Rainbow Solutions to $x + y = z$ in $[m] \times [n]$

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Rainbows in $[m] \times [n]$

The set $[m] \times [n]$ is the set of ordered pairs of positive integers (i, j) where $i \leq m$ and $j \leq n$.

We will be looking at coloring the set $[m] \times [n]$ and finding solutions to $x + y = z$.

A **rainbow solution** is a solution where each element in the solution is a different color.

[4]x[5]		j				
		1	2	3	4	5
i	1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)
	2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)
	3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)
	4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)

As illustrated in the $[4] \times [5]$ array above,

$(1,3)+(2,2)=(3,5)$ is a **solution**.

$(1,1)+(1,2)=(2,3)$ is a **rainbow solution**.

Our Goal:

Use as many colors as possible to color a set $[m] \times [n]$ while avoiding rainbow solutions.

Equivalently, find the least number of colors needed to color $[m] \times [n]$ and guarantee a rainbow solution.

Definitions and Basic Lemmas

Start in the leftmost column or topmost row and travel down and right at 45°. This forms a **diagonal**. D_1 is just the element $(m, 1)$. The m^{th} diagonal (**main diagonal**), D_m, starts at $(1, 1)$.

A color is **distinct** in a diagonal if it only appears in that diagonal and no other diagonal.

An element of the diagonal D_k and an element of D_{k+1} that are colored distinctly and are not adjacent forms a **jump**.

Landing Lemma:

If a point is in the k^{th} diagonal D_k and a point is in the ℓ^{th} diagonal D_ℓ, then their sum is in $D_{k+\ell-m}$ (where m comes from $[m] \times [n]$).

Lower Bound

Lower Bound: Show that the set $[m] \times [n]$ can have $(m + n)$ colors without rainbow solutions.

Color $[m] \times [n]$ with $m + n$ colors as below, this avoids rainbow solutions, as desired.

All elements that are not on the bottom or right side are the same color.

[m]x[n]		j				
		1	2	3	...	n
i	1	(1,1)	(1,2)	(1,3)	(1,...)	(1,n)
	2	(2,1)	(2,2)	(2,3)	(2,...)	(2,n)
	...	(...,1)	(...,2)	(...,3)	(...,...)	(...,n)
	m	(m,1)	(m,2)	(m,3)	(m,...)	(m,n)

Each element on the bottom or right side is a **distinct** color.

Upper Bound

Upper Bound: Show $[m] \times [n]$ can never have $(m + n + 1)$ colors without a rainbow solution. We have shown this is true for $m = 2, 3, 4, 5$ and any integer n.

[6] x [9]		j								
		1	2	3	4	5	6	7	8	9
i	1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)	(1,8)	(1,9)
	2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)	(2,8)	(2,9)
	3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)	(3,7)	(3,8)	(3,9)
	4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)	(4,7)	(4,8)	(4,9)
	5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)	(5,7)	(5,8)	(5,9)
	6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)	(6,7)	(6,8)	(6,9)

One Distinct Color Lemma: Each (non-main) diagonal can only have one color.

Upper Left and Bottom Right Lemma: No distinct color can appear above and to the left of, or below and to the right of another distinct color.

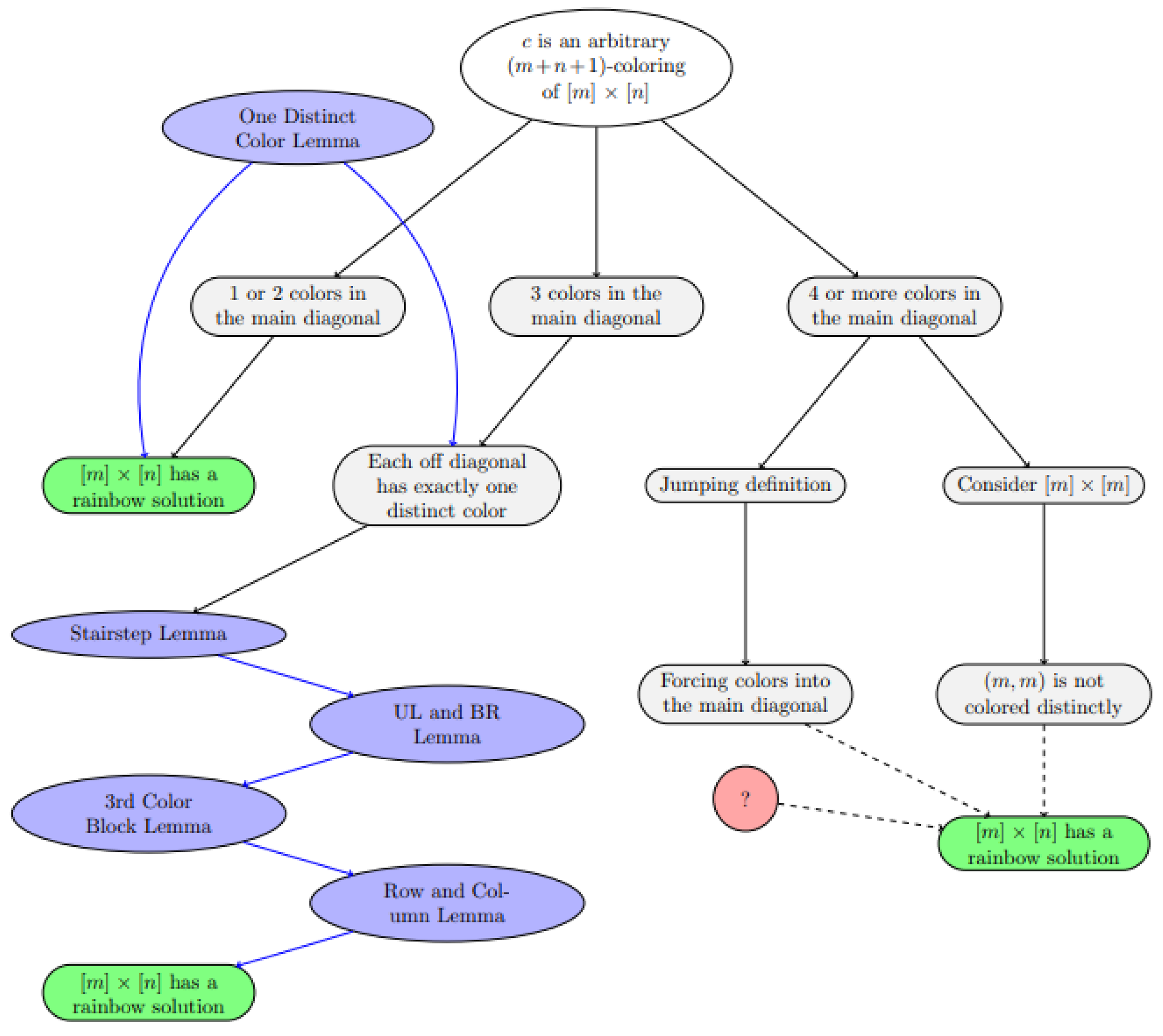
Stairstep Lemma: If an element (i, j) of a diagonal D_k is colored distinctly, then the distinct color in D_{k+1} must be directly above or to the right of (i, j).

Row and Column Lemma: The first several rows and columns have a distinct color.

Figure for Definitions and Basic Lemmas

[4] x [9]		j								
		1	2	3	4	5	6	7	8	9
i	1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)	(1,8)	(1,9)
	2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)	(2,8)	(2,9)
	3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)	(3,7)	(3,8)	(3,9)
	4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)	(4,7)	(4,8)	(4,9)

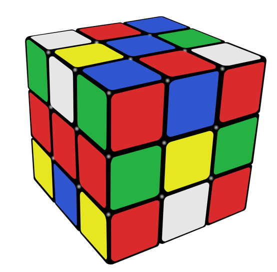
Proof Tree



THE “PROOF TREE”

Future Research

- Finish $[m] \times [n]$
- $x+y = z$ in $[m] \times [n] \times [\ell]$
- $w+x+y = z$ in $[m] \times [n]$



Applications

- checking software/hardware reliability
- automated theorem proving
- approximation algorithms

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