Exact Solutions to Kaup-Kupershmidt Equation by Projective Riccati Equations Method

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Abstract

In this paper we consider the Kaup-Kupershmidt equation (K-K), and we obtain exact solutions by the projective Riccati equation method.

Keywords: Nonlinear differential equation; Travelling Wave Solution; Projective Riccati Equation Method

1 Introduction

In this paper we present exact solutions for the well-known Kaup-Kupershmidt equation (KK)[8] by the projective Riccati equations method [1][3][4][6]. This equation is a particular case of the new system [7]

$$\begin{cases} u_t = u_{xxxxx} - 20uu_{xxx} - 50u_x u_{xx} + 80u^2 u_x - w_x \\ w_t = -6wu_{xxx} - 2u_{xx} w_x + 96wuu_x + 16w_x u^2 \end{cases} \tag{1}$$

when $w(x,t) = 0$. More exactly, in this case the system (1) is reduced to the (KK) equation [7]

$$u_t = u_{xxxxx} - 20uu_{xxx} - 50u_x u_{xx} + 80u^2 u_x, \tag{2}$$

which is a particular case of a class of fifth order PDEs with four parameters [6]

$$u_t + \omega u_{xxxxx} + \gamma uu_{xxx} + \beta u_x u_{xx} + \alpha u^2 u_x = 0 \tag{3}$$

which has solutions in some cases:

- for $\alpha = 30$, $\beta = 20$, $\gamma = 10$, $\rho = 1$. See [6].

- for $\alpha = 5$, $\beta = 5$, $\gamma = 5$, $\rho = 1$. Equation (3) reduces to Sawada and Kotera (SK) equation. See [6].
- for $\alpha = 20$, $\beta = 25$, $\gamma = 10$, $\rho = 1$. See [6].

In this work we obtain solutions of the equation (3) for the values $\alpha = -80$, $\beta = 50$, $\gamma = 20$, $\omega = -1$. This gives us the Kaup-Kupershmidt equation

2 Kaup-Kupershmidt equation

The Kaup-Kupershmidt equation [6] reads

$$u_t - u_{xxxxx} + 20uu_{xxx} + 50u_x u_{xx} - 80u^2 u_x = 0 \tag{4}$$

We search exact solutions to equation (4) in the form

$$\begin{cases} u(x,t) = v(\xi) \\ \xi = x + \lambda t, \end{cases} \tag{5}$$

As a result we have that equation (4) reduces to the nonlinear ordinary differential equation (ODE) given by

$$\lambda v'(\xi) - v^{(5)}(\xi) + 20v(\xi)v'''(\xi) + 50v'(\xi)v''(\xi) - 80[v(\xi)]^2 v'(\xi) = 0. \tag{6}$$

To obtain exact solution for the equation (6) we use the projective Riccati equation method [4] which we may describe in the following five steps:

First Step:

We consider solutions of (6) in the form

$$v(\xi) = a_0 + a_1\sigma(\xi) + b_1\tau(\xi), \tag{7}$$

where $\sigma(\xi)$, $\tau(\xi)$ satisfy the system

$$\begin{cases} \sigma'(\xi) = e\sigma(\xi)\tau(\xi) \\ \tau'^2(\xi) - \mu\sigma(\xi) + r. \end{cases} \tag{8}$$

It is easy to see that the first integral of this system is given by

$$\tau^2 = -e[r - 2\mu\sigma(\xi) + \frac{\mu^2 + \rho}{r}\sigma^2(\xi)], \tag{9}$$

where $\rho = \pm 1$ and $e = \pm 1$.

Second Step:

Substituting (7), along with (8) and (9) into (6) and collecting all terms with the same power with respect to $\sigma(\xi)$ and $\tau(\xi)$, we get a polynomial in the two variables $\sigma(\xi)$ and $\tau(\xi)$. Equaling the coefficients of these polynomial to zero we get the following algebraic system:

1. $120\, e\, (\mu^2 + \rho)^2\, a_1 = 0.$

2. $120\, (\mu^2 + \rho)^3\, b_1 = 0.$

3. $-40\, e\, (\mu^2 + \rho)^2\, (9\, e\, \mu - 11\, a_1)\, b_1 = 0.$

4. $20\, (\mu^2 + \rho)\, \left(r\, (12\, e\, \mu - 11\, a_1)\, a_1 + 11\, e\, (\mu^2 + \rho)\, {b_1}^2\right) = 0.$

5. $-(e\, (r^2 - \lambda + 20\, a_0\, (e\, r + 4\, a_0))\, a_1) - 10\, (2\, e\, r\, \mu + 16\, \mu\, a_0 - 8\, r\, a_1)\, {b_1}^2 = 0.$

6. $e\, r\, (e\, r\, (3\, e\, \mu - 7\, a_1) + 4\, a_0\, (3\, e\, \mu - 4\, a_1))\, a_1 +$
   $(16\, (\mu^2 + \rho)\, a_0 + r\, (21\, e\, \mu^2 + 44\, e\, (3\, \mu^2 + \rho) - 2\, e\, (67\, \mu^2 + 18\, \rho) - 32\, \mu\, a_1))\, {b_1}^2 = 0.$

7. $a_1\, (-3\, r\, (5\, \mu^2 + 2\, \rho) - 12\, e\, (\mu^2 + \rho)\, a_0 + r\, (27\, e\, \mu - 8\, a_1)\, a_1)$
   $-e\, (\mu^2 + \rho)\, (39\, e\, \mu - 24\, a_1)\, {b_1}^2 = 0.$

8. $10\, \left(-8\, \mu\, {a_0}^2 + 2\, r\, a_0\, (-e\, \mu + 8\, a_1) + e\, r\, \left(2\, r\, a_1 + 8\, \mu\, {b_1}^2\right)\right) +$
   $(\lambda - r^2)\, \mu = 0.$

9. $-\lambda\, (\mu^2 + \rho) + r^2\, (31\, \mu^2 + 16\, \rho) + 80\, (\mu^2 + \rho)\, {a_0}^2 +$
   $20\, r\, a_0\, (3\, e\, \mu^2 + 12\, e\, (3\, \mu^2 + \rho) - 8\, e\, (4\, \mu^2 + \rho) - 24\, \mu\, a_1)$
   $+40\, r\, \left(r\, a_1\, (-7\, e\, \mu + 4\, a_1) - 2\, e\, (3\, \mu^2 + \rho)\, {b_1}^2\right) = 0.$

10. $-8\, (\mu^2 + \rho)\, a_0\, (3\, e\, \mu - 4\, a_1) + r(-18\, \mu^3 - 15\, \mu\, \rho +$
    $a_1\, (11\, e\, \mu^2 + 88\, e\, (3\, \mu^2 + \rho) - 3\, e\, (61\, \mu^2 + 17\, \rho) - 40\, \mu\, a_1)) +$
    $24\, e\, \mu\, (\mu^2 + \rho)\, {b_1}^2 = 0.$

11. $12e\, (\mu^2 + \rho)\, a_0 + r\, (27\mu^2 + 36\, (5\mu^2 + \rho) - 24\, (7\mu^2 + \rho) - 110e\mu a_1 + 24{a_1}^2) - 8e\, (\mu^2 + \rho)\, {b_1}^2 = 0.$

Third Step :

(This is the more difficult step.) Solving the previous system respect to unknowns variables r, μ, a_0, a_1, b_1 we get the followings results:
With $\mu^2+\rho=0$, $(\rho=\pm 1)$ we obtain

1. For $e=1$

   - $a_1=\frac{3}{8}\mu$, $a_0=\pm\frac{\sqrt{\lambda}}{4}$, $r=\mp 4\sqrt{\lambda}$, $b_1=0$.
   - $a_1=3\mu$, $a_0=\pm\frac{\sqrt{\lambda}}{2\sqrt{11}}$, $r=\mp\frac{\sqrt{\lambda}}{11}$, $b_1=0$.

2. For $e=-1$

   - $a_1=\mp\frac{3}{8}\mu$, $a_0=\pm\frac{15\sqrt{\lambda}}{116}$, $r=\pm\frac{4\sqrt{\lambda}}{29}$, $b_1=0$.
   - $a_1=\mp 3\mu$, $a_0=\pm\frac{2\sqrt{\lambda}}{\sqrt{449}}$, $r=\mp\frac{3\sqrt{\lambda}}{\sqrt{449}}$, $b_1=0$.

Fourth Step:

We consider the solutions of the system obtained in the Second Step.

1. Case I:
   If $r=\mu=0$ then

$$\tau_1(\xi)=-\frac{1}{e\xi},\quad \sigma_1(\xi)=\frac{C}{\xi}. \tag{10}$$

2. Case II:
   If $e=1$ and $\rho=-1$

$$\begin{cases}\tau_2=\dfrac{\sqrt{r}\tan(\sqrt{r}\xi)}{\mu\sec(\sqrt{r}\xi)+1} & (r>0)\\ \sigma_2=\dfrac{r\sec(\sqrt{r}\xi)}{\mu\sec(\sqrt{r}\xi)+1} & (r>0).\end{cases} \tag{11}$$

3. Case III:
   If $e=-1$ and $\rho=-1$

$$\begin{cases}\tau_3=\dfrac{\sqrt{r}\tanh(\sqrt{r}\xi)}{\mu\operatorname{sech}(\sqrt{r}\xi)+1} & (r>0)\\ \sigma_3=\dfrac{r\operatorname{sech}(\sqrt{r}\xi)}{\mu\operatorname{sech}(\sqrt{r}\xi)+1} & (r>0).\end{cases} \tag{12}$$

4. Case IV:
   If $e=-1$ and $\rho=1$

$$\begin{cases} \tau_4 = \dfrac{\sqrt{r}\coth(\sqrt{r}\xi)}{\mu \text{csch}(\sqrt{r}\xi)+1} & (r>0) \\ \sigma_4 = \dfrac{r\,\text{csch}(\sqrt{r}\xi)}{\mu\,\text{csch}(\sqrt{r}\xi)+1} & (r>0). \end{cases} \tag{13}$$

5. Case V:
   If $e=1$ and $\rho=1$

$$\begin{cases} \tau_5 = \dfrac{-\sqrt{-r}\coth(\sqrt{-r}\xi)}{\mu \text{csch}(\sqrt{-r}\xi)+1} & (r<0) \\ \sigma_5 = \dfrac{r\,\text{csch}(\sqrt{-r}\xi)}{\mu\,\text{csch}(\sqrt{-r}\xi)+1} & (r<0). \end{cases} \tag{14}$$

Fifth Step :

From (11) −(14) and the Third Step (solutions of the algebraic system) we get solutions for (4) by using (5). By space reasons we do not consider all. Some of them for $\lambda>0$ are

e	ρ	μ	a_0	a_1	r	u
1	−1	±1	$\pm\frac{\sqrt{\lambda}}{4}$	$\pm\frac{3}{8}$	$4\sqrt{\lambda}$	$\pm\frac{\sqrt{\lambda}}{4}\pm\frac{3}{8}\frac{r\sec(\sqrt{r}(x+\lambda t))}{\mu\sec(\sqrt{r}(x+\lambda t))+1}$
1	−1	±1	$\pm\frac{\sqrt{\lambda}}{2\sqrt{11}}$	±3	$\frac{\sqrt{\lambda}}{11}$	$\pm\frac{\sqrt{\lambda}}{2\sqrt{11}}\pm 3\frac{r\sec(\sqrt{r}(x+\lambda t))}{\mu\sec(\sqrt{r}(x+\lambda t))+1}$
−1	−1	±1	$\pm\frac{15\sqrt{\lambda}}{116}$	$\mp\frac{3}{8}$	$\frac{4\sqrt{\lambda}}{29}$	$\pm\frac{15\sqrt{\lambda}}{116}\mp\frac{3}{8}\frac{r\text{sech}(\sqrt{r}(x+\lambda t))}{\mu\text{sech}(\sqrt{r}(x+\lambda t))+1}$
−1	−1	±1	$\pm\frac{2\sqrt{\lambda}}{\sqrt{449}}$	∓3	$\frac{3\sqrt{\lambda}}{\sqrt{449}}$	$\pm\frac{2\sqrt{\lambda}}{\sqrt{449}}\mp 3\frac{r\text{sech}(\sqrt{r}(x+\lambda t))}{\mu\text{sech}(\sqrt{r}(x+\lambda t))+1}$

3 Conclusions

In this paper, by using the projective Riccati equations method and the help of symbolic computation system *Mathematica*, we obtained some exact solutions to equation (4).

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