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EFFECTIVE TOPOLOGICAL SPACES III: FORCING AND DEFINABILITY

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Introduction

This paper is a continuation of Effective Topological Spaces I: A Definability Theory [7], and Effective Topological Spaces II: A Hierarchy [8]. Here we study the interrelations between our definability in Part I and forcing. We refer the reader to [7] for all definitions and terminology.

Here, we start by defining strong forcing in Section 16; in Section 17 we define weak forcing. Both 16 & 17 contain preliminary facts which are needed for the main results. In Section 18 we find useful links between strong and weak forcing. In Section 19 we discover a crucial relationship between our notion of satisfaction in [7] and the weak forcing defined in Section 17. In Section 20 we introduce generic models and in Section 21 we prove the usual 'Truth & Forcing' Theorem for them. Section 22 contains a brief discussion of Boolean valued models and we close with some remarks in Section 23.

16. Strong forcing

In this section we introduce the notion of *strong forcing* and prove a proposition for it. The proposition may be thought of as a version of *monotonicity* and *consistency* for strong forcing.

Definition. To the language L_X over Δ add an infinite set C of constant symbols. A *forcing condition* is a finite function p with $\mathrm{dom}(p) \subseteq C$ and $\mathrm{range}(p) \subseteq \Delta$. Given two conditions p and q define

$$p \leqslant q \quad \text{iff} \quad \forall c \in \mathrm{dom}\, q\; p(c) \subseteq q(c).$$

For convenience we adopt the notation that p' denotes a *stronger* condition than p; i.e., $p' \leqslant p$. Thus $p \geqslant p' \geqslant p'' \geqslant \cdots$.

The *strong forcing relation* between the conditions and L_X over Δ is defined

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inductively as follows:

If $\{c_1, \ldots, c_n\} \subseteq \mathrm{dom}(p)$, then

$$p \Vdash^* \mathcal{U}(c_1, \ldots, c_n) \quad \text{iff} \quad p(c_1) \times \cdots \times p(c_n) \subseteq \mathcal{U}. \tag{16.1}$$

$$p \Vdash^* \psi \vee \phi \quad \text{iff} \quad p \Vdash^* \psi \text{ or } p \Vdash^* \phi. \tag{16.2}$$

$$p \Vdash^* \neg\psi \quad \text{iff} \quad \forall p'\, p' \nVdash^* \psi. \tag{16.3}$$

$$p \Vdash^* \exists x\, \psi(x) \quad \text{iff} \quad \exists c\, p \Vdash^* \psi(c). \tag{16.4}$$

The strong forcing relation is both monotone and consistent in the following sense.

Proposition 16.1. (i) *If* $p \Vdash^* \phi$ *and* $q \leqslant p$, *then* $q \Vdash^* \phi$.
(ii) *No condition strong forces both* ϕ *and* $\neg\phi$.

Proof. (i) The proof of (i) proceeds by induction.

Case 1: $\phi = \mathcal{U}(c_1, \ldots, c_n)$. In this case

$$\begin{aligned} p \Vdash^* \mathcal{U}(c_1, \ldots, c_n) \quad &\text{iff} \quad p(c_1) \times \cdots \times p(c_n) \subseteq \mathcal{U} \\ &\text{implies} \quad q(c_1) \times \cdots \times q(c_n) \subseteq \mathcal{U} \qquad (\text{since } q \leqslant p) \\ &\text{iff} \quad q \Vdash^* \mathcal{U}(c_1, \ldots, c_n). \end{aligned}$$

Case 2: $\phi = \psi \vee \theta$.

$$\begin{aligned} p \Vdash^* \psi \vee \theta \quad &\text{iff} \quad p \Vdash^* \psi \text{ or } p \Vdash^* \theta \\ &\text{implies} \quad q \Vdash^* \psi \text{ or } q \Vdash^* \theta \qquad (\text{by inductive hypothesis}) \\ &\text{iff} \quad q \Vdash^* \psi \vee \theta. \end{aligned}$$

Case 3: $\phi = \neg\psi$. In this case observe

$$\begin{aligned} p \Vdash^* \neg\psi \quad &\text{iff} \quad \forall p'\, p' \nVdash^* \psi \\ &\text{implies} \quad \forall q'\, q' \nVdash^* \psi \qquad (\text{because } q' \leqslant p) \\ &\text{iff} \quad q \Vdash^* \neg\psi. \end{aligned}$$

Case 4: $\phi = \exists x\, \psi(x)$.

$$\begin{aligned} p \Vdash^* \exists x\, \psi(x) \quad &\text{implies} \quad \exists c\, p \Vdash^* \psi(c) \\ &\text{implies} \quad \exists c\, q \Vdash^* \psi(c) \qquad (\text{by the inductive hypothesis and since } \mathrm{dom}\, p \subseteq \mathrm{dom}\, q) \\ &\text{implies} \quad q \Vdash^* \exists x\, \psi(x). \end{aligned}$$

This completes the proof of part (i).

To prove (ii), suppose $p \Vdash^* \neg\phi$. Then by definition $\forall p'\, p' \nVdash^* \phi$. In particular (setting $p' = p$) $p \nVdash^* \phi$. □

17. Weak forcing

In this section we define the *forcing* relation (denoted by $\Vdash$) and prove some basic facts.

Definition. We say a condition p *forces* a formula φ iff p strong forces the double negation of φ. That is

$$p \Vdash \varphi \quad \text{iff} \quad p \Vdash^* \neg\neg\varphi.$$

The following lemma establishes several useful links between the forcing relation $\Vdash$ and the strong forcing relation $\Vdash^*$.

Lemma 17.1. (i) *$p \Vdash \varphi$ iff $\forall p' \exists p'' \Vdash^* \varphi$*

(ii) *Given a condition p, $\exists p' p' \Vdash \varphi$ iff $\exists p' p' \Vdash^* \varphi$.*

Proof. The proof of (i) proceeds simply by unfolding the definition of strong forcing.

$$\begin{aligned} p \Vdash \varphi \quad &\text{iff} \quad p \Vdash^* \neg\neg\varphi \\ &\text{iff} \quad \forall p' p' \nVdash^* \neg\varphi \\ &\text{iff} \quad \forall p' \exists p'' p'' \Vdash^* \varphi. \end{aligned}$$

To prove part (ii) first suppose $p' \Vdash \varphi$. Then by (i) there is a $q \leqslant p'$ so that $q \Vdash^* \varphi$. So there exists $q \leqslant p$ such that $q \Vdash^* \varphi$.

Conversely, if $p' \Vdash^* \varphi$, then $\forall p'' \forall p''' p''' \Vdash^* \varphi$ (let $q = p'''$ in Proposition 16.1). That is $\forall p'' \exists p''' p''' \Vdash^* \varphi$. Hence $p' \Vdash \varphi$ (by part (i)). □

At this point it is not difficult to see that the forcing relation $\Vdash$ satisfies analogues of the monotonicity, density and consistency theorems (see [7]). In particular we obtain:

Proposition 17.2. (i) *$p \Vdash \varphi$ and $q \leqslant p$, then $q \Vdash \varphi$.*

(ii) *$p \Vdash \varphi$ iff $\forall p' \exists p'' \Vdash \varphi$.*

(iii) *No condition forces both φ and $\neg\varphi$.*

Proof. (i) Since $p \Vdash \varphi$, it follows that $\forall p' \exists p'' p'' \Vdash^* \varphi$ by Lemma 17.1. Since $q \leqslant p$ we may set p' equal to any q' to obtain $\forall q' \exists q'' q'' \Vdash^* \varphi$. So $q \Vdash \varphi$.

To prove (ii) notice

$$\begin{aligned} p \Vdash \varphi \quad &\text{iff} \quad \forall p' \exists p'' p'' \Vdash^* \varphi \qquad &\text{(Lemma 17.1(i))} \\ &\text{iff} \quad \forall p' \exists p'' p'' \Vdash \varphi \qquad &\text{(Lemma 17.2(ii)).} \end{aligned}$$

Finally to prove (iii), suppose $p \Vdash \neg\varphi$. Then by definition $p \Vdash^* \neg\neg\neg\varphi$. So by Lemma 17.1(ii), $p \nVdash^* \neg\neg\varphi$. Consequently, by definition again $p \nVdash \varphi$. □

18. Useful links between strong and weak forcing

In order to place the main connection between forcing and our notion of satisfaction, we need the following technical information.

Proposition 18.1.

(i) $p \Vdash \mathcal{U}(c_1, \ldots, c_n)$ iff $\forall p' \exists p'' p''(c_1) \times \cdots \times p''(c_n) \subseteq \mathcal{U}$.

(ii) $p \Vdash \psi \vee \theta$ iff $\forall p' \exists p'' [p'' \Vdash \psi$ or $p'' \Vdash \theta]$.

(iii) $p \Vdash \neg\psi$ iff $\forall p' p' \nVdash \psi$.

(iv) $p \Vdash \exists x\, \psi(x)$ iff $\forall p' \exists p'' \exists c\, p'' \Vdash \psi(c)$.

Proof.

(i) $p \Vdash \mathcal{U}(c_1, \ldots, c_n)$ iff $\forall p' \exists p'' p'' \Vdash^* \mathcal{U}(c_1, \ldots, c_n)$ (Lemma 17.1)
iff $\forall p' \exists p'' p''(c_1) \times \cdots \times p''(c_n) \subseteq \mathcal{U}$.

(ii) $p \Vdash \psi \vee \theta$ iff $\forall p' \exists p'' (p'' \Vdash^* \psi \vee \theta)$ (Lemma 17.1(i))
iff $\forall p' \exists p'' (p'' \Vdash^* \psi$ or $p'' \Vdash^* \theta)$ (Definition (16.2))
iff $\forall p' \exists p'' (p'' \Vdash \psi$ or $p'' \Vdash \theta)$ (Lemma 17.1(ii)).

(iii) $p \Vdash \neg\psi$ iff $p \Vdash^* \neg\neg\neg\psi$ (Definition)
iff $\forall p' p' \nVdash^* \neg\psi$ (Definition (16.3))
iff $\forall p' p' \nVdash \psi$ (Definition).

(iv) $p \Vdash \exists x\, \psi(x)$ iff $\forall p' \exists p'' p'' \Vdash^* \exists x\, \psi$ (Lemma 17.1(ii))
iff $\forall p' \exists p'' \exists c\, p'' \Vdash^* \psi(c)$ (Definition (16.4))
iff $\forall p' \exists p'' \exists c\, p'' \Vdash \psi(c)$ (Lemma 17.1(ii)).

19. Forcing and satisfaction

For all of the definitions regarding our notions of satisfaction, we refer the reader to [7].

Our main result in this section is that for well behaved formulas, forcing and satisfaction are related. Specifically,

Theorem 19.1. *Let $\varphi(x_1, \ldots, x_n)$ be a definition and let*

$$p = \begin{pmatrix} c_1 \cdots c_n \\ \alpha_1 \cdots \alpha_n \end{pmatrix} \quad \text{and} \quad s = \begin{pmatrix} x_1 \cdots x_n \\ \alpha_1 \cdots \alpha_n \end{pmatrix}$$

be a condition and an assignment respectively such that $p(c_i) = s(x_i)$ for $i = 1, \ldots, n$.

Then $p \Vdash \varphi(c_1, \ldots, c_n)$ *iff* $\models \varphi[s]$.

Proof. The proof is by induction on the complexity of φ.

Case 1: $\varphi = \mathcal{U}(x_1, \ldots, x_n)$. By Proposition 18.1 we have

$$\begin{aligned} p \Vdash \mathcal{U}(c_1, \ldots, c_n) \quad &\text{iff} \quad \forall p' \, \exists p'' \, p''(c_1) \times \cdots \times p''(c_n) \subseteq \mathcal{U} \\ &\text{iff} \quad \forall s' \, \exists s'' \, s''(x_1) \times \cdots \times s''(x_n) \subseteq \mathcal{U} \\ &\qquad (\text{since } s(x_i) = p(c_i) \text{ for } i = 1, \ldots, n) \\ &\text{iff} \quad \models \mathcal{U}[s]. \end{aligned}$$

Case 2: $\varphi = \psi \vee \theta$.

$$\begin{aligned} p \Vdash \psi \vee \theta \quad &\text{iff} \quad \forall p' \, \exists p'' \, [p'' \Vdash \psi \text{ or } p'' \Vdash \theta] \qquad (\text{Proposition 18.1}) \\ &\text{iff} \quad \forall s' \, \exists s'' \, [\models \psi[s''] \text{ or } \models \theta[s'']] \\ &\qquad (s(x_i) = p(c_i) \text{ and the inductive hypothesis}) \\ &\text{iff} \quad \models (\psi \vee \theta)[s]. \end{aligned}$$

Case 3: $\varphi = \neg\psi$.

$$\begin{aligned} p \Vdash \neg\psi \quad &\text{iff} \quad \forall p' \, p' \nVdash \psi && (\text{Proposition 18.1}) \\ &\text{iff} \quad \forall s' \, \not\models \psi[s'] && (\text{induction hypothesis}) \\ &\text{iff} \quad \forall s' \, \exists s'' \models \neg\psi[s''] && (\text{Theorem 8.2}) \\ &\text{iff} \quad \models \neg \psi[s]. \end{aligned}$$

Case 4: $\varphi = \exists x \, \psi$.

$$\begin{aligned} p \Vdash \exists x \, \psi \quad &\text{iff} \quad \forall p' \, \exists p'' \, \exists c \, p'' \Vdash \psi(c, c_1, \ldots, c_n) \\ &\text{iff} \quad \forall p' \, \exists p'' \, \exists c \models \psi\begin{bmatrix} x & x_1 \cdots x_n \\ p''(c) & p''(c_1) \cdots p''(c_n) \end{bmatrix}. \end{aligned} \tag{1}$$

This last equivalence is by the inductive hypothesis which applies because the assignment

$$t = \begin{bmatrix} x & x_1 \cdots x_n \\ p''(c) & p''(c_1) \cdots p''(c_n) \end{bmatrix}$$

is such that $t(x) = p''(c)$ and $t(x_i) = p''(c_i)$ for $i = 1, \ldots, n$.

We further claim that (1) iff

$$\forall s' \, \exists s'' \, \exists \alpha \models \psi[s'' \cup (\tbinom{x}{\alpha})]. \tag{2}$$

To see that (1) implies (2) just define $p'(c_i) = s'(x_i)$ and use (1) to find p''. Define $s''(x_i) = p''(c_i)$ and define $\alpha = p''(c)$.

To prove (2) implies (1) define $s'(x_i) = p'(c_i)$ and use (2) to find s'' and α. Let c

be a constant symbol not in $\operatorname{dom} p'$ and define

$$p''(x_i) = \begin{cases} s''(x_i) & \text{if } d = c_i, \\ \alpha & \text{if } d = c, \\ p'(d) & \text{if } d \in \operatorname{dom} p' - \{c, c_1, \ldots, c_n\}. \end{cases}$$

So far we have

$$\begin{aligned} p \Vdash \exists x\, \psi \quad &\text{iff} \quad \forall s'\, \exists s''\, \exists \alpha \vDash \psi[s'' \cup (\tbinom{x}{\alpha})] \\ &\text{iff} \quad \vDash \exists x\, \psi[s] \qquad \text{(by definition of } \vDash\text{)}. \qquad \square \end{aligned}$$

Corollary 19.2. *Let $\varphi(x_1, \ldots, x_n)$ be a definition and s an assignment with $\{x_1, \ldots, x_n\} \subseteq \operatorname{dom} s$. Define K_s (the set of $(n+1)$-tuples 'consistent' with s) to be the set*

$$K_s = \{(p, c_1, \ldots, c_n) : \forall i\, [1 \leqslant i \leqslant n \rightarrow p(c_i) = s(x_i)]\}.$$

Then the following are equivalent:

(i) $\vDash \varphi[s]$,

(ii) $\exists (p, c_1, \ldots, c_n) \in K_s\, p \Vdash \varphi(c_1, \ldots, c_n)$,

(iii) $\forall (p, c_1, \ldots, c_n) \in K_s\, p \Vdash \varphi(c_1, \ldots, c_n)$.

Proof. Observe that the proof of the above theorem is independent of the particular choice of the condition. □

20. Generic models

In the next section, we intended to show some interconnections between weak forcing and classical satisfaction in certain generic models of X. In this section we describe such models first.

Definition. Let $\mathbb{P}$ denote the class of forcing conditions corresponding to the class C of constants. A *filter* over $\mathbb{P}$ is a set F of conditions in $\mathbb{P}$ so that

(1) if $p \in F$ and $p \leqslant q$, then $q \in F$, and

(2) if $p_1, p_2 \in F$, then there is $p \in F$ so that $p \leqslant p_1$ and $p \leqslant p_2$.

If S is a sublanguage of the language obtained by adding C to L_X, we say F is *S-generic* provided F is a filter over $\mathbb{P}$ also satisfying

(3) if φ is a sentence in S, then $\exists p \in F\, (p \Vdash \varphi$ or $p \Vdash \neg\varphi)$.

S-generic filters are usually denoted by the letter G.

Definition. If $\mathcal{U}$ is an open subset of X^n and F is a filter over $\mathbb{P}$, then we define $\mathcal{U}^F$ to be the set

$$\mathcal{U}^F = \{(c_1, \ldots, c_n) : \exists p \in F\, p \Vdash \mathcal{U}(c_1, \ldots, c_n)\}.$$

Also let $X[F]$ denote the structure with universe C and relations $\mathcal{U}^F$; i.e., $X[F] = \langle C; \mathcal{U}^F \rangle_{\mathcal{U} \subseteq X^n \text{open}}$.

Thus when we write $X[F] \models \varphi$ we mean that as a structure $X[F]$ satisfies φ, a first-order sentence in L_X, in the classical sense of model theory (see Chang–Keisler [0]).

21. Truth and forcing

In this section we prove the traditional 'Truth & Forcing' Theorem for truth in $X[G]$, where G is a generic filter, and $\Vdash$ the forcing relation of previous sections.

Theorem 21.1. *Let S be a sublanguage of the language obtained by adding the class C of constant symbols to L_X. If G is S-generic and if $\varphi(x_1, \ldots, x_n)$ is in S, then*

$$X[G] \models \varphi(c_1, \ldots, c_n) \quad \textit{iff} \quad \exists p \in G\, p \Vdash \varphi(c_1, \ldots, c_n).$$

Proof. The proof is by induction on φ.

Case 1: $\varphi = \mathcal{U}(x_1, \ldots, x_n)$. In this case

$X[G] \models \mathcal{U}(c_1, \ldots, c_n)$ iff $(c_1, \ldots, c_n) \in \mathcal{U}^G$ (definition of classical satisfaction)

iff $\exists p \in G\, p \Vdash \mathcal{U}(c_1, \ldots, c_n)$ (definition of $\mathcal{U}^G$).

Case 2: $\varphi(c_1, \ldots, c_n) = \psi \vee \theta$.

$X[G] \models \psi \vee \theta$ iff $X[G] \models \psi$ or $X[G] \models \theta$

iff $\exists p \in G\, p \Vdash \psi$ or $\exists p \in G\, p \Vdash \theta$ (induction hypothesis)

(1) iff $\exists p \in G\, [p \Vdash \psi \text{ or } p \Vdash \theta]$

(2) iff $\exists p \in G\, \forall p'\, \exists p''\, [p'' \Vdash \psi \text{ or } p'' \Vdash \theta]$.

To see that (1) implies (2) apply monotonicity (Proposition 17.2); namely if $p'' \leq p$ and $p \Vdash \psi$, then $p'' \Vdash \psi$.

To prove (2) implies (1), we exploit the genericity of G over S. Suppose we are given $p \in G$ so that $\forall p'\, \exists p''\, [p'' \Vdash \psi$ or $p'' \Vdash \theta]$. Since G is S-generic there are q_0 and q_1 both in G so that q_0 *decides* ψ (i.e., $q_0 \Vdash \psi$ or $q_0 \Vdash \neg\psi$) and q_1 decides θ. Since G is a filter there is a $p' \in G$ so that $p' \leq q_0$ and $p' \leq q_1$. By Proposition 17.2 (monotonicity), p' decides both ψ and θ. By our assumption there is a p'' so that $p'' \Vdash \psi$ or $p'' \Vdash \theta$. Again by Proposition 17.2, p' and p'' decide ψ and θ in the same way (for example: if $p'' \Vdash \psi$, then by monotonicity and consistency $p' \nVdash \neg\psi$ and since p' decides $\psi, p' \Vdash \psi$). Hence $p' \Vdash \psi$ or $p' \Vdash \theta$. Thus we have shown $\exists p' \in G\,[p' \Vdash \psi$ or $p \Vdash \theta]$ which is equivalent to (1). Hence (2) implies (1).

Furthermore by Proposition 18.1

$$(2) \quad \text{iff} \quad \exists p \in G\, p \Vdash \psi \vee \theta.$$

This completes Case 2.

Case 3: $\varphi(c_1, \ldots, c_n) = \neg\psi$.

$$\begin{aligned} X[G] \models \neg\psi \quad &\text{iff} \quad X[G] \not\models \psi \\ &\text{iff} \quad \forall p \in G\, p \not\Vdash \psi \qquad \text{(induction hypothesis)} \\ &\text{iff} \quad \exists p \in G\, p \Vdash \neg\psi \qquad \text{(by the genericity of } G \text{ over } S \text{ and the definition of a filter)} \end{aligned}$$

Case 4: $\varphi(c_1, \ldots, c_n) = \exists x\, \psi$.

$$X[G] \models \exists x\, \psi(x) \quad \text{iff} \quad \exists a \in C\, X[G] \models \psi(a)$$

(3) $\quad$ iff $\exists a \in C\, \exists p \in G\, p \Vdash \psi(a)$ (induction hypothesis)

(4) $\quad$ iff $\exists p \in G\, \forall p'\, \exists p'', a\, p'' \Vdash \psi(a)$.

. That (3) implies (4) follows from monotonicity. To see that (4) implies (3) we use the genericity of G as well as monotonicity and consistency.

Suppose $p \in G$ and $\forall p'\, \exists p'', a\, p'' \Vdash \psi(a)$. Pick $p' \in G$ so that p' decides $\psi(a)$. Then find $p'' \leqslant p'$ so that $p'' \Vdash \psi(a)$. By monotonicity and consistency it follows that $p' \Vdash \psi(a)$ too. So $\exists p' \in G\, p' \Vdash \psi(a)$ which is equivalent to (3).

Finally by Proposition 18.1

$$(4) \quad \text{iff} \quad \exists p \in G\, p \Vdash \exists x\, \psi. \qquad \square$$

The existence of S-generic filters depends largely on the language S and the space X. When $S = \emptyset$, then every filter is S-generic. Otherwise the existence of an S-generic filter depends on our ability to find a filter G which contains an element from each of the sets $D\varphi = \{p \in \mathbb{P} : p \Vdash \varphi \text{ or } p \Vdash \neg\varphi\}$, as φ ranges through the sentences of S. When S is countable and each $D\varphi$ is dense in $\mathbb{P}$, such a G can always be constructed using the axiom of choice. When S is presented effectively, G may be constructed effectively. When S is uncountable, it is possible that no S-generic filters exist without further set-theoretical hypothesis. For example, when $X = \mathbb{R}$ and Δ is the usual basis for $\mathbb{R}$ and S is the entire language obtained by adding C to L_X, then any S-generic filter is essentially Cohen generic over the universe.

Corollary 21.2. *Let $\varphi(x_1, \ldots, x_n)$ be a definition in S and let G be an S-generic filter. If* $s = \begin{pmatrix} x_1 \cdots x_n \\ \alpha_1 \cdots \alpha_n \end{pmatrix}$, *then*

$$\models\varphi[s] \quad \textit{iff} \quad \forall\langle c_1, \ldots, c_n\rangle \in (\alpha_1 \times \cdots \times \alpha_n)^G X[G] \models \varphi(c_1, \ldots, c_n)$$

(i.e., $\models\varphi[s]$ holds iff $\varphi(x_1, \ldots, x_n)$ is true in $X[G]$ of all the points in s).

Proof. Let $\models\varphi[s]$ and let $\langle c_1, \ldots, c_n\rangle \in (\alpha_1 \times \cdots \times \alpha_n)^G$. By Section 20, there is a $p \in G$ so that $p \Vdash (\alpha_1 \times \cdots \times \alpha_n)(c_1, \ldots, c_n)$. By Proposition 18.1, this means $p(c_i) \subseteq \alpha_i$ for each $i = 1, \ldots, n$. Therefore $p \leqslant \begin{pmatrix} c_1, \ldots, c_n \\ \alpha_1, \ldots, \alpha_n \end{pmatrix}$. However, since $\models\varphi[s]$, then $\begin{pmatrix} c_1, \ldots, c_n \\ \alpha_1, \ldots, \alpha_n \end{pmatrix} \Vdash \varphi(c_1, \ldots, c_n)$ by Theorem 19.1. Hence $p \Vdash \varphi(c_1, \ldots, c_n)$ by Proposition 17.2. So by Truth and Forcing, $X[G] \Vdash \varphi(c_1, \ldots, c_n)$.

Conversely, suppose $\forall\langle c_1, \ldots, c_n\rangle \in (\alpha_1 \times \cdots \times \alpha_n)^G X[G] \models \varphi(c_1, \ldots, c_n)$. Given any $\alpha_1' \times \cdots \times \alpha_n' \subseteq \alpha_1 \times \cdots \times \alpha_n$, find $\langle c_1, \ldots, c_n\rangle \in (\alpha_1' \times \cdots \times \alpha_n')^G$. For some $p \in G$, $p \Vdash (\alpha_1' \times \cdots \times \alpha_n')(c_1, \ldots, c_n)$ by Section 20; i.e. $p(c_i) \subseteq \alpha_i'$ for $i = 1, \ldots, n$. Notice also $\langle c_1, \ldots, c_n\rangle \in (\alpha_1 \times \cdots \times \alpha_n)^G$ and so by hypothesis, $X[G] \models \varphi(c_1, \ldots, c_n)$. By Truth and Forcing there is a $q \in G$ so that $q \Vdash \varphi(c_1, \ldots, c_n)$. Let r be a condition in G which is stronger than p and q. Define $\alpha_i'' = r(c_i)$ for $i = 1, \ldots, n$. Since $r \leqslant p$, $\alpha_i'' \subseteq \alpha_i'$ and since $r \leqslant q$, $r \Vdash \varphi(c_1, \ldots, c_n)$. Consequently $\models\varphi\begin{pmatrix} x_1, \ldots, x_n \\ \alpha_1'', \ldots, \alpha_n'' \end{pmatrix}$ by Theorem 19.1. Hence we have shown:

$$\forall \alpha_1' \times \cdots \times \alpha_n' \, \exists \alpha_1'' \times \cdots \times \alpha_n'' \models \varphi\begin{pmatrix} x_1 \cdots x_n \\ \alpha_1'' \cdots \alpha_n'' \end{pmatrix}.$$

Hence $\forall s' \, \exists s'' \models \varphi[s'']$. Hence $\models\varphi[s]$ by Density. □

22. The Boolean valued model and supports

In this section we shall see that the Boolean value of a formula is essentially its support (see [7]). The Boolean algebra associated with this notion of forcing is the regular open algebra of ${}^{C}X$ given the product topology. Here ${}^{C}X$ denotes the space of all functions from C into X. We denote this algebra by $\mathbb{B}$.

Definition. Let $\varphi(x_1, \ldots, x_n)$ be a definition and $c_1, \ldots, c_n$ be constants in C. The Boolean value of $\varphi(c_1, \ldots, c_n)$ is the supremum in $\mathbb{B}$ of $\{p : p \Vdash \varphi(c_1, \ldots, c_n)\}$. Here we identify the forcing condition p with the corresponding basic open subset of ${}^{C}X$. We write

$$\|\varphi(c_1, \ldots, c_n)\| = \bigvee \{p : p \Vdash \varphi(c_1, \ldots, c_n)\}.$$

Definition. Given $c_1, \ldots, c_n$ in C and $f \in {}^C X$, define $\pi_{\langle c_1,\ldots,c_n\rangle}(f) = \langle f(c_1), \ldots, f(c_n)\rangle$. Thus $\pi_{\langle c_1,\ldots,c_n\rangle}$ projects ${}^C X$ onto X^n.

Theorem 22.1. *Let $\varphi(x_1, \ldots, x_n)$ be a definition and $\pi = \pi_{\langle c_1,\ldots,c_n\rangle}$. Then $|\varphi(x_1, \ldots, x_n)|$ is the image of $\|\varphi(c_1, \ldots, c_n)\|$ under π; i.e., $|\varphi(x_1, \ldots, x_n)| = \pi(\|\varphi(c_1, \ldots, c_n)\|)$.*

Proof. Let $p \Vdash \varphi(c_1, \ldots, c_n)$ and $\{c_1, \ldots, c_n\} \subseteq \operatorname{dom} p$. Define $s(x_i) = p(c_i)$ for each $i = 1, \ldots, n$. Since $p \Vdash \varphi(c_1, \ldots, c_n)$, it follows that $\models\varphi[s]$. If we think of p as a basic open subset of ${}^C X$ and as such if f is a point in p, then for every $i = 1, \ldots, n$, $f(i) \in p(c_i) = s(x_i)$. So in fact $\pi(f) \in s(x_1) \times \cdots \times s(x_n) \subseteq |\varphi(x_1, \ldots, x_n)|$. Thus we have $\pi(\|\varphi(c_1, \ldots, c_n)\|) \subseteq \|\varphi(x_1, \ldots, x_n)\|$.

To prove the reverse inclusion, let $\mathcal{U} \subseteq |\varphi(x_1, \ldots, x_n)|$. Find an assignment s so that $s(x_1) \times \cdots \times s(x_n) \subseteq \mathcal{U}$ and $\models\varphi[s]$. Define for $i = 1, \ldots, n$, $p(c_i) = s(x_i)$. Then $p \Vdash \varphi(c_1, \ldots, c_n)$. Hence as an element of $\mathbb{B}$, $p \subseteq \|\varphi(c_1, \ldots, c_n)\|$. Now if $f \in p$, then

$$\pi(f) = \langle f(c_1), \ldots, f(c_n)\rangle \in p(c_1) \times \cdots \times p(c_n) = s(x_1) \times \cdots \times s(x_n) \subseteq \mathcal{U}.$$

Thus $\pi(\|\varphi(c_1, \ldots, c_n)\|) \cap \mathcal{U} \neq \emptyset$ for every open $\mathcal{U} \subseteq |\varphi(x_1, \ldots, x_n)|$. Therefore $|\varphi(x_1, \ldots, x_n)| \subseteq \pi(\|\varphi(c_1, \ldots, c_n)\|)$. □

23. Conclusion

The gist of Part III has been that the point-free notion of topological satisfaction is in essence a forcing relation over an algebra of regular open sets. The crux of this equivalence is the characterization "$\models\varphi[{}^x_\alpha]$ iff $({}^c_\alpha) \Vdash \varphi(c)$" proved in Section 19. Hence the point-free logic of a topological space X is equivalent to the classical logic of a corresponding generic space $X[G]$ in the sense that $\models_X\varphi({}^x_\alpha)$ iff $\forall c \in \alpha^G\ X[G] \models \varphi(c)$. Consequently $\models_X\varphi({}^x_\alpha)$ has the interpretation that $\varphi(x)$ is true of the generic points in α.

References

[0] C.C. Chang and H.J. Keisler, Model Theory (North-Holland, Amsterdam, 1973).

[1] G. Torado Di Francia, The Investigation of the Physical World (Cambridge Univ. Press, Cambridge, 1976).

[2] I. Kalantari, Major subsets in effective topology, Proc. European Summer Meeting of the Association for Symbolic logic, Patras, Greece.

[3] I. Kalantari and A. Leggett, Simplicity in effective topology, J. Symbolic Logic 47 (1) (1982) 169–183.

[4] I. Kalantari and A. Leggett, Maximality in effective topology, J. Symbolic Logic 48 (1) (1983) 100–112.

[5] I. Kalantari and J. Remmel, Degrees of unsolvability in effective topology, J. Symbolic Logic 48 (3) (1983) 610–622.

[6] I. Kalantari and A. Retzlaff, Recursive construction in topological spaces, J. Symbolic Logic 44 (1979) 609–625.

[7] I. Kalantari and G. Weitkamp, Effective topological spaces I: A definability theory, Ann. Pure Appl. Logic 29 (1985) 1–27.

[8] I. Kalantari and G. Weitkamp, Effective topological spaces II: A hierarchy Ann. Pure Appl. Logic 29 (1985) 207–224.

[9] S.C. Kleene, Recursive predicates and quantifiers, Trans. Amer. Math. Soc. 53 (1943) 41–73.

[10] S.C. Kleene, Arithmetic predicates and function quantifiers, Trans. Amer. Math. Soc. 79 (1955) 312–340.

[11] S.C. Kleene, Hierarchies of number-theoretic predicates, Bull. Amer. Math. Soc. 61 (1955) 193–213.

[12] S.C. Kleene, Recursive functionals and quantifiers of finite types I, Trans. Amer. Math. Soc. 91 (1959) 1–52.

[13] S.C. Kleene, Recursive functionals and quantifiers of finite types II, Trans. Amer. Math. Soc. 108 (1963) 106–142.

[14] G. Weitkamp, Analytic sets having incomparable Kleene degrees, J. Symbolic Logic 47 (4) (1982) 860–868.