ic} = 70 MN/m$^{3/2}$, n = 3.75, $C = 9.22 \cdot 10^{-12}$ m$^{6.625}$/MN$^{3.75}$.

Solution:

Here the stress intensity range becomes (Case 7 in Appendix 3 in the textbook, terms containing a/t may be neglected here)

$$\Delta K_I = \sigma_\infty \sqrt{\pi a}\, f_7\left(\frac{1}{2}, 0\right) = 0.896\, \sigma_\infty \sqrt{\pi a} \qquad \text{(a)}$$

Paris' law then becomes

$$\frac{da}{dN} = C\,(\Delta K_I)^n = C\,(0.896\,\sigma_\infty \sqrt{\pi a}\,)^{3.75} \qquad \text{(b)}$$

Separating and integrating give

$$\int_{a_{\text{initial}}}^{a_{\text{final}}} \frac{da}{(\sqrt{a}\,)^{3.75}} = \int_0^N C\,(0.896\,\sigma_\infty \sqrt{\pi}\,)^{3.75}\, dN \qquad \text{(c)}$$

Thus

$$\frac{1}{C\,(0.896\sqrt{\pi}\,)^{3.75}} \frac{-1}{0.875}\left(\frac{1}{a_{\text{final}}^{0.875}} - \frac{1}{a_{\text{initial}}^{0.875}}\right) = (\sigma_\infty)^{3.75} N \qquad \text{(d)}$$

or, rewritten,

$$3.75 \log\sigma_\infty + \log N = \log\left\{\frac{1}{C\,(0.896\sqrt{\pi})^{3.75}} \frac{-1}{0.875}\left(\frac{1}{a_{\text{final}}^{0.875}} - \frac{1}{a_{\text{initial}}^{0.875}}\right)\right\} \qquad \text{(e)}$$

which gives, for $a = a_{\text{final}}$ = 5, 10, 15, and 20 mm, respectively,

$$3.75 \log\sigma_\infty + \log N = 12.44,\ 12.57,\ 12.62, \text{ and } 12.64 \qquad \text{(f)}$$

(b) The critical stress at the different crack lengths is, from (a), (and if LEFM can be used)

$$K_{\text{Imax}} = (\,\Delta K_{\text{I}} =)\; 0.896\,\sigma_\infty \sqrt{\pi\, a} = \frac{1}{1.5}\, K_{\text{Ic}} = 46.7\ \text{MN/m}^{3/2} \tag{g}$$

Check for LEFM. One has

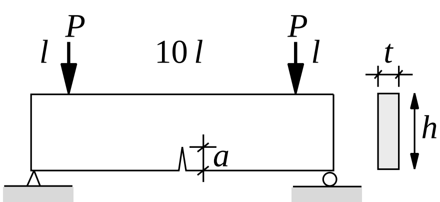
$$2.5\left(\frac{K_{\text{Ic}}}{s \cdot \sigma_{\text{Y}}}\right)^2 = 0.00378\ \text{m} \tag{h}$$

implying that one may use LEFM.
With (g), the four crack lengths 5, 10, 15, and 20 mm give, respectively,

$$\sigma_\infty = 416,\ 294,\ 240,\ \text{and}\ 208\ \text{MPa} \tag{i}$$

Answer: (a) $3.75\log\sigma_\infty + \log N = 12.44$, 12.57, 12.62, and 12.64 for a = 5, 10, 15, and 20 mm, respectively, and
(b) σ_∞ = 416, 294, 240, and 208 MPa, respectively.

6/6.



A beam with an edge crack is loaded with two forces $P = P_0 \sin\omega t$, see figure. Determine a lower limit of the cyclic life of the beam (assume that no other cracks are present). The material follows Paris' crack propagation law. Use safety factor s = 1.5.

Numerical data: crack depth a_0 = 0.005 m, thickness t = 0.01 m, height h = 0.10 m, length l = 0.10 m, yield limit σ_{Y} = 1400 MPa, threshold value ΔK_{th} = 4 MN/m$^{3/2}$, fracture toughness K_{Ic} = 70 MN/m$^{3/2}$, and crack propagation parameters n = 4 and $C = 1\cdot 10^{-13}$ m^7/MN^4. Load amplitude is P_0 = 0.042 MN.

Solution:

Between the two forces P, the bending moment in the beam is constant and it is Pl. The maximum value of the bending moment is $P_0\, l$ and the minimum value contributing to the stress range is zero (it is assumed that the negative part of the moment will not contribute to the crack growth). Thus, the range of the bending moment contributing to the stress range is $P_0\, l$.
The stress intensity factor becomes (Case 6 in Appendix 3 in the textbook)

$$K_{\text{I}} = \frac{6\,P\,l}{t\,W^2}\sqrt{\pi\, a}\; f_6\!\left(\frac{a}{W}\right) \tag{a}$$

For crack length $a = a_0$ = 0.005 m one has $a / W = a_0 / h$ = 1/20 = 0.05, which gives $f_6(0.05)$ = 1.08.

Determine the crack length a_c at failure. One obtains

$$K_I = \frac{6\,P_0\,l}{t\,W^2}\sqrt{\pi\,a_c}\,f_6\left(\frac{a_c}{W}\right) = \frac{K_{Ic}}{s} \tag{b}$$

Here $f_6(a_c / W)$ is unknown so one has to try (to guess) a value of f_6. Try $f_6(a_c / W) = 1.06$ to see what happens (we see from the diagram that f_6 decreases in the region where $a / W = 0.05$). It gives, for stress and fracture toughness expressed in MPa,

$$K_I = \frac{6 \cdot 0.042 \cdot 0.1}{0.01 \cdot 0.10^2}\sqrt{\pi\,a_c}\,1.06 = \frac{70}{1.5} \tag{c}$$

Solving for a_c gives $a_c = 0.0097$ m.
This value of a_c gives $f_6(a_c / W) = f_6(0.097) = 1.06$, which is the same as the value assumed. Therefore, the critical crack length $a_c = 0.0097$ is selected for the following calculations.

Paris' law now gives

$$\frac{da}{dN} = C\,(\Delta K_I)^n = C\left(\frac{6\,P_0\,l}{t\,W^2}\sqrt{\pi\,a}\,f_6\left(\frac{a}{W}\right)\right)^4 \tag{d}$$

Again, we have the problem of not knowing the value of the function $f_6 = f_6(a)$ when the crack grows, *i.e.* when a changes. From the above calculations (and from the form of the function f_6 seen in the diagram) it is concluded that f_6 varies between 1.08 and 1.06 in the crack length interval obtained here. Therefore, to be on the safe side, select $f_6 = 1.08$ for the whole interval (this is a conservative estimation, as it will give a too large crack propagation rate). Now equation (d) will give (with stresses in MPa, see the dimension of the constant C)

$$\frac{da}{a^2} = C\left(\frac{6\,P_0\,l}{t\,W^2}\sqrt{\pi}\,1.08\right)^4 dN \tag{e}$$

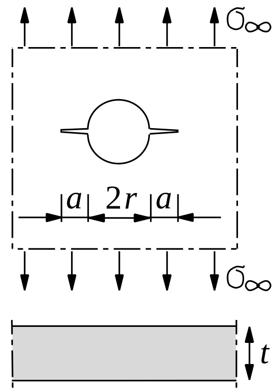
or

$$\int_{0.005}^{0.0097}\frac{da}{a^2} = N \cdot C\left(\frac{6\,P_0\,l}{t\,W^2}\sqrt{\pi}\,1.08\right)^4 \tag{f}$$

from which $N = 17\,900$ is solved. As we have used a too large value of f_6 this will be a lower limit of the cyclic life of the structure. The cyclic life is expected to be larger than the life calculated. Thus, it is expected that N will be slightly more than 17 900 cycles. (An upper limit of the fatigue life is estimated if $f_6 = 1.06$ is used. Then one obtains, see equation (f), $N_{upper} = 17\,900\,(1.08/1.06)^4 = 19\,290$ cycles.)

Answer: The fatigue life is expected to be slightly more than $N = 17\,900$ cycles (but less than 19290 cycles). Critical crack length is $a_{crit} = 0.0097$ m.

6/7.



In a large plate of a ship's hull two cracks of approximately the same size were discovered at a hole, see figure. The plate is loaded in cyclic tension with $\sigma_{min} = 0.2\sigma_\infty$ and $\sigma_{max} = \sigma_\infty = 60$ MPa. The loading frequency is 6 cycles per minute. Determine the remaining life (in hours) of the plate. It is assumed that the crack propagation is symmetric with respect to the hole.

The material follows Paris' law for crack propagation. Use safety factor s = 1.5. The geometry function f used to determine the stress intensity factor may in this example, and for the crack length used here, be approximated to

$$f\left(\frac{a}{r}\right) = 1.42\left(\frac{a}{r}\right)^{-1/6}$$

(Thus, this is an approximation of function f_3 in Appendix 3 of the textbook.)

Numerical data: $r = a_0 = t = 0.08$ m, $\sigma_Y = 600$ MPa, $K_{Ic} = 90$ MN/m$^{3/2}$, $n = 3$, $C = 1\cdot 10^{-11}$ m$^{11/2}$/MN^3.

Solution:

In this example, the stress intensity factor may be written, approximately,

$$K_I = \sigma_0\sqrt{\pi a}\, f\left(\frac{a}{r}\right) = \sigma_0\sqrt{\pi a}\,\frac{1.42}{(a/r)^{1/6}} = 1.42\sqrt{\pi}\,\sigma_0\, a^{1/3} r^{1/6} \qquad \text{(a)}$$

(Note that the function f given here is an approximation of the function f_3 given in Appendix 3 in the textbook.)

Determine the critical crack length a_{cr}. Equation (a) gives, by use of $K_{Imax} = K_{Ic} / s$,

$$a = a_{cr} = \left(\frac{K_{Ic}}{s}\,\frac{1}{1.42\sqrt{\pi}\,\sigma_\infty\, r^{1/6}}\right)^3 = \left(\frac{90}{1.5\cdot 1.42\sqrt{\pi}\,60\cdot 0.08^{1/6}}\right)^3 = 0.22175 \text{ m} \qquad \text{(b)}$$

May linear elastic fracture mechanics (LEFM) be used here? One has

$$2.5\left(\frac{K_{Ic}}{s\cdot\sigma_Y}\right)^2 = 0.025 \text{ m} \qquad \text{(c)}$$

Yes, all three measures a, t, and $W - a$ are larger than 0.025 m, so LEFM may be used. The critical crack length is a_{cr} as calculated in (b).

The stress intensity range becomes $\Delta\sigma_\infty = \sigma_{max} - \sigma_{min} = 0.8\sigma_\infty$, which gives

$$\Delta K_I = 1.42\sqrt{\pi}\,0.8\sigma_\infty\, a^{1/3}\, r^{1/6} \tag{d}$$

Paris' law gives

$$\frac{da}{dN} = C\,(\Delta K_I)^n = C\,(1.42\sqrt{\pi}\,0.8\sigma_\infty\, a^{1/3}\, r^{1/6})^3 \tag{e}$$

Thus

$$\int_{a_0}^{a_{cr}} \frac{da}{a} = \int_0^N C\,(1.42\sqrt{\pi}\,0.8\sigma_\infty\, r^{1/6})^3\, dN \tag{f}$$

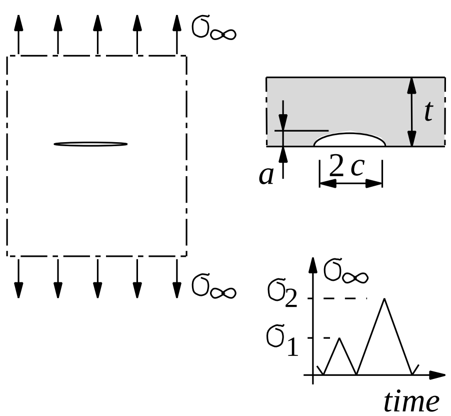
giving

$$N = \frac{1}{C\,(1.42\sqrt{\pi}\,0.8\sigma_\infty\, r^{1/6})^3} \ln\frac{a_{cr}}{a_0} = 204\,400 \text{ cycles} \tag{g}$$

Remaining fatigue life is $t = N/(6\cdot 60) = 568$ h.

Answer: Fatigue life N = 204 000 cycles gives that the remaining life is expected to be 568 hours.

6/8.



A large plate of thickness t contains a half-elliptical surface crack, see figure. The load consists of a repeated sequence of two cycles as shown in the figure. Determine the expected remaining life of the plate expressed in number of sequences to failure. It is assumed that the crack keeps its form during the crack propagation. The material follows Paris' law. Use safety factor $s = 1.4$.

Numerical data: a_0 = 0.002 m, c_0 = 0.004 m, t = 0.100 m, σ_Y = 1200 MPa, K_{Ic} = 70 MN/m$^{3/2}$, ΔK_{th} = 6 MN/m$^{3/2}$, n = 4, $C = 1\cdot 10^{-13}$ m^7/MN^4, σ_1 = 200 MPa and σ_2 = 400 MPa .

Solution:

A half-elliptical surface crack (Case 7 in Appendix 3 in the textbook) gives the stress intensity factor, for $a/c = 0.5$,

$$K_I = \sigma_0\sqrt{\pi a}\; f_7\!\left(\frac{a}{c}, 0\right) = \sigma_0\sqrt{\pi a}\; 0.896 = 0.90\,\sigma_0\sqrt{\pi a} \tag{a}$$

The range of the stress intensity factor becomes

$$\Delta K_I = 0.90\,\Delta\sigma_0\sqrt{\pi a} \tag{b}$$

Will the lower stress level σ_1 = 200 MPa contribute to the crack growth?
Let $a = a_0$ = 0.002 m and $\Delta\sigma_0 = \sigma_1$ = 200 MPa. One then obtains from (b)

$$\Delta K_\mathrm{I} = 0.90 \cdot 200\,\sqrt{\pi \cdot 0.002} = 14.3\ \mathrm{MN/m^{3/2}} \tag{c}$$

The threshold value is ΔK_th = 6 MN/m$^{3/2}$, giving $\Delta K_\mathrm{I} > \Delta K_\mathrm{th}$.

This implies that the crack will propagate at the lower stress cycle σ_1, and thereby also at the higher stress level σ_2.

Determine next the crack length a_crit at fracture. Assume that linear elastic fracture mechanics (LEFM) is valid. Fracture will then occur when $K_\mathrm{Imax} = K_\mathrm{Ic}/s$. It gives

$$K_\mathrm{Imax} = 0.90\,\sigma_0\sqrt{\pi \cdot a_\mathrm{crit}} = \frac{K_\mathrm{Ic}}{s} \tag{d}$$

Fracture will occur when the highest stress is loading the structure, *i.e.* when the stress is σ_2. For $\sigma_0 = \sigma_2$ one obtains from (d) (if linear elastic fracture mechanics is valid)

$$a_\mathrm{crit} = \frac{1}{\pi}\left(\frac{K_\mathrm{Ic}}{s \cdot 0.90 \cdot \sigma_2}\right)^2 = \frac{1}{\pi}\left(\frac{70}{1.4 \cdot 0.90 \cdot 400}\right)^2 = 0.0061\ \mathrm{m} \tag{e}$$

Is linear elastic fracture mechanics valid?

$$2.5\left(\frac{K_\mathrm{Ic}}{s \cdot \sigma_\mathrm{Y}}\right)^2 = 2.5\left(\frac{70}{1.4 \cdot 1200}\right)^2 = 0.0043\ \mathrm{m} \tag{f}$$

which is less than a_crit, t, and $W - a$. Thus, linear elastic fracture mechanics can be used.

It is now known that the crack will grow from a_0 to a_crit, and both cycles in the sequence will contribute to the crack propagation. Per *sequence* of two cycles, the crack propagation will be

$$\frac{\mathrm{d}a}{\mathrm{d}N_\mathrm{s}} = C\sum(\Delta K_{\mathrm{I}i})^n = C\sum(0.90\,\Delta\sigma_{0i}\sqrt{\pi \cdot a}\,)^n$$

$$= C\left\{(0.90\,\sigma_1\sqrt{\pi \cdot a}\,)^n + (0.90\,\sigma_2\sqrt{\pi \cdot a}\,)^n\right\} \tag{g}$$

Note that N_s stands for the number of sequences (not number of cycles).
Separation of variables gives

$$\int_{a_0}^{a_\mathrm{crit}} \frac{\mathrm{d}a}{(\sqrt{a}\,)^n} = C\left\{(0.90\,\sigma_1\sqrt{\pi}\,)^n + (0.90\,\sigma_2\sqrt{\pi}\,)^n\right\}\int_0^{N_\mathrm{s}} \mathrm{d}N_\mathrm{s} \tag{h}$$

which gives

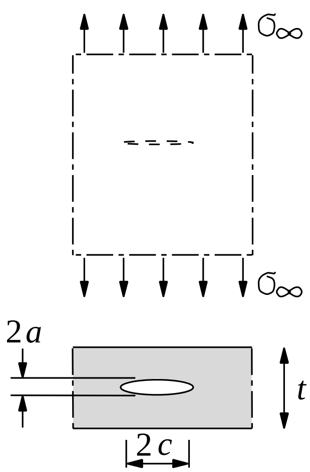
$$a_{\text{crit}}^{1-n/2} - a_0^{1-n/2} = \left(1 - \frac{n}{2}\right) C\ (0.90\sqrt{\pi})^n\ \{\ \sigma_1^n + \sigma_2^n\} \cdot N_s \qquad \text{(i)}$$

Entering a_{crit}, a_0, σ_1, σ_2, and the material parameters C and n in (i), and then solving for N_s, give

$$N_s = 19\,080 \text{ sequences} \qquad \text{(j)}$$

Answer: Expected fatigue life is N_s = 19 000 sequences, giving N = 38 000 cycles to failure.

6/9.



A large plate of thickness t contains an embedded elliptical crack, see figure. Determine the cyclic life (the number of cycles to failure) of the plate if the load varies between $0.5\sigma_\infty$ and σ_∞.

The influence of the mean value of the stress intensity factor has to be taken into account. The crack is assumed to keep its form during the crack propagation (*i.e.*, ratio a/c is constant during the crack growth).

(This is Problem 6/4(b) once again, but now the influence of the mean value is taken into account.)

The material follows Paris' law for crack propagation. Use safety factor s = 1.4.

Numerical data: crack length a_0 = 0.001 m, c_0 = 0.002 m, t = 0.10 m, σ_Y = 1200 MPa, ΔK_{th} < 6 MN/m$^{3/2}$, K_{Ic} = 60 MN/m$^{3/2}$, n = 4.0 and $C_1 = 5 \cdot 10^{-13}$ m^7/MN^4, γ = 0.7, σ_∞ = 400 MPa (it is assumed that the material parameters C and n in Paris' law are the same here as in Problem 6/4).

Solution:

Following the solution to Problem 6/4, one obtains the stress intensity factor for an embedded elliptical crack as

$$K_{\text{I}} = \sigma_\infty \sqrt{\pi a}\ f_8\left(\frac{a}{c}, \frac{a}{t}\right) = \sigma_\infty \sqrt{\pi a}\ f_8\left(\frac{1}{2}, 0\right) = 0.826\ \sigma_\infty \sqrt{\pi a} \qquad \text{(a)}$$

The stress intensity range ΔK_{I} becomes

$$\Delta K_{\text{I}} = 0.826\ \Delta\sigma_\infty \sqrt{\pi a} \qquad \text{(b)}$$

The initial crack length a_0 = 0.001 m gives the stress intensity range

$$\Delta K_{\mathrm{I}} = 0.826\,\Delta\sigma_{\infty}\sqrt{\pi\,a} = 0.826 \cdot 200\,\sqrt{\pi\,0.001} = 9.26\ \mathrm{MN/m}^{3/2} > \Delta K_{\mathrm{th}} \qquad \text{(c)}$$

Thus, the crack will grow at the stress range given.

Determine the critical crack length a_c. Failure will occur (if linear elastic fracture mechanics theory, LEFM, may by used) when

$$K_{\mathrm{Imax}} = \frac{K_{\mathrm{Ic}}}{s} \quad \text{giving} \quad 0.826 \cdot 400\,\sqrt{\pi\,a_{\mathrm{c}}} = \frac{60}{1.4}\ \mathrm{MN/m}^{3/2} \qquad \text{(d)}$$

which gives a_c = 0.0054 m at failure (or, to be exact, the crack will start to grow rapidly at this crack length, and the failure will come after a very limited number of loading cycles after this crack length has been reached).

Check the conditions for LEFM. One has

$$2.5\left(\frac{K_{\mathrm{Ic}}}{s\ \sigma_{\mathrm{Y}}}\right)^2 = 2.5\left(\frac{60}{1.4 \cdot 1200}\right)^2 = 0.0032\ \mathrm{m} \qquad \text{(e)}$$

Thus, a_c > 0.0032 m and LEFM can be used (also t and $W - a$ are large enough).

Paris' law, with correction for the mean stress value according to Walker, now gives

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \frac{C_1}{(1-R)^{n(1-\gamma)}}\,(\Delta K_{\mathrm{I}})^n = C\ (0.826\,\Delta\sigma_0\,\sqrt{\pi a}\,)^n \qquad \text{(f)}$$

where stress ratio R is R = 0.5. The factor C becomes $C = C_1/(1-R)^{n(1-\gamma)}$ = $1.1487 \cdot 10^{-12}\ \mathrm{m^7/MN^4}$.

Separating and integrating give

$$\int_{a_0}^{a_{\mathrm{cr}}} \frac{1}{(\sqrt{a}\,)^n}\,\mathrm{d}a = \int_0^N C\ (0.826\,\Delta\sigma_0\,\sqrt{\pi}\,)^n\ \mathrm{d}N \qquad \text{(g)}$$

giving

$$N = \frac{a_{\mathrm{cr}}^{1-n/2} - a_0^{1-n/2}}{(1-n/2)\ C\ (0.826\,\Delta\sigma_0\,\sqrt{\pi}\,)^n} \qquad \text{(h)}$$

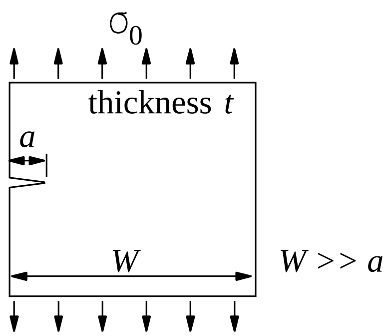
Thus,

$$N^{(\mathrm{a})} = \frac{0.0054^{-1} - 0.001^{-1}}{(-1)\ 1.1487 \cdot 10^{-12}\ (0.826 \cdot 200\,\sqrt{\pi}\,)^4} = 96\,500\ \text{cycles} \qquad \text{(i)}$$

The expected fatigue life now becomes N = 96 500 cycles, to be compared with N = 221 700 cycles in Problem 6/4. These two numbers differ by the factor $(1-R)^{n(1-\gamma)}$ = 0.435, which means that in this case the mean value plays an important role for the fatigue life of the structure.

Answer: Expected fatigue life is N = 96 500 cycles.

6/10.



Determine the number of loading cycles required to make the crack in the figure grow by 1 mm. The structure is loaded with stress cycles of constant amplitude. The stress range $\Delta\sigma_0$ is $\Delta\sigma_0$ = 100 MPa. Use some different initial crack lengths a_i, say a_i = 1, 5, 10, 20, 40, 60, and 80 mm.

For crack growth calculations use both the un-modified Paris law:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = C\,(\Delta K_\mathrm{I})^n$$

and the modified version, taking the closeness to the threshold value into account:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = C\,\{(\Delta K_\mathrm{I})^n - (\Delta K_\mathrm{th})^n\}$$

Compare the results. Assume that the constants C and n are the same in the two formulae: $C = 2.5\cdot 10^{-12}$ (units in MN and meter) and $n = 3.2$. The material has the fatigue threshold value ΔK_th = 6 MN/m$^{3/2}$.

Solution:

First investigate if crack propagation is obtained for the shortest crack length a_i = 1 mm. The intensity range for the component with the edge crack is (Case 5 in Appendix 3 in the textbook, use that $W >> a$)

$$\Delta K_\mathrm{I} = \Delta\sigma_0\sqrt{\pi a}\; f_5\left(\frac{a}{W}\right) = 1.12\,\Delta\sigma_0\sqrt{\pi a} \tag{a}$$

The stress range $\Delta\sigma_0$ = 100 MPa gives, for crack length a_i = 1 mm,

$$\Delta K_\mathrm{I} = \Delta\sigma_0\sqrt{\pi a}\; f_5\left(\frac{a}{W}\right) = 1.12\cdot 100\sqrt{\pi\cdot 0.001} = 6.28\ \mathrm{MN/m}^{/2} \tag{b}$$

which is larger than the threshold value ΔK_th = 6 MN/m$^{3/2}$. It is concluded that the crack will grow for all crack lengths a_i.

Paris' law gives

$$\frac{\mathrm{d}a}{\mathrm{d}N} = C\,(\Delta K_\mathrm{I})^n \tag{c}$$

where ΔK_I is obtained from (a).

Separating and solving for N give

$$N = N^{(\mathrm{d})} = \frac{1}{C}\int_{a_i}^{a_i+0.001} \frac{\mathrm{d}a}{(1.12\cdot 100\sqrt{\pi})^{3.2}\, a^{1.6}} \tag{d}$$

The modified crack propagation law gives

$$N = N^{(\mathrm{e})} = \frac{1}{C}\int_{a_i}^{a_i+0.001} \frac{\mathrm{d}a}{(1.12\cdot 100\sqrt{\pi})^{3.2}\, a^{1.6} - (\Delta K_{\mathrm{th}})^{3.2}} \tag{e}$$

Numerical solution (for example by use of MATLAB) of (d) and (e) gives results according to the table below. It is noted that only when the stress intensity range is very close to the threshold value there is a large difference between the two results (about one million cycles). A difference was expected, since the crack growth rate tends to zero at the threshold value, whereas the unmodified Paris law predicts a finite crack growth rate also at the threshold value. Already at the crack length 5 mm the difference in crack growth rate between the two formulae is small; only a few per cent.

Making the crack grow from 1 mm to 81 mm would, according to the unmodified Paris law, require $1.73{\cdot}10^6$ cycles, whereas the modified version predicts $2.89{\cdot}10^6$ cycles. The main part of this difference (one million cycles) depends on the slower crack growth the first millimetre.

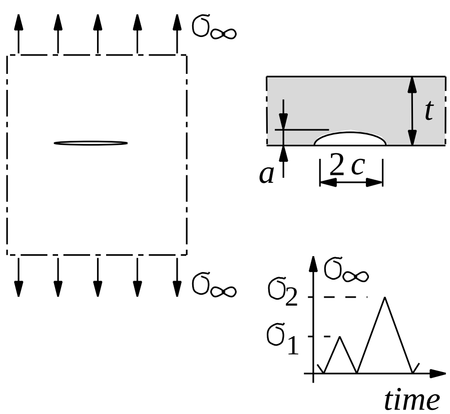
Table: Number of loading cycles required to make the crack grow 1 mm for different initial lengths a_i of the crack. The fatigue life $N^{(\mathrm{d})}$ is calculated with the un-modified Paris law whereas $N^{(\mathrm{e})}$ is calculated with the modified crack propagation law given in the problem. It is seen that the two formulae give different results only when the stress intensity range $\Delta K_{\mathrm{I}}(a_{\mathrm{i}})$ is very close to the threshold value ΔK_{th} (here ΔK_{th} = 6 MN/m$^{3/2}$).

a_{i} (m)	0.001	0.005	0.010	0.020	0.040	0.060	0.080
$N^{(\mathrm{d})}$ (cycles)	$0.63{\cdot}10^6$	73 628	26 056	8 923	3 001	1 579	1 000
$N^{(\mathrm{e})}$ (cycles)	$1.68{\cdot}10^6$	78 105	26 592	8 985	3 008	1 581	1 000
$\Delta K_{\mathrm{I}}(a_{\mathrm{i}})$ (MN/m$^{3/2}$)	6.27	14.0	19.9	28.1	39.7	48.6	56.1

In the table is also given the stress intensity range $\Delta K_{\mathrm{I}}(a_{\mathrm{i}})$ for the initial length a_{i} of the crack. It is noted that the fracture toughness K_{Ic} of the material must be so large that after reduction with the safety factor s the fracture toughness should be larger than $\Delta K_I(a_i + 0.001) = \Delta K_I(0.081) = 56.5$ MN/m$^{3/2}$. Thus, K_{Ic} must be such that $K_{Ic} > s \cdot 56.5$ MN/m$^{3/2}$.

Answer: The un-modified propagation law gives
$N = 0.63 \cdot 10^6$, 73 600, 26 060, 8 900, 3 000, 1 580, 1000 cycles,
and the modified formula gives
$N = 1.68 \cdot 10^6$, 78 100, 26 600, 9 000, 3 000, 1 580, 1000 cycles, respectively.

6/11.



A large plate of thickness t contains a half-elliptical surface crack, see figure. The load consists of a repeated sequence of two cycles as shown in the figure. The crack is assumed to keep its form during the crack propagation. The material follows Paris' law. The crack is detected when the crack length a_0 is $a_0 = 0.002$ m. It is decided that the crack can be allowed to grow to the length $a_{final} = 0.006$ m before the crack has to be repaired.

(a) Determine the expected number of cycles (twice the number of sequences) required to make the crack grow from a_0 to a_{final}. Use Paris' law to determine the crack propagation "cycle by cycle".

(b) What number of cycles would be obtained if the root mean square value (*i.e.* ΔK_{rms}) were used as an equivalent measure of the stress intensity range?

Numerical data: crack depth $a_0 = 0.002$ m, $c_0 = 0.004$ m, $t = 0.100$ m, $\sigma_Y =$ 1200 MPa, $K_{Ic} = 70$ MN/m$^{3/2}$, $\Delta K_{th} = 6$ MN/m$^{3/2}$, $n = 4$, $C = 1 \cdot 10^{-13}$ m^7/MN^4, $\sigma_1 = 200$ MPa and $\sigma_2 = 400$ MPa.

Solution:
A half-elliptical surface crack (Case 7 in Appendix 3 in the textbook) gives the stress intensity factor (for $a/c = 0.5$)

$$K_I = \sigma_0 \sqrt{\pi a}\, f_7\left(\frac{a}{c}\right) = 0.896\, \sigma_0 \sqrt{\pi a} \qquad \text{(a)}$$

The range of the stress intensity factor becomes

$$\Delta K_I = 0.896\, \Delta\sigma_0 \sqrt{\pi a} \qquad \text{(b)}$$

Will the lower stress level σ_1 = 200 MPa contribute to the crack growth?
Let $a = a_0$ = 0.002 m and $\Delta\sigma_0 = \sigma_1$ = 200 MPa. One obtains

$$\Delta K_{\mathrm{I}} = 0.896 \cdot 200\sqrt{\pi \cdot 0.002} = 14.2\ \mathrm{MN/m}^{3/2} \qquad \text{(c)}$$

The threshold value is ΔK_{th} = 6 MN/m$^{3/2}$, giving $\Delta K_{\mathrm{I}} > \Delta K_{\mathrm{th}}$.
This implies that the crack will propagate for the lower stress cycle σ_1, and thereby also for the higher stress cycle σ_2.

Check the stress intensity factor for the higher stress σ_2 and crack length a_{final} = 6 mm. One obtains

$$K_{\mathrm{I}} = 0.896 \cdot 400\sqrt{\pi \cdot 0.006} = 49.2\ \mathrm{MN/m}^{3/2} \qquad \text{(d)}$$

Thus, at the planned repair of the crack, the stress intensity factor is K_{I} = 49.2 MN/m$^{3/2}$, which is 70 per cent of the critical value K_{Ic} = 70 MN/m$^{3/2}$ (the "safety factor" is s = 70 / 49.2 = 1.42).

It is given in the problem that the crack will grow from a_0 to a_{final}. Both cycles in the sequence will contribute to the crack growth. Then, *per loading sequence* the crack growth will be

$$\frac{\mathrm{d}a}{\mathrm{d}N_{\mathrm{s}}} = C\ \sum(\Delta K_{\mathrm{I}\,i})^n = C\ \sum(0.896\,\Delta\sigma_{0\,i}\sqrt{\pi \cdot a}\,)^n$$

$$= C\left\{(0.896\,\sigma_1\sqrt{\pi \cdot a}\,)^n + (0.896\,\sigma_2\sqrt{\pi \cdot a}\,)^n\right\} \qquad \text{(e)}$$

Note that N_{s} here stands for the number of sequences (not number of cycles).
Separation of variables gives

$$\int_{a_0}^{a_{\mathrm{final}}} \frac{\mathrm{d}a}{(\sqrt{a}\,)^n} = C\left\{(0.896\,\sigma_1\sqrt{\pi}\,)^n + (0.896\,\sigma_2\sqrt{\pi}\,)^n\right\}\int_0^{N_{\mathrm{s}}} \mathrm{d}N_{\mathrm{s}} \qquad \text{(f)}$$

which gives

$$a_{\mathrm{final}}^{1-n/2} - a_0^{1-n/2} = \left(1 - \frac{n}{2}\right) C\ (0.896\sqrt{\pi}\,)^n\ \{\sigma_1^n + \sigma_2^n\} \cdot N_{\mathrm{s}} \qquad \text{(g)}$$

Entering a_{final}, a_0, σ_1, σ_2, and the material parameters C and n in (g), and then solving for N_{s}, give

$$N_{\mathrm{s}} = 19\,265\ \text{ sequences,} \quad \text{giving} \quad N = 38\,500\ \text{ cycles} \qquad \text{(h)}$$

(b) Using the rms value as an equivalent stress intensity range, one obtains

$$\Delta K_{\text{rms}} = \sqrt{\frac{\sum\{ (\Delta K_i)^2 \, n_i \}}{\sum n_i}} = 0.896\sqrt{\pi\, a}\,\sqrt{\frac{200^2 + 400^2}{2}}$$

$$= 0.896\sqrt{\pi\, a}\cdot 316.23 \text{ MN/m}^{3/2} \tag{i}$$

Thus, the equivalent stress range becomes $\Delta\sigma_{\text{eq}}^{(\text{rms})}$ = 316.23 MPa.

Paris' law then gives

$$a_{\text{final}}^{1-n/2} - a_0^{1-n/2} = \left(1 - \frac{n}{2}\right) C\,(0.92\sqrt{\pi})^n \left\{ (\Delta\sigma_{\text{eq}}^{(\text{rms})})^n \right\} \cdot N \tag{j}$$

giving N = 52 400 cycles.

Thus, the rms value here predicts 52 400 / 38 500 = 1.36 times too many cycles. During the 52 400 – 38 500 ≈ 13 900 extra cycles, the crack will grow (far) beyond the 6 mm limit where the crack should have been repaired.

Conclusion: The conclusion is that the rms value can *not* be used for crack propagation calculations.

(c) *Extra problem*:

Use N = 52 400 cycles (26 200 sequences) to see what crack length would have been obtained if Paris' law were used as in problem (a). One obtains

$$a_{\text{final}}^{1-n/2} - a_0^{1-n/2} = \left(1 - \frac{n}{2}\right) C\,(0.896\sqrt{\pi})^n \,\{ \sigma_1^n + \sigma_2^n \} \cdot 26\,200 \tag{k}$$

giving a_{final} = 0.0214 m = 21.4 mm.

This crack length gives the stress intensity factor, at stress σ_2,

$$K_{\text{I}} = 0.896\,\sigma_2\sqrt{\pi\, a} = 0.896 \cdot 400\,\sqrt{\pi \cdot 0.0214} = 92.9 \text{ MN/m}^{3/2} \tag{l}$$

which is much larger than the fracture toughness K_{Ic} of the material. The structure would fail long before the 52 400 cycles had been applied.

Comment:

The importance (influence) of the high amplitude loading cycles (here σ_2) is diminished when the rms value is used. Instead, use

$$\Delta K_{\text{I equivalent}} = \left\{ \frac{\sum\{ (\Delta K_{\text{I}i})^n \, n_i \}}{\sum n_i} \right\}^{1/n} \tag{m}$$

where n is the factor in Paris' law (and n_i is number of cycles at each stress level). One then obtains the equivalent stress intensity range

$$\Delta K_{\text{I equivalent}} = \left\{ \frac{\sum \{ (\Delta K_{\text{I}i})^n \, n_i \}}{\sum n_i} \right\}^{1/n} = 0.896\sqrt{\pi\, a} \left\{ \frac{200^4 + 400^4}{2} \right\}^{1/4}$$

$$= 0.896\sqrt{\pi\, a}\ 341.5 \text{ MN/m}^{3/2} \qquad \text{(n)}$$

This value of the stress intensity range gives

$$a_{\text{final}}^{1-n/2} - a_0^{1-n/2} = \left(1 - \frac{n}{2}\right) C\ (0.896\sqrt{\pi}\,)^n\ \{341.5^4\} \cdot 38\,500 \qquad \text{(o)}$$

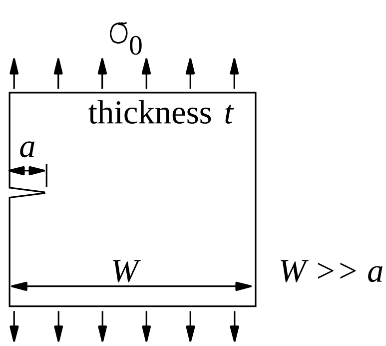
which gives $a_{\text{final}} = 0.006$ m, as it should.

Answer: (a) 38 500 cycles will propagate the crack to length 6 mm, (b) using the rms value of the stress intensity range, 52 400 cycles are obtained, which is 36 per cent too many cycles. That many cycles would propagate the crack to a length that would be longer than the critical value of the crack length. Thus, failure would occur during the loading if the rms value were used in the calculations.

6/12.

The irregular loading sequence of a machine component has been analysed and reduced to the 60 stress range cycles given in the table below. Upon inspection of the component, edge cracks of depth a = 1 mm (or longer) will be detected and eliminated. The maximum permissible crack length in the component is a = 5 mm, see figure.

Stress range $\Delta\sigma_0$ (MPa)	285	270	200	92
Number of cycles N	1	3	12	44



Determine the maximum number of loading sequences the component may be loaded with between two inspections. Assume that cracks of length 1 mm may remain in the structure after an inspection.

The material has the fatigue threshold value ΔK_{th} = 8 MN/m$^{3/2}$.

For crack growth calculations Paris' law may be used:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = C\,(\Delta K_{\mathrm{I}})^n$$

where $C = 2.5 \cdot 10^{-12}$ (units in MN and meter) and $n = 3.2$.

Solution:

First investigate if all stress cycles will contribute to the crack propagation. The stress intensity range for the component with the edge crack is (Case 5 in Appendix 3 in the textbook, use that $W >> a$)

$$\Delta K_{\mathrm{I}} = \Delta\sigma_0 \sqrt{\pi\, a}\; f_5\left(\frac{a}{W}\right) = 1.12\, \Delta\sigma_0 \sqrt{\pi\, a} \tag{a}$$

The stress range $\Delta\sigma_0$ = 92 MPa gives

$$\Delta K_{\mathrm{I}} = \Delta\sigma_0 \sqrt{\pi\, a}\; f_5\left(\frac{a}{W}\right) = 1.12 \cdot 92\, \sqrt{\pi \cdot 0.001} = 5.77\ \mathrm{MN/m}^{/2} \tag{b}$$

which is below the threshold value ΔK_{th} = 8 MN/m$^{3/2}$. It is concluded that at least initially the lower stress range cycles $\Delta\sigma_0$ = 92 MPa do not contribute to the crack propagation.

Check also for the next stress range cycle: $\Delta\sigma_0$ = 200 MPa. One obtains

$$\Delta K_{\mathrm{I}} = \Delta\sigma_0 \sqrt{\pi\, a}\; f_5\left(\frac{a}{W}\right) = 1.12 \cdot 200\, \sqrt{\pi \cdot 0.001} = 12.6\ \mathrm{MN/m}^{/2} \tag{c}$$

This cycle will induce crack propagation at all crack lengths.

Determine at which crack length the stress range cycles $\Delta\sigma_0$ = 92 MPa will start to contribute to the crack propagation. One has

$$\Delta K_{\mathrm{I}} = \Delta\sigma_0 \sqrt{\pi\, a}\; f_5\left(\frac{a}{W}\right) = 1.12 \cdot 92\, \sqrt{\pi \cdot a_1} = 8\ \mathrm{MN/m}^{/2} \tag{d}$$

giving a_1 = 1.92 mm.

Thus, as long as the crack is shorter than 1.92 mm, only the cycles with stress range 200 MPa and larger will make that the crack grows. When the crack has reached 1.92 mm all cycles (all four stress ranges) will be damaging to the structure (*i.e.* give crack propagation).

The crack growth must be calculated i two steps: first the growth from 1 mm to 1.92 mm, and then the growth from 1.92 mm to 5 mm.

Paris' law gives

$$\frac{\mathrm{d}a}{\mathrm{d}N} = C\,(\Delta K_{\mathrm{I}})^n \tag{e}$$

Using notation N_s for number of loading sequences, one obtains

$$\frac{\mathrm{d}a}{\mathrm{d}N_s} = C\,\sum(\Delta K_{\mathrm{I}\,i})^n = C\,(1.12\sqrt{\pi}\,)^n\,\{1\cdot 285^n + 3\cdot 270^n + 12\cdot 200^n\}\,(\sqrt{a}\,)^n \tag{f}$$

Separating and integrating give

$$\int_{0.001}^{0.00192} \frac{\mathrm{d}a}{a^{1.6}} = C\,(1.12\sqrt{\pi}\,)^n\,\{1\cdot 285^n + 3\cdot 270^n + 12\cdot 200^n\}\,N_s \tag{g}$$

Solving for N_s gives $N_s = N_{s1} = 2867$ sequences, giving $N = N_1 = 2867\cdot 60 =$ 172 000 cycles (of these 172 000 cycles $2867\cdot 44 =$ 126 100 cycles do not contribute to the crack growth).

When the crack has reached length 1.92 mm, all stress range cycles contribute to the crack propagation. One obtains

$$\frac{\mathrm{d}a}{\mathrm{d}N_s} = C\,\sum(\Delta K_{\mathrm{I}\,i})^n$$

$$= C\,(1.12\sqrt{\pi}\,)^n\,\{1\cdot 285^n + 3\cdot 270^n + 12\cdot 200^n + 44\cdot 92^n\}\,(\sqrt{a}\,)^n \tag{h}$$

Separating and integrating give

$$\int_{0.00192}^{0.005} \frac{\mathrm{d}a}{a^{1.6}} = C\,(1.12\sqrt{\pi}\,)^n\,\{1\cdot 285^n + 3\cdot 270^n + 12\cdot 200^n + 44\cdot 92^n\}\,N_s \tag{i}$$

Solving for N_s now gives $N_s = 2254$ sequences (giving $N = N_2 = 2254\cdot 60 =$ 135 240 cycles).

In total, 2867 + 2254 = 5121 sequences, giving 307 000 cycles may be allowed between two inspections.

Paris' law has been used for crack length a = 5 mm. The largest stress range cycle is $\Delta\sigma_0$ = 285 MPa. This gives the stress intensity range

$$\Delta K_{\mathrm{I}} = 1.12\,\Delta\sigma_0\,\sqrt{\pi\,a} = 1.12\cdot 285\,\sqrt{\pi\cdot 0.005} = 40\ \mathrm{MN/m}^{3/2} \tag{j}$$

It is concluded that after reduction with a safety factor s ($s > 1$) the fracture toughness K_{Ic} of the material must be at least 40 MN/m$^{3/2}$. Thus, one must have $K_{\mathrm{Ic}} > s\cdot 40$ MN/m$^{3/2}$.

Alternative solution:
Solve the problem by use of the equivalent stress range $\Delta\sigma_{eq}$.

For crack length less than 1.92 mm, the equivalent stress range becomes

$$\Delta\sigma_{\text{equivalent}} = \left\{ \frac{\sum\{ (\Delta\sigma_{0i})^n \, n_i \}}{\sum n_i} \right\}^{1/n} = \left\{ \frac{1 \cdot 285^{3.2} + 3 \cdot 270^{3.2} + 12 \cdot 200^{3.2}}{16} \right\}^{1/3.2}$$

$$= 223.85 \text{ MPa} \tag{k}$$

Paris' law gives

$$\frac{\mathrm{d}a}{\mathrm{d}N} = C \, (\Delta K_{\mathrm{I}})^n = C \, (1.12 \, \Delta\sigma_{\text{equivalent}} \sqrt{\pi a}\,)^n \tag{l}$$

Separating and integrating give

$$\int_{0.001}^{0.00192} \frac{\mathrm{d}a}{(\sqrt{a}\,)^n} = C \, (1.12 \cdot 223.85 \sqrt{\pi}\,)^n \cdot N \tag{m}$$

Solving for N gives N = 45 867 cycles. Dividing N by 16 (the number of damaging cycles in a sequence) gives the number of sequences N_s to make the crack grow from length 1 mm to length 1.92 mm. One obtains $N_s = N_{s1}$ = 45 867 / 16 = 2867 sequences, which is in agreement with the result obtained above.

For crack length larger than 1.92 mm, the equivalent stress range becomes

$$\Delta\sigma_{\text{equivalent}} = \left\{ \frac{\sum\{ (\Delta\sigma_{0i})^n \, n_i \}}{\sum n_i} \right\}^{1/n}$$

$$= \left\{ \frac{1 \cdot 285^{3.2} + 3 \cdot 270^{3.2} + 12 \cdot 200^{3.2} + 44 \cdot 92^{3.2}}{60} \right\}^{1/3.2} = 155.13 \text{ MPa} \tag{n}$$

Separating and integrating Paris' law give

$$\int_{0.00192}^{0.005} \frac{\mathrm{d}a}{(\sqrt{a}\,)^n} = C \, (1.12 \cdot 155.13 \sqrt{\pi}\,)^n \cdot N \tag{o}$$

Solving for N gives N = 135 247 cycles. Dividing N by 60 (the number of damaging cycles in a sequence) gives the number of sequences N_s to make the crack grow from length 1.92 mm to length 5 mm. One obtains $N_s = N_{s2}$ = 135 247 / 60 = 2254 sequences, which is in agreement with the result obtained above.

Answer: In total, N_s = 2867 + 2254 = 5121 sequences may be allowed between two inspections, giving 307 000 cycles. For crack length a < 1.92 mm the stress range cycle $\Delta\sigma_0$ = 92 MPa will not contribute to the crack propagation.

Solutions to problems in
T Dahlberg and A Ekberg: Failure, Fracture, Fatigue - An Introduction.
Studentlitteratur, Lund 2002, ISBN 91-44-02096-1.

Chapter 7

An introduction to fatigue

No problems in this chapter

Empty page.

Solutions to problems in
T Dahlberg and A Ekberg: Failure, Fracture, Fatigue - An Introduction.
Studentlitteratur, Lund 2002, ISBN 91-44-02096-1.

Chapter 8

Stress-based fatigue design

Problems with solutions

8/1.
A generator is rotating with a speed of 3000 rpm (revolutions per minute). Determine the number of loading cycles N the axle is subjected to due to its own weight. The machine is operating 7000 hours per year during 20 years.

Solution:
3000 revolutions per minute give 3000 loading cycles per minute. It gives $3000 \cdot 60 \cdot 7000$ loading cycles per year, giving $3000 \cdot 60 \cdot 7000 \cdot 20 = 25.2 \cdot 10^9$ loading cycler per 20 years.

Answer: Number N of loading cycles is $N = 25.2 \cdot 10^9$ cycles (thus, it may be exposed to fatigue).

8/2.
Estimate the fatigue limit in tension/compression for alternating and pulsating loading of a steel (approximately SS 2225) with the ultimate strength $\sigma_U = 810$ MPa. Sketch the Haigh diagram.
At an investigation of a similar material the relationships

$$\sigma_{FL} = 0.468\,\sigma_U \pm 50 \text{ MPa} \quad \text{and} \quad \sigma_{FLP} \cong 0.85\,\sigma_{FL}$$

were found. Use these relations and other “thumb rules” to estimate the fatigue limits (alternating and pulsating) of the material.

Solution:
The relationship $\sigma_{FL} = 0.468\sigma_U \pm 50$ MPa gives, for alternating loading:
$\sigma_{FL} \cong 0.468 \cdot 810 \pm 50$ MPa $= 380 \pm 50$ MPa,
i.e., the fatigue limit σ_{FL} in tension/compression at alternating loading may, according to this relationship, be expected to fall in the range 330 MPa to 430 MPa.
For pulsating loading the relationship between pulsating and alternating

loading, $\sigma_{FLP} \cong 0.85\sigma_{FL}$, gives $\sigma_{FLP} = 0.85\sigma_{FL} = 0.85\,(380 \pm 50)$ MPa = 320 ± 42 MPa. Thus, the fatigue limit in tension at pulsating loading is expected to fall in the range from 280 ± 280 MPa (from 320 – 42 MPa) to 360 ± 360 MPa (from 320 + 42 MPa).

Also, compare with other materials.

Material data for SIS 2225-03, -04 and -05 gives, for rotating bending, that σ_{FLB} should be in the range 350 MPa to 460 MPa. The engineering rule $\sigma_{FL} \cong 0.8\sigma_{FLB}$ gives that σ_{FL} should fall in the range 280 MPa to 370 MPa.

Another engineering rule says that σ_{FLR} (= σ_{FLB}) decreases from $0.45\sigma_U$ for mild steel to $0.38\sigma_U$ for the harder materials. The most unfavourable case, $\sigma_{FLR} = 0.38\sigma_U$ gives $\sigma_{FLR} = \sigma_{FLB} \cong 308$ MPa, while the case $\sigma_{FLR} = 0.45\sigma_U$ gives $\sigma_{FLR} = \sigma_{FLB} \cong 365$ MPa. This gives, using $\sigma_{FL} \cong 0.8\sigma_{FLB}$, that σ_{FL} should be in the range 250 MPa to 290 MPa.

Finally, select, hopefully "on the safe side", for alternating loading σ_{FL} = 280 MPa and for pulsating loading $\sigma_{FLP} = 0.85\sigma_{FL}$, which gives σ_{FLP} = 240 ± 240 MPa. (In practice, fatigue limits will probably be higher than the values estimated here.)

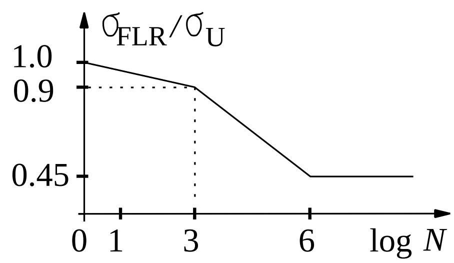
Answer: The fatigue limit σ_{FL} at alternating loading is estimated to 280 MPa, approximately, and the fatigue limit σ_{FLP} at pulsating loading is estimated to 240 ± 240 MPa, approximately.

8/3.

Sketch the Wöhler (SN, Stress-Number) curve for a normalised 37Mn Si5 steel bar with an ultimate strength of σ_U = 810 MPa. The bar is subjected to a rotating bending moment. At an investigation of a similar material the following fatigue lives were found:

$\sigma_{FLR} \cong 0.45\sigma_U$ yields $N \geq 10^6$ cycles,
$\sigma_{FLR} \cong 0.90\sigma_U$ yields N = 1000 cycles, and
$\sigma_{FLR} = \sigma_U$ yields N =1 cycle.

Solution and answer: See figure.

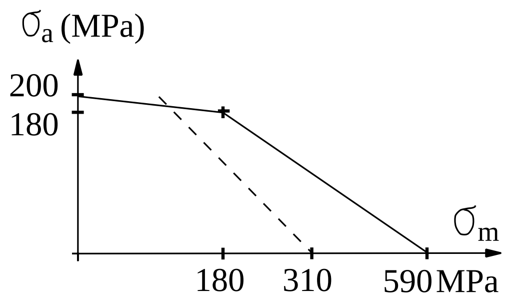


8/4.

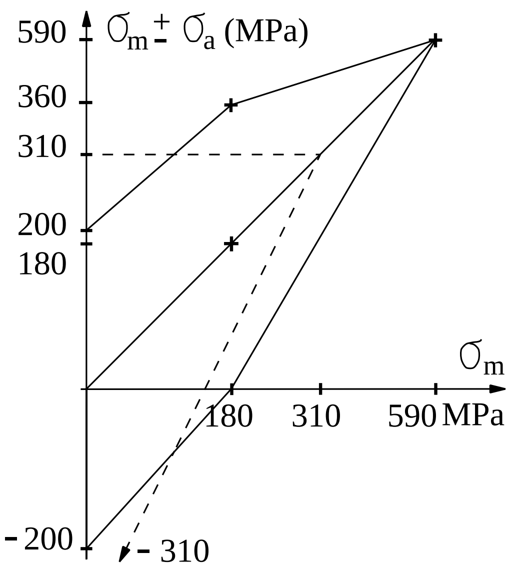
(a) Draw the fatigue diagram according to Haigh for the normalised steel SS 1650-01 subjected to loading in pure tension/compression.

(b) Draw the fatigue diagram according to Goodman for the normalised steel SS 1650-01 subjected to loading in pure tension/compression.

Solution:



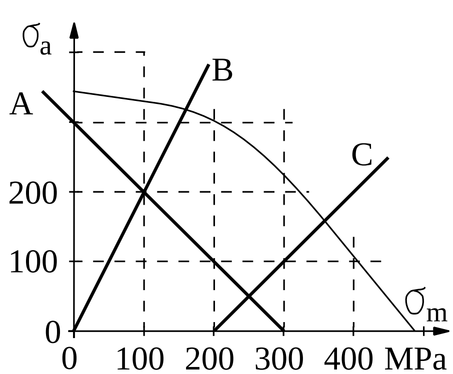
(a) Material data gives
$\sigma_{FL} = \pm\ 200$ MPa, $\sigma_{FLP} = 180 \pm 180$ MPa,
$\sigma_Y = 310$ MPa and $\sigma_U = 590$ MPa,
from which the Haigh diagram can be constructed, see figure.



(b) The same data gives the Goodman diagram according to the second figure. It is noted that this diagram is more complicated to draw than the Haigh diagram (one has both mean value σ_m and amplitude σ_a on the ordinate). It does not, however, contain more information than the Haigh diagram. (From now on, only the Haigh diagram will be used.)

Answer: See figures.

8/5.



Constant-amplitude fatigue strengths for materials subjected to different cyclic loading conditions are often expressed in Haigh diagrams. In the Haigh diagram given three straight lines (A, B and C) are shown. Each line corresponds to constant loading conditions. The loading varies sinusoidally with mean value σ_m, amplitude σ_a, maximum value σ_{max} (= $\sigma_m + \sigma_a$), minimum value σ_{min} (= $\sigma_m - \sigma_a$), and stress ratio $R = \sigma_{min}/\sigma_{max}$. Determine which one of these five loading variables is constant in the three cases shown in the figure.

Solution:
For line A one notices that the sum $\sigma_m + \sigma_a$ is a constant (= 300 MPa). Thus, line A gives that stress σ_{max} is constant.

For line B one notices that ratio σ_a / σ_m is constant (= 2). Thus, stress ratio R is constant.

For line C one notices that mean stress σ_m = 200 MPa + σ_a, giving $\sigma_m - \sigma_a$ = 200 MPa = σ_{min}. Thus, σ_{min} is constant.

Answer: Line A: stress σ_{max} is constant, line B: stress ratio R is constant, line C: stress σ_{min} is constant.

8/6.
Two axles with the same geometry (but of different size) are machined from the normalised steel SS 1650-01 (σ_U = 590 MPa). The axle diameters are 10 mm and 100 mm, respectively. One axle was machined from a raw material with a diameter 15 mm and the other from a raw material with diameter 120 mm. The loading is rotating bending.
Estimate the ratio of the reduced fatigue limits of the two axles if they have the same surface finish.

Solution:
In Case (a), axle of diameter 10 mm, and in Case (b), axle of diameter 100 mm, the following reduction factors are obtained:
due to the size (volume) of the raw material (15 mm and 120 mm, respectively): (a) $\lambda^{(a)} = 1.0$, and (b) $\lambda^{(b)} \cong 0.8$,
due to loaded volume: (a) $\delta^{(a)} = 1.0$, and (b) $\delta^{(b)} \cong 0.9$,
due to surface finish: (a) $\kappa^{(a)} = \kappa$, and (b) $\kappa^{(b)} = \kappa$, where κ is unknown, but the same in the two cases.

One obtains for Case (a) that $\lambda^{(a)} \cdot \delta^{(a)} \cdot \kappa^{(a)} = 1.0 \cdot \kappa$, and for Case (b) that $\lambda^{(b)} \cdot \delta^{(b)} \cdot \kappa^{(b)} \cong 0.72 \cdot \kappa$

The reduced fatigue limits σ_{FL}^{red} give

$$\frac{\sigma_{FL}^{\text{red, Case (b)}}}{\sigma_{FL}^{\text{red, Case (a)}}} = \frac{0.72 \cdot \kappa \cdot \sigma_{FL}}{1.0 \cdot \kappa \cdot \sigma_{FL}} = 0.72$$

Answer: The ratio of the two reduced fatigue limits is, approximately, 0.72.

8/7.

Assume that the fatigue limits of polished test specimens subjected to alternating loading in tension/compression are given by (ultimate strength is $R_m = \sigma_U$)

$$\sigma_{FL} = \frac{R_m}{2} \qquad \text{when } R_m \text{ is less than 1000 MPa, and}$$

$$\sigma_{FL} = 500 + \frac{R_m - 1000}{5} \qquad \text{when } R_m \text{ is greater than 1000 MPa.}$$

Estimate how the fatigue limits of different materials depend on the ultimate strength. Use some different surface roughnesses, for example a polished surface, a machined surface and a surface with a notch (a standard notch, see Figure 8.7).

Solution:

Factor κ for reduction of fatigue limit due to surface finish, and formulae above, give

	$R_m = \sigma_U =$	500	1000	1500 MPa
Polished specimen				
(κ = 1.0)	$\sigma_{FL}^{pol} =$	250	500	600 MPa
Machined specimen				$0.68 \cdot 600 =$
	$\sigma_{FL}^{red} = \kappa \cdot \sigma_{FL}^{pol} \cong$	$0.87 \cdot 250 = 220$	$0.78 \cdot 500 = 390$	410 MPa
Notch				$0.4 \cdot 600 =$
	$\sigma_{FL}^{red} = \kappa \cdot \sigma_{FL}^{pol} \cong$	$0.8 \cdot 250 = 200$	$0.59 \cdot 500 = 295$	240 MPa

It is concluded that high-strength steels are very sensitive to surface irregularities (surface finish) and notches.

Using the reduction factor K_r, where $\kappa = 1/K_r$, the KTH Handbook, Sundström (1998), gives for some values of the surface roughness measure R_a

$R_m = \sigma_U =$	500	1000	1500 MPa
Polished specimen			
$\sigma_{FL}^{pol} =$	250	500	600 MPa
$R_a = 10\ \mu m$			$0.60 \cdot 600 =$
$\sigma_{FL}^{red} = \kappa \cdot \sigma_{FL}^{pol} \cong$	$0.90 \cdot 250 = 225$	$0.65 \cdot 500 = 325$	360 MPa
$R_a = 100\ \mu m$			$0.25 \cdot 600 =$
$\sigma_{FL}^{red} = \kappa \cdot \sigma_{FL}^{pol} \cong$	$0.62 \cdot 250 = 155$	$0.33 \cdot 500 = 167$	150 MPa

Also here it is noted that the high-strength steels are sensitive to surface irregularities (surface finish) and notches.

Answer: Factor κ for reduction of fatigue limit due to surface finish, and the two formulae given in the problem, give reduced fatigue limits as given in the tables above. It is concluded that high-strength steels are very sensitive to surface roughness and to notches.

8/8.

The mean value of the fatigue limit of a welded 4 mm thick aluminium plate has been found to be 92 MPa and the standard deviation of the spread of the fatigue limit measurements is 6 MPa. Determine the stress level at which the probability of fatigue failure is 0.1 per cent. The spread of the fatigue limit is assumed to have a normal (Gaussian) distribution. For the normal distribution function $\Phi(x)$ one has

$\Phi(x) =$	0.50	0.90	0.99	0.999	0.9999
$x =$	0	1.28	2.33	3.09	3.72

Solution:

The mean value of the fatigue limit (here 92 MPa) gives the failure probability 50 per cent.

The standard deviation s is $s = 6$ MPa.

The failure probability 0.1 per cent is obtained at the stress level

$\sigma = \sigma_{mean} - 3.09 \cdot s = 92 - 3.09 \cdot 6 = 73.5$ MPa (where 3.09 was obtained from the table).

This implies that in mean one plate out of 1000 plates loaded at the stress level 73.5 MPa is expected to fail due to fatigue.

(At stress lever $\sigma_{mean} - 2.33 \cdot s$ = 78 MPa the failure probability is one per cent, which means that one plate out of 100 is expected to fail due to fatigue if the plates are loaded at this stress level, and so on.)

Answer: At stress level 73.5 MPa the probability of fatigue failure is expected to be 0.1 per cent.

8/9.

The fatigue lives of welded beams have been determined at seven (n = 7) identical tests. Stress range 250 MPa was used. The numbers of cycles to failure were

49 000, 61 000, 71 000, 81 000, 88 000, 110 000, and 135 000.

Determine the allowable number of cycles at the stress level given, if the probability of failure is not allowed to exceed 1 per cent with confidence C = 0.95.

The table below gives the "k-factors" for different values of n and P and with confidence C = 0.95.

Tolerance limits for the normal distribution (from Råde and Westergren (1998)).

The table gives factors k_1 and k_2 such that the following kind of statements can be made:

At least the proportion P is less than $\bar{x} + k_1 s$ with confidence 0.95.

At least the proportion P is greater than $\bar{x} - k_1 s$ with confidence 0.95.

At least the proportion P is between $\bar{x} - k_2 s$ and $\bar{x} + k_2 s$ with confidence 0.95.

n is sample size, $\bar{x}$ is sample mean, and s is sample standard deviation.

	P = 0.90		P = 0.95		P = 0.99	
n	k_1	k_2	k_1	k_2	k_1	k_2
6	3.006	3.733	3.707	4.422	5.062	5.758
7	2.755	3.390	3.399	4.020	4.641	5.241
8	2.582	3.156	3.188	3.746	4.353	4.889
9	2.454	2.986	3.031	3.546	4.143	4.633
10	2.355	2.856	2.911	3.393	3.981	4.437
15	2.068	2.492	2.566	2.965	3.520	3.885
20	1.926	2.319	2.396	2.760	3.295	3.621
25	1.838	2.215	2.292	2.638	3.158	3.462

Solution:

First calculate the logarithm of the number of cycles N to fatigue failure. It gives $x_i = \log N_i$ = 4.69, 4.79, 4.85, 4.91, 4.94, 5.04, and 5.13

Assume that x_i (*i.e.* $\log N_i$) has a normal distribution. This means that it is assumed that the x_i:s are samples taken from a normally distributed process. Determine the mean value $\bar{x}$ of x_i and the standard deviation s. One obtains

$$\bar{x} = \frac{\sum^{n} x_i}{n} = 4.9074$$

(which gives $\bar{N} = 10^{4.9074}$ = 80 789 cycles) and

$$s = \sqrt{\frac{1}{n-1}\sum^{n}(x_i - \bar{x})^2} = 0.1496$$

The probability of fatigue failure should be 1 per cent at confidence level 95 per cent. A table of statistics then gives, using number of samples n = 7, probability P = 0.99, and confidence level C = 0.95:

$$k = 4.641$$

Allowable value of x at failure probability 0.01 (= $1 - P$) becomes

$$x = x_{\text{allowable}} = \bar{x} - k \cdot s = 4.9074 - 4.641 \cdot 0.1496 = 4.2131$$

Number of cycles $N_{\text{allowable}}$ that can be allowed at failure probability 1 per cent is

$$N_{\text{allowable}} = 10^{x_{\text{allowable}}} = 10^{4.2131} = 16\,334$$

Answer: Allowable number of cycles is, for 1 per cent failure probability, $N_{\text{allowable}}$ = 16 300 cycles.

8/10.

A large plate has a small elliptical hole in it. The ratio of the axes of the hole is 2 to 1. Determine the stress concentration factor K_t when the plate is loaded in parallel to the

(a) minor axis,

(b) major axis.

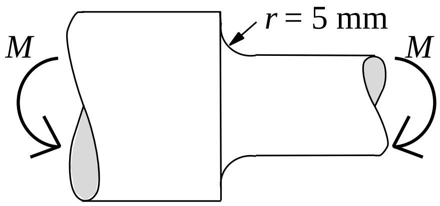
Solution:

Handbook (for example Appendix 2 in the textbook) gives, $K_t = (1 + 2b/a)$. Using $b/a = 2$ and $b/a = 1/2$ one obtains K_t = 5 and 2, respectively.

Comment: A large stress concentration implies a large probability of fatigue failure if the loading varies with time. Consequently, notches and other geometric irregularities should be placed so that the stress concentration is avoided or made as small as possible (*i.e.*, use large radii at the geometric irregularities).

Answer: The stress concentration factor K_t is (a) $K_t = 5$, and (b) $K_t = 2$.

8/11.



A ground (surface finish $R_a = 3\,\mu$m) axle of steel SS 1650-01 (σ_U = 590 MPa) has been manufactured from a circular bar of 200 mm diameter. The axle has a change of diameter from 180 mm to 160 mm, with the shoulder fillet radius $r = 5$ mm, see figure. Determine the maximum rotating bending moment M the axle may be subjected to if a safety factor of 2.0 with respect to fatigue failure should be used.

Solution:

The material has fatigue limit $\sigma_{FLB} = \pm$ 270 MPa when loaded with an alternating bending moment. The ultimate strength of the material is σ_U = 590 MPa.

The factors reducing the fatigue limit are

$\lambda = 0.80$ (diameter 200 mm),

$\delta = 1$ here, because the fatigue notch factor K_f will be used and this factor takes over the role of δ (else, one would have had $\delta = 0.87$),

and $\kappa = 0.92$ (ground axle with ultimate strength σ_U = 590 MPa).

Stress concentration gives $K_t = 2.25$ (here $D/d = 1.125$ and $r/d = 0.03$).

The fatigue notch factor K_f becomes

$$K_f = 1 + q\,(K_t - 1) = 1 + 0.84 \cdot (2.25 - 1) = 2.05$$

where $q = 0.84$ has been taken from Figure 8.11 in the textbook.

Using safety factor $s_a = 2.0$ one obtains

$$K_f\,\sigma_{\text{allowable}}^{\text{nom}} = \frac{\lambda\delta\kappa}{s_a}\sigma_{FLB} \quad \text{which gives} \quad \sigma_{\text{allowable}}^{\text{nom}} = \frac{\lambda\delta\kappa}{s_a K_f}\sigma_{FLB} = 0.18 \cdot 270 \text{ MPa}$$

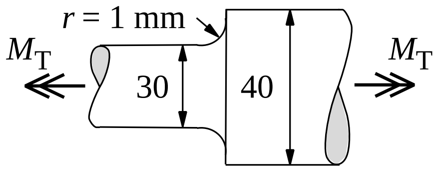
Using

$$\sigma_{\text{allowable}}^{\text{nom}} = \frac{32M}{\pi d^3} = 0.18 \cdot 270 \cdot 10^6 \text{ Pa}$$

one obtains the bending moment M = 19 kNm.

Answer: Maximum allowable bending moment M is M = 19 kNm.

8/12.



An axle has a change of diameter from 40 mm to 30 mm with a shoulder fillet with radius r = 1 mm, see figure. The fillet is ground (R_a = 5 μm). The material is SS 1550-01 with an ultimate strength $\sigma_U = R_m$ = 490 MPa and a fatigue limit (in torsion) $\tau_{FLT} = \pm$ 140 MPa.

Determine the maximum alternating torque M_T the axle may be subjected to if a factor of safety 3.0 should be used?

Solution:

The shear stress at the shoulder fillet is

$$\tau^{\text{shoulder}} = K_f \tau^{\text{nom}} = K_f \frac{16 M_T}{\pi d^3}$$

Figures 8.7, 8, 9, and 11, and Appendix 2 in the textbook give $\kappa = 0.94$, $\lambda = 0.90$, $\delta = 1$, $q = 0.70$, and $K_t = 1.85$.

The fatigue notch factor becomes

$$K_f = 1 + q\,(K_t - 1) \cong 1 + 0.70 \cdot (1.85 - 1) = 1.6$$

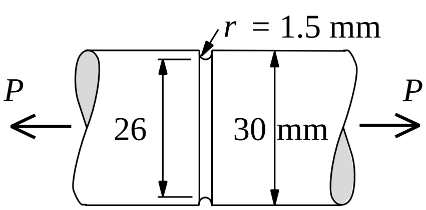
One obtains

$$1.6 \frac{16 M_T}{\pi d^3} = \frac{1}{s} \kappa \lambda \delta \tau^{\text{polished}} = \frac{1}{3} \cdot 0.94 \cdot 0.90 \cdot 1 \cdot 140 \cdot 10^6 \text{ Pa}$$

which, using d = 0.030 m, gives $M_T \cong 130$ Nm.

Answer: The maximum allowable torque is M_T = 130 Nm.

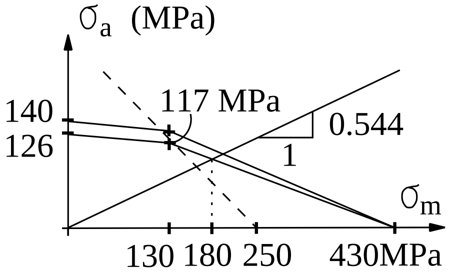
8/13.



An axle has a groove (ground, R_a = 7 μm) according to the figure. The axle is subjected to a tensile loading $P = 18 \pm 12$ kN. The material is 1450-01 with σ_U = 430 MPa, σ_Y = 250 MPa, σ_{FL} = 140 MPa and $\sigma_{FLP} = 130 \pm 130$ MPa.

Determine the safety factor with respect to fatigue failure if the ratio of the mean value and the amplitude of the loading stress is constant.

Solution:



Reduce the fatigue data. One obtains

$$\sigma_{FL}^{red} = \kappa\lambda\delta\sigma_{FL} = 0.95 \cdot 0.95 \cdot 1 \cdot 140 = 126 \text{ MPa}$$

$$\sigma_{FLP}^{red} = \kappa\lambda\delta\sigma_{FLP} = 0.95 \cdot 0.95 \cdot 1 \cdot 130 = 117 \text{ MPa}$$

Draw the Haigh diagram, see figure.
The notch (the groove) gives the stress concentration factor $K_t \cong 2.6$.

The fatigue notch factor K_f becomes (notch sensitivity factor q from Figure 8.11 in the textbook)

$$K_f = 1 + q\,(K_t - 1) \cong 1 + 0.7 \cdot (2.6 - 1) = 2.12$$

Stresses in the material at the notch become:
mean value $\sigma_m = 2.6\,P_m / A$ and amplitude $\sigma_a = 2.12\,P_a / A$ (the cross-sectional area A is calculated using the smallest diameter, d = 26 mm, at the notch).

Ratio σ_a / σ_m becomes

$$\frac{\sigma_a}{\sigma_m} = \frac{2.12 \cdot 12}{2.6 \cdot 18} = 0.544$$

A straight line with this slope is entered into the Haigh diagram. The intersection of this line with the reduced fatigue limit of the material gives that the allowable mean stress σ_m is $\sigma_m \cong 180$ MPa. Allowable mean value P_m of the tensile force P can now be calculated. One has

$$K_t \cdot \frac{4P_m}{\pi d^2} = \sigma_m = 180 \cdot 10^6 \text{ Pa}, \quad \text{which gives} \quad P_m = P_{allowable} = 36.7 \text{ kN}$$

Without safety factor the mean value of the froce can be $P_{allowable}$ = 36.7 kN. But, according to the problem, the applied load was $P = P_{applied}$ = 18 kN.
The safety factor then is

$$s = \frac{P_{allowable}}{P_{applied}} = \frac{36.7}{18} \cong 2.0$$

Answer: The safety factor is s = 2.0, approximately.

8/14.

When testing smooth specimens (polished) made from a high strength steel with yield limit σ_Y = 1510 MPa and ultimate strength σ_U = 1800 MPa, fatigue limits at $N = 10^7$ cycles were obtained at the following stress levels:

$$800 \pm 400, \quad 400 \pm 560, \quad 0 \pm 720, \quad -400 \pm 860, \text{ and } -800 \pm 850 \text{ MPa.}$$

Draw the Haigh diagram for notched cylindrical specimens of diameter D = 11 mm. The notch is 1.5 mm deep with a radius in the trough of 0.5 mm and it is polished.

Solution:

Determine the reduction factors κ, λ, δ, the stress concentration factor K_t, and the fatigue notch factor K_f.

Diameter $D = 11$ mm gives $\lambda \approx 0.98 \approx 1.0$.

No bending or torsion give $\delta = 1.0$.

Polished surface give $\kappa = 1.0$.

Thus $\kappa\lambda\delta = 1.0$, which means that no reduction due to surface finish and volume is needed here.

Stress concentration:

$D / d = 11/8 = 1.38$ and $\rho / d = r / d = 0.5/8 = 0.06$ give $K_t = 2.8$.

Fatigue notch factor $K_f = 1 + q\,(K_t - 1) = 1 + 0.95\,(2.8 - 1) = 2.7$.

To make the Haigh diagram valid for the notched specimen, the mean value (corresponding to a static load) should be reduced by the factor K_t and the amplitude by the factor K_f.

The point of fatigue limit 800 ± 400 then "moves" to $800 / 2.8 \pm 400 / 2.7 = 286 \pm 148$ MPa. For the points given, the following fatigue limits are obtained:

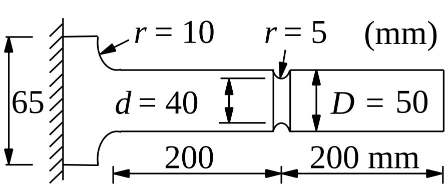
$$286 \pm 148, \quad 143 \pm 207, \quad 0 \pm 267, \quad -143 \pm 318, \text{ and } -286 \pm 315 \text{ MPa.}$$

The yield strength of the material is $\sigma_Y = 1510$ MPa. If yielding is to be avoided, the Haigh diagram should be limited at stress $\sigma_Y / K_t = 1510 / 2.8 = 540$ MPa both on the abscissa and on the ordinate.

Answer: $\sigma'_m = \sigma_m / K_t$, $\sigma'_a = \sigma_a / K_f$ and $\sigma'_m + \sigma'_a < \sigma_Y / K_t$, where $K_t = 2.8$ and $K_f = 2.7$, which give fatigue limits 286 ± 148; 143 ± 207; 0 ± 267; -143 ± 318; and -286 ± 315 MPa.

8/15.

The right end of the cantilever beam (see figure) is moved up and down; the displacement upwards is $2a$ and downwards a. Determine the maximum allowable value of a with respect to fatigue failure. Use safety factor $s = 2.0$.



The surface of the beam is polished. The material has the modulus of elasticity $E = 206$ GPa, the yield limit 390 MPa and the ultimate strength 590 MPa. Fatigue limits are ± 270 MPa at alternating bending and 240 ± 240 MPa at pulsating bending.

Solution:

The force P that loads the free (right) end of the beam is obtained from

$$\delta = \frac{PL^3}{3EI} \quad \text{which gives} \quad P = \frac{3EI}{L^3}\,\delta$$

Here $\delta = 0.5a \pm 1.5a$, L = 400 mm and I (= $\pi D^4/64$) is the second moment of area of the circular beam cross section with diameter D = 50 mm (implying that here the influence of the notch on the beam deflection is neglected).

At the notch the bending moment $M_{\text{b, notch}}$ is obtained as

$$M_{\text{b, notch}} = P\,\frac{L}{2} = \frac{3EI}{2L^2}\,\delta = \frac{3EI}{2L^2}\,(0.5\,a \pm 1.5\,a)$$

At the shoulder the bending moment becomes

$$M_{\text{b, shoulder}} = P\,L = \frac{3EI}{L^2}\,\delta = \frac{3EI}{L^2}\,(0.5\,a \pm 1.5\,a)$$

Stress concentration at the notch: K_t^{notch} = 1.8 and q = 0.83 give K_f^{notch} = 1.66.

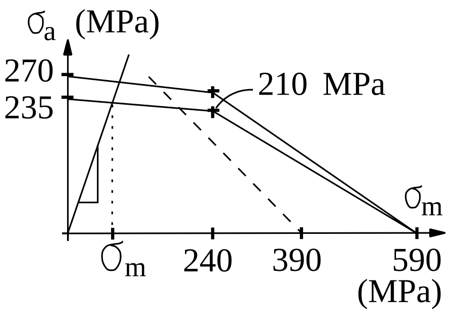
Stress concentration at the shoulder: K_t^{shoulder} = 1.4 and q = 0.88 give K_f^{shoulder} = 1.35.

The stresses at the notch and at the shoulder may now be determined (as a function of deflection a). One obtains

$$\sigma^{\text{notch}} = \frac{M_{\text{b, notch}}}{W_{\text{b, notch}}} = \frac{3EI}{2L^2\,W_{\text{b, notch}}}\,(K_t^{\text{notch}}\,0.5\,a \pm K_f^{\text{notch}}\,1.5\,a)$$

and $$\sigma^{\text{shoulder}} = \frac{M_{\text{b, shoulder}}}{W_{\text{b, shoulder}}} = \frac{3EI}{L^2\,W_{\text{b, shoulder}}}\,(K_t^{\text{shoulder}}\,0.5\,a \pm K_f^{\text{shoulder}}\,1.5\,a)$$

where W_b is the section modulus in bending.



Reduction factors: $\kappa = 1$, $\lambda = 0.87$, $\delta = 1$

The reduced fatigue limit at alternating loading becomes

$\sigma_{\text{FLB,red}} = 0.87 \cdot 270 = 235$ MPa

and at pulsating loading it becomes

$\sigma_{\text{FLBP,red}} = 240 \pm 0.87 \cdot 240 = 240 \pm 210$ MPa.

This gives a Haigh diagram according to the figure.

The loading is such that the ratio of the amplitude to the mean value is constant. This gives *at the notch*:

$$\frac{\sigma_a^{\text{notch}}}{\sigma_m^{\text{notch}}} = \frac{1.66 \cdot 1.5}{1.8 \cdot 0.5} = 2.77$$

Enter a straight line of slope 2.77 in the Haigh diagram. This line intersects the reduced fatigue limit at $\sigma_m \cong 82$ MPa. Fatigue failure is thus expected when the mean value of the stress is $\sigma_m = 82$ MPa (the stress amplitude is then $2.77\sigma_m = 227$ MPa).

The mean value σ_m of the stress at the notch will now be determined from the calculated bending moment. Using the stress concentration factor $K_t^{\text{notch}} = 1.8$ on the nominal mean stress applied, and using the safety factor $s = 2$ on the reduced fatigue limit $\sigma_m = 82$ MPa determined, one obtains

$$1.8\,\sigma_{\text{nominal}} = \sigma_m \cdot \frac{1}{s} \quad \text{which gives} \quad \left(1.8\,\frac{M_{\text{b, notch}}^{\text{mean}}}{W_{\text{b, notch}}} =\right) 1.8\,\frac{3EI}{2L^2}\,0.5\,a\,\frac{32}{\pi d^3} = 82 \cdot 10^6 \cdot \frac{1}{2}\ \text{Pa}$$

From this $a = 0.00048$ m $= 0.48$ mm is solved.

At the shoulder is obtained:

$$\frac{\sigma_a^{\text{shoulder}}}{\sigma_m^{\text{shoulder}}} = \frac{1.35 \cdot 1.5}{1.4 \cdot 0.5} = 2.9$$

Enter a straight line of slope 2.9 in the Haigh diagram. The line intersects the reduced fatigue limit at $\sigma_m \cong 78$ MPa. Fatigue failure is thus expected when the mean value of the stress is $\sigma_m = 78$ MPa (the stress amplitude is then $2.9\sigma_m = 226$ MPa).

The mean value σ_m of the stress at the shoulder will now be determined from the calculated bending moment. Using the stress concentration factor $K_t^{\text{shoulder}} = 1.4$ on the nominal stress applied to the beam, and using the safety factor $s = 2$ on the reduced fatigue limit, one obtains

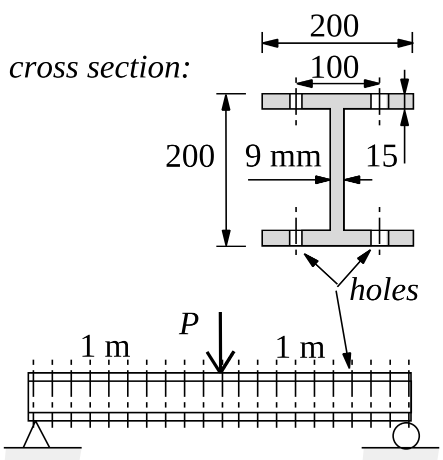
$$\left(1.4\,\sigma_{\text{nominal}} = 1.4\,\frac{M_{\text{b, shoulder}}^{\text{mean}}}{W_{\text{b, shoulder}}} =\right) 1.4\,\frac{3EI}{L^2}\,0.5\,a\,\frac{32}{\pi D^3} = 78 \cdot 10^6 \cdot \frac{1}{2} \quad \left(= \sigma_m \cdot \frac{1}{s}\right)$$

From this $a = 0.00058$ m $= 0.58$ mm is solved.

It is concluded that the notch is the most critical part of the structure. The stresses at the notch will be limiting for the displacement a.

Answer: Maximum allowable displacement a at the free beam end will be $a = 0.48$ mm.

8/16.



A beam with cross section HE200B is made of the material 1312-00. The beam is simply supported at its ends and loaded by a force $P = P_0 \pm P_0$ at its centre point.
Determine, with respect to fatigue failure of the beam, the maximum allowable value of P_0. Each flange has two rows of holes (machined, R_a = 20 µm) with diameter 8 mm. The holes are drilled 100 mm from each other.
Material data: σ_U = 360 MPa, σ_Y = 240 MPa, σ_{FL} = 110 MPa and σ_{FLP} = 110 ± 110 MPa.

Solution:
The largest bending moment in the beam will appear at the mid-point of the beam, and it is (use total beam length 2*l*)

$$M = \frac{P \cdot l}{2} = (P_0 \pm P_0)\,\frac{l}{2} \qquad \text{where } l = 1 \text{ m}$$

The beam HE200B has the section modulus $W_b = 570 \cdot 10^{-6}\,\text{m}^3$ in bending.

If the holes had not been there, the stress σ_0 in the lower flange of the beam would have been

$$\sigma_0 = \frac{M}{W_b}$$

Due to the holes, stress concentration will appear.

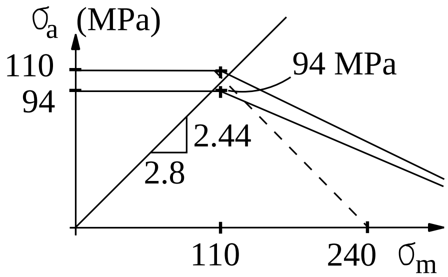
It is assumed that the holes are so widely separated that when calculating the stress concentration factor at one hole, the disturbance of the stress distribution from the other holes may be neglected.
Study half of the flange width. This can be done because of symmetry. Using r/a = 4/50 and B/a = 100/50 the stress concentration factor K_t is obtained as $K_t \cong 2.8$.

The fatigue notch factor K_f becomes $K_f = 1 + q\,(K_t - 1) \cong 1 + 0.8(2.8 - 1) = 2.44$.

This gives the maximum stress σ at a hole situated where the bending moment is the largest. One obtains

$$\sigma = (2.8 \pm 2.44)\,\frac{B}{B - 2r}\,\frac{P_0\,l}{2W_b}$$

where *B* is half the width of the flange, thus *B* = 100 mm.



Reduction of fatigue limits:
The flange thickness is 15 mm, the fatigue notch factor K_f is used, where $K_f > 1$, and the surface of the hole is machined. This give
$\lambda \cong 0.95$, $\delta = 1$, and $\kappa \cong 0.90$.
The reduced fatigue limit at a pulsating load then becomes

$$\sigma_{FLP}^{red} = \lambda\,\delta\,\kappa \cdot \sigma_{FLP} = 0.95 \cdot 1 \cdot 0.90 \cdot 110\text{ MPa} = 94\text{ MPa}$$

The allowable stress amplitude at the hole is 94 MPa (with no safety factor). It gives

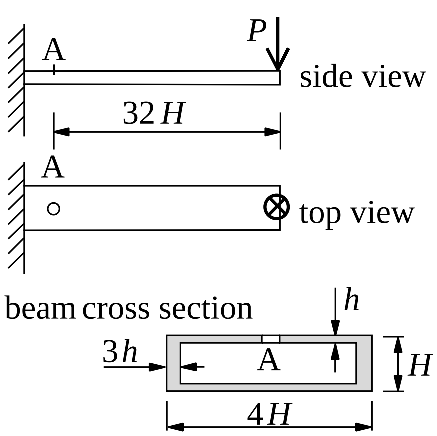
$$2.44\,\frac{B}{B-2r}\,\frac{P_0\,l}{2W_b} = 2.44\,\frac{0.100 \cdot P_0 \cdot 1}{0.092 \cdot 2 \cdot 570 \cdot 10^{-6}} = 94 \cdot 10^6\ \text{Pa}$$

which gives $P_0 \cong 40$ kN.

The stress amplitude was used as the design stress. The mean stress σ_{mean} (= 2.8·94 / 2.44 = 108 MPa) is less than 110 MPa, implying that the straight line giving the service stress intersects the horizontal branch of the fatigue limit curve just slightly to the left of the breaking point at 110 MPa. Thus, it was correct to use the stress amplitude σ_a = 94 MPa as the design stress.

Answer: The force $P = P_0 \pm P_0$ may be allowed to be, approximately, P = 40 ± 40 kN (no safety factor included here).

8/17.



The structure in the figure is loaded by a force $P = P_0 \pm P_0$. Determine, with respect to fatigue failure at hole A, the maximum allowable value of P_0. The hole is machined (R_a = 7 μm) and it has a diameter of 20 mm. Material data (for SS 1510-00): σ_U = 600 MPa, σ_Y = 320 MPa, σ_{FL} = 230 MPa and σ_{FLP} is unknown (use thumb rule $\sigma_{FLP} \cong 0.85\sigma_{FL}$).
Geometrical data: H = 200 mm and h = 20 mm (cf. Chapter 1, Section 1.4.8).

Solution:

The nominal stress at the hole becomes $\sigma^{nom} = Mz\,/\,I$

where $M = P\cdot 32H$, $z = H\,/2$ and
$$I = \frac{4H\,H^3}{12} - \frac{(4H-6h)\,(H-2h)^3}{12}$$

One obtains $\sigma^{nom} = 2125P = 2125(P_0 \pm P_0)$ N/m^2 (P in N).

The stress concentration factor K_t at the hole is $K_t = 3.0$ (here the case "a small hole in a large plate subjected to uniaxial loading" is used). This gives the fatigue notch factor

$$K_f = 1 + q(K_t - 1) = 1 + 0.87\cdot(3.0 - 1) = 2.74$$

The factors reducing the fatigue limit due to surface finish and volume become: $\kappa \cong 0.86$, $\lambda \cong 0.90$ and $\delta = 1$.

Pulsating loading implies that the fatigue limit σ_{FLP} should be used. One obtains

$$\sigma_{FLP} = 0.85\,\sigma_{FL}$$

By use of the reduction factors one obtains

$$\sigma_{FLP}^{red} = \kappa\lambda\delta\cdot\sigma_{FLP} = \kappa\lambda\delta\cdot 0.85\;\sigma_{FL} = 0.86\cdot 0.90\cdot 1\cdot 0.85\cdot 230\,\text{MPa} = 151\,\text{MPa}$$

The amplitude P_0 of the load gives the stress amplitude $K_f\cdot\sigma^{nom} = 2.74\cdot\sigma^{nom}$.
Let $K_f\cdot\sigma^{nom} = \sigma_{FLP}^{red}$. It gives

$$2.74\cdot 2125P_0 = 151\cdot 10^6, \quad \text{which gives } P_0 = 26 \text{ kN.}$$

Answer: Without safety factor, the force P may be $P = 26 \pm 26$ kN.

Damage accumulation, counting of load cycles

8/18.

A component of a road vehicle is subjected to a repeated loading sequence. One sequence contains the following loading (*i.e.* stress) amplitudes (the loading is alternating)

$$\sigma_a = 200,\ 180,\ 150,\ \text{and } 100 \text{ MPa}$$

The number of loading cycles at each stress level is

$$n = 15,\ 20,\ 150,\ \text{and } 3000, \text{ respectively}$$

The Wöhler curve of the material is, in this stress range, given by the relation

$$\sigma_a = -\,55\log N + 430 \text{ MPa}$$

Determine the damage accumulation D due to one loading sequence, and then determine how many loading sequences the component might resist before fatigue failure.

Solution:

The Wöhler curve $\sigma_a = -55 \log N + 430$ of the material gives the fatigue life N at a given stress amplitude σ_a.

One obtains

$$N = 10^{(430-\sigma_a)/55}$$

At the different stress levels the following damages are obtained

σ_a (MPa)	N_i	n_i	damage n_i / N_i
200	15 199	15	15 / 15 199
180	35 112	20	20 / 35 112
150	123 285	150	150 / 123 285
100	10^6	3 000	3 000 / 10^6

Damage D due to one sequence is

$$D = \frac{15}{15\,199} + \frac{20}{35\,112} + \frac{150}{123\,285} + \frac{3\,000}{10^6} = 5.77 \cdot 10^{-3}$$

The expected number of sequences N_s to fatigue failure is $N_s = 1/D \cong 173$

Answer: Damage D due to one sequence is $D = 5.77 \cdot 10^{-3}$, and expected number of sequences N_s to fatigue failure is $N_s \cong 173$.

8/19.

For a welded plate, a number of points on the Wöhler curve have been determined. At the probability of failure p = 50 per cent and at the stress ratio R = 0 the following points were obtained:

σ_{max} (MPa)	189	182	161	131	107
N_f (cycles)	$1{\cdot}10^4$	$3{\cdot}10^4$	$1{\cdot}10^5$	$3{\cdot}10^5$	$1{\cdot}10^6$

(a) Use linear regression to fit a straight line in a σ_{max}–log$N$ diagram to these points. Estimate by use of linear damage accumulation (the Palmgren-Miner rule) how many sequences (as given below) the plate might resist before fatigue failure is expected. One sequence contains the following stress levels and cycles:

σ_{max} (MPa)	120	150	180	160	140
n (cycles)	100	50	20	40	60

(b) According to Jarfall (1977), see reference in textbook, the relationship

$$\sigma^k \cdot N = \text{constant}$$

could give a better result at spectrum loading (*i.e.* loading with different, possibly random, amplitudes). This relation gives a straight line in a $\log\sigma_{max}$–$\log N$ diagram. Fit a straight line in a $\log\sigma_{max}$–$\log N$ diagram to the data given and estimate by the formula so obtained the number of sequences to fatigue failure.

Solution:

(a) Fit a straight line in a σ_{max}–$\log N$ diagram to the measured data. Let the coordinate y be $y = \sigma_{max}$ and the coordinate $x = \log N$. This gives

$x =$	4	4.47712	5	5.47712	6
$y =$	189	182	161	131	107

Fit the straight line $y = Ax + B$ to the points.

Linear regression gives (n is number of points)

$$A = \frac{n\,\Sigma x \cdot y - \Sigma x \cdot \Sigma y}{n\,\Sigma x^2 - (\Sigma x)^2} = -\,43.034945 \quad \text{and} \quad B = \frac{\Sigma y - A\,\Sigma x}{n} = 368.78089$$

The coefficient of correlation r becomes

$$r = \frac{n\,\Sigma x \cdot y - \Sigma x \cdot \Sigma y}{\sqrt{[n\,\Sigma x^2 - (\Sigma x)^2]\;[n\,\Sigma y^2 - (\Sigma y)^2]}} = -\,0.98$$

which means (r close to – 1) that the points fall almost on a straight line.

The Wöhler curve of the material thus becomes

$$\sigma_{max} = -\,43.03 \log N + 368.78 \text{ MPa}$$

From this is solved: $$N = 10^{(368.78 - \sigma_{max})/43.03}$$

For the different stress levels the fatigue life and the damage are obtained as

σ_{max}	N_i	n_i	damage n_i / N_i
120	603 815	100	$0.1656136 \cdot 10^{-3}$
150	121 282	50	$0.4122625 \cdot 10^{-3}$
180	24 360	20	$0.8209972 \cdot 10^{-3}$
160	71 028	40	$0.5631617 \cdot 10^{-3}$
140	207 093	60	$0.2897249 \cdot 10^{-3}$

The damage D due to one sequence becomes $D = \Sigma (n_i / N_i) = 2.25176 \cdot 10^{-3}$. Expected number of sequences to fatigue failure is $1 / D = 444$.

(b) Fit a straight line in a $\log\sigma_{max}$–$\log N$ diagram to the measured data. Let $y = \log\sigma_{max}$ and $x = \log N$; one then obtains

$x =$	4	4.47712	5	5.47712	6
$y =$	2.276	2.260	2.207	2.117	2.029

This gives the line $\log\sigma_{max} = A \log N + B$, where $A = -0.12755$ and $B = 2.8146$ (and the coefficient of correlation r now is $r = -0.9723$).

The Wöhler curve gives

$$N = 10^{(B - \log \sigma_{max}) / -A}$$

from which is obtained

σ_{max}	N_i	n_i	damage n_i / N_i
120	582893	100	$0.1715581 \cdot 10^{-3}$
150	101353	50	$0.4933274 \cdot 10^{-3}$
180	24270	20	$0.8240612 \cdot 10^{-3}$
160	61108	40	$0.6545826 \cdot 10^{-3}$
140	174077	60	$0.3446760 \cdot 10^{-3}$

The damage D due to one loading sequence becomes $D = \Sigma (n_i / N_i) = 2.4882 \cdot 10^{-3}$, which gives the expected number of sequences N_s to fatigue failure: $N_s = 1 / D \cong 402$.

Answer: Number of sequences to fatigue failure becomes (a) 444, and (b) 402, approximately.

8/20.

A part of a structure is subjected to sequences of vibrations. Each sequence consists of free damped vibrations giving rise to stresses with the stress ratio $R = -1$, *i.e.*, the mean value of each stress cycle is zero. In the point of the material to be investigated, the stress amplitude of the first cycle in each sequence is σ_0. The damping of the vibrations is characterized by the logarithmic decrement δ. The logarithmic decrement δ is defined as $\delta = \ln(\sigma_i / \sigma_{i+1})$ where σ_i is the amplitude of the vibration cycle i and σ_{i+1} is the amplitude of the cycle $i + 1$, see figure (a). The Wöhler curve of the material is, for stress amplitudes $\sigma_a < \sigma_0$, assumed to be a straight line in a diagram with $\log\sigma_a$ on the ordinate (the "y" axis) and $\log N$ on the abscissa (the "x" axis), *cf*. equation (8.2b). The slope of the curve is $-1/m$ in the diagram. The material has no fatigue limit. Determine the number of vibration sequences the structure may be expected to survive. The number of cycles to failure at the stress amplitude σ_0 would be N_0.

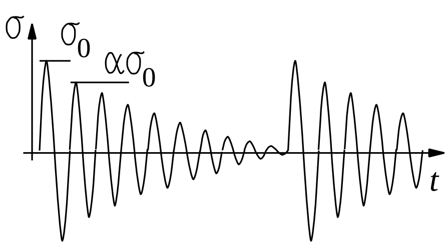


Figure (a)

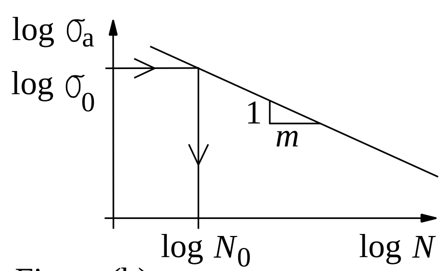


Figure (b)

Solution:

The Wöhler curve is a straight line of slope $-1/m$ in a $\log\sigma_a$–$\log N$ diagram. The equation of the curve may be written

$$\log\sigma_a = -\frac{1}{m}\log N + C \tag{a}$$

The constant C is determined from the condition that the fatigue life is N_0 cycles when the stress amplitude is σ_0. This gives

$$\log\sigma_0 = -\frac{1}{m}\log N_0 + C \tag{b}$$

By inserting C as given by relation (b) into the expression (a), the equation of the Wöhler curve takes the form

$$\log\frac{\sigma_a}{\sigma_0} = \frac{1}{m}\log\frac{N_0}{N} \tag{c}$$

The stress amplitude of the different stress cycles following the first stress cycle (with amplitude σ_0) in each sequence will now be determined together with the fatigue life at each stress amplitude. Let $\sigma_1 = \alpha\sigma_0$, $\sigma_2 = \alpha\sigma_1 = \alpha^2\sigma_0$, ..., $\sigma_i = \alpha^i \sigma_0$, and so on. The definition of the logarithmic decrement yields

$$\delta = \ln\frac{\sigma_i}{\sigma_{i+1}} = \ln\frac{\sigma_i}{\alpha\sigma_i} \qquad \text{(d)}$$

Solve (d) for α. It gives $\alpha = \exp(-\delta)$ (here $\exp(-\delta)$ stands for $e^{-\delta}$).

Determine the cyclic life $N = N_i$ at stress level $\sigma_i = \alpha^i \sigma_0$. The relation (c) gives

$$\log\frac{\alpha^i \sigma_0}{\sigma_0} = \frac{1}{m}\log\frac{N_0}{N_i} \qquad \text{(e)}$$

This relation gives $N_i = N_0 / \alpha^{i \cdot m}$.
The accumulated damage D due to *one* loading sequence becomes

$$D = \frac{1}{N_0} + \frac{1}{N_1} + \frac{1}{N_2} + \ldots = \frac{1}{N_0} + \frac{\alpha^m}{N_0} + \frac{\alpha^{2m}}{N_0} + \frac{\alpha^{3m}}{N_0} + \ldots = \frac{1}{N_0}(1 + \alpha^m + \alpha^{2m} + \alpha^{3m} + \ldots.) \qquad \text{(f)}$$

Use the series expansion

$$1 + x + x^2 + x^3 + \ldots. = \frac{1}{1-x} \quad \text{when} \quad |x| < 1 \qquad \text{(g)}$$

This gives the damage D of *one* sequence of vibration as:

$$D = \frac{1}{N_0(1-\alpha^m)} \qquad \text{(h)}$$

Let I be the total number of sequences the structure may be subjected to. The accumulated damage then is $I \cdot D$. Fatigue failure is expected when the accumulated damage becomes unity, thus when $I \cdot D = 1$. This gives

$$1 = I \cdot \frac{1}{N_0(1-\alpha^m)} \qquad \text{(i)}$$

Solve for I, which gives

$$I = N_0(1-\alpha^m) = \frac{1}{D}$$

Using numerical values $\delta = 0.05$ (giving $\alpha \cong 0.95$) and $m = 8$ one obtains

$$I = 0.33\, N_0$$

Hence, the stress amplitude σ_0 alone would have given a cyclic life of N_0 loading cycles. The free damped vibration after the first cycle (with amplitude σ_0) reduces, in this case, the fatigue life of the structure to one third of N_0.

Answer: The number I of sequences to fatigue failure is expected to be

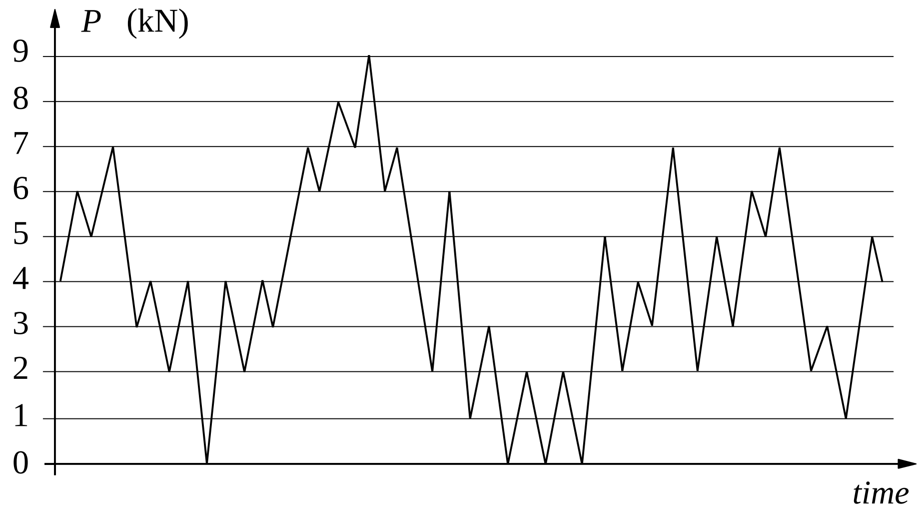
$$I = \frac{1}{D} = N_0(1-\alpha^m) \quad \text{where} \quad \alpha = e^{-\delta}$$

If $\delta = 0.05$ and $m = 8$ one obtains $I = 1/D = 0.33\, N_0$ (solution also given in the Example in Section 8.2.3 in the textbook).

8/21.

At a random loading of a structure, the following time sequence was recorded (force in kN, see also figure):

4 6 5 7 3 4 2 4 0 4 2 4 3 7 6 8 7 9 6 7 2 6 1 3 0 2 0
2 0 5 2 4 3 7 2 5 3 6 5 7 2 3 1 5 4 (= static level at rest)



Determine

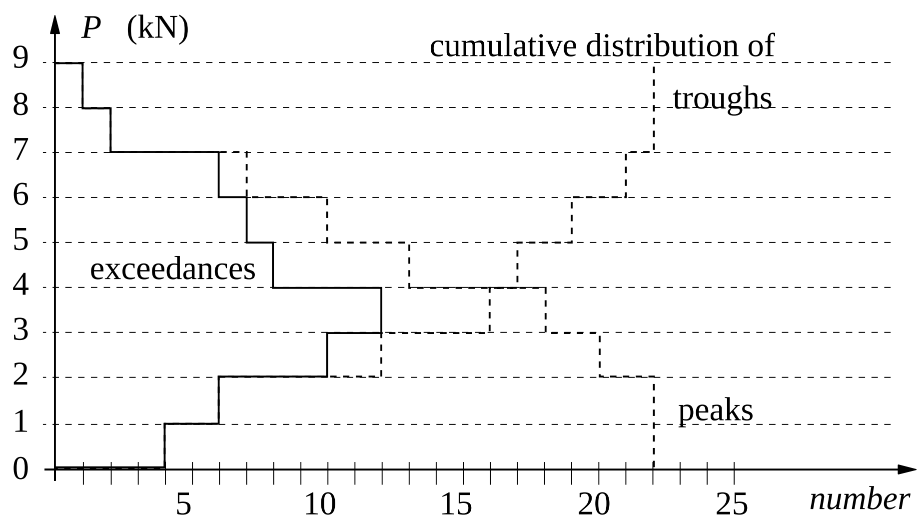
(a) the distribution of the peaks (number of peaks larger than or equal to a certain level),

(b) the distribution of the troughs (number of troughs smaller than or equal to a certain level), and

(c) the distribution of exceedances (the load spectrum).

Solution: See answer.

Answer:



8/22.

At a random loading of a structure, the following time sequence was recorded (force in kN, same sequence as in Problem 8/21, see also figure in that problem):

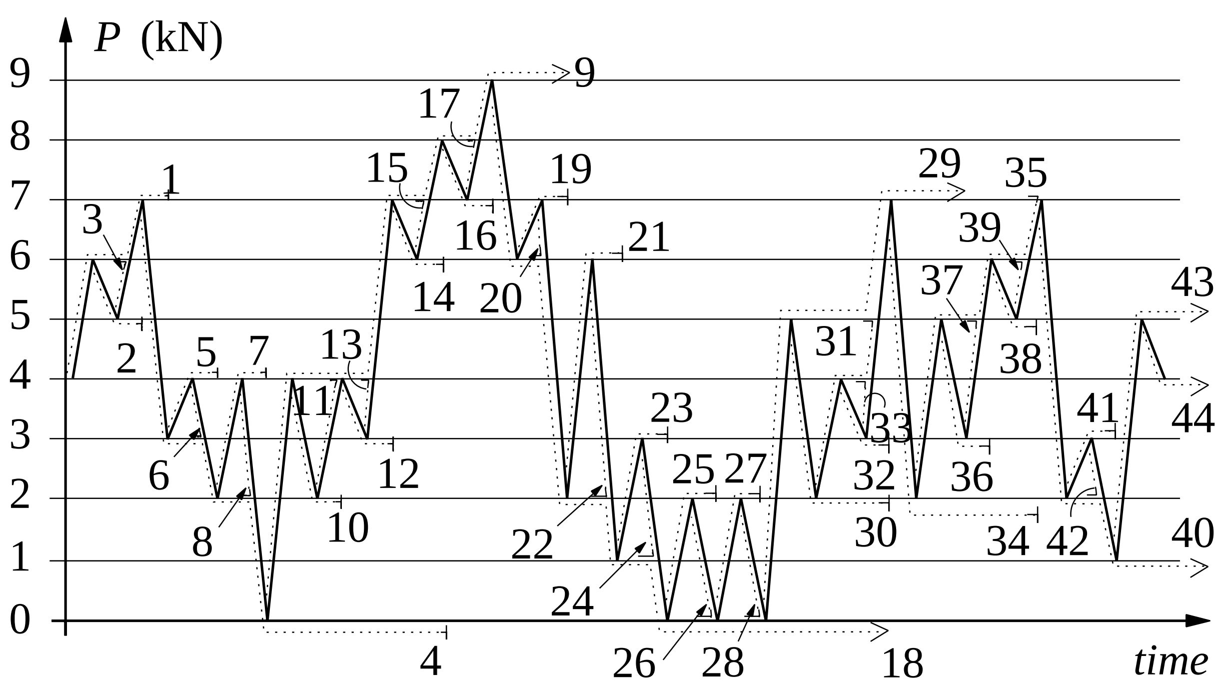
4 6 5 7 3 4 2 4 0 4 2 4 3 7 6 8 7 9 6 7 2 6 1 3 0 2 0
2 0 5 2 4 3 7 2 5 3 6 5 7 2 3 1 5 4 (= static level at rest)

Identify loading cycles by use of the rain flow count method. Make a list of those cycles whose amplitude is larger than, or equal to, 1 kN.

Solution:

This solution is given for the load sequence as it is given in the problem. It should be noted, however, that the solution could be slightly simplified if the sequence is rearranged so that it starts at its largest maximum or its smallest minimum. Here, fore example, the part of the sequence from the starting point at P = 4 kN to the minimum at P = 0 could be cut off and moved to the end of the sequence given. If this is done, the final result will be the same, but the problem of obtaining half cycles that must be matched to each other, see below, will be avoided.

Now, let one drop of rain-water start to run (flow) from each maximum value and each minimum value of the loading sequence (the starting point and the final point here form a minimum). A total of 44 drops are needed, see the figure.



Make a list of the run-ways of all drops. For each drop, write down its number (the drop number), its stating point (load level in kN) and its end point (load level in kN). It gives

Drop No	starts at (kN)	stops at (kN)
1	4	7
2	6	5
3	5	6
4	7	0
5	3	4
6	4	3
7	2	4
8	4	2
9	0	9
10	4	2
11	2	4
12	4	3
13	3	4
14	7	6
15	6	7
16	8	7
17	7	8
18	9	0
19	6	7
20	7	6
21	2	6
22	6	2
23	1	3
24	3	1
25	0	2
26	2	0
27	0	2
28	2	0
29	0	7
30	5	2
31	2	5
32	4	3
33	3	4
34	7	2
35	2	7
36	5	3
37	3	5
38	6	5
39	5	6
40	7	1
41	2	3
42	3	2
43	1	5
44	5	4

Collect pairs of drops so that one pair forms a closed loop. Write down the two drop numbers, the minimum value (in kN) of the closed loop and the maximum value (in kN). One obtains

Drops No	minimum value	maximum value
(1	4	7)
2, 3	5	6
4, 29	7	0
5, 6	3	4
7, 8	2	4
9, 18	0	9
10, 11	2	4
12, 13	3	4
14, 15	6	7
16, 17	7	8
19, 20	6	7
21, 22	2	6
23, 24	1	3
25, 26	0	2
27, 28	0	2
30, 31	2	5
32, 33	3	4
34, 35	2	7
36, 37	3	5
38, 39	5	6
(40	7	1)
41, 42	2	3
(43	1	5)
(44	5	4)

Here 20 closed loops have been identified. Due to the fact that the sequence does not starts at the largest maximum or the smallest minimum (see the introduction to the solution above) there will be some single drops left: drops No 1, 40, 43, and 44. Also these drops may, however, be grouped into closed loops in the following way:
Divide the running way of drop No 1 into two parts, 1a and 1b. It gives

Drop No	starts	stops
1a	4	5
1b	5	7
40	7	1
43	1	5
44	5	4

If we think of a continuation of the given sequence as a repetition of the first sequence, then the run-way of the drop No 43 would continue the same way as the drop No 1 has from 5 kN and onwards. The drop No 43 would then start at 1 kN, initially it will run to 5 kN and then further “on the next roof” to 7 kN. Thus drops 43 and 1b together form half a cycle from 1 kN to 7 kN. This half-cycle may be matched to the drop No 40 to form a full loading cycle (one loop). The drop 1a now stops at 5 kN and it is seen that this part may be matched to the drop No 44. The five one-way drop ways above may thus be combined to give two closed loops. Again, the numbers of the two drops, the minimum value (in kN) of the closed loop and the maximum value (in kN) are written down. One obtains

Drop way No	minimum value	maximum value
1a, 44	4	5
43, 1b, 40	1	7

In total 22 loops (cycles) have been obtained. These cycles may now be used in a fatigue analysis. The ranges (twice the amplitude) of he loading cycles are given in the tables above. Sometimes cycles of a small amplitude do not take part in the fatigue process. If cycles with an *amplitude* less than 1 kN are neglected, then 12 loading cycles are left to form the basis of a fatigue damage analysis. These are

Drops No	minimum value	maximum value
4, 27	0	7
7, 8	2	4
9, 18	0	9
10, 11	2	4
21, 22	2	6
23, 24	1	3
25, 26	0	2
28, 29	0	2
30, 31	2	5
35, 40	2	7
36, 37	3	5
43, 1b, 34	1	7

Answer: Twelve cycles with an amplitude larger than or equal to 1 kN are obtained (and ten cycles with an amplitude less than 1 kN). Minimum and maximum values of the twelve cycles are: (0,7), (2,4), (0,9), (2,4), (2,6), (1,3), (0,2), (0,2), (2,5), (2,7), (3,5), and (1,7).

Empty page.

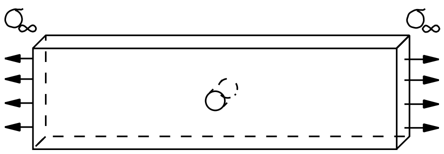
Solutions to problems in
T Dahlberg and A Ekberg: Failure, Fracture, Fatigue - An Introduction.
Studentlitteratur, Lund 2002, ISBN 91-44-02096-1.

Chapter 9

Strain-based fatigue design

Problems with solutions

9/1.



A flat bar of material SAE 1045 with a rectangular cross section has a small circular hole at its longitudinal centre axis. The bar is subjected to a remote alternating stress $\sigma_\infty = \pm 300$ MPa.

(a) Estimate by use of Neuber's method and the Morrow equation the number of cycles to fatigue failure. Assume that $K_f = 0.9K_t$.
(b) For the maximum stress and strain at the hole draw the hysteresis loop when $\sigma_\infty = \pm 300$ MPa.
(c) Determine the fatigue life if $\sigma_\infty = 100 \pm 200$ MPa.

Solution:
(Compare with the problem solved in Section 9.3.3 in the textbook.)

(a) *Number of cycles to fatigue failure*
If the stress state at the small hole in the flat bar were purely elastic, then the stress concentration factor would have been $K_t = 3$. The fatigue notch factor K_f is

$$K_f = 1 + q\,(K_t - 1) \tag{a}$$

No information for calculation of the notch sensitivity factor q is given here. Instead, $K_f = 0.9K_t = 2.7$ was given. Thus, use $K_f = 2.7$.

Due to the high stresses close to the hole, the material at the hole will yield locally. Taking this into consideration, the stress concentration factor K_σ and the strain concentration factor K_ε may be determined from equations (9.9a,b) in the textbook. For low stresses, the second term in the material relation (9.2a,b) can be disregarded. This means that here, where the stresses far away from the hole are relatively low (*i.e.* mainly within the elastic range), the second term will be small when compared to the first term. Expressions (9.9a,b) in the textbook then become

$$K_\sigma = \frac{\sigma_{max}}{\sigma_\infty} \quad \text{and} \quad K_\varepsilon = \frac{\varepsilon_{max}}{\varepsilon_\infty} = \frac{\varepsilon_{max}}{\sigma_\infty / E} \tag{b,c}$$

where, as can be seen, only the first term (*i.e.* Hooke's law) in (9.2a,b) has been used in (c).

The Neuber hyperbola becomes

$$\sigma \cdot \varepsilon = K_\sigma \, \sigma_\infty \cdot K_\varepsilon \, \varepsilon_\infty = \frac{K_f^2 \, \sigma_\infty^2}{E} \tag{d}$$

The material data given earlier (see the textbok) for the material SAE 1045 and the nominal stress amplitude σ_∞ = 300 MPa now determine the Neuber hyperbola. The local stress amplitude σ_a and strain amplitude ε_a at the hole are obtained as the intersection point between the Neuber hyperbola and the material stress-strain relation for cyclic loading. One obtains

$$\begin{cases} \sigma_a \cdot \varepsilon_a = \dfrac{K_f^2 \sigma_\infty^2}{E} = \dfrac{2.7^2 \cdot 300^2}{200\,000} = 3.2805 \\ \varepsilon_a = \dfrac{\sigma_a}{E} + \left(\dfrac{\sigma_a}{K'} \right)^{1/n'} = \dfrac{\sigma_a}{200\,000} + \left(\dfrac{\sigma_a}{1344} \right)^{1/0.18} \end{cases} \tag{e,f}$$

This system of equations gives $\sigma_a = \sigma_{max}$ = 499 MPa and $\varepsilon_a = \varepsilon_{max}$ = 0.006574.

The stress and strain concentration factors K_σ and K_ε then become

$$K_\sigma = \frac{\sigma_{max}}{\sigma_\infty} = \frac{499}{300} = 1.663 \quad \text{and} \quad K_\varepsilon = \frac{\varepsilon_{max}}{\sigma_\infty / E} = \frac{0.006574}{300/200\,000} = 4.3828 \tag{g,h}$$

(Verify: $K_\sigma \cdot K_\varepsilon = 1.663 \cdot 4.383 = 7.29 = 2.7^2 = K_f^2$, as it should.)

Now the number of cycles to fatigue crack initiation (or fatigue failure) may be determined. According to Morrow, equations (9.4) and (9.7) in the textbook, and by use of the mean stress σ_m = 0, one obtains

$$\varepsilon_a = \frac{\sigma'_f - \sigma_m}{E} (2N)^b + \varepsilon'_f (2N)^c = \frac{1227}{200\,000} (2N)^{-0.095} + 1.0 (2N)^{-0.66} \tag{i}$$

Using ε_a = 0.006574, the fatigue life $2N$ = 4600 reversals to failure is obtained, giving N = 2300 cycles to fatigue failure.

(b) *Display the hysteresis loop for the material at the hole, when the remote loading stress is* $\sigma_\infty = \pm 300$ MPa

In problem (a) above the stress and strain amplitudes at the hole have been calculated when the remote stress amplitude is $\sigma_\infty = 300$ MPa. This implies that when the remote stress has its maximum ($\sigma_\infty = 300$ MPa), the upper end point of the hysteresis loop is situated at $\sigma = \sigma_{max}$ = 499 MPa and $\varepsilon = \varepsilon_{max}$ =

0.006574, see point A in the figure below. When the remote stress has its minimum ($\sigma_\infty = -300$ MPa), the stress and strain at the stress concentration (*i.e.* at the hole) is $\sigma = -\sigma_{max} = -499$ MPa and $\varepsilon = -\varepsilon_{max} = -0.006574$. These values give the lower end point of the hysteresis loop, see point B in the figure.

Choose point A as starting point of the hysteresis loop, see figure below. A change $\Delta\sigma_\infty$ of the remote stress (the stress far away from the stress concentration) will cause a change of stress $\Delta\sigma$ and a change of strain $\Delta\varepsilon$ at the stress concentration (the hole). The changes $\Delta\sigma$ and $\Delta\varepsilon$ are obtained by use of the Neuber hyperbola (9.13d) and the material relation for *changes* of stress and strain (9.3a). Using $\Delta\sigma_\infty$ = 300 MPa one obtains

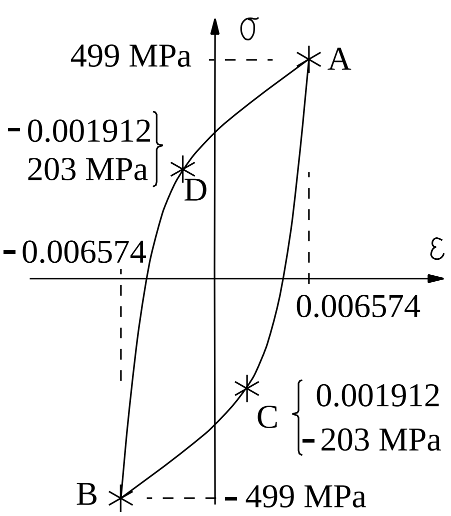
$$\begin{cases} \Delta\sigma \cdot \Delta\varepsilon = \dfrac{K_f^2\,(\Delta\sigma_\infty)^2}{E} = \dfrac{2.7^2 \cdot 300^2}{200\,000} = 3.2805 \\ \Delta\varepsilon = \dfrac{\Delta\sigma}{E} + 2\left(\dfrac{\Delta\sigma}{2K'}\right)^{1/n'} = \dfrac{\Delta\sigma}{200\,000} + 2\left(\dfrac{\Delta\sigma}{2\cdot 1344}\right)^{1/0.18} \end{cases} \tag{j,k}$$

From this system of equations $\Delta\varepsilon = 0.004670$ and $\Delta\sigma = 703$ MPa are solved.

The lower branch of the hysteresis loop, starting at point A, will pass through the point C. The co-ordinates of point C are obtained from

$$\begin{cases} \sigma_C = \sigma_A - \Delta\sigma = 499 \text{ MPa} - 703 \text{ MPa} = -204 \text{ MPa} \\ \varepsilon_C = \varepsilon_A - \Delta\varepsilon = 0.006574 - 0.004670 = 0.001904 \end{cases} \tag{l,m}$$

For the student: verify that the change of the remote stress $\Delta\sigma_\infty$ = 600 MPa will take this branch of the hysteresis loop to the point B.



We now turn to the upper (*i.e.* the left) branch of the hysteresis loop. At a change of stress $\Delta\sigma_\infty$ = 300 MPa the upper branch of the hysteresis loop, starting at point B, will pass through point D. The coordinates of point D are obtained from

$$\begin{cases} \sigma_D = \sigma_B + \Delta\sigma = -499 \text{ MPa} + 703 \text{ MPa} = 204 \text{ MPa} \\ \varepsilon_D = \varepsilon_B + \Delta\varepsilon = -0.006574 + 0.004670 \\ \quad = -0.001904 \end{cases} \tag{n,o}$$

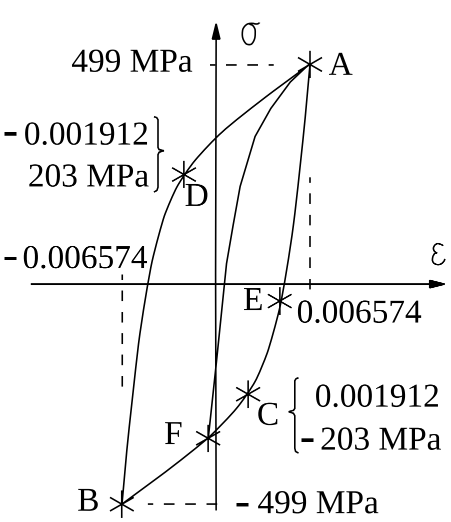
By use of these values (and perhaps some more) the hysteresis loop may be drawn, see figure.

Comment: It is noticed that when the loop passes the point C the external loading (far away from the hole) will give a nominal stress σ_∞ equal to zero, *i.e.* the bar has been loaded a number of times up to stress σ_∞ = 300 MPa and then unloaded by the stress $\Delta\sigma_\infty$ = 300 MPa so that the total stress far away from the stress concentration is σ_∞ = 0. Thus, when the bar is unloaded (*i.e.* when σ_∞ = 0), the residual stress at the hole is σ_C (compressive) and the residual strain is ε_C. The residual stress is compressive because the material has been "stretched out" (by plastic deformation) at the hole and it was "too long" when the bar was unloaded. On the other hand, when unloading from the remote stress σ_∞ = – 300 MPa to stress σ_∞ = 0, the residual stress at the hole will become σ_D (in tension) and the residual strain is ε_D. In this case the material was "compressed" at the hole so it was made "too short" when the bar was unloaded from σ_∞ = – 300 MPa to σ_∞ = 0. This explains why the residual stress is in tension at point D when the remote stress is σ_∞ = 0. (The remote stress σ_∞ = 0 indicates that the axial force in the bar is zero. This is the case also at the hole. For equilibrium to be fulfilled at the cross section of the bar at the hole, there must be residual stresses both in tension and in compression. This implies that residual stresses in tension close to the hole will be balanced by residual stresses in compression further away from the hole, and *vice versa*.)

(c) *Fatigue life when* σ_∞ = 100 ± 200 MPa

The same bar as above (K_f = 2.7) will be investigated, but now the bar is loaded to a nominal (remote) stress σ_∞ = 100 ± 200 MPa. The nominal (remote) stress σ_∞ will now vary between 300 MPa and – 100 MPa. This implies that the stress at the hole will vary between σ_A (when σ_∞ = 300 MPa) and a point on the hysteresis loop below σ_C (the stress at the hole is σ_C when the remote stress is σ_∞ = 0).

Determine some more points on the hysteresis loop that is obtained when σ_∞ = 100 ± 200 MPa.



Select, once again, the starting point of the loop at point A where σ_A = 499 MPa and ε_A = 0.006574. Determine the point on the loop (the stress at the hole) when $\Delta\sigma_\infty$ = 200 MPa. It gives, as in (j,k) above,

$$\begin{cases} \Delta\sigma \cdot \Delta\varepsilon = \dfrac{K_f^2\,(\Delta\sigma_\infty)^2}{E} = \dfrac{2.7^2 \cdot 200^2}{200\,000} = 1.458 \\ \Delta\varepsilon = \dfrac{\Delta\sigma}{200\,000} + 2\left(\dfrac{\Delta\sigma}{2 \cdot 1344}\right)^{1/0.18} \end{cases} \qquad \text{(p,q)}$$

This system of equations gives $\Delta\sigma$ = 519 MPa and $\Delta\varepsilon$ = 0.002810. The point E on the hysteresis loop is now obtained from

$$\begin{cases} \sigma_E = \sigma_A - \Delta\sigma = 499 \text{ MPa} - 519 \text{ MPa} = -20 \text{ MPa} \\ \varepsilon_E = \varepsilon_A - \Delta\varepsilon = 0.006574 - 0.002810 = 0.003764 \end{cases} \qquad \text{(r,s)}$$

Finally, determine the turning point of the new loop (the stress at the hole) when $\Delta\sigma_\infty$ = 400 MPa (the remote stress then is σ_∞ = – 100 MPa). It gives, as in (j,k) above,

$$\begin{cases} \Delta\sigma \cdot \Delta\varepsilon = \dfrac{K_f^2 (\Delta\sigma_\infty)^2}{E} = \dfrac{2.7^2 \cdot 400^2}{200\,000} = 5.832 \\ \Delta\varepsilon = \dfrac{\Delta\sigma}{200\,000} + 2\left(\dfrac{\Delta\sigma}{2 \cdot 1344}\right)^{1/0.18} \end{cases} \qquad \text{(t,u)}$$

This gives $\Delta\sigma$ = 829 MPa and $\Delta\varepsilon$ = 0.007038.

The turning point F on the new hysteresis loop is now obtained from

$$\begin{cases} \sigma_F = \sigma_A - \Delta\sigma = 499 \text{ MPa} - 829 \text{ MPa} = -330 \text{ MPa} \\ \varepsilon_F = \varepsilon_A - \Delta\varepsilon = 0.006574 - 0.007038 = -0.000464 \end{cases} \qquad \text{(v,w)}$$

Using point F as starting point, some points on the upper branch of the hysteresis loop may be determined from the stress changes already calculated. This is left to the student as an exercise.

The number of loading cycles to crack initiation is obtained, according to Morrow, from

$$\varepsilon_a = \frac{\sigma'_f - \sigma_m}{E}\,(2N)^b + \varepsilon'_f\,(2N)^c \qquad \text{(x)}$$

The mean value σ_m of the stress becomes $\sigma_m = (\sigma_A + \sigma_F)/2 = (499 - 330)/2$ = 84.5 MPa, and the strain *amplitude* will be ε_a = $(\varepsilon_A - \varepsilon_F)/2$ = (0.006574 + 0.000464) / 2 = 0.003519. By use of these values in (x), and by use of the parameter values given, one obtains

$$\varepsilon_a = 0.003519 = \frac{1227 - 84.5}{200\,000}\,(2N)^{-0.095} + 1.0\,(2N)^{-0.66} \qquad \text{(y)}$$

From this $2N$ = 23 070 reversals, giving N = 11 500 cycles, to fatigue failure is solved. (This result may be compared with the result obtained in Section 9.3.3 for the same structure. There the remote stress σ_∞ = 200 ± 200 MPa gave the fatigue life N = 6 500 cycles to failure.)

Answer: (Compare with the problem solved in Section 9.3.3 in the textbook.)
(a) At remote stress $\sigma_\infty = \pm\, 300$ MPa the number of cycles N to fatigue failure is $N = 2300$, approximately,
(b) stress and strain at the hole are (end points of the hysteresis loop)
$\sigma_a = \pm\, 499$ MPa and $\varepsilon_a = \pm\, 0.006574$, respectively, and
(c) fatigue life is expected to be N = 11 500 cycles to failure (according to Morrow) at remote stress $\sigma_\infty = 100 \pm 200$ MPa.

9/2.
In a structure a notch with stress concentration factor $K_t = 2.8$ has been found. Assume that the material follows the cyclic stress-strain curve

$$\varepsilon_a = \frac{\sigma_a}{E} + \left(\frac{\sigma_a}{K'}\right)^{1/n'} \qquad \left(\text{and for changes:} \qquad \Delta\varepsilon = \frac{\Delta\sigma}{E} + 2\left(\frac{\Delta\sigma}{2K'}\right)^{1/n'}\right)$$

where E = 210 GPa, K' = 1200 MPa and n' = 0.19.

The structure is loaded so that the nominal stress some distance away from the notch varies between – 50 MPa and + 250 MPa. (In addition to this stress, the stress concentration will be added at the notch.)

Determine the expected number of load cycles to fatigue failure.
Use fatigue notch factor $K_f = K_t$, the reduction of the cross-sectional area at rupture is Ψ = 65 per cent, and the ultimate strength of the material is σ_U = 470 MPa.

Solution:
The material relation and the Neuber hyperbola give for stress and strain ranges (stress range is $\Delta\sigma_\infty = 250 - (-50) = 300$ MPa). As we are going to use the Coffin-Manson rule, only ranges are of interest here. The influence of the mean values is disregarded. One obtains

$$\Delta\varepsilon = \frac{\Delta\sigma}{E} + 2\left(\frac{\Delta\sigma}{2\,K'}\right)^{1/n'} = \frac{\Delta\sigma}{210\,000} + 2\left(\frac{\Delta\sigma}{2\cdot 1200}\right)^{1/0.19} \qquad \text{(a)}$$

Neuber:

$$\Delta\sigma\cdot\Delta\varepsilon = \frac{(K_f\cdot\Delta\sigma_\infty)^2}{E} = \frac{(2.8\cdot 300)^2}{210\,000} = 3.36 \qquad \text{(b)}$$

This system of equations gives $\Delta\sigma$ = 650 MPa and $\Delta\varepsilon$ = 0.005169.

Coffin-Manson's fatigue law reads

$$\Delta\varepsilon = 3.5\,\frac{\sigma_U}{E}\,N^{-0.12} + D^{0.6}N^{-0.6} \quad \text{where} \quad D = \ln\frac{1}{1-\Psi} = \ln\frac{1}{1-0.65} = 1.0498 \qquad \text{(c)}$$

Numerical values give

$$0.005169 = 3.5\,\frac{470}{210\,000}\,N^{-0.12} + 1.0498^{0.6}\,N^{-0.6} \qquad \text{(d)}$$

from which N = 19 200 cycles (approximately) is solved.
Thus, failure is expected after, approximately, 19 000 cycles.

Answer: Fatigue failure is expected after N = 19 000 cycles (approximately).

9/3.

A test specimen is subjected to a remote nominal stress σ_∞ = 500 ± 200 MPa. The specimen has a notch with stress concentration factor K_t and fatigue notch factor K_f such that $K_t = K_f = 2.8$. Assume that the material follows the cyclic stress-strain relationships (for amplitudes and changes, respectively)

$$\varepsilon_a = \frac{\sigma_a}{E} + \left(\frac{\sigma_a}{K'}\right)^{1/n'} \qquad \text{and} \qquad \Delta\varepsilon = \frac{\Delta\sigma}{E} + 2\left(\frac{\Delta\sigma}{2K'}\right)^{1/n'}$$

Determine the stress (mean value and amplitude) and the strain (mean value and amplitude) in the material at the place of stress and strain concentration. Then use the calculated values in a low-cycle fatigue analysis to determine the expected fatigue life N of the test specimen.

The material has the following cyclic properties:
E = 206 GPa, K' = 1750 MPa, n' = 0.11, σ_Y' = 850 MPa, σ_f' = 1400 MPa, ε_f' = 0.60, b = – 0.10, and c = – 0.55.

Solution:

Determine the hysteresis loop obtained when the remote stress is varying between 700 MPa and 300 MPa.
Comment: When σ_∞ = 700 MPa, the stress at the stress concentration should be more than 700 MPa, of course, and less than 2.8·700 = 1960 MPa, which is the stress one should have had if the material were fully elastic.

The upper end point of the hysteresis loop is obtained from the intersection of the Neuber hyperbola and the cyclic stress-strain relation. One obtains

$$\sigma_a \cdot \varepsilon_a = \frac{K_f^2 \cdot \sigma_\infty^2}{E} = \frac{2.8^2 \cdot 700^2}{206\,000} = 18.65 \qquad \text{(a)}$$

$$\varepsilon_a = \frac{\sigma_a}{E} + \left(\frac{\sigma_a}{K'}\right)^{1/n'} = \frac{\sigma_a}{206\,000} + \left(\frac{\sigma_a}{1750}\right)^{1/0.11} \qquad \text{(b)}$$

From these two equations the upper end point (the turning point) of the hysteresis loop will be found. One obtains

$$\sigma_{max} = 1077 \text{ MPa} \quad \text{and} \quad \varepsilon_{max} = 0.017319 \tag{c,d}$$

The stress concentration factor K_σ and the strain concentration factor K_ε may now be determined (if desired). One obtains

$$K_\sigma = \frac{1077}{700} = 1.5386 \quad \text{and} \quad K_\varepsilon = \frac{\varepsilon_{max}}{\varepsilon_\infty} = \frac{\varepsilon_{max}}{\sigma_\infty / E} = \frac{0.017319}{700/206\,000} = 5.096 \tag{e,f}$$

One finds that $K_\sigma \cdot K_\varepsilon$ = 7.8418, and that $K_f^2 = 2.8^2$ = 7.84, as it should.

A change $\Delta\sigma_\infty$ = 400 MPa (*i.e.* twice the amplitude 200 MPa) of the remote stress σ_∞ causes changes of $\Delta\sigma$ and $\Delta\varepsilon$ at the stress concentration. The changes $\Delta\sigma$ and $\Delta\varepsilon$ at the stress concentration are obtained from the intersection of the Neuber hyperbola and the stress-strain relation for changes. One obtains

$$\Delta\sigma \cdot \Delta\varepsilon = \frac{K_f^2 \cdot (\Delta\sigma_\infty)^2}{E} = \frac{2.8^2 \cdot 400^2}{206\,000} = 6.119 \tag{g}$$

$$\Delta\varepsilon = \frac{\Delta\sigma}{E} + 2\left(\frac{\Delta\sigma}{2K'}\right)^{1/n'} = \frac{\Delta\sigma}{206\,000} + 2\left(\frac{\Delta\sigma}{2 \cdot 1750}\right)^{1/0.11} \tag{h}$$

This gives the changes of stress and strain at the stress concentration as

$$\Delta\sigma = 1116 \text{ MPa} \quad \text{and} \quad \Delta\varepsilon = 0.005479 \tag{i,j}$$

Comment: One notices that $2.8 \cdot 400$ = 1120 MPa, which is the change one would have obtained if the material were fully elastic. Here we have got the change $\Delta\sigma$ = 1116 MPa in (i), which implies that almost all deformation at the notch is elastic at this change of the remote stress. The second term on the right hand side of (h) is thus, in this case, much smaller than the first term. The reason why we have such a large range for the elastic deformation here is that the material is yielding in tension when the change of load is applied, and from that starting point the elastic range is twice the yield limit of the material until yielding starts at compression.

The lower end point of the hysteresis loop may now be determined. One obtains

$$\sigma_{min} = \sigma_{max} - \Delta\sigma = 1077 - 1116 \text{ MPa} = -39 \text{ MPa} \quad \text{and} \tag{k}$$

$$\varepsilon_{min} = \varepsilon_{max} - \Delta\varepsilon = 0.017319 - 0.005479 = 0.01184 \tag{l}$$

The mean value and the amplitude of the stress and the strain at the notch can now be calculated. One obtains

$$\sigma_{mean} = \frac{\sigma_{max} + \sigma_{min}}{2} = \frac{1077 - 39 \text{ MPA}}{2} = 519 \text{ MPa} \tag{m}$$

$$\sigma_{a} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{1077 + 39 \text{ MPA}}{2} = 558 \text{ MPa} \tag{n}$$

$$\varepsilon_{mean} = \frac{\varepsilon_{max} + \varepsilon_{min}}{2} = \frac{0.017319 + 0.01184}{2} = 0.01450 \tag{o}$$

$$\varepsilon_{a} = \frac{\varepsilon_{max} - \varepsilon_{min}}{2} = \frac{0.017319 - 0.01184}{2} = 0.0027395 \tag{p}$$

The fatigue life N is obtained from the Morrow expression:

$$\varepsilon_a = \frac{\sigma_f' - \sigma_{mean}}{E}(2N)^b + \varepsilon_f'(2N)^c \tag{q}$$

which gives

$$0.0027395 = \frac{1400 - 519}{206\,000}(2N)^{-0.10} + 0.60(2N)^{-0.55} \tag{r}$$

From this, N = 33 400 cycles is solved.

Thus, the expected number of cycles to fatigue failure is N = 33 400 cycles.

Answer: Failure is expected after N = 33 400 cycles (approximately).

9/4.

A structure has a notch with the fatigue notch factor K_f. The structure is designed for cyclic loading, and (maximum) 10 000 loading cycles is expected during the life of the structure. The remote loading (the loading far away from the notch) is $\sigma_\infty = \pm 250$ MPa. Determine the highest value of the fatigue notch factor K_f that can be allowed.

Material data: The material can be considered linear elastic, ideally plastic with modulus of elasticity E = 210 GPa and yield strength σ_Y = 350 MPa. The ultimate strength of the material is σ_U = 380 MPa and the ductility D = 1.05.

Solution:

Use the Coffin-Manson rule to calculate the maximum strain amplitude allowed. It gives

$$\varepsilon_a = 1.75\frac{\sigma_U}{E}N^{-0.12} + 0.5\,D^{0.6}\,N^{-0.6} \tag{a}$$

Numerical values, as given above, give

$$\varepsilon_a = 1.75 \frac{380}{210000} 10\,000^{-0.12} + 0.5 \cdot 1.05^{0.6}\, 10\,000^{-0.6}$$

$$= 0.0030983 \qquad \text{(b)}$$

The stress at the notch can not be higher than σ_Y = 350 MPa. (In fact, it could be slightly higher than 350 MPa, as the ultimate strength of the material is 380 MPa. Thus, some deformation hardening could take place in the material, but this is not taken into account here.) The Neuber hyperbola

$$\sigma \cdot \varepsilon = \frac{K_f^2\, \sigma_\infty^2}{E} \qquad \text{(c)}$$

gives, with σ_{max} = σ_Y = 350 MPa,

$$350 \cdot 0.003098 = \frac{K_f^2\ 250^2}{210\,000} \qquad \text{(d)}$$

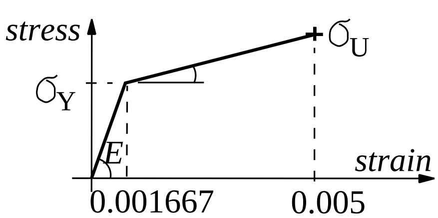
From this the maximum value of the fatige notch factor is solved. One obtains

$$K_f = 1.91 \qquad \text{(e)}$$

Answer: Maximum allowable value of the fatige notch factor is $K_f = 1.91$.

9/5.

A structure has a notch with the fatigue notch factor K_f. The structure is designed for cyclic loading, and (maximum) 10 000 loading cycles is expected during the life of the structure. The remote loading (the loading far away from the notch) is $\sigma_\infty = \pm\, 200$ MPa. Determine the highest value of the fatigue notch factor K_f that can be allowed.



Material data: The material behaviour at cyclic loading may be regarded as linear elastic, deformation hardening (see figure). The modulus of elasticity is E = 210 GPa, the yield strength σ_Y = 350 MPa, the ultimate strength of the material is σ_U = 380 MPa, at which stress the strain (at failure) is 0.005. The ductility is D = 1.05.

Solution:

Use the Coffin-Manson rule to calculate the maximum strain amplitude allowed. It gives

$$\varepsilon_a = 1.75 \frac{\sigma_U}{E} N^{-0.12} + 0.5\, D^{0.6}\, N^{-0.6} \qquad \text{(a)}$$

Numerical values, as given above, give

$$\varepsilon_a = 1.75 \frac{380}{210000} 10\,000^{-0.12} + 0.5 \cdot 1.05^{0.6}\, 10\,000^{-0.6}$$

$$= 0.0030983 \qquad \text{(b)}$$

This strain is larger than the strain 0.001667 (the elastic limit, see figure), and less than 0.005, so it is concluded that the Neuber hyperbola must intersect the strain-hardening branch of the stress-strain curve given. The equation of this branch is $\sigma = 335 + 9000 \cdot \varepsilon$ (stress in MPa)

Enter the strain ε = 0.003098 into this equation. It gives stress σ = 363 MPa. Thus, the maximum stress σ = 363 MPa can be allowed at the notch.

The Neuber hyperbola

$$\sigma \cdot \varepsilon = \frac{K_f^2\, \sigma_\infty^2}{E} \qquad \text{(c)}$$

gives, with σ_{max} = 363 MPa,

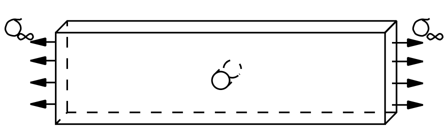
$$363 \cdot 0.003098 = \frac{K_f^2\;\; 200^2}{210\,000} \qquad \text{(d)}$$

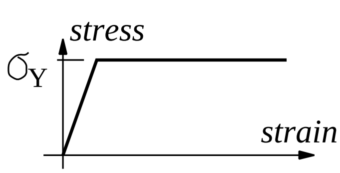
From this the maximum allowable value of the fatige notch factor is solved. One obtains $K_f = 2.43$ (e)

Answer: The largest fatigue notch factor that can be allowed is $K_f = 2.43$.

9/6.

A flat bar of a linear elastic, *ideally plastic* material with a rectangular cross section has a small circular hole at its centre axis. The stress concentration factor at the hole is K_t = 3.0 and the fatigue notch factor is K_f = 2.8.





The bar is subjected to an alternating stress

$$\sigma_\infty = \pm 200 \text{ MPa.}$$

Use the Neuber method to estimate the stress and the stain at the hole, and then use the Morrow equation to estimate the number of cycles to fatigue failure. Note that the material is assumed to be ideally plastic also at cyclic loading, *i.e.* no deformation hardening or softening is present. The modulus of elasticity is E = 200 GPa and the yield strength is σ_Y = 400 MPa. Further, σ_f'= 1200 MPa, ε_f'= 1.0, b = – 0.1 and c = – 0.62.

Solution:
Due to the high stresses at the hole, the material will yield locally. The stress concentration factor K_σ and the strain concentration factor K_ε may be written

$$K_\sigma = \frac{\sigma_{max}}{\sigma_\infty} \qquad \text{and} \quad K_\varepsilon = \frac{\varepsilon_{max}}{\varepsilon_\infty} = \frac{\varepsilon_{max}}{\sigma_\infty / E} \tag{a,b}$$

(Hooke's law is valid for stresses below 400 MPa, thus $\varepsilon_\infty = \sigma_\infty / E$)
The Neuber hyperbola becomes

$$\sigma \cdot \varepsilon = \frac{K_f^2 \sigma_\infty^2}{E} \tag{c}$$

The material data given and the nominal stress amplitude σ_∞ = 200 MPa now determine the Neuber hyperbola. The local (maximum) stress and strain amplitudes σ_a and ε_a, respectively, at the hole are obtained as the intersection of the Neuber hyperbola with the material stress-strain relation. One obtains

$$\begin{cases} \sigma_a \cdot \varepsilon_a = \dfrac{K_f^2 \sigma_\infty^2}{E} = \dfrac{2.8^2 \cdot 200^2}{200\,000} = 1.568 \\ \varepsilon_a \quad \text{is unknown,} \quad \text{but} \quad \sigma_a = \sigma_Y = 400 \text{ MPa} \end{cases} \tag{d,e}$$

This gives $\sigma_a = \sigma_{max}$ = 400 MPa and, from (d), $\varepsilon_a = \varepsilon_{max}$ = 0.00392.

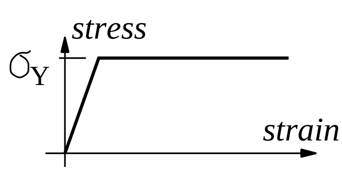
Now the number of cycles to fatigue failure may be determined. Using the mean stress σ_m = 0, Morrow's formula gives

$$\varepsilon_a = \frac{\sigma'_f - \sigma_m}{E} (2N)^b + \varepsilon'_f (2N)^c = \frac{1200}{200\,000} (2N)^{-0.1} + 1.0 (2N)^{-0.62} \tag{f}$$

Using ε_a = 0.00392, the fatigue life N = 13800 (or 13827) cycles is obtained.

Answer: The fatigue life N = 13 800 cycles is expected.

9/7.
A test specimen with a notch (stress concentration factor K_t = 3.0 and fatigue notch factor K_f = 2.5) is subjected to a remote stress $\sigma_\infty = \pm 300$ MPa. The material is assumed to be linear elastic, ideally plastic (also at cyclic loading) with modulus of elasticity E = 210 GPa, and yield limit σ_Y = 650 MPa, see figure.



Determine the expected number of loading cycles to fatigue failure of the specimen.
Data: Cross-section reduction Ψ at fracture is Ψ = 65 per cent and fracture strength is σ_U = 650 MPa.

Solution:

The Neuber hyperbola gives (for amplitudes)

$$\varepsilon_a \cdot \sigma_a = \frac{(K_f \sigma_\infty)^2}{E} \quad \text{where} \quad \sigma_\infty = 300 \text{ MPa} \tag{a}$$

As the material is ideally plastic, it is concluded that the stress cannot exceed σ_Y = 650 MPa at the point where the stress concentration appears. Using this (*i.e.* $\sigma_a = \sigma_Y$ = 650 MPa) in equation (a), one obtains

$$\varepsilon_a = \frac{1}{\sigma_a}\frac{(K_f \sigma_\infty)^2}{E} = \frac{1}{650}\frac{(2.5 \cdot 300)^2}{210\,000} = 0.00412 \tag{b}$$

The Coffin-Manson relationship gives, with ε_a = 0.00412,

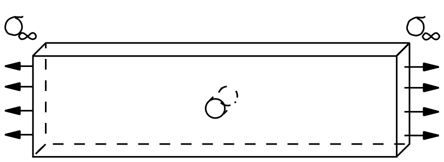
$$\Delta\varepsilon = 2 \cdot \varepsilon_a = 2 \cdot 0.00412 = 3.5\frac{650}{210\,000} N^{-0.12} + D^{0.6} N^{-0.6} \tag{c}$$

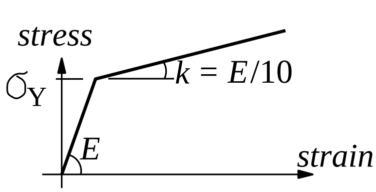
Where $D = \ln[1/(1-\Psi)] = 1.04982$.
Solving (c) for N gives N = 8330 cycles.

Answer: Fatigue failure is expected after 8300 cycles, approximately.

9/8.

A flat bar of a linear elastic, *deformation hardening* plastic material has a rectangular cross section. A small circular hole has been drilled in the bar at its centre axis. The stress concentration factor at the hole is K_t = 3.0 and the fatigue notch factor is K_f = 2.8.





The bar is subjected to an alternating remote stress

$$\sigma_\infty = \pm 200 \text{ MPa}.$$

Estimate the number of cycles to fatigue failure by use of Neuber's method and Morrow's equation. Note that the material is assumed to be linearly elastic, linearly deformation hardening also at cyclic loading as shown in the figure. Use modulus of elasticity E = 200 GPa, slope $k = E/10$, and the yield strength σ_Y = 400 MPa. Further, σ_f'= 1200 MPa, ε_f'= 1.0, b = – 0.1, and c = – 0.62.

Solution:

Due to the high stresses near the hole, the material will yield locally. The stress concentration factor K_σ and the strain concentration factor K_ε may be written

$$K_\sigma = \frac{\sigma_{\max}}{\sigma_\infty} \qquad \text{and} \qquad K_\varepsilon = \frac{\varepsilon_{\max}}{\varepsilon_\infty} = \frac{\varepsilon_{\max}}{\sigma_\infty / E} \tag{a,b}$$

(Hooke's law is valid for stresses below 400 MPa; thus $\varepsilon_\infty = \sigma_\infty / E$ in (b).)

The Neuber hyperbola becomes

$$\sigma \cdot \varepsilon = \frac{K_f^2 \sigma_\infty^2}{E} \tag{c}$$

The material data given and the nominal stress amplitude σ_∞ = 200 MPa now determine the Neuber hyperbola. The local stress and strain amplitudes σ_a and ε_a, respectively, at the hole are obtained as the intersection of the Neuber hyperbola and the material stress-strain relationship given in the figure. One obtains

$$\begin{cases} \sigma_a \cdot \varepsilon_a = \dfrac{K_f^2 \sigma_\infty^2}{E} = \dfrac{2.8^2 \cdot 200^2}{200\,000} = 1.568 \\ \sigma_a = 360 + k \cdot \varepsilon_a \quad \text{for} \quad \sigma_a > \sigma_Y = 400 \text{ MPa} \end{cases} \tag{d,e}$$

This gives σ_a = σ_{max} = 432.5 MPa and, from (d), ε_a = ε_{max} = 0.003625.

Now the number of cycles to fatigue failure may be determined. According to Morrow, and by use of mean stress σ_m = 0, one obtains

$$\varepsilon_a = \frac{\sigma'_f - \sigma_m}{E} (2N)^b + \varepsilon'_f (2N)^c = \frac{1200}{200\,000} (2N)^{-0.1} + 1.0 (2N)^{-0.62} \tag{f}$$

Using ε_a = 0.003625, the fatigue life N = 17575 (or 17600) cycles is obtained.

Answer: The fatigue life N = 17 600 cycles (approximately) is expected.

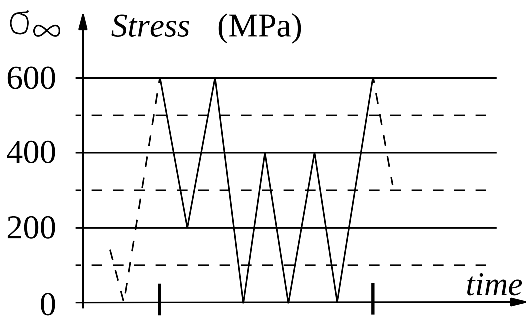
9/9.

A test specimen is loaded with a repeated stress sequence with a nominal stress σ_∞ according to the figure. The test specimen contains a notch with stress concentration factor K_t = 2.7 and fatigue notch factor K_f = 2.5. Assume that the material follows the cyclic stress-strain curve

$$\varepsilon_a = \frac{\sigma_a}{E} + \left(\frac{\sigma_a}{K'} \right)^{1/n'} \qquad \left(\text{and at changes:} \quad \Delta\varepsilon = \frac{\Delta\sigma}{E} + 2 \left(\frac{\Delta\sigma}{2K'} \right)^{1/n'} \right)$$

(a) Determine the stresses and strains at the notch needed for a fatigue life analysis (*i.e.* determine the mean values of the stress cycles and the amplitudes of the strain cycles at the notch). Use the rain-flow count method for the cycle counting.

(b) Determine the expected fatigue life (the expected number of loading sequences before failure) of the test specimen.



The material has the following cyclic properties:
E = 206 GPa, K' = 1750 MPa, n' = 0.11,
σ_Y = 850 MPa, σ_f' = 1500 MPa,
ε_f' = 0.60, $b = -0.10$, and $c = -0.55$.

Solution:

(a) *Stresses and strains at the notch*

The mean value and the amplitude of the stress and strain at the notch are obtained from the turning points of the hysteresis loops. Determine these points.

The stress and strain at the notch, when σ_∞ = 600 MPa, are obtained from the relationships (the Neuber hyperbola and the cyclic material relation for amplitudes):

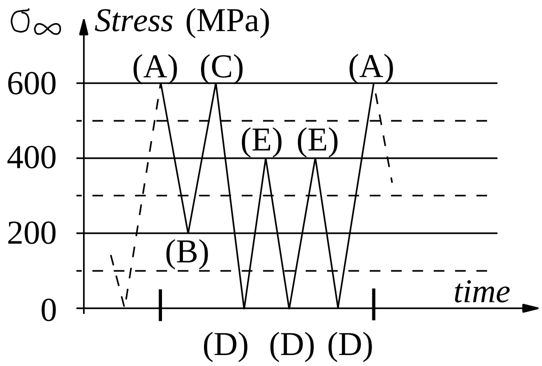
$$\begin{cases} \sigma_a \cdot \varepsilon_a = \dfrac{K_f^2(\sigma_\infty)^2}{E} = \dfrac{2.5^2 \cdot 600^2}{206\,000} = 10.922 \\ \varepsilon_a = \dfrac{\sigma_a}{E} + \left(\dfrac{\sigma_a}{K'}\right)^{1/n'} = \dfrac{\sigma_a}{206\,000} + \left(\dfrac{\sigma_a}{1750}\right)^{1/0.11} \end{cases} \qquad \text{(a,b)}$$

It is assumed that after a number of loading sequences, the stress and the strain stabilize as given by the Ramberg-Osgood’s stress-strain relation even if the loading is not alternating here. Therefore, the remote stress 600 MPa is used as an amplitude when the upper turning point of the hysteresis loop is determined.

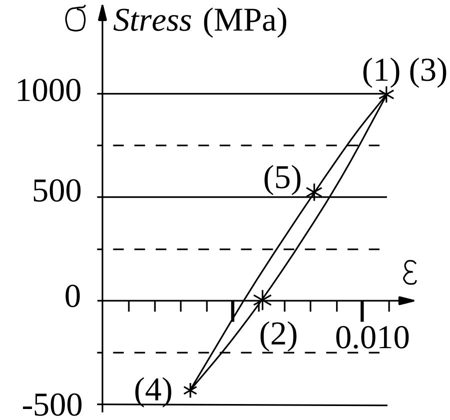
From (a,b) the upper turning point of the hysteresis loop is found. This point is marked (1) in the figure to the right below. One obtains

$$\sigma_1 = 999 \text{ MPa and } \varepsilon_1 = 0.010943$$

Thus, each time the remote stress σ_∞ reaches the level 600 MPa (points (A) and (C) in the figure to the left), the stress and the strain at the notch become σ_1 = 999 MPa and ε_1 = 0.010943, respectively (point (1) in the figure to the right).



Remote (nominal) stress giving stress concentration at the notch



Stress and strain at the notch

A *change* $\Delta\sigma_\infty$ = 400 MPa of the remote stress, from level (A) to level (B) in the figure, gives

$$\begin{cases} \Delta\sigma \cdot \Delta\varepsilon = \dfrac{K_f^2\,(\Delta\sigma_\infty)^2}{E} = \dfrac{2.5^2 \cdot 400^2}{206\,000} = 4.854 \\ \Delta\varepsilon = \dfrac{\Delta\sigma}{E} + 2\left(\dfrac{\Delta\sigma}{2\cdot K'}\right)^{1/n'} = \dfrac{\Delta\sigma}{206\,000} + 2\left(\dfrac{\Delta\sigma}{2\cdot 1750}\right)^{1/0.11} \end{cases} \qquad \text{(c,d)}$$

This gives the stress and strain *changes* at the notch:

$$\Delta\sigma = 998 \text{ MPa} \quad \text{and} \quad \Delta\varepsilon = 0.004865$$

The stress and the stain at the notch, when σ_∞ = 200 MPa at (B), become

$$\sigma_2 = \sigma_1 - \Delta\sigma = 999 - 998 \text{ MPa} = 1 \text{ MPa} \quad \text{and}$$
$$\varepsilon_2 = \varepsilon_1 - \Delta\varepsilon = 0.010943 - 0.004865 = 0.006078$$

This point has been marked as point (2) in the stress-strain diagram.

After this the loading stress σ_∞ turns upwards the amount $\Delta\sigma_\infty$ = 400 MPa up to stress level σ_∞ = 600 MPa, see point (C). The stress at the notch then increases $\Delta\sigma$ = 998 MPa from σ_2 to σ_3 = 1 MPa + 998 MPa = 999 MPa = σ_1, and the strain increases from ε_2 to ε_3 = 0.006078 – 0.004865 = 0.010943 = ε_1, giving point (3) in the diagram. Point (3) coincides with point (1).

A stress *change* $\Delta\sigma_\infty$ = 600 MPa of the remote stress, from level (C) to level (D), gives

$$\begin{cases} \Delta\sigma \cdot \Delta\varepsilon = \dfrac{K_f^2\,(\Delta\sigma_\infty)^2}{E} = \dfrac{2.5^2 \cdot 600^2}{206\,000} = 10.922 \\ \Delta\varepsilon = \dfrac{\Delta\sigma}{E} + 2\left(\dfrac{\Delta\sigma}{2 \cdot K'}\right)^{1/n'} = \dfrac{\Delta\sigma}{206\,000} + 2\left(\dfrac{\Delta\sigma}{2 \cdot 1750}\right)^{1/0.11} \end{cases} \qquad \text{(e,f)}$$

This gives the stress and strain *changes* at the notch:

$$\Delta\sigma = 1438 \text{ MPa} \quad \text{and} \quad \Delta\varepsilon = 0.007596$$

The stress and the strain at the notch when $\sigma_\infty = 0$ (σ_∞ is situated at point (D)) become (see point (4) in the figure)

$$\sigma_4 = \sigma_3 - \Delta\sigma = 999 - 1438 \text{ MPa} = -439 \text{ MPa and}$$
$$\varepsilon_4 = \varepsilon_3 - \Delta\varepsilon = 0.010943 - 0.007596 = 0.003347$$

After this two loops with stress changes $\Delta\sigma_\infty = 400$ MPa follow. At next maximum, *i.e.* when σ_∞ has come to level (E), the stress and the strain at the notch become (see point (5) in the figure)

$$\sigma_5 = \sigma_4 + \Delta\sigma = -439 + 998 \text{ MPa} = 559 \text{ MPa and}$$
$$\varepsilon_5 = \varepsilon_4 + \Delta\varepsilon = 0.003347 + 0.004865 = 0.008212$$

The final stress change, from level (D) back to (A), gives the branch from (4) back to (1) in the hysteresis loop.

To summarize, these values give the following stresses and strains at the notch (not all these values were asked for in the problem):

No of cycles	σ_{min}	σ_{max}	σ_{mean}	σ_{ampl}	ε_{min}	ε_{max}	ε_{mean}	ε_{ampl}
1	1	999	500	499	0.006078	0.010943	0.008510	0.002433
1	– 439	999	280	719	0.003347	0.010943	0.007145	0.003798
2	– 439	559	60	499	0.003347	0.008212	0.005780	0.002433

Thus, at the notch the following is obtained:
- one cycle between the points (1) and (2) in the stress-strain diagram, corresponding to cycle (A)-(B)-(C) for the remote stress,
- one cycle between the points (3) and (4) in the figure, corresponding to cycle (C)-(D)-(A) for the remote stress, and
- two cycles between the points (4) and (5) in the figure, corresponding to cycles (D)-(E)-(D)-(E)-(D) for the remote stress.

(b) *Fatige life*

The fatigue life N_i at each strain amplitude, with the corresponding mean value of the stress, is obtained from the Morrow relationship

$$\varepsilon_{ai} = \frac{\sigma_f' - \sigma_{mi}}{E}(2N_i)^b + \varepsilon'_f(2N_i)^c \tag{g}$$

which gives

$$0.002433 = \frac{1500 - 500}{206\,000}(2N_1)^{-0.10} + 0.60(2N_1)^{-0.55} \qquad \text{giving} \quad N_1 = 63\,650$$

$$0.003798 = \frac{1500 - 280}{206\,000}(2N_2)^{-0.10} + 0.60(2N_2)^{-0.55} \qquad \text{giving} \quad N_2 = 20\,355$$

$$0.002433 = \frac{1500 - 60}{206\,000}(2N_3)^{-0.10} + 0.60(2N_3)^{-0.55} \qquad \text{giving} \quad N_3 = 195\,400$$

The accumulated damage from one sequence is (use the Palmgren-Miner damage accumulation rule)

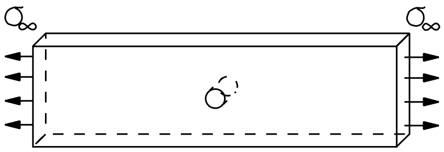
$$D = \frac{1}{63\,650} + \frac{1}{20\,355} + \frac{2}{195\,400} = \frac{1}{13\,320} \tag{h}$$

From this, $S = 1/D = 13\,320$ sequences to fatigue failure is obtained.

Answer: One expects $S = 1/D = 13\,300$ sequences to fatigue failure.

Extra problem, a version of 9/1.

9/1x.



A flat bar of material SAE 1045 with a rectangular cross section has a small circular hole at its centre axis. The bar is subjected to an alternating stress $\sigma_\infty = \pm 300$ MPa.

(a) Estimate the number of cycles to fatigue failure by use of Neuber's method. Assume that $K_f = 0.9K_t$.
(b) Draw the hysteresis loop for $\sigma_\infty = \pm 300$ MPa.
(c) Determine the fatigue life if $\sigma_\infty = 150 \pm 150$ MPa.

Solution:

(a) *Number of cycles to fatigue crack initiation*

If the stress state at the small hole in the flat bar was purely elastic, then the stress concentration factor would have been $K_t = 3$. The fatigue notch factor K_f is

$$K_f = 1 + q\,(K_t - 1) \tag{a}$$

No information for calculation of the notch sensitivity factor q is given here. Instead, $K_f = 0.9K_t = 2.7$ was given. Thus, use $K_f = 2.7$.

Due to the high stresses close to the hole, the material will yield locally. Taking this into consideration, the stress concentration factor K_σ and the strain concentration factor K_ε may be determined from equations (9.9a,b). If the second term in the material relation (9.2a,b) is disregarded (for stresses far away from the hole the second term is supposed to be small as compared with the first term) one obtains (9.9a,b) as

$$K_\sigma = \frac{\sigma_{max}}{\sigma_\infty} \quad \text{and} \quad K_\varepsilon = \frac{\varepsilon_{max}}{\varepsilon_\infty} = \frac{\varepsilon_{max}}{\sigma_\infty / E} \tag{b,c}$$

The Neuber hyperbola becomes

$$\sigma \cdot \varepsilon = \frac{K_f^2 \sigma_\infty^2}{E} \tag{d}$$

The material data given earlier for the material SAE 1045 and the nominal stress amplitude σ_∞ = 300 MPa now determine the Neuber hyperbola. The local stress amplitude σ_a and strain amplitude ε_a at the hole are obtained as the intersection point between the Neuber hyperbola and the material stress-strain relation (for cyclic loading). One obtains

$$\begin{cases} \sigma_a \cdot \varepsilon_a = \dfrac{K_f^2\,\sigma_\infty^2}{E} = \dfrac{2.7^2 \cdot 300^2}{200\,000} = 3.2805 \\ \varepsilon_a = \dfrac{\sigma_a}{E} + \left(\dfrac{\sigma_a}{K'}\right)^{1/n'} = \dfrac{\sigma_a}{200\,000} + \left(\dfrac{\sigma_a}{1344}\right)^{1/0.18} \end{cases} \tag{e,f}$$

This system of equations gives $\sigma_a = \sigma_{max} = 499$ MPa and $\varepsilon_a = \varepsilon_{max} = 0.006574$.

The stress and strain concentration factors K_σ and K_ε then become

$$K_\sigma = \frac{\sigma_{max}}{\sigma_\infty} = \frac{499}{300} = 1.663 \quad \text{and} \quad K_\varepsilon = \frac{\varepsilon_{max}}{\sigma_\infty / E} = \frac{0.006574}{300/200\,000} = 4.3828 \tag{g,h}$$

(Verify: $K_\sigma \cdot K_\varepsilon = 1.663 \cdot 4.383 = 7.29 = 2.7^2 = K_f^2$, as it should.)

Now the number of cycles to fatigue failure may be determined. According to Morrow, (9.4) and (9.7), and by use of mean stress $\sigma_m = 0$, one obtains

$$\varepsilon_a = \frac{\sigma'_f - \sigma_m}{E}(2N)^b + \varepsilon'_f(2N)^c = \frac{1227}{200\,000}(2N)^{-0.095} + 1.0\,(2N)^{-0.66} \tag{j}$$

Using $\varepsilon_a = 0.006574$, the fatigue life $2N = 4600$ reversals to failure is obtained, giving $N = 2300$ cycles to fatigue failure.

(b) *Display the hysteresis loop for the material at the notch, when the remote loading is* $\sigma_\infty = \pm 300$ MPa

According to problem (a) above the upper turning point of the hysteresis loop is situated at $\sigma = \sigma_{max} = 499$ MPa and $\varepsilon = \varepsilon_{max} = 0.006574$, see point A in the figure below. The lower turning point is situated at $\sigma = -\sigma_{max} = -499$ MPa and $\varepsilon = -\varepsilon_{max} = -0.006574$, see point B in the figure.

Choose the point A as starting point of the hysteresis loop, see figure below. A nominal change of stress $\Delta\sigma_\infty$ (far away from the stress concentration) will cause a change of stress $\Delta\sigma$ and a change of strain $\Delta\varepsilon$ at the stress concentration. The changes $\Delta\sigma$ and $\Delta\varepsilon$ are obtained by use of the Neuber hyperbola and the material relation for *changes* of stress and strain. Using $\Delta\sigma_\infty = 300$ MPa one obtains

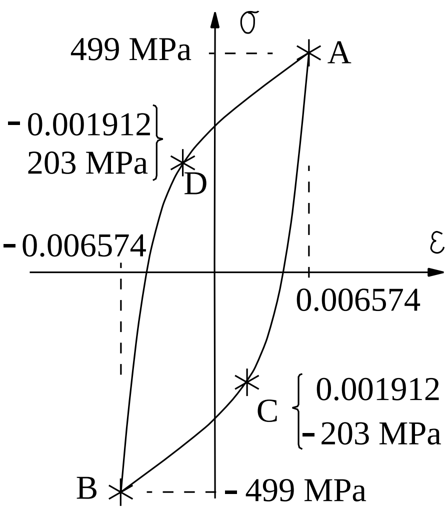
$$\begin{cases} \Delta\sigma \cdot \Delta\varepsilon = \dfrac{K_f^2\,(\Delta\sigma_\infty)^2}{E} = \dfrac{2.7^2 \cdot 300^2}{200\,000} = 3.2805 \\ \Delta\varepsilon = \dfrac{\Delta\sigma}{E} + 2\left(\dfrac{\Delta\sigma}{2K'}\right)^{1/n'} = \dfrac{\Delta\sigma}{200\,000} + 2\left(\dfrac{\Delta\sigma}{2 \cdot 1344}\right)^{1/0.18} \end{cases} \tag{k,l}$$

From this system of equations $\Delta\varepsilon = 0.004662$ and $\Delta\sigma = 702$ MPa are solved.

The lower branch of the hysteresis loop, starting at point A, will pass through the point C. The co-ordinates of point C are obtained from

$$\begin{cases} \sigma_C = \sigma_A - \Delta\sigma = 499 \text{ MPa} - 702 \text{ MPa} = -203 \text{ MPa} \\ \varepsilon_C = \varepsilon_A - \Delta\varepsilon = 0.006574 - 0.004662 = 0.001912 \end{cases} \quad \text{(m,n)}$$

Verify that the change of stress $\Delta\sigma_\infty = 600$ MPa will take this branch of the hysteresis loop to the point B.



At a change of stress $\Delta\sigma_\infty = 300$ MPa the upper branch of the hysteresis loop, starting at point B, will pass through point D. The coordinates of point D are obtained from

$$\begin{cases} \sigma_D = \sigma_B + \Delta\sigma = -499 \text{ MPa} + 702 \text{ MPa} = 203 \text{ MPa} \\ \varepsilon_D = \varepsilon_B + \Delta\varepsilon = -0.006574 + 0.004662 \\ \quad = -0.001912 \end{cases} \quad \text{(o,p)}$$

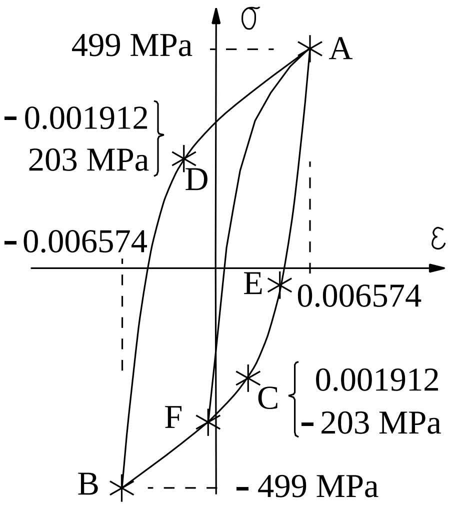
By use of these values (and perhaps some more) the hysteresis loop may now be drawn, see the figure.

Comment

It is noticed that when the loop passes the point C the external loading (far away from the hole) will give a nominal stress σ_∞ equal to zero, *i.e.* the bar has been loaded (a number of times) up to stress $\sigma_\infty = 300$ MPa and then unloaded by the stress $\Delta\sigma_\infty = 300$ MPa so that the total stress far away from the stress concentration will be $\sigma_\infty = 0$. At the stress concentration, however, the residual stress σ_C and a residual strain ε_C will remain.

(c) *Fatigue life when* $\sigma_\infty = 150 \pm 150$ MPa

The same bar as above ($K_f = 2.7$) will be investigated, but now the bar is loaded to a nominal stress $\sigma_\infty = 150 \pm 150$ MPa. The nominal (remote) stress σ_∞ will now vary between 0 and 300 MPa. This implies that the stress at the hole will vary between σ_C and σ_A according to problem (b).



As an extra exercise the hysteresis loop between A and C will be drawn. Select, once again, the starting point of the loop at point A where σ_A = 499 MPa and ε_A = 0.006574. Determine one more point on the loop. Here $\Delta\sigma_\infty$ = 200 MPa will be chosen. It gives, as in (j,k) above,

$$\begin{cases} \Delta\sigma \cdot \Delta\varepsilon = \dfrac{K_f^2\,(\Delta\sigma_\infty)^2}{E} = \dfrac{2.7^2 \cdot 200^2}{200\,000} = 1.458 \\ \Delta\varepsilon = \dfrac{\Delta\sigma}{200\,000} + 2\left(\dfrac{\Delta\sigma}{2 \cdot 1344}\right)^{1/0.18} \end{cases} \qquad \text{(q,r)}$$

This gives $\Delta\sigma$ = 519 MPa and $\Delta\varepsilon$ = 0.002810. The point E on the hysteresis loop is now obtained from

$$\begin{cases} \sigma_E = \sigma_A - \Delta\sigma = 499 \text{ MPa} - 519 \text{ MPa} = -20 \text{ MPa} \\ \varepsilon_E = \varepsilon_A - \Delta\varepsilon = 0.006574 - 0.002810 = 0.003764 \end{cases} \qquad \text{(s,t)}$$

The point F on the loop (branch starting at C, see the figure) gets the coordinates

$$\begin{cases} \sigma_F = \sigma_C + \Delta\sigma = -203 \text{ MPa} + 519 \text{ MPa} = 316 \text{ MPa} \\ \varepsilon_F = \varepsilon_C + \Delta\varepsilon = 0.001912 + 0.002810 = 0.004722 \end{cases} \qquad \text{(u,v)}$$

The number of loading cycles to fatigue failure is obtained, according to Morrow, from

$$\varepsilon_a = \frac{\sigma'_f - \sigma_m}{E}\,(2N)^b + \varepsilon'_f\,(2N)^c \qquad \text{(w)}$$

The mean value σ_m of the stress becomes $\sigma_m = (\sigma_A + \sigma_C)/2 = (499 - 203)/2$ = 148 MPa, and the strain *amplitude* will be $\varepsilon_a = (\varepsilon_A - \varepsilon_C)/2$ = (0.006574 – 0.001912) / 2 = 0.002331. By use of these values in (v), and by use of the given parameter values, one obtains

$$\varepsilon_a = 0.002331 = \frac{1227 - 148}{200\,000}\,(2N)^{-0.095} + 1.0\,(2N)^{-0.66} \qquad \text{(x)}$$

From this $2N$ = 95 600 reversals, giving N = 47 800 cycles, to fatigue failure is solved. (This result may be compared with the result N = 6 500 cycles to fatigue failure obtained when $\sigma_\infty = 200 \pm 200$ MPa, obtained in Section 9.3.1.)

Answer: (Compare with the solution given in example above)
(a) Number of cycles N to crack initiation is 2300, approximately,
(b) stress and strain at hole is $\sigma_a = \pm 499$ MPa and $\varepsilon_a = \pm 0.006574$, respectively
(c) N_f = 47 800 cycles (according to Morrow).