

Quasi-Ergodicity of Transient Patterns in Stochastic Reaction-Diffusion Equations

(PREPRINT)

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05/05/2023

Abstract

We study transient patterns appearing in a class of SPDE using the framework of quasi-stationary and quasi-ergodic measures. In particular, we prove the existence and uniqueness of quasi-stationary and quasi-ergodic measures for a class of reaction-diffusion systems perturbed by additive cylindrical noise. We obtain convergence results in L^2 and almost surely, and demonstrate an exponential rate of convergence to the quasi-stationary measure in an L^2 norm. These results allow us to qualitatively characterize the behaviour of these systems in neighbourhoods of an invariant manifold of the corresponding deterministic systems at some large time $t > 0$, conditioned on remaining in the neighbourhood up to time t . The approach we take here is based on spectral gap conditions, and is not restricted to the small noise regime.

Keywords: Quasi-stationary measures · Spectral gap · Stochastic reaction-diffusion equations · Pattern formation · Travelling waves · Spiral waves

Acknowledgements

The author would like to thank Jürgen Jost, for his continuing patience and support during the preparation of this document. This work was funded by the International Max Planck Research School for Mathematics in the Sciences.

Statements and Declarations

The author has no conflicts of interest to declare.

1 Introduction

The purpose of this article is to provide a framework with which to discuss the stability and long time behaviour of stochastic perturbations of spatiotemporal patterns appearing in reaction-diffusion equations. This topic has recently been of great interest to physicists, biologists, and mathematicians. For instance, travelling waves and stochastic perturbations thereof are important features of models from neuroscience [25, 37, 52, 66], population genetics [39, 67, 68], ecology [72, 79], and many other disciplines. Meanwhile, understanding stochastically perturbed spiral waves in excitable media has led to insights on cardiac arrhythmias and how to treat them [5, 38, 63].

Stochastic perturbations of a pattern in a reaction-diffusion system usually destroy the pattern at some finite time. Hence, while the unperturbed pattern may be stable, the perturbed pattern is often only *metastable*. In recent decades, there has been much interest in characterizing metastable behaviour in a rigorous mathematical framework, for instance in [16, 35, 52, 56, 59, 60, 80]. In this paper, we approach the study of metastable patterns in stochastic reaction-diffusion systems using the theory of *quasi-stationary* and *quasi-ergodic measures*, as defined for instance in [19]. As described more precisely below, these measures characterize the long time behaviour of a metastable pattern prior to its destruction. The connection between metastability and quasi-ergodicity has been studied previously, for instance in discrete and one-dimensional settings in [12, 13] and [50, 53], respectively. To the author's knowledge, the results of this paper are the first contribution studying metastability via quasi-stationary measures in an infinite dimensional setting.

The remainder of this document is structured as follows. In Section 1.1, we outline sufficient conditions for the existence and uniqueness of quasi-stationary and quasi-ergodic measures in SPDE, while Section 1.2 presents an incomplete review of relevant literature. In Section 2, we provide results on quasi-ergodicity of general continuous Markov processes in separable Hilbert space. As we do not assume a modified Doeblin condition as in [19], nor do we assume the existence of bounded integral kernels for the Markov semigroups which we study as in [18, 49, 70, 71, 86], we obtain convergence results that hold in an L^2 and almost sure sense, rather than uniformly. In Section 3, we prove that a large class of semilinear SPDEs satisfy the hypotheses of Section 2, and therefore admit unique quasi-stationary and quasi-ergodic measures. In Section 4, we conclude the paper with a cursory discussion of how quasi-ergodic measures relate to metastable patterns in SPDEs.

1.1 Setup and Results

Let $O \subset \mathbb{R}^d$ be a spatial domain and let H be a separable Hilbert space of functions $f : O \rightarrow \mathbb{R}^n$ for some $n \in \mathbb{N}$. Consider the following evolution equation on H ,

$$\partial_t x = Lx + N(x), \tag{1}$$

where L and N are linear and nonlinear operators, respectively, on H . We impose the following conditions on (1), which we verify for the case where $L = \Delta$ is the Laplace operator and N is a polynomial in Example below. Under these assumptions, unique mild solutions to (1) are known to exist [51, Theorem 3.3.3].

Assumption 1. *The PDE (1) satisfies the following.*

- (a) N is a nonlinearity defined on a set $D(N)$ that is densely and continuously embedded in H , $D(N)$ possesses its own Banach space structure, and $N : D(N) \rightarrow H$ is locally Lipschitz and Fréchet differentiable as an operator on $D(N)$.
- (b) $L : D(L) \subset H \rightarrow H$ generates a strongly continuous semigroup $(\Lambda_t)_{t \geq 0}$ on H such that $\Lambda_t(H) \subset D(N)$ for all $t > 0$, and $(\Lambda_t)_{t \geq 0}$ restricted to $D(N)$ is a strongly continuous semigroup.
- (c) There exists $\omega > 0$ such that $\|\Lambda_t\|_H \leq e^{-\omega t}$ and $\|\Lambda_t\|_{D(N)} \leq e^{-\omega t}$.
- (d) $\Lambda_t : H \rightarrow E$ is compact, and $\|\Lambda_t\|_{H \rightarrow E}$ is locally square integrable in $t > 0$.
- (e) There is a stable normally hyperbolic invariant manifold Γ of (1), in the sense of [7, Condition (H3)].

Our primary interest is in how the solutions to (1) near the invariant manifold Γ behave under perturbations by noise. Specifically, consider the following stochastic perturbation of (1),

$$dX = (LX + N(X)) dt + \sigma B dW. \quad (2)$$

The noise amplitude is $\sigma \geq 0$, B^2 is a symmetric positive bounded linear operator with bounded inverse, and $W = (W_t)_{t \geq 0}$ is a cylindrical Wiener process on H .

For each $x \in H$, all stochastic objects are considered over a fixed ambient probability space $(\Omega, \mathcal{F}, \mathbb{P}_x)$, where \mathbb{P}_x is the probability measure associated with (2) with initial condition $X_0 = x$. Let \mathbb{E}_x denote expectation with respect to \mathbb{P}_x . Let $(\mathcal{F}_t)_{t \geq 0}$ be the filtration associated with $(W_t)_{t \geq 0}$. For any probability measure ν on H let

$$\mathbb{P}_\nu[\cdot] := \int_H \mathbb{P}_x[\cdot] \nu(dx),$$

and let \mathbb{E}_ν be expectation with respect to \mathbb{P}_ν . For any measure ν on H and ν -measurable function $f : H \rightarrow \mathbb{R}$, we denote $\nu(f) := \int f(x) \nu(dx)$.

We now provide sufficient conditions for the existence and uniqueness of mild solutions to (2). By a *mild solution* to (2), we mean a continuous Markov process $(X_t)_{t \geq 0}$ satisfying

$$X_t = \Lambda_t X_0 + \int_0^t \Lambda_{t-s} N(X_s) ds + \sigma \int_0^t \Lambda_{t-s} B dW_s, \quad X_0 \in H, t > 0.$$

Assumption 2. *The SPDE (2) satisfies the following.*

- (a) $Y_t := \int_0^t \Lambda_{t-s} B dW_s$ is a well-defined continuous $D(N)$ -valued process.
- (b) For all $t > 0$,

$$\text{Tr} \left(\int_0^t \Lambda_s B^2 \Lambda_s^* ds \right) < \infty. \quad (3)$$

It should be emphasized that while we allow for non-trace class noise, our assumptions guarantee that the SPDE (2) is non-singular, so that we need not be concerned with the renormalization theory developed *e.g.* in [28, 45, 46].

In Theorem 3.1 below, we see that Assumptions 1 & 2 imply the existence and uniqueness of local in time $D(N)$ -valued mild solutions to (2). In Theorem 3.2, we see that these assumptions also imply that (2) with N cutoff to Γ_δ possesses a unique invariant measure, henceforth denoted μ , such that $\mu(D(N)) = 1$. We let Γ_δ be a δ -neighbourhood of Γ defined in the $D(N)$ topology. Following these considerations, we make the following additional assumption.

Assumption 3. *Assumptions 1 & 2 hold, so that there exists a unique invariant measure μ of (2) with N replaced by $N|_{\Gamma_\delta}$, and there exists $\delta > 0$ such that the following hold.*

(a) Γ_δ is contained in the support of μ , so that $\mu(\Gamma_\delta) > 0$.

(b) Denoting the Lipschitz constant of $N|_{\Gamma_\delta}$ by κ , we have $\kappa \in (0, \omega)$.

Example. The principal example we have in mind is as follows. Let $O \subset \mathbb{R}^d$ be a sufficiently regular bounded spatial domain and set $H := L^2(O, \mathbb{R}^n)$. Let $L = d\Delta$ with Dirichlet boundaries for some $d > 0$, or $L = d\Delta - a$ for constants $a, d > 0$ with periodic boundaries. For some vector of polynomials $p : \mathbb{R}^n \rightarrow \mathbb{R}^n$, define

$$N(f)(\xi) = p(f(\xi)) \quad \text{for } f \in D(N) := C_b(O, \mathbb{R}^n), \quad \xi \in O.$$

It is straightforward to check that this example satisfies Assumption 1(a), (b), (b), (d), due to the compactness of $\Lambda_t : H \rightarrow D(N)$ and the $L^2 \rightarrow L^\infty$ ultracontractivity of $(\Lambda_t)_{t \geq 0}$ [32, Theorem 9.3]. In this example, we suppose that (1) admits a stable travelling wave solution, as in the FitzHugh-Nagumo equation on $O = [0, L]$ with periodic boundaries [4, 40, 57]. Specifically, we suppose that there exists $\hat{x} \in C_b(O; \mathbb{R}^n)$ and $c \in \mathbb{R}^d$ such that

$$x_t(\xi) = \hat{x}(\xi - ct), \quad \xi \in O, \quad t > 0,$$

is a solution of (1), and $\Gamma := \{\hat{x}(\cdot - ct) : t \in \mathbb{R}\}$ is a nonlinearly stable invariant manifold of (1). Hence there exists $\delta > 0$ such that if

$$\sup_{\xi \in O} \|x_0(\xi) - \hat{x}(\xi - cs)\|_{\mathbb{R}^d} \leq \delta$$

for some $s \in \mathbb{R}$ and $(x_t)_{t \geq 0}$ is the solution to (1) with initial condition x_0 , then

$$\lim_{t \rightarrow \infty} d(x_t, \Gamma) = 0,$$

where $d(x, A)$ is the distance between a point x and the closest point in a set A . So long as we work in one spatial dimension, we may take $B = I$ and verify Assumption 2 as in [30, Theorem 11.3.1]. If we take B such that $\text{Tr } B < \infty$, then we may take $O \subset \mathbb{R}^d$ for $d > 1$, and verify that Assumption 2 is still satisfied. In this example, by taking sufficiently small δ we can verify Assumption 3 so long as either of the constants a, d in $L = d\Delta - a$ are large enough for Assumption 3(b) to hold.

Now, we return to the abstract SPDE (2). Fix $\delta > 0$ such that Assumption 3 holds. As we are interested in the dynamics of (2) in Γ_δ , it is natural to define

$$\tau := \inf \{t > 0 : X_t \in H/\Gamma_\delta\}.$$

We say that a solution of (2) is Γ -like (with tolerance $\delta > 0$) at time $t > 0$ if $X_t \in \Gamma_\delta$. We would like to be able to answer the following questions.

- Q1.** (Stability) If $X_0 \in \Gamma$, how long does a Γ -like solution persist?
- Q2.** (Similarity) If X_t is a Γ -like solution over a (random) time interval $[0, \tau)$, how does its behaviour differ from the behaviour of solutions to (1) in Γ over this time interval?
- Q3.** (Ergodicity) When, and how, can we understand the ‘‘average’’ behaviour of a Γ -like solution?

A brief, non-exhaustive survey of some of the literature related to these questions is presented in Section 1.2. In this paper, we address these questions – primarily Q3 – using the theory of quasi-stationary and quasi-ergodic measures. We obtain the following result on the existence and uniqueness of quasi-stationary and quasi-ergodic measures for (2).

Theorem A. *Let Assumption 3 hold. Then, there exist unique positive $\varphi, \varphi^* \in L^2(\Gamma_\delta, \mu)$, given by the principal eigenfunctions of the sub-Markov semigroup of (2) with killing at H/Γ_δ , such that the following statements hold.*

(i) $\alpha(dx) := \varphi^*(x)\mu(dx)$ is the unique probability measure on Γ_δ satisfying

$$\mathbb{P}_\alpha[X_t \in A | t < \tau] = \alpha(A) \tag{4}$$

for all measurable $A \subset \Gamma_\delta$. Moreover, for $f \in L^2(\Gamma_\delta, \mu)$, the following convergence result holds in L^2_μ and μ -almost surely:

$$\lim_{t \rightarrow \infty} \mathbb{E}_x[f(X_t) | t < \tau] = \alpha(f). \tag{5}$$

(ii) $\beta(dx) := \varphi(x)\varphi^*(x)\mu(dx)$ is the unique probability measure on Γ_δ such that for any $\epsilon > 0$, and bounded measurable $f : H \rightarrow \mathbb{R}$,

$$\mathbb{P}_x \left[\left\| \frac{1}{t} \int_0^t f(X_s) ds - \beta(f) \right\| > \epsilon | t < \tau \right] \xrightarrow[t \rightarrow \infty]{} 0, \tag{6}$$

the above convergence holding in L^2_μ and μ -almost surely over $x \in \Gamma_\delta$.

The proof of Theorem A follows from Theorems 2.1 & 3.1 below. A probability measure satisfying (4) is referred to as a *quasi-stationary measure*, while a probability measure satisfying (6) is referred to as a *quasi-ergodic measure*. As will be seen in Section 2, the functions φ, φ^* are given by the principal eigenfunctions of the sub-Markov generator of (2) in $L^2(\Gamma_\delta, \mu)$. We obtain the following result on the rate of convergence to the quasi-stationary measure α , the proof of which follows from Theorems 2.2 & 3.1 below.

Theorem B. *Let Assumption 3 hold, and let $\varphi, \varphi^*, \alpha$ be as in Theorem A. Then, defining $\alpha^*(dx) := \varphi(x)\mu(dx)$, for any $f \in L^2(E, \mu)$ there exists $T > 0$ and a constant $C_f > 0$ such that*

$$\|\mathbb{E}_x [f(Z_t) | t < \tau] - \alpha(f)\|_{L^1_{\alpha^*}} \leq C_f e^{-\gamma t} \quad \forall t \geq T.$$

Building on the above results, we prove the existence of a Q -process for (2) in Γ_δ . Roughly speaking, this Q -process is a homogeneous Markov process in Γ_δ given by $(X_t)_{t \geq 0}$ conditioned on never exiting Γ_δ . In practice, Q -processes may be analyzed via Doob's h -transform [73, pp. 296], for instance as seen in [23, Theorem 6.16(iv)], though this is not a technique used in this document. Since we do not prove uniform convergence to the quasi-stationary measure, and instead rely on the spectral properties of the sub-Markov generator of (2) in Γ_δ , we cannot use the theory developed *e.g.* in [19]. The proof of the following theorem is provided in Theorems 2.3 & 3.1.

Theorem C. *Let Assumption 3 hold for some fixed $\delta > 0$. Then, for each $s \geq 0$ and $A \in \mathcal{F}_s$, the limit*

$$\mathbb{Q}_x[A] := \lim_{t \rightarrow \infty} \mathbb{P}_x[A | t < \tau], \quad A \in \mathcal{F}_s, s \geq 0 \quad (7)$$

is defined for μ -almost all $x \in \Gamma_\delta$. With respect to $(\mathbb{Q}_x)_{x \in \Gamma_\delta}$, the solution process $(X_t)_{t \geq 0}$ is a homogeneous Markov process in Γ_δ with unique ergodic measure β .

Before proving Theorems A, B, & C, we discuss the existence of quasi-stationary and quasi-ergodic measures in a more general Hilbert space setting in Section 2, below, and return to the question of quasi-ergodicity of Γ -like solutions of (2) in Section 3. First, we review the literature related to the problem at hand.

1.2 Literature

As described in Example , are principally interested in the metastability of travelling waves in stochastic reaction-diffusion equations, a Previous studies on this topic include Bressloff & Weber [16], Eichinger, Gnann, & Kuehn [35], Hamster & Hupkes [47, 48], Inglis & MacLaurin [52], Krüger & Stannat [55, 56], Lang [59], Lang & Stannat [60], MacLaurin [64], and Stannat [80]. We also expect the theory developed in this paper to apply to spiral waves on bounded discs in \mathbb{R}^2 with Dirichlet boundary conditions, such as those studied by Xin [84], and with significant modifications to spiral waves on \mathbb{R}^2 , as studied in PDE by Beyn & Lorenz [11], Sandstede, Scheel, & Wulff [77], and in SPDEs by Kuehn, MacLaurin, & Zucal [58].

Other studies have addressed metastability of patterns in SPDE using techniques related to the theory of large deviations [41]. The recent work of Salins, Budhiraja, & Dupuis [76] provides a general overview of large deviation theory for SPDEs, while Barret [6] and Berglund & Gentz [9] have obtained Eyring-Kramers formula for SPDEs using finite dimensional approximation methods. The results of [6, 9] provide quantitative estimates on the exit time up to which a metastable pattern in an SPDE persists. However, these exit time estimates are only valid in a small noise regime, making them irrelevant for many applications to the life sciences, where small noise approximations are often bad. Many biological systems are composed of populations of interacting particles/agents, and in practice the

population size of such systems is rarely large enough for a deterministic limit to be a good approximation of the dynamics. The resulting *demographic noise* – *i.e.* noise arising from the fact that the system is composed of finitely many agents – can have significant effects on the dynamics of a systems. See [17, 24, 42, 72, 74] for examples from various fields of biology.

Metastability has been studied using quasi-stationary and quasi-ergodic measures in finite dimensional systems in [12, 13, 50, 53]. An introduction to quasi-ergodicity can be found in the textbook of Collet *et al.* [23]. The questions of existence and uniqueness of quasi-stationary and quasi-ergodic measures are discussed in their greatest generality in the work of Champagnat & Villemonais [19, 20, 21, 22]. In [19], exponential convergence to quasi-stationary measures in total variation norm is shown to be equivalent to a modified Doeblin condition, while [22] studies Lyapunov conditions for the existence of quasi-stationary and quasi-ergodic measures. Castro *et al.* [18], Hening & Kolb [49], Hening *et al.* [50], and Ji *et al.* [53], Lelièvre *et al.* [61], Pinsky [70, 71], Zhang, Li, & Song [86], and others have studied quasi-ergodicity by exploiting the spectral properties of the sub-Markov semigroup of a killed Markov process. Their arguments are similar in spirit to those of the present document. However, [18, 49, 70, 71, 86] work in finite dimensional settings, and implicitly or explicitly assume the existence of a bounded density of the sub-Markov transition kernel with respect to some reference measure, which we do not assume here. Meanwhile [50, 53, 61] make explicit use of the finite dimensional nature and gradient structure of the systems they study.

To the author's knowledge, there is almost no work on quasi-stationary and quasi-ergodic measures of SPDEs. Liu *et al.* [62] prove quasi-stationarity of subcritical superprocesses, the distribution of which is governed by an SPDE [82]. However, the arguments of [62] are only at the level of the particle process, and hence do not generalize to other SPDEs which are not dual to a particle system. To the author's knowledge, this paper therefore represents the first results on the existence and uniqueness of quasi-stationary and quasi-ergodic measures for general semilinear SPDEs.

2 General Sub-Markov Semigroups

In this section, we establish conditions for the existence and uniqueness of quasi-stationary and quasi-ergodic measures for a general irreducible continuous Markov process $(Z_t)_{t \geq 0}$ on a separable Hilbert space H . Our strategy is to demonstrate that if the Markov semigroup of $(Z_t)_{t \geq 0}$ is irreducible and compact in some topology, then these properties are inherited by the sub-Markov semigroup of $(Z_t)_{t \geq 0}$ with killing at the boundary of any bounded connected subdomain of H . This allows us to conclude that the top of the spectrum of this sub-Markov generator is a simple eigenvalue. These considerations are proven in Section 2.1. Then, in Section 2.2 we prove quasi-ergodicity of the process with killing. We emphasize that we make no assumptions on the existence or boundedness of an integral kernel density for our sub-Markov semigroup, nor do we assume any Doeblin type or Lyapunov condition on our Markov process. As a consequence we obtain convergence results that hold in an L^p and almost sure sense, rather than uniformly.

Before proceeding, we introduce the general setup and assumptions of this section. In Section 3, we verify that these assumptions hold for SPDEs satisfying Assumption 3. Let

$(Z_t)_{t \geq 0}$ be a continuous $(\mathcal{F}_t)_{t \geq 0}$ -adapted Markov process on a separable Hilbert space H . Let the Markov semigroup of $(Z_t)_{t \geq 0}$ be defined as

$$P_t f(x) := \mathbb{E}_x [f(Z_t)], \quad f \in BM(H), \quad x \in H,$$

where $BM(E)$ be the set of bounded measurable functions $f : H \rightarrow \mathbb{R}$. In our general setting, $(P_t)_{t \geq 0}$ is not necessarily strongly continuous on $BM(H)$. We therefore look to find an extension of $BM(H)$ onto which $(P_t)_{t \geq 0}$ extends to a strongly continuous semigroup. To this end, we make the following assumptions.

Assumption 4. *There exists an invariant measure μ of $(P_t)_{t \geq 0}$ such that P_t extends to a compact operator on $L^2(H, \mu)$ for each $t > 0$.*

Assumption 5. *The semigroup $(P_t)_{t \geq 0}$ is irreducible, in the sense that*

$$P_t 1_F(x) > 0 \quad \text{for all measurable } F \subset H, \quad x \in H, \quad t > 0. \quad (8)$$

Now, fix a bounded connected subset E of H contained in the support of μ . For $E \subset \text{supp } \mu$ and $p \geq 1$, define the L^p_μ -norm of $f \in BM(E)$ as

$$\|f\|_{L^p_\mu}^p := \int_E \|f(x)\|^p \mu(dx). \quad (9)$$

We then define $L^p(E, \mu)$ as the closure of $BM(E)$ under the L^p_μ -norm. Since we have specified $E \subset \text{supp } \mu$, these spaces are nontrivial. Supposing that the initial distribution of $(Z_t)_{t \geq 0}$ has support in E , define the stopping time

$$\tau := \inf \{t > 0 : Z_t \in H/E\}.$$

For $t \geq 0$ the sub-Markov semigroup of $(Z_t)_{t \geq 0}$ killed on H/E is

$$Q_t f(x) := \mathbb{E}_x [f(Z_t) 1_{\{t < \tau\}}], \quad f \in C_b(E).$$

Assumption 6. *For all $x \in E$ and $t > 0$, we have $\mathbb{P}_x[\tau < \infty] = 1$ and $\mathbb{P}_x[t < \tau] > 0$.*

2.1 Spectral Properties of the Sub-Markov Semigroup

To prove the quasi-ergodicity of $(Z_t)_{t \geq 0}$ with killing outside of E , we study the spectrum of $(Q_t)_{t \geq 0}$. First, we need the following lemma.

Lemma 2.1. *Under Assumption 4, $(Q_t)_{t \geq 0}$ extends to a strongly continuous semigroup of compact operators on $L^2(E, \mu)$.*

Proof. We first prove strong continuity. For $f \in BM(E)$, let \bar{f} be the extension of f to H by zero. For arbitrary $p \geq 1$, $f \in BM(E)$, and all $t > 0$, $x \in E$, we have

$$|Q_t f(x)|^p \leq \mathbb{E}_x [|f(Z_t)|^p 1_{t < \tau}] \leq \mathbb{E}_x [|\bar{f}(Z_t)|^p] \leq P_t (|\bar{f}|^p)(x),$$

by Jensen's inequality. Hence, by the μ -invariance of $(P_t)_{t \geq 0}$,

$$\begin{aligned} \int_E |Q_t f(x)|^p \mu(dx) &\leq \int_H P_t(|\bar{f}|^p)(x) \mu(dx) \\ &= \int_H |\bar{f}(x)|^p \mu(dx) = \int_E |f(x)|^p \mu(dx). \end{aligned}$$

Thus $\|Q_t f\|_{L_\mu^p} \leq \|f\|_{L_\mu^p}$ for $p \geq 1$, so $\|Q_t f\|_{L_\mu^\infty} \leq \|f\|_{L_\mu^\infty}$. It follows that $Q_t f$ is bounded μ -almost surely by $f \in BM(E)$ uniformly in $t \geq 0$, and we may apply the dominated convergence theorem to find that

$$\frac{1}{2} \lim_{t \rightarrow 0} \|Q_t f - f\|_{L_\mu^2} \leq \lim_{t \rightarrow 0} \left(\int_E f(x)^2 \mu(dx) - \int_E f(x) Q_t f(x) \mu(dx) \right) = 0.$$

Compactness of Q_t follows from from [3, Theorem 2.3], observing that Q_t is dominated by P_t , in the sense that for all positive $f \in L^p(E, \mu)$,

$$Q_t f(x) = \mathbb{E}_x [f(Z_t) 1_{t < \tau}] \leq \mathbb{E}_x [f(Z_t)] = P_t f(x), \quad x \in E, t > 0.$$

□

Before proceeding, we must borrow a few ideas from the theory of Banach lattices. Since these concepts are not the main focus of this paper, we refer to [8, Section 10.3] for the relevant definitions. The proof of the following result is similar to an argument in [18, Theorem 4.5], with a subtle difference due to the L^2 setting in which we work.

Lemma 2.2. *If $(P_t)_{t \geq 0}$ is an irreducible semigroup, in the sense of (8), then $(Q_t)_{t \geq 0}$ is an ideal irreducible semigroup on the Banach lattice $L^2(E, \mu)$, in the sense that for each $t > 0$, the only closed ideals in $L^2(E, \mu)$ that are Q_t -invariant are $L^2(E, \mu)$ and $\{0\}$.*

Proof. Let I be a closed ideal in $L^2(E, \mu)$. By [8, Proposition 10.15], there must exist a measurable set $A \subset E$ such that

$$I = \{f \in L^2(E, \mu) : f|_A = 0 \text{ } \mu\text{-almost surely}\}.$$

If $\mu(A) = \mu(E)$, then I consists solely of the zero function in $L^2(E, \mu)$, while $\mu(A) = 0$ implies that $I = L^2(E, \mu)$. Suppose that $0 < \mu(A) < \mu(E)$, and take $f \in I/\{0\}$ such that $f \geq 0$. Hence there exists a real number $\epsilon > 0$ and a set $B \subset A$ of positive measure such that $f(x) > \epsilon$ for μ -almost all $x \in B$. Hence taking $0 < s < t$,

$$Q_t 1_B(x) = \mathbb{E}_x [1_B(Z_t) 1_{t < \tau}] = \mathbb{E}_x [P_{t-s} 1_B(Z_s) 1_{t < \tau}],$$

by the Markov property. Since $P_{t-s} 1_B(Z_s) > 0$ \mathbb{P}_x -almost surely and $1_{t < \tau} > 0$ on a set of \mathbb{P}_x -positive measure, we see that $Q_t 1_B(x) > 0$ for μ -almost all $x \in E$. Since $\epsilon 1_B \leq f$ and Q_t is a positive operator, we have

$$0 < \epsilon Q_t 1_B(x) \leq Q_t f(x) \quad \text{for } \mu\text{-almost all } x \in E.$$

In particular, this implies that $Q_t f(x) > 0$ for μ -almost all $x \in A$, so that $Q_t f \notin I$. Therefore if $0 < \mu(A) < \mu(E)$, it is impossible for I to be Q_t -invariant. Hence for arbitrary $t > 0$, the only Q_t -invariant ideals in $L^2(E, \mu)$ are $\{0\}$ and $L^2(E, \mu)$. □

The generator of $(P_t)_{t \geq 0}$ in $L^2(H, \mu)$ is denoted \mathcal{L} , while the generator of $(Q_t)_{t \geq 0}$ in $L^2(E, \mu)$ is denoted \mathcal{L}_E . For each $t > 0$, Q_t has an adjoint in $L^2(E, \mu)$, denoted Q_t^* . Define

$$s(Q_t) := \sup \text{Spec } Q_t, \quad s(\mathcal{L}_E) := \sup \text{Spec } \mathcal{L}_E.$$

Given Lemma 2.1 & 2.2, we have the following classical result.

Proposition 2.1. *$s(\mathcal{L}_E)$ is a simple eigenvalue of \mathcal{L}_E with positive eigenvector.*

Proof. Since for each $t > 0$ the operator Q_t is compact, \mathcal{L}_E must have compact resolvent [36, Theorem II.4.29]. Since each Q_t is positive, \mathcal{L}_E is resolvent positive [8, Corollary 11.4]. Moreover, since Q_t is a positive compact irreducible operator, [33] implies that $s(Q_t) > 0$, and therefore by the spectral mapping theorem [69, Theorem 2.4]

$$s(\mathcal{L}_E) = \frac{1}{t} \ln s(Q_t) > -\infty.$$

By [8, Proposition 12.15], there exists $\varphi \in D(A)$ such that $\varphi > 0$ and $\mathcal{L}_E \varphi = s(\mathcal{L}_E) \varphi$. As $s(\mathcal{L}_E)$ is an eigenvalue of \mathcal{L}_E , it must be a pole of the resolvent operator of \mathcal{L}_E . By [8, Theorem 14.12(d)], the order of this pole has multiplicity equal to one. Consequently, the eigenspace of $s(\mathcal{L}_E)$ is equal to $\text{span}\{\varphi\}$. \square

2.2 Quasi-Ergodicity of Markov Processes on Separable Hilbert Space

Proposition 2.1 in hand, we are able to prove results on the quasi-ergodicity of $(Z_t)_{t \geq 0}$. Let φ, φ^* , be the eigenfunctions from Proposition 2.1, normalized such that

$$\int_E \varphi(x) \mu(dx) = \int_E \varphi^*(x) \mu(dx) = 1,$$

and define

$$M := \langle \varphi, \varphi^* \rangle_{L^2_\mu} \in (0, 1].$$

We have the following lemma on the asymptotic behaviour of $(Q_t)_{t \geq 0}$. The proof idea is similar to Birkhoff [14, Lemma 3]. In the case where H is finite dimensional and Q_t possesses a uniformly bounded integral kernel with respect to μ , similar results are proven in Pinsky [70, Theorem 3], and also in [54, 86]. Our weaker assumptions lead us to convergence results in L^2 and almost surely, rather than uniformly.

Lemma 2.3. *Let Assumptions 4, 5, & 6 hold. Then, there exists $\gamma > 0$ such that for all $f \in L^2(E, \mu)$ we have*

$$\left\| e^{\lambda_1 t} Q_t f(\cdot) - M^{-1} \int \varphi(\cdot) \varphi^*(y) f(y) \mu(dy) \right\|_{L^2_\mu} \leq e^{-\gamma t} (1 + M^{-1}) \|f\|_{L^2_\mu}. \quad (10)$$

Moreover, $e^{\lambda_1 t} Q_t f(\cdot)$ converges μ -almost surely to $M^{-1} \int \varphi(\cdot) \varphi^*(y) f(y) \mu(dy)$.

Proof. By Proposition 2.1 we may split $L^2(E, \mu) = \mathcal{H}_1 \otimes \mathcal{H}_{rem}$, where $\mathcal{H}_1 = \text{span}\{\varphi\}$ and both of $\mathcal{H}_1, \mathcal{H}_{rem}$ are Q_t -invariant, see Deimling [34, Theorem 8.9]. Since $e^{-\lambda_1 t} = \sup \text{Spec}(Q_t)$ and there is a gap between $e^{-\lambda_1 t}$ and the rest of the spectrum of Q_t , there exists $\gamma > 0$ such that

$$e^{\lambda_1 t} \|Q_t|_{\mathcal{H}_{rem}}\|_{L^2_\mu} \leq e^{-\gamma t} \quad \text{for large } t > 0. \quad (11)$$

Moreover, for $f \in L^2(E, \mu)$ there exists $\psi_f \in \mathcal{H}_{rem}$ and $c_f \in \mathbb{R}$ such that

$$f = c_f \varphi + \psi_f.$$

Observing that

$$\begin{aligned} 0 &= \lim_{t \rightarrow \infty} \int_E (e^{\lambda_1 t} Q_t f(x) - c_f \varphi(x)) \varphi^*(x) \mu(dx) \\ &= \lim_{t \rightarrow \infty} \int_E e^{\lambda_1 t} f(x) Q_t^* \varphi^*(x) \mu(dx) - c_f M \\ &= \lim_{t \rightarrow \infty} \int_E f(x) \varphi^*(x) \mu(dx) - c_f M, \end{aligned}$$

we obtain $c_f = M^{-1} \langle f, \varphi^* \rangle_{L^2_\mu}$. By Hölder's inequality,

$$|c_f| \leq M^{-1} \|f\|_{L^2_\mu}, \quad \|\psi_f\|_{L^2_\mu} \leq (1 + M^{-1}) \|f\|_{L^2_\mu},$$

and hence

$$\begin{aligned} &\left\| e^{\lambda_1 t} Q_t f(x) - M^{-1} \int_E \varphi(x) \varphi^*(y) f(y) \mu(dy) \right\|_{L^2_\mu} \\ &= \left\| e^{\lambda_1 t} c_f Q_t \varphi(x) + e^{\lambda_1 t} Q_t \psi_f - M^{-1} \varphi(x) \langle f, \varphi^* \rangle_{L^2_\mu} \right\|_{L^2_\mu} \\ &= \left\| \left(c_f - M^{-1} \langle f, \varphi^* \rangle_{L^2_\mu} \right) \varphi(x) + e^{\lambda_1 t} Q_t \psi_f \right\|_{L^2_\mu} \\ &= \left\| e^{\lambda_1 t} Q_t \psi_f \right\|_{L^2_\mu} \leq e^{-\gamma t} (1 + M^{-1}) \|f\|_{L^2_\mu}, \end{aligned}$$

completing the proof of (10).

To prove that $e^{\lambda_1 t} Q_t f(\cdot)$ converges μ -almost surely to $M^{-1} \int \varphi(\cdot) \varphi^*(y) f(y) \mu(dy)$, fix $f \in L^2(E, \mu)$ and define

$$A_t(\cdot) := \left| e^{\lambda_1 t} Q_t f(\cdot) - M^{-1} \int \varphi(\cdot) \varphi^*(y) f(y) \mu(dy) \right|,$$

so that $\|A_t\|_{L^2_\mu} \leq C_f e^{-\gamma t}$ for some $C_f > 0$. By Chebyshev's inequality, for any $n \in \mathbb{N}$ and $t, s > 0$ we have

$$\begin{aligned} \mu \{x \in E : |A_t(x) - A_s(x)| > n^{-1}\} &\leq n^2 \|A_t - A_s\|_{L^2_\mu}^2 \\ &\leq 2n^2 \min \left\{ \|A_s\|_{L^2_\mu}^2, \|A_t\|_{L^2_\mu}^2 \right\} \\ &\leq 2n^2 C_f \exp(-\gamma \min\{s, t\}). \end{aligned}$$

Therefore $(A_t)_{t \geq 0}$ is Cauchy μ -almost surely, and so converges μ -almost surely. By (10), the μ -almost sure limit of $(A_t)_{t \geq 0}$ must be zero, completing the proof. \square

We are now ready to prove the main result of this section. The following results consist of “conditional” limiting theorems for 1. the distribution of $(Z_t)_{t \geq 0}$, 2. the time-averaged dynamics of $(Z_t)_{t \geq 0}$, and 3. the distribution of a sequence of measurements of $(Z_t)_{t \geq 0}$ taken at different times prior to killing. Compare the following results with those of [86, Section 3], noting in particular that we work in an L^p framework due to the fact that we make no assumptions on the existence or boundedness of a density for the transition kernel of $(Q_t)_{t \geq 0}$.

Theorem 2.1. *Let Assumptions 4, 5, & 6 hold. Defining $\alpha(dx) := \varphi^*(x)\mu(dx)$ and $\beta(dx) := \varphi(x)\varphi^*(x)\mu(dx)$, the following results hold.*

1. For $f \in L^2(E, \mu)$ we have

$$\lim_{t \rightarrow \infty} \mathbb{E}_x [f(Z_t) | t < \tau] = \alpha(f) \quad (12)$$

in L^2_μ and μ -almost surely. Moreover, α is the unique quasi-stationary measure of $(Z_t)_{t \geq 0}$. That is, α is the only measure on E such that for any measurable $A \subset E$,

$$\mathbb{P}_\alpha [Z_t \in A | t < \tau] = \alpha(A), \quad t > 0.$$

2. For arbitrary $f \in L^2(E, \mu)$, $\epsilon > 0$, and μ -almost all $x \in E$, we have

$$\lim_{t \rightarrow \infty} \mathbb{P}_x \left[\left| \frac{1}{t} \int_0^t f(Z_s) ds - \beta(f) \right| > \epsilon | t < \tau \right] = 0. \quad (13)$$

Moreover, β is the unique quasi-ergodic measure of $(Z_t)_{t \geq 0}$ on E .

3. For $f, g \in L^2(E, \mu)$ and $0 < a < b < 1$ we have in L^2_μ and μ -almost surely that

$$\lim_{t \rightarrow \infty} \mathbb{E}_x [f(Z_{at})g(Z_t) | t < \tau] = \beta(f)\alpha(g), \quad (14)$$

$$\lim_{t \rightarrow \infty} \mathbb{E}_x [f(Z_{at})g(Z_{bt}) | t < \tau] = \beta(f)\beta(g). \quad (15)$$

4. For $f \in L^2(E, \mu)$, it holds in L^2_μ and μ -almost surely that

$$\lim_{t \rightarrow \infty} \lim_{T \rightarrow \infty} \mathbb{E}_x [f(Z_t) | T < \tau] = \beta(f). \quad (16)$$

Proof. To see (12), note that Lemma 2.3 implies

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathbb{E}_x [f(Z_t) | t < \tau] &= \lim_{t \rightarrow \infty} \frac{Q_t f(x)}{Q_t 1(x)} \\ &= \frac{M^{-1} \int_E \varphi(x)\varphi^*(y)f(y)\mu(dy)}{M^{-1} \int_E \varphi(x)\varphi^*(y)\mu(dy)} = \alpha(f), \end{aligned}$$

the above limits holding μ -almost surely and in L^2_μ . To see that α is in fact a quasi-stationary measure, we compute for μ -almost all $x \in E$

$$\begin{aligned}
\mathbb{E}_\alpha [f(Z_t) | t < \tau] &= \frac{\int_E Q_t f(y) \alpha(dy)}{\mathbb{P}_\alpha[t < \tau]} \\
&= \frac{1}{\mathbb{P}_\alpha[t < \tau]} \lim_{s \rightarrow \infty} \frac{Q_s(Q_t f)(x)}{\mathbb{P}_x[s < \tau]} \\
&= \frac{1}{\mathbb{P}_\alpha[t < \tau]} \lim_{s \rightarrow \infty} \frac{Q_s(Q_t f)(x)}{\mathbb{P}_x[s + t < \tau]} \frac{\mathbb{P}_x[s + t < \tau]}{\mathbb{P}_x[s < \tau]} \\
&= \frac{1}{\mathbb{P}_\alpha[t < \tau]} \lim_{s \rightarrow \infty} \mathbb{E}_x [f(X_{s+t}) | s + t < \tau] \mathbb{E}_x [Q_t 1(X_s) | s < \tau] \\
&= \frac{1}{\mathbb{P}_\alpha[t < \tau]} \int_E f(y) \alpha(dy) \int_E Q_t 1(y) \alpha(dy) = \int_E f(y) \alpha(dy).
\end{aligned}$$

Supposing α_0 were a second quasi-stationary measure, then

$$\int_E f(y) \alpha_0(dy) = \lim_{t \rightarrow \infty} \mathbb{E}_{\alpha_0} [f(Z_t) | t < \tau] = \int_E f(y) \alpha(dy),$$

so that $\alpha_0 = \alpha$ by duality.

We now prove (14). First note that, by the Markov property,

$$\mathbb{E}_x [f(Z_{at})g(Z_t) | t < \tau] = \frac{e^{\lambda_1 t} Q_{at} (f(\cdot) Q_{t-at} g(\cdot)) (x)}{e^{\lambda_1 t} Q_t 1(x)}. \quad (17)$$

From Lemma 2.3, we have that $(e^{\lambda_1 t} Q_t 1(x))$ converges to $M^{-1} \varphi(x)$ as $t \rightarrow \infty$ μ -almost surely and in L^2_μ . We now show that the limit of the numerator in (17) is $M^{-1} \varphi(x) \beta(f) \alpha(g)$ in L^2_μ and μ -almost surely. To see this, we define

$$h_t(x) := f(x) e^{\lambda_1(t-at)} Q_{t-at} g(x),$$

and compute

$$\begin{aligned}
&e^{\lambda_1 t} Q_{at} (f(\cdot) Q_{t-at} g(\cdot)) (x) - M^{-1} \varphi(x) \beta(f) \alpha(g) \\
&= e^{\lambda_1 t} Q_{at} (f(\cdot) Q_{t-at} g(\cdot)) (x) - \varphi(x) M^{-2} \int \varphi(y) \varphi^*(y) f(y) \mu(dy) \int \varphi^*(z) g(z) \mu(dz) \\
&= e^{\lambda_1 at} Q_{at} h_t(x) - M^{-1} \int \varphi(x) \varphi^*(y) f(y) M^{-1} \int \varphi(y) \varphi^*(z) g(z) \mu(dz) \mu(dy) \\
&= e^{\lambda_1 at} Q_{at} h_t(x) \\
&\quad - M^{-1} \int \varphi(x) \varphi^*(y) f(y) \left(M^{-1} \int \varphi(y) \varphi^*(z) g(z) \right. \\
&\quad \quad \left. - e^{\lambda_1(t-at)} Q_{t-at} g(y) + e^{\lambda_1(t-at)} Q_{t-at} g(y) \right) \mu(dy) \\
&= e^{\lambda_1 at} Q_{at} h_t(x) - M^{-1} \int \varphi(x) \varphi^*(y) h_t(y) \mu(dy) \\
&\quad + M^{-1} \int \varphi(x) \varphi^*(y) f(y) \left(e^{\lambda_1(t-at)} Q_{t-at} g(y) - M^{-1} \int \varphi(y) \varphi^*(z) g(z) \mu(dz) \right) \mu(dy). \quad (18)
\end{aligned}$$

Taking the L_μ^2 norm and applying Hölder's inequality, we have

$$\begin{aligned}
& \left\| e^{\lambda_1 at} Q_{at} h_t(x) - M^{-1} \int \varphi(x) \varphi^*(y) h_t(y) \mu(dy) \right\|_{L_\mu^2} \\
& + M^{-1} \|\varphi\|_{L_\mu^2} \left| \int \varphi^*(y) f(y) \left(e^{\lambda_1(t-at)} Q_{t-at} g(y) - M^{-1} \int \varphi(y) \varphi^*(z) g(z) \mu(dz) \right) \mu(dy) \right| \\
& \leq \left\| e^{\lambda_1 at} Q_{at} h_t(x) - M^{-1} \int \varphi(x) \varphi^*(y) h_t(y) \mu(dy) \right\|_{L_\mu^2} \\
& \quad + M^{-1} \|\varphi\|_{L_\mu^2} \|\varphi^* f\|_{L_\mu^2} \left\| e^{\lambda_1(t-at)} Q_{t-at} g(\cdot) - M^{-1} \int \varphi(\cdot) \varphi^*(z) g(z) \mu(dz) \right\|_{L_\mu^2} \\
& \leq e^{-\gamma t} (1 + M^{-1}) \|h_t\|_{L_\mu^2} + M^{-1} \|\varphi\|_{L_\mu^2} \|\varphi^* f\|_{L_\mu^2} e^{-\gamma(t-at)} (1 + M^{-1}) \|g\|_{L_\mu^2}.
\end{aligned} \tag{19}$$

Note that

$$\begin{aligned}
\|h_t\|_{L_\mu^2} &= \|f(\cdot) e^{\lambda_1(t-at)} Q_{t-at} g(\cdot)\|_{L_\mu^2} \\
&\leq \|f\|_{L_\mu^2} e^{\lambda_1(t-at)} \|Q_{t-at} g\|_{L_\mu^2} \leq \|f\|_{L_\mu^2} \|g\|_{L_\mu^2},
\end{aligned}$$

so that (19) tends to zero as $t \rightarrow \infty$. Hence

$$\lim_{t \rightarrow \infty} e^{\lambda_1 t} Q_{at} (f(\cdot) Q_{t-at} g(\cdot))(x) = M^{-1} \varphi(x) \beta(f) \alpha(g). \tag{20}$$

in L_μ^2 . Since the rate of convergence in (20) is exponential, the same argument as in Lemma 2.3 can be used to show that the convergence in (20) holds μ -almost surely. We have therefore proven that $\lim_{t \rightarrow \infty} \mathbb{E}_x [f(Z_{at}) g(Z_t) | t < \tau] = \beta(f) \alpha(g)$ in L_μ^2 and μ -almost surely. The proofs of (15) & (16) are similar.

To prove (13), note that from (17) we have

$$\mathbb{E}_x \left[\frac{1}{t} \int_0^t f(Z_s) ds | t < \tau \right] = \frac{\frac{1}{t} \int_0^t e^{\lambda_1 s} Q_s (f(\cdot) Q_{t-s} 1(\cdot))(x) ds}{e^{\lambda_1 t} Q_t 1(x)}.$$

Then, defining $h_t(x) := f(x) Q_{t-s} 1(x)$, observe that

$$\begin{aligned}
& \frac{1}{t} \int_0^t e^{\lambda_1 s} Q_s (f(\cdot) Q_{t-s} 1(\cdot))(x) ds - M^{-1} \varphi(x) \beta(f) \\
&= \frac{1}{t} \int_0^t e^{\lambda_1 s} Q_s (f(\cdot) e^{\lambda_1(t-s)} Q_{t-s} 1(\cdot))(x) - M^{-1} \varphi(x) \beta(f) ds \\
&= \frac{1}{t} \int_0^t e^{\lambda_1 s} Q_s (f(\cdot) e^{\lambda_1(t-s)} Q_{t-s} 1(\cdot))(x) \\
&\quad - M^{-1} \int \varphi(x) \varphi^*(y) f(y) M^{-1} \int \varphi(y) \varphi^*(z) 1(z) \mu(z) \mu(dy) ds \\
&= \frac{1}{t} \int_0^t e^{\lambda_1 s} Q_s h_t(x) - M^{-1} \int \varphi(x) \varphi^*(y) h_t(y) \mu(dy) \\
&\quad - M^{-1} \int \varphi(x) \varphi^*(y) \left(M^{-1} \int \varphi(y) \varphi^*(z) 1(z) \mu(dz) - e^{\lambda_1(t-s)} Q_{t-s} 1(y) \right) ds.
\end{aligned}$$

As the L_μ^2 norm of h_t is bounded uniformly in $t > 0$, the same argument as used to prove (14) implies that

$$\lim_{t \rightarrow \infty} \left\| \mathbb{E}_x \left[\frac{1}{t} \int_0^t f(Z_s) ds | t < \tau \right] - \beta(f) \right\|_{L_\mu^2} = 0.$$

Since the above convergence can be seen to occur at an exponential rate,

$$\lim_{t \rightarrow \infty} \mathbb{E}_x \left[\frac{1}{t} \int_0^t f(Z_s) ds \mid t < \tau \right] = \beta(f)$$

μ -almost surely, again using the argument as in Lemma 2.3. Applying Markov's inequality completes the proof. \square

Now, we prove that the rate of convergence to the quasi-stationary measure α in Theorem 2.1.1 is exponential in $L^2(E, \alpha^*)$, where $\alpha^*(dx) := \varphi(x)\mu(dx)$. If $(Z_t)_{t \geq 0}$ is reversible with respect to μ , we of course have $\alpha = \alpha^*$.

Theorem 2.2. *Let Assumptions 4, 5, & 6 hold, and define the probability measure $\alpha^*(dx) := \varphi(x)\mu(dx)$. Then, for any $f \in L^2(E, \mu)$ there exist $T > 0$ and $C_f > 0$ such that*

$$\|\mathbb{E}_x [f(Z_t) \mid t < \tau] - \alpha(f)\|_{L^1_{\alpha^*}} \leq C_f e^{-\gamma t} \quad \forall t \geq T.$$

Proof. First, note that since $\frac{Q_t f}{Q_t 1}$ converges to $\alpha(f)$ in L^2_μ , for any $K > 0$ there exists $T > 0$ such that for all $t \geq T$, we have

$$\begin{aligned} \left\| \frac{M^{-1}}{Q_t 1} Q_t f - e^{\lambda t} Q_t f \right\|_{L^1_\mu} &= \left\| (M^{-1} \varphi - e^{\lambda t} Q_t 1) \frac{e^{\lambda t} Q_t f}{Q_t 1} \right\|_{L^1_\mu} \\ &\leq \|e^{\lambda t} Q_t 1 - M^{-1} \varphi\|_{L^2_\mu} \left\| \frac{e^{\lambda t} Q_t f}{Q_t 1} \right\|_{L^2_\mu} \\ &\leq C'_f e^{-(\gamma - \lambda)t} \end{aligned}$$

for some $C'_f > 0$. From this we compute

$$\begin{aligned} \|\mathbb{E}_x [f(Z_t) \mid t < \tau] - \alpha(f)\|_{L^1_{\alpha^*}} &= \left\| \frac{\varphi}{Q_t 1} Q_t f - \alpha(f) \varphi \right\|_{L^1_\mu} \\ &\leq M \left\| \frac{M^{-1} \varphi}{Q_t 1} Q_t f - e^{\lambda t} Q_t f + e^{\lambda t} Q_t f - M^{-1} \alpha(f) \varphi \right\|_{L^1_\mu} \\ &\leq M \left\| \frac{M^{-1} \varphi}{Q_t 1} Q_t f - e^{\lambda t} Q_t f \right\|_{L^1_\mu} + M \|e^{\lambda t} Q_t f - M^{-1} \alpha(f) \varphi\|_{L^2_\mu} \\ &\leq M C'_f e^{-(\gamma - \lambda)t} + M C''_f e^{-\gamma t}, \end{aligned}$$

where C''_f is as in Lemma 2.3, completing the proof. \square

Since we have not proven that the rate of convergence to α is uniform, the proof of existence of a Q -process found in Champagnat & Villemonais [19] does not immediately translate to our setting. Nevertheless, we prove the existence and uniqueness of the Q -process in Theorem 2.3 below using arguments inspired by their work.

Theorem 2.3. *Under Assumptions 4, 5, & 6, for μ -almost all $x \in E$ there exists a probability measure \mathbb{Q}_x defined set-wise as the limit*

$$\mathbb{Q}_x(A) := \lim_{t \rightarrow \infty} \mathbb{P}_x(A \mid t < \tau), \quad A \in \mathcal{F}_s, \quad s \geq 0. \quad (21)$$

Moreover, the limit in (21) holds in L^2_μ . With respect to $(\mathbb{Q}_x)_{x \in E}$, the solution process $(Z_t)_{t \geq 0}$ is a homogeneous Markov process in E with unique ergodic measure β

Proof. For $x \in E$ and $t > 0$, we define an auxiliary probability measure \mathbb{Q}_x^t on Ω as

$$\mathbb{Q}_x^t(d\omega) := \frac{1_{t < \tau}(\omega)}{\mathbb{E}_x[1_{t < \tau}]} \mathbb{P}_x(d\omega).$$

By the Markov property of $(Z_t)_{t \geq 0}$ with respect to \mathbb{P}_x , for fixed $s > 0$ and any $t \geq s$

$$\frac{\mathbb{E}_x[1_{t < \tau} | \mathcal{F}_s]}{\mathbb{E}_x[1_{t < \tau}]} = \frac{1_{s < \tau} \mathbb{P}_{Z_s}[t - s < \tau]}{\mathbb{P}_x[t < \tau]}.$$
 (22)

By Lemma 2.3 with $\mathbb{P}_x[t < \tau] = Q_t 1(x)$, we have in L_μ^2 and μ -almost surely that

$$M^{-1}\varphi(x) = \lim_{t \rightarrow \infty} \frac{\mathbb{P}_x[t < \tau]}{\mathbb{P}_\alpha[t < \tau]}.$$

Hence for fixed $s > 0$, (22) converges μ -almost surely as $t \rightarrow \infty$ to

$$M_s(x) := 1_{s < \tau} e^{\lambda_1 s} \frac{\varphi(Z_s)}{\varphi(x)}.$$

Also, we may compute

$$\mathbb{E}_x[M_s(x)] = e^{\lambda_1 s} \varphi(x)^{-1} \mathbb{E}_x[1_{s < \tau} \varphi(Z_s)] = e^{\lambda_1 s} \varphi(x)^{-1} Q_s \varphi(x) = 1.$$
 (23)

Now, we claim that the definition of $(M_t)_{t \geq 0}$ and (23) imply that for each $A_s \in \mathcal{F}_s$, $s \geq 0$, it holds that

$$\lim_{t \rightarrow \infty} \mathbb{Q}_x^t(A_s) = \mathbb{E}_x[1_{A_s} M_s] \quad \mu\text{-almost surely.}$$
 (24)

This claim is similar to [75, Theorem 2.1]. For $s \geq 0$, $A_s \in \mathcal{F}_s$, and $\omega \in A_s$ note that $1_{t < \tau}(\omega) = \mathbb{E}_x[1_{t < \tau} | \mathcal{F}_s](\omega)$, and hence

$$\begin{aligned} \lim_{t \rightarrow \infty} \int_{A_s} \left\| \frac{1_{t < \tau}(\omega)}{\mathbb{E}_x[1_{t < \tau}]} - M_s(\omega) \right\| \mathbb{P}_x(d\omega) &\leq \lim_{t \rightarrow \infty} \int_{A_s} \left\| \frac{1_{t < \tau}(\omega)}{\mathbb{E}_x[1_{t < \tau}]} \right\| \mathbb{P}_x(d\omega) \\ &\quad - \lim_{t \rightarrow \infty} \int_{A_s} \left\| \frac{\mathbb{E}[1_{t < \tau} | \mathcal{F}_s](\omega)}{\mathbb{E}[1_{t < \tau}]} \right\| \mathbb{P}_x(d\omega) \\ &= 0 \end{aligned}$$

for μ -almost all $x \in E$. By Scheffé's Lemma [78], it then follows that

$$\lim_{t \rightarrow \infty} \mathbb{Q}_x^t(A_s) = \int_{A_s} M_s(\omega) \mathbb{P}_x(d\omega) = \mathbb{E}_x[1_{A_s} M_s] \quad \mu\text{-almost surely,}$$

proving (24). It follows that \mathbb{Q}_x is well-defined for μ -almost all $x \in E$, and

$$\left. \frac{d\mathbb{Q}_x}{d\mathbb{P}_x} \right|_{\mathcal{F}_s} = M_s(x) \quad \text{for } s > 0.$$

We now show that $(Z_t)_{t \geq 0}$ is a Markov process with respect to $(\mathbb{Q}_x)_{x \in E}$. Indeed, using the definition of conditional expectations we have

$$\begin{aligned} M_s \mathbb{E}_x^\mathbb{Q}[f(Z_t) | \mathcal{F}_s] &= \mathbb{E}_x^\mathbb{Q}[M_t f(Z_t) | \mathcal{F}_s] \\ &= \mathbb{E}_x^\mathbb{Q}[M_t f(Z_t) | Z_s] = M_s \mathbb{E}_x^\mathbb{Q}[f(Z_t) | Z_s], \end{aligned}$$

the second equality following from the definition of M_t and the Markov property of $(\mathbb{P}_x)_{x \in E}$. This proves that $(Z_t)_{t \geq 0}$ is a Markov process with respect to $(\mathbb{Q}_x)_{x \in E}$. The fact that $(Z_t)_{t \geq 0}$ with respect to $(\mathbb{Q}_x)_{x \in E}$ has a unique ergodic measure given by β follows from the fourth statement of Theorem 2.1 . \square

3 Application to a class of SPDEs

We now demonstrate that Assumption 3 guarantees that a cutoff version of (2) satisfies Assumptions 4, 5, & 6, and therefore possesses unique quasi-stationary and quasi-ergodic measures in Γ_δ . We then show, in Proposition 3.2, that the quasi-stationary and quasi-ergodic measure of this cutoff version of (2) are a quasi-stationary and quasi-ergodic measure of (2) itself. Letting $N_\delta : D(N) \rightarrow D(N)$ be a Fréchet differentiable function such that

- (i) $N_\delta(x) = N(x)$ for $x \in \Gamma_\delta$,
- (ii) for some $\delta' > \delta$ we have $N_\delta(x) = 0$ for $x \in D(N)/\Gamma_{\delta'}$, and
- (iii) $\text{Lip } N_\delta = \text{Lip } N|_{\Gamma_\delta} = \kappa$,

we consider

$$dX' = (LX' + N_\delta(X')) dt + \sigma B dW. \quad (25)$$

Before proceeding, we establish a mild solution theory for (2) & (25). A solution theory of the associated Ornstein-Uhlenbeck process,

$$dY_t = LY_t dt + \sigma B dW_t, \quad (26)$$

and the first variational equation of (25),

$$\partial_t \eta_t = L\eta_t + DN_\delta(X'_t)\eta_t, \quad (27)$$

is also needed to apply the results of [29]. The solution theory of (2) and (25) follows from a relatively standard fixed point argument, similar to that in [31, Theorem 7.7].

Proposition 3.1. *Under Assumptions 1 & 2, we have the following results.*

- (i) For each $x \in H$, there exists a unique mild solution $(X'_t)_{t \geq 0}$ to (25),

$$X'_t = \Lambda_t x + \int_0^t \Lambda_{t-s} N_\delta(X'_s) ds + \sigma \int_0^t \Lambda_{t-s} B dW_s, \quad t \geq 0. \quad (28)$$

- (ii) For each $x \in H$ there exists a unique mild solution $(X_t)_{t \in [0, \tau]}$ to (2),

$$X_t = \Lambda_t x + \int_0^t \Lambda_{t-s} N(X_s) ds + \sigma \int_0^t \Lambda_{t-s} B dW_s, \quad t \in [0, \tau]. \quad (29)$$

- (iii) There exists a unique mild solution to (26),

$$Y_t = \Lambda_t Y_0 + \sigma \int_0^t \Lambda_{t-s} B dW_s, \quad t \geq 0. \quad (30)$$

(iv) *There exists a unique solution $(\eta_t)_{t \geq 0}$ to the first variational equation (27).*

Proof. That $(Y_t)_{t \geq 0}$ in (30) is a unique solution to (26) is a straightforward consequence of Assumption 2(a,b).

We now prove existence and uniqueness of mild solutions to (25). Since these arguments are largely known under Assumption 2, we sketch the proof. In this case, N_δ is globally Lipschitz on $D(N)$ with Lipschitz constant $\kappa > 0$. Fix an arbitrary $T > 0$ and $X_0 \in D(N)$. Let $(Y_t)_{t \geq 0}$ be defined as in (30) with $Y_0 = 0$, and for some E -valued Markov process $(X'_t)_{t \geq 0}$ define $V_t := X'_t - Y_t$. Note that $(X'_t)_{t \geq 0}$ satisfies (28) for $t \in [0, T]$ if and only if

$$V_t = \Lambda_t X_0 + \int_0^t \Lambda_{t-s} N_\delta(V_s + Y_s) ds \quad \text{for } t \in [0, T]. \quad (31)$$

Let C_T be the space of continuous paths from $[0, T]$ to $D(N)$, and define $U : C_T \rightarrow C_T$ by

$$U(v)(t) = \Lambda_t X_0 + \int_0^t \Lambda_{t-s} N_\delta(v_s + Y_s) ds, \quad v \in C_T, t \in [0, T].$$

Due to the local boundedness of $(\Lambda_t)_{t \geq 0}$ and N on $D(N)$, the local inversion theorem [31, Lemma 9.2] implies that there exists small $T > 0$ such that (31) admits a unique solution $(V_t)_{t \in [0, T]}$, and hence there exists a unique local mild solution $(X'_t)_{t \in [0, T]}$ to (25).

To extend this solution globally in time, we exploit the fact that N_δ is Lipschitz to obtain

$$e^{-\omega t} \|V_t\|_{D(N)} \leq \|X_0\|_{D(N)} + \int_0^t \kappa e^{-\omega s} \|Y_s\|_{D(N)} ds + \int_0^t \kappa e^{-\omega s} \|V_s\|_{D(N)} ds.$$

We may then apply Grönwall's inequality to conclude that V_t cannot blowup in finite time, and we may extend the solution to (31) to $t \in [0, \infty)$ for $X_0 \in D(N)$. Extending this solution theory to allow for $X_0 \in H$ can be achieved as in [31, Theorem 7.15]. The same arguments apply to (25) for $t < \tau$. \square

Remark 1. Assumption 2 fails when $B = I$, $L = \Delta$, and $d \geq 2$. In this case, to make sense of solutions to (2) one must perform a renormalization procedure [28, 45, 46], as discussed for stochastic reaction-diffusion equations by Berglund & Kuehn [10], and for many other classes of SPDEs elsewhere in the literature. It is presently unclear as to how much of the following discussion holds for (2) when it must be renormalized.

The following is the main result of this section, proving Theorem A above. Henceforth, let $(X'_t)_{t \geq 0}$ denote the mild solution to (25), and let $(P_t)_{t \geq 0}$ denote the corresponding Markov semigroup,

$$P_t : BM(H) \rightarrow BM(H), \quad P_t f(x) := \mathbb{E}_x [f(X'_t)].$$

Let $(X_t)_{t \in [0, \tau)}$ denote the mild solution of (2).

Theorem 3.1. *Under Assumption 3, the solution to (25) satisfies Assumptions 4, 5 & 6. In particular, the following properties of $(X'_t)_{t \geq 0}$ and $(P_t)_{t \geq 0}$ hold.*

- (i) $(P_t)_{t \geq 0}$ is irreducible, in the sense of (8).
- (ii) $(X'_t)_{t \geq 0}$ possesses a unique invariant measure μ .

(iii) $(P_t)_{t \geq 0}$ extends to a strongly continuous semigroup of compact operators on $L^p(H, \mu)$ for $p > 1$.

As a consequence, $(X'_t)_{t \geq 0}$ possesses a unique quasi-stationary measure, a unique quasi-ergodic measure, and a unique Q -process on Γ_δ , which are also the unique quasi-stationary measure, unique quasi-ergodic measure, and unique Q -process of the solution to (2).

The proof that (2) satisfies Assumption 6 follows from [31, Theorem 12.19]. We prove the remainder of Theorem 3.1 in a series of lemmas, beginning with a proof of the stochastic irreducibility and strong Feller property of $(P_t)_{t \geq 0}$.

Lemma 3.1. *Assumption 3 implies that $(P_t)_{t \geq 0}$ is irreducible in the sense of (8). Moreover, $(P_t)_{t \geq 0}$ is a strong Feller semigroup, in the sense that for any bounded measurable function $f : E \rightarrow \mathbb{R}$, the function $P_t f : E \rightarrow \mathbb{R}$ is bounded and continuous.*

Proof. Under Assumption 2, the strong Feller property follows from [26, Theorem 3.1]. Stochastic irreducibility follows from Propositions 2.8 & 2.11 of Maslowski [65]. \square

Lemma 3.2. *Assumption 3 implies that (25) possesses a unique invariant measure μ on H . Moreover, μ satisfies a logarithmic Sobolev inequality.*

Proof. By Lemma 3.1, $(P_t)_{t \geq 0}$ is irreducible and strong Feller. Moreover,

$$\begin{aligned} \mathbb{E}_x [\|X'_t\|_{D(N)}] &\leq e^{-\omega t} \mathbb{E}_x [\|X_0\|_{D(N)}] + \int_0^t e^{-\omega(t-s)} \mathbb{E}_x [\|N_\delta(X'_s)\|_{D(N)}] ds \\ &\quad + \sigma \mathbb{E}_x [\|Y_t\|_{D(N)}] \\ &\leq c_1 e^{-\omega t} + c_2 (1 - e^{-\omega t}) \end{aligned}$$

for constants $c_1, c_2 > 0$. Hence [44, Theorem 2.5 & Remark 3.2] implies the existence and uniqueness of an invariant measure μ for (25).

We now sketch the proof of the logarithmic Sobolev inequality, following ideas from [27, Proposition 3.30]. First, note that Assumption 2 implies

$$\|DX'_t\|_{D(N)} \leq e^{-(\kappa-\omega)t},$$

and therefore for any $f \in C_b^1(H)$ we have for some constant $C_f > 0$ that

$$\begin{aligned} \|DP_t(f^2)\|^2 &= \|2 \mathbb{E}_x [f(X'_t) Df(X'_t) DX'_t]\|^2 \\ &\leq 2C_f e^{-(\kappa-\omega)t}. \end{aligned}$$

Consequently, taking into account μ -invariance and the integration by parts formula [29, Theorem 3.6],

$$\begin{aligned} \partial_t \int P_t f^2 \ln P_t f^2 d\mu &= \int \mathcal{L} P_t f^2 \ln P_t f^2 d\mu + \int \mathcal{L} P_t f^2 d\mu \\ &= -\frac{1}{2} \int \frac{1}{P_t f^2} \|BDP_t f^2\|^2 d\mu \\ &\geq -C_f e^{-(\kappa-\omega)t} \int P_t \|BDf\|^2 d\mu = -C_f \mu(\|BDf\|^2) e^{-(\kappa-\omega)t}. \end{aligned}$$

Integrating the above over a finite time interval $[0, t]$, we obtain

$$\int P_t f^2 \ln P_t f^2 d\mu \Big|_0^t \geq (\kappa - \omega)^{-1} (1 - e^{-(\kappa - \omega)t}) \mu(\|BDf\|^2),$$

and taking $t \rightarrow \infty$ yields the logarithmic Sobolev inequality

$$\mu(f^2) \ln \mu(f^2) - \mu(f^2 \ln f^2) \geq (\kappa - \omega)^{-1} \mu(\|BDf\|^2).$$

□

To prove that each P_t is compact on $L^p(H, \mu)$, we make use of a result from [29]. To this end we must introduce the Yosida approximations of (25) and (27), which are essential to the proof of the integration by parts formula necessary for the arguments [29] in the case where N_δ is not a bounded operator on H . The Yosida approximations of (2) and (27) are constructed by replacing N_δ with its Yosida approximation $N_{\delta, \alpha}$. The Yosida approximation of dissipative operators is well-defined, and while N_δ itself is not dissipative, note that $N_\delta - \kappa$ is, where $\kappa = \text{Lip } N_\delta$. Hence, following the construction in [31, Appendix D.3], we define for small $\alpha > 0$

$$J_\alpha(x) := (I - \alpha(N_\delta - \kappa I))^{-1}(x), \quad x \in H,$$

and

$$(N_{\delta, \alpha} - \kappa I)(x) = (N_\delta - \kappa I)(J_\alpha(x)) = \frac{1}{\alpha}(J_\alpha(x) - x), \quad x \in H.$$

Then, the Yosida approximation of (25) is

$$dX^\alpha = (LX^\alpha + N_{\delta, \alpha}(X^\alpha)) dt + \sigma dW, \quad (32)$$

while the Yosida approximation of (27) is

$$\partial_t \eta^\alpha = L\eta_t + DN_{\delta, \alpha}(X_t)\eta_t. \quad (33)$$

Lemma 3.3. *If Assumption 3 holds, then the Yosida approximations (32) and (33) possess unique mild solutions, which we denote $(X_t^\alpha)_{t \geq 0}$ and $(\eta_t^\alpha)_{t \geq 0}$. Moreover, X_α converges to X' almost surely on $[0, \infty) \times H$, and η^α converges almost surely to η on $[0, \infty) \times H$.*

Proof. The Yosida approximations $N_{\delta, \alpha}$ are Lipschitz and bounded on H , so that the existence and uniqueness of solutions to (32) & (33) follow from the same arguments as in Proposition 3.1.

We sketch the proof of the convergence of $(X_t^\alpha)_{t \geq 0}$ to $(X_t')_{t \geq 0}$. Observe that

$$\begin{aligned} \|X_t' - X_t^\alpha\|_{D(N)} &= \left\| \int_0^t \Lambda_{t-s} (N_\delta(X_s') - N_{\delta, \alpha}(X_s^\alpha)) ds \right\|_{D(N)} \\ &\leq \int_0^t e^{-\omega(t-s)} \|N_\delta(X_s') - N_\delta(X_s^\alpha)\|_{D(N)} ds \\ &\quad + \int_0^t e^{-\omega(t-s)} \|N_\delta(X_s^\alpha) - N_{\delta, \alpha}(X_s^\alpha)\|_{D(N)} ds. \end{aligned} \quad (34)$$

Using the fact that $N_{\delta,\alpha}(x) = N_\delta(J_\alpha(x))$, by [31, Proposition D.11] and the Lipschitz property of N_δ we have

$$\|N_\delta(x) - N_{\delta,\alpha}(x)\|_{D(N)} = \|(N_\delta - \kappa I)(x) - (N_{\delta,\alpha} - \kappa I)(x)\|_{D(N)} \leq 2\kappa\alpha\|(N_\delta - \kappa)(x)\|_{D(N)}.$$

Therefore from (34) we may apply Grönwall's inequality and take $\alpha \rightarrow 0$ to conclude the result. A similar argument applies for the convergence of $(\eta_t^\alpha)_{t \geq 0}$ to $(\eta_t)_{t \geq 0}$. \square

Finally, to apply the results of [29], we must prove a result relating the invariant measure μ of (25) to the invariant measure of (26).

Lemma 3.4. *Under Assumption 3, the Ornstein-Uhlenbeck process governed by (26) possesses a unique invariant measure ν , which is a Gaussian measure with mean zero and covariance operator*

$$Q_\infty := \frac{\sigma^2}{2} \int_0^\infty \Lambda_t B B^* \Lambda_t^* dt.$$

Moreover, the density $\rho := d\mu/d\nu$ exists, and $\ln \rho \in W^{1,2}(H, \mu)$.

Proof. The existence of the invariant measure ν follows from [27, Theorem 2.34]. To prove the existence of ρ and that $\ln \rho \in W^{1,2}(H, \mu)$, first observe that

$$\int_H \|N_\delta(x)\|_{D(N)}^2 \mu(dx) \leq \sup_{x \in \Gamma_\delta} \|N(x)\|_{D(N)}^2 \mu(\Gamma_\delta) < \infty. \quad (35)$$

Letting μ_α be the unique invariant measure of the Yosida approximation (32), the existence of $\rho_\alpha = d\mu_\alpha/d\nu$ follows from [15]. Moreover, [15, Claim 3] implies that $D \ln \rho_\alpha = N_{\delta,\alpha}$, where $D \ln \rho_\alpha$ denotes the Fréchet derivative of $\ln \rho_\alpha$, and hence

$$\int_H \|D \ln \rho_\alpha(x)\|^2 \mu(dx) = \int_H \|N_{\delta,\alpha}(x)\|^2 \mu(dx) < \infty,$$

implying that $\ln \rho_\alpha \in W^{1,2}(H, \mu)$. For $\alpha > 0$, let $(P_t^\alpha)_{t \geq 0}$ denote the Markov semigroup of $(X_t^\alpha)_{t \geq 0}$. Since X_t^α converges to X_t in $D(N)$ for each $t > 0$ as $\alpha \rightarrow 0$, by the dominated convergence theorem we have

$$\lim_{\alpha \rightarrow 0} \mu_\alpha(A) = \lim_{\alpha \rightarrow 0} \lim_{t \rightarrow \infty} P_t^\alpha 1_A(x) = \lim_{t \rightarrow \infty} P_t 1_A(x) = \mu(A).$$

Hence by the Vitali-Hahn-Sack Theorem [85, Chapter II.2], $\rho = d\mu/d\nu$ exists. By (35) we may apply the dominated convergence theorem to conclude that $\ln \rho \in W^{1,2}(H, \mu)$. \square

We are now able to prove the fourth statement in Theorem 3.1

Lemma 3.5. *Let Assumption 3 hold. For each $p \in [1, \infty)$, $(P_t)_{t \geq 0}$ extends to a strongly continuous semigroup of compact operators on $L^p(H, \mu)$.*

Proof. Follows from [29, Theorem 5.1]. \square

We have therefore proven Theorem 3.1 for the solution to (2) with N replaced by N_δ , and hence this system with a cutoff nonlinearity possesses a unique quasi-stationary measure, a unique quasi-ergodic measure, and a unique Q -process in Γ_δ defined μ -almost surely in Γ_δ . However, our interest is ultimately in (2) with a non-cutoff nonlinearity. The following result demonstrates that the quasi-stationary distribution, quasi-ergodic distribution, and Q -process of (2) with cutoff nonlinearity in Γ_δ are also the quasi-stationary measure, quasi-ergodic measure, and Q -process of (2) with non-cutoff nonlinearity.

Proposition 3.2. *Let Assumption 3 hold, and let α and β be the unique quasi-stationary and quasi-ergodic measure of (25), guaranteed to exist by Theorems 3.1 & 2.1. Then, α and β are the unique quasi-stationary and quasi-ergodic measure of (2). Moreover, (2) admits a unique Q -process in Γ_δ , equal to the Q -process of (25).*

Proof. For $t < \tau$ we of course have $N(X_t) = N_\delta(X_t)$ and $N(X'_t) = N_\delta(X'_t)$. Since $(X_t)_{t \geq 0}$ and $(X'_t)_{t \geq 0}$ are driven by the same Wiener process, it follows that

$$\|X_t - X'_t\| \leq \int_0^t e^{-\omega(t-s)} \kappa \|X_s - X'_s\| ds,$$

where $\kappa > 0$ is the Lipschitz constant of N_δ . Grönwall's inequality then implies $\|X_t - X'_t\| = 0$ for $t < \tau$. In particular, $\mathbb{P}_x[X_t \in \cdot \mid t < \tau] = \mathbb{P}_x[X'_t \in \cdot \mid t < \tau]$. \square

4 Applications

In this section, we sketch how the results of Section 3 may be used to study the effects of noise on the behaviour of metastable patterns in SPDEs. We first study how noise may affect the average “position” of a pattern. Here, the position of a pattern is rigorously defined using a generalization of the *isochron map*, introduced by Winfree [83] to define the phase of stochastic oscillators in biological systems. Loosely speaking, the isochron map uniquely projects a given point in the vicinity of a deterministic stable invariant manifold Γ to a point on Γ , in a manner that is consistent with the deterministic dynamics. The generalization of the isochron map used here allows us to define the *isochronal phase* of patterns such as stochastic travelling waves and stochastic spiral waves, giving a precise notion of the position of such a stochastically perturbed pattern. This is done in Section 4.1. Using the isochron map, in Section 4.2 we sketch how one may obtain results on noise-induced deviations in the *velocity* of a spatiotemporal pattern. To do so we restrict our attention to the case where the pattern Γ is a limit cycle.

4.1 Quasi-Ergodicity of the Isochronal Phase

Suppose that Γ is a stable normally hyperbolic invariant manifold of (2) which is compact, smooth, and m -dimensional for some $m \in \mathbb{N}$. Let $(t, x) \mapsto \phi_t(x)$ denote the flow map of (1), so that

$$\phi_t(x) = \Lambda_t x + \int_0^t \Lambda_{t-s} N(\phi_s(x)) ds$$

for $x \in H$. Let $(X_t)_{t \geq 0}$ denote the unique mild solution to (2). To study the long-term behaviour of $(X_t)_{t \geq 0}$ in the vicinity of Γ_δ , we reduce this process to a finite dimensional process on Γ in a manner that is consistent with the deterministic dynamics.

Lemma 4.1. *Suppose that for all $t > 0$ the flow map $\phi_t : \Gamma_\delta \rightarrow \Gamma_\delta$ is bi-Lipschitz. Then, for each $x \in \Gamma_\delta$ there exists a unique point $\pi(x) \in \Gamma$ such that*

$$\|\phi_t(x) - \phi_t(\pi(x))\| \xrightarrow[t \rightarrow \infty]{} 0. \quad (36)$$

The map $\pi : \Gamma_\delta \rightarrow \Gamma$ is referred to as the isochron map of Γ . Moreover, if ϕ_t is C^2 , then π is C^2 (in the topology of $D(N)$).

Proof. A simple proof follows from Cantor's Intersection Theorem, as in [2, Theorem 3.1]. \square

Remarkably, even though (2) only admits a mild solution when the noise is not trace class, the isochron map possesses certain regularizing properties that allow one to prove a strong Itô formula.

Lemma 4.2. *Let $V = L + N$, and let $D\pi$ and $D^2\pi$ denote the first and second Fréchet derivatives of π in the topology of $D(N)$, respectively. Then, for some orthonormal basis $\{e_k\}_{k \in \mathbb{N}}$ of H it holds that*

$$\pi(X_t) - \pi(X_0) = \int_0^t D\pi(X_s)V(X_s) + \frac{\sigma^2}{2} \sum_{k \in \mathbb{N}} D^2\pi(X_s)[Be_k, Be_k] ds + \int_0^t D\pi(X_s)B dW_s, \quad (37)$$

and all of the above integrals are well defined.

Proof. See [2, Theorem 3.5]. \square

Now, so long as (2) satisfies Assumption 3, we may apply Theorem A to conclude that (2) admits a unique quasi-ergodic measure β in Γ_δ . We may then define a measure on Γ via pushforward under the isochron map,

$$\pi_*\beta(A) := \beta(\pi^{-1}(A)).$$

Using the change of variables formula and Theorems 2.1 & 3.1, we have the following result, providing a measure that indicates how the noise affects the average position of our stochastically perturbed pattern. This may be compared with [2, Theorem C], obtained using different methods.

Theorem 4.1. *For any $p \geq 1$, $\epsilon > 0$, and bounded $g \in L^2(\Gamma, \pi_*\beta)$, we have*

$$\lim_{t \rightarrow \infty} \mathbb{P}_x \left[\left| \frac{1}{t} \int_0^t g(\pi_s) ds - (\pi_*\beta)(g) \right|^p > \epsilon \mid t < \tau \right] = 0. \quad (38)$$

4.2 Quasi-Asymptotic Frequencies of Stochastic Oscillators

Now, assume that Γ is a periodic (in time) solution of (1). In this case, we refer to a Γ -like solutions as a *stochastic oscillator*. For instance, Γ could be a family of periodic solutions corresponding to a limit cycle solution of (1) without diffusion (*i.e.* of the ODE $\dot{x} = N(x)$), or a stable pulse-type travelling wave solution of (1) when O is a periodic spatial domain, as in [4]. Let the period of Γ under the flow of (1) be $T > 0$. Specifically, for $(t, x) \in [0, \infty) \times H$ we have

$$\gamma_{t+T} = \gamma_t, \quad \text{and} \quad \phi_s(\gamma_t) = \gamma_{s+t}, \quad \forall t \in \mathbb{R}, s \geq 0.$$

For a sufficiently small neighbourhood Γ_δ of Γ , define the isochron map $\pi : \Gamma_\delta \rightarrow [0, T]$ as in (36). The stochastic integral in (37), being a finite dimensional local Markov process with linear quadratic variation, has a time average equal to zero – see for instance [81] for details. Applying the second statement of Theorem 2.1, we find that

$$c_\sigma := \lim_{t \rightarrow \infty} \mathbb{E}_x \left[\frac{1}{t} \pi(X_t) \mid t < \tau \right] = \int_E \pi'(x) V(x) + \frac{\sigma^2}{2} \sum_{k \in \mathbb{N}} \pi''(x) [Be_k, Be_k] \beta(dx) \quad (39)$$

exists as a deterministic limit. We refer to this c_σ as the *quasi-asymptotic frequency* of (2) in Γ_δ .

Based on the properties of the isochron map, it can be shown as in [1] that

$$c_\sigma = c_0 + \sigma^2(b_0 + b_\sigma),$$

where $c_0 = 1/T$ is the frequency of Γ under the deterministic flow of (1), and b_0 is a constant independent of the noise amplitude. Moreover,

$$b_\sigma = \frac{1}{2} \int_{\Gamma_\delta} \sum_{k \in \mathbb{N}} \pi''(x) [Be_k, Be_k] \delta_\sigma(dx),$$

where $\delta_\sigma := \beta - \eta$ is the σ -dependent difference measure of the quasi-ergodic measure β and the deterministic invariant measure η of Γ under the deterministic flow of (1). Thus, we see that the quasi-asymptotic frequency c_σ depends “almost” quadratically on σ in some interval $[\sigma_0, \sigma_1] \subset [0, \infty)$ if and only if $db_\sigma/d\sigma$ is “almost” zero for all $\sigma \in [\sigma_0, \sigma_1]$. This might be considered a refinement of the results of Giacomin, Poquet, & Shapira [43], who suggest that this asymptotic frequency should depend quadratically on $\sigma > 0$. The approach taken here allows for a straightforward of their results to the infinite dimensional setting.

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