PROJECT I - MATH 800
SPRING 2015

(1) Problem 19/page 23;
Solution: Let
$$w_j = \frac{\bar{z}_j}{(\sum_{k=1}^{n} |z_k|^2)^{1/2}}.$$
Clearly $\sum_{j=1}^{n} |w_j|^2 = 1$ and hence we should have $|\sum_{j=1}^{n} z_j w_j| \leq 1$. But
$$\sum_{j=1}^{n} z_j w_j = \sum_{j=1}^{n} \frac{|z_j|^2}{(\sum_{k=1}^{n} |z_k|^2)^{1/2}} = (\sum_{k=1}^{n} |z_k|^2)^{1/2}.$$
Thus,
$$\sum_{k=1}^{n} |z_k|^2 \leq 1.$$
(2) Problem 29/page 24;
(3) Problem 34/page 25;
Solution: We have
$$\overline{\frac{\partial f}{\partial z}} = \frac{1}{2}\overline{(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y})} = \frac{1}{2}(\frac{\partial \bar{f}}{\partial x} + i\frac{\partial \bar{f}}{\partial y}) = \frac{\partial \bar{f}}{\partial \bar{z}}.$$
(4) Problem 47/page 26;
Solution: Let $f = u + iv$. Then
$$g = \log|f| = \frac{1}{2}\log|f|^2 = \frac{1}{2}\log(u^2 + v^2).$$
Thus,
$$g_x = \frac{2uu_x + 2vv_x}{u^2 + v^2}, g_y = \frac{2uu_y + 2vv_y}{u^2 + v^2}.$$
Next,
$$\begin{aligned} g_{xx} &= 2\frac{(u_x^2 + v_x^2 + uu_{xx} + vv_{xx})(u^2 + v^2) - 2(uu_x + vv_x)^2}{(u^2 + v^2)^2} \\ g_{yy} &= 2\frac{(u_y^2 + v_y^2 + uu_{yy} + vv_{yy})(u^2 + v^2) - 2(uu_y + vv_y)^2}{(u^2 + v^2)^2} \end{aligned}$$
Adding them together and using the harmonicity of $u, v : u_{xx} + u_{yy} = 0 = v_{xx} + v_{yy}$ and the Cauchy Riemann equations $u_x = v_y, u_y = -v_x$ yields that $g_{xx} + g_{yy} = 0$.
(5) Show that the functions
$$f(x, y) = \frac{y}{x^2 + y^2}; g(x, y) = -\frac{x}{x^2 + y^2}$$

satisfy $f_y = g_x$ for each $\mathbf{R}^2 \setminus \{0\}$, but on the other hand **there is no** C^2 funtion h on $\{(x,y) : 0 < x^2 + y^2 < 1\}$ so that

$$h_x = f, h_y = g.$$

Explain why this does not contradict the generalized version of Theorem 1.5.1 that we have established in class.

Hint: To show the non-existence of h argue by contradiction, by considering the path integral

$$\int_{x^2+y^2=1} f(x,y)dx + g(x,y)dy.$$

Solution: The proof of $f_y = g_x$ is by inspection. Assume that there is an h, so that $h_x = f, h_y = g$. Then,

$$\int_{x^2+y^2=1} f(x,y)dx + g(x,y)dy =$$

$$\int_0^{2\pi} h_x(\cos(t),\sin(t))(-\sin(t)) + h_y(\cos(t),\sin(t))(\cos(t)))dt =$$

$$\int_0^{2\pi} \frac{d}{dt} h(\cos(t),\sin(t))dt = h(1,0) - h(1,0) = 0.$$

On the other hand,

$$\int_{x^2+y^2=1} f(x,y)dx + g(x,y)dy = \int_0^{2\pi} (-\sin^2(t) - \cos^2(t))dt = -2\pi \neq 0,$$

a contradiction. This is not in a contradiction with Theorem 1.5.1, because the functions f, g are not well-defined at $x = y = 0$, in the middle of the domain.

(6) Problem 55/page 27 without the counterexample. I will discuss the counterexample later.

Solution: Let F_1, F_2 be the anti-derivatives on U_1, U_2 respectively. On $U = U_1 \cap U_2$, we have

$$F_1' = f = F_2'.$$

Hence, there is a constant, say C, so that $F_2 = F_1 + C$. Define

$$F(z) = \begin{cases} F_1(z) + C & z \in U_1 \\ F_2(z) & z \in U_2 \end{cases}$$

The function F is consistently defined on U. Moreover, $F'(z) = f(z)$.