CHAPTER 8
Integration Techniques, L'Hôpital's Rule, and Improper Integrals

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Section 8.1 Basic Integration Rules

1. (a) $\frac{d}{dx}\left[2\sqrt{x^2+1}+C\right] = 2\left(\frac{1}{2}\right)(x^2+1)^{-1/2}(2x)$

$= \frac{2x}{\sqrt{x^2+1}}$

(b) $\frac{d}{dx}\left[\sqrt{x^2+1}+C\right] = \frac{1}{2}(x^2+1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+1}}$

(c) $\frac{d}{dx}\left[\frac{1}{2}\sqrt{x^2+1}+C\right] = \frac{1}{2}\left(\frac{1}{2}\right)(x^2+1)^{-1/2}(2x)$

$= \frac{x}{2\sqrt{x^2+1}}$

(d) $\frac{d}{dx}[\ln(x^2+1)+C] = \frac{2x}{x^2+1}$

$\int \frac{x}{\sqrt{x^2+1}}\,dx$ matches (b).

2. (a) $\frac{d}{dx}\left[\ln\sqrt{x^2+1}+C\right] = \frac{1}{2}\left(\frac{2x}{x^2+1}\right) = \frac{x}{x^2+1}$

(b) $\frac{d}{dx}\left[\frac{2x}{(x^2+1)^2}+C\right] = \frac{(x^2+1)^2(2)-(2x)(2)(x^2+1)(2x)}{(x^2+1)^4}$

$= \frac{2(1-3x^2)}{(x^2+1)^3}$

(c) $\frac{d}{dx}[\arctan x + C] = \frac{1}{1+x^2}$

(d) $\frac{d}{dx}[\ln(x^2+1)+C] = \frac{2x}{x^2+1}$

$\int \frac{x}{x^2+1}\,dx$ matches (a).

3. (a) $\frac{d}{dx}\left[\ln\sqrt{x^2+1}+C\right] = \frac{1}{2}\left(\frac{2x}{x^2+1}\right) = \frac{x}{x^2+1}$

(b) $\frac{d}{dx}\left[\frac{2x}{(x^2+1)^2}+C\right] = \frac{(x^2+1)^2(2)-(2x)(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{2(1-3x^2)}{(x^2+1)^3}$

(c) $\frac{d}{dx}[\arctan x + C] = \frac{1}{1+x^2}$

(d) $\frac{d}{dx}[\ln(x^2+1)+C] = \frac{2x}{x^2+1}$

$\int \frac{1}{x^2+1}\,dx$ matches (c).

4. (a) $\frac{d}{dx}[2x\sin(x^2+1)+C)] = 2x[\cos(x^2+1)(2x)] + 2\sin(x^2+1) = 2[2x^2\cos(x^2+1)+\sin(x^2+1)]$

(b) $\frac{d}{dx}\left[-\frac{1}{2}\sin(x^2+1)+C\right] = -\frac{1}{2}\cos(x^2+1)(2x) = -x\cos(x^2+1)$

(c) $\frac{d}{dx}\left[\frac{1}{2}\sin(x^2+1)+C\right] = \frac{1}{2}\cos(x^2+1)(2x) = x\cos(x^2+1)$

(d) $\frac{d}{dx}[-2x\sin(x^2+1)+C] = -2x[\cos(x^2+1)(2x)] - 2\sin(x^2+1) = -2[2x^2\cos(x^2+1)+\sin(x^2+1)]$

$\int x\cos(x^2+1)\,dx$ matches (c).

5. $\int (3x - 2)^4\, dx$

$u = 3x - 2,\ du = 3\,dx,\ n = 4$

Use $\int u^n\, du$.

6. $\int \frac{2t - 1}{t^2 - t + 2}\, dt$

$u = t^2 - t + 2,\ du = (2t - 1)\, dt$

Use $\int \frac{du}{u}$.

7. $\int \frac{1}{\sqrt{x}\left(1 - 2\sqrt{x}\right)}\, dx$

$u = 1 - 2\sqrt{x},\ du = -\frac{1}{\sqrt{x}}\, dx$

Use $\int \frac{du}{u}$.

8. $\int \frac{2}{(2t - 1)^2 + 4}\, dt$

$u = 2t - 1,\ du = 2\,dt,\ a = 2$

Use $\int \frac{du}{u^2 + a^2}$.

9. $\int \frac{3}{\sqrt{1 - t^2}}\, dt$

$u = t,\ du = dt,\ a = 1$

Use $\int \frac{du}{\sqrt{a^2 - u^2}}$.

10. $\int \frac{-2x}{\sqrt{x^2 - 4}}\, dx$

$u = x^2 - 4,\ du = 2x\,dx,\ n = -\frac{1}{2}$

Use $\int u^n\, du$.

11. $\int t \sin t^2\, dt$

$u = t^2,\ du = 2t\,dt$

Use $\int \sin u\, du$.

12. $\int \sec 3x \tan 3x\, dx$

$u = 3x,\ du = 3\,dx$

Use $\int \sec u \tan u\, du$.

13. $\int (\cos x) e^{\sin x}\, dx$

$u = \sin x,\ du = \cos x\, dx$

Use $\int e^u\, du$.

14. $\int \frac{1}{x\sqrt{x^2 - 4}}\, dx$

$u = x,\ du = dx,\ a = 2$

Use $\int \frac{du}{u\sqrt{u^2 - a^2}}$.

15. Let $u = x - 4,\ du = dx$.

$$\int 6(x - 4)^5\, dx = 6\int (x - 4)^5\, dx = 6\,\frac{(x - 4)^6}{6} + C$$
$$= (x - 4)^6 + C$$

16. Let $u = t - 9,\ du = dt$.

$$\int \frac{2}{(t - 9)^2}\, dt = 2\int (t - 9)^{-2}\, dt = \frac{-2}{t - 9} + C$$

17. Let $u = z - 4,\ du = dz$.

$$\int \frac{5}{(z - 4)^5}\, dz = 5\int (z - 4)^{-5}\, dz = 5\,\frac{(z - 4)^{-4}}{-4} + C$$
$$= \frac{-5}{4(z - 4)^4} + C$$

18. Let $u = t^3 - 1,\ du = 3t^2\, dt$.

$$\int t^2\sqrt[3]{t^3 - 1}\, dt = \frac{1}{3}\int (t^3 - 1)^{1/3}(3t^2)\, dt$$
$$= \frac{1}{3}\,\frac{(t^3 - 1)^{4/3}}{4/3} + C$$
$$= \frac{(t^3 - 1)^{4/3}}{4} + C$$

19. $$\int \left[v + \frac{1}{(3v - 1)^3}\right] dv = \int v\, dv + \frac{1}{3}\int (3v - 1)^{-3}(3)\, dv$$
$$= \frac{1}{2}v^2 - \frac{1}{6(3v - 1)^2} + C$$

20. $$\int \left[x - \frac{3}{(2x + 3)^2}\right] dx = \int x\, dx - \frac{3}{2}\int (2x + 3)^{-2}(2)\, dx$$
$$= \frac{x^2}{2} - \frac{3}{2}\,\frac{(2x + 3)^{-1}}{-1} + C$$
$$= \frac{x^2}{2} + \frac{3}{2(2x + 3)} + C$$

21. Let $u = -t^3 + 9t + 1,\ du = (-3t^2 + 9)\, dt = -3(t^2 - 3)\, dt$.

$$\int \frac{t^2 - 3}{-t^3 + 9t + 1}\, dt = -\frac{1}{3}\int \frac{-3(t^2 - 3)}{-t^3 + 9t + 1}\, dt$$
$$= -\frac{1}{3}\ln\left|-t^3 + 9t + 1\right| + C$$

22. Let $u = x^2 + 2x - 4$, $du = 2(x + 1)\,dx$.

$$\int \frac{x+1}{\sqrt{x^2 + 2x - 4}}\,dx = \frac{1}{2}\int (x^2 + 2x - 4)^{-1/2}(2)(x+1)\,dx$$
$$= \sqrt{x^2 + 2x - 4} + C$$

23. $$\int \frac{x^2}{x-1}\,dx = \int (x+1)\,dx + \int \frac{1}{x-1}\,dx$$
$$= \frac{1}{2}x^2 + x + \ln|x - 1| + C$$

24. $$\int \frac{2x}{x-4}\,dx = \int 2\,dx + \int \frac{8}{x-4}\,dx$$
$$= 2x + 8\ln|x - 4| + C$$

25. Let $u = 1 + e^x$, $du = e^x\,dx$.

$$\int \frac{e^x}{1+e^x}\,dx = \ln(1 + e^x) + C$$

26. $$\int \left(\frac{1}{3x-1} - \frac{1}{3x+1}\right)dx = \frac{1}{3}\int \frac{1}{3x-1}(3)\,dx - \frac{1}{3}\int \frac{1}{3x+1}(3)\,dx$$
$$= \frac{1}{3}\ln|3x - 1| - \frac{1}{3}\ln|3x + 1| + C = \frac{1}{3}\ln\left|\frac{3x-1}{3x+1}\right| + C$$

27. $$\int (1 + 2x^2)^2\,dx = \int (4x^4 + 4x^2 + 1)\,dx = \frac{4}{5}x^5 + \frac{4}{3}x^3 + x + C = \frac{x}{15}(12x^4 + 20x^2 + 15) + C$$

28. $$\int x\left(1 + \frac{1}{x}\right)^3 = \int x\left(1 + \frac{3}{x} + \frac{3}{x^2} + \frac{1}{x^3}\right)dx = \int \left(x + 3 + \frac{3}{x} + \frac{1}{x^2}\right)dx = \frac{1}{2}x^2 + 3x + 3\ln|x| - \frac{1}{x} + C$$

29. Let $u = 2\pi x^2$, $du = 4\pi x\,dx$.

$$\int x(\cos 2\pi x^2)\,dx = \frac{1}{4\pi}\int (\cos 2\pi x^2)(4\pi x)\,dx$$
$$= \frac{1}{4\pi}\sin 2\pi x^2 + C$$

30. $$\int \sec 4x\,dx = \frac{1}{4}\int \sec(4x)(4)\,dx$$
$$= \frac{1}{4}\ln|\sec 4x + \tan 4x| + C$$

31. Let $u = \pi x$, $du = \pi\,dx$.

$$\int \csc(\pi x)\cot(\pi x)\,dx = \frac{1}{\pi}\int \csc(\pi x)\cot(\pi x)\pi\,dx$$
$$= -\frac{1}{\pi}\csc(\pi x) + C$$

32. Let $u = \cos x$, $du = -\sin x\,dx$.

$$\int \frac{\sin x}{\sqrt{\cos x}}\,dx = -\int (\cos x)^{-1/2}(-\sin x)\,dx$$
$$= -2\sqrt{\cos x} + C$$

33. Let $u = 5x$, $du = 5\,dx$.

$$\int e^{5x}\,dx = \frac{1}{5}\int e^{5x}(5)\,dx = \frac{1}{5}e^{5x} + C$$

34. Let $u = \cot x$, $du = -\csc^2 x\,dx$.

$$\int \csc^2 x e^{\cot x}\,dx = -\int e^{\cot x}(-\csc^2 x)\,dx = -e^{\cot x} + C$$

35. Let $u = 1 + e^x$, $du = e^x\,dx$.

$$\int \frac{2}{e^{-x}+1}\,dx = 2\int \left(\frac{1}{e^{-x}+1}\right)\left(\frac{e^x}{e^x}\right)dx$$
$$= 2\int \frac{e^x}{1+e^x}\,dx$$
$$= 2\ln(1 + e^x) + C$$

36. $$\int \frac{5}{3e^x - 2}\,dx = 5\int \left(\frac{1}{3e^x - 2}\right)\left(\frac{e^{-x}}{e^{-x}}\right)dx$$
$$= 5\int \frac{e^{-x}}{3 - 2e^{-x}}\,dx$$
$$= \frac{5}{2}\int \frac{1}{3 - 2e^{-x}}(2e^{-x})\,dx$$
$$= \frac{5}{2}\ln|3 - 2e^{-x}| + C$$

37. $\displaystyle\int \frac{\ln x^2}{x}\,dx = 2\int (\ln x)\frac{1}{x}\,dx = 2\,\frac{(\ln x)^2}{2} + C = (\ln x)^2 + C$

38. Let $u = \ln(\cos x)$, $du = \dfrac{-\sin x}{\cos x}\,dx = -\tan x\,dx$.

$$\int (\tan x)(\ln \cos x)\,dx = -\int (\ln \cos x)(-\tan x)\,dx = \frac{-[\ln(\cos x)]^2}{2} + C$$

39.
$$\begin{aligned}\int \frac{1+\sin x}{\cos x}\,dx &= \int \frac{1+\sin x}{\cos x}\cdot\frac{1-\sin x}{1-\sin x}\,dx\\ &= \int \frac{1-\sin^2 x}{\cos x(1-\sin x)}\,dx\\ &= \int \frac{\cos^2 x}{\cos x(1-\sin x)}\,dx\\ &= -\int \frac{-\cos x}{1-\sin x}\,dx\\ &= -\ln|1-\sin x| + C, \qquad (u = 1-\sin x)\end{aligned}$$

Alternate Solution:
$$\begin{aligned}\int \frac{1+\sin x}{\cos x}\,dx &= \int (\sec x + \tan x)\,dx\\ &= \ln|\sec x + \tan x| + \ln|\sec x| + C\\ &= \ln|\sec x(\sec x + \tan x)| + C\end{aligned}$$

40.
$$\begin{aligned}\int \frac{1+\cos\alpha}{\sin\alpha}\,d\alpha &= \int \csc\alpha\,d\alpha + \int \cot\alpha\,d\alpha\\ &= -\ln|\csc\alpha + \cot\alpha| + \ln|\sin\alpha| + C\end{aligned}$$

41.
$$\begin{aligned}\frac{1}{\cos\theta-1} &= \frac{1}{\cos\theta-1}\cdot\frac{\cos\theta+1}{\cos\theta+1} = \frac{\cos\theta+1}{\cos^2\theta-1}\\ &= \frac{\cos\theta+1}{-\sin^2\theta} = -\csc\theta\cdot\cot\theta - \csc^2\theta\end{aligned}$$
$$\begin{aligned}\int \frac{1}{\cos\theta-1}\,d\theta &= \int(-\csc\theta\cot\theta - \csc^2\theta)\,d\theta\\ &= \csc\theta + \cot\theta + C\\ &= \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} + C\\ &= \frac{1+\cos\theta}{\sin\theta} + C\end{aligned}$$

42.
$$\begin{aligned}\int \frac{2}{3(\sec x-1)}\,dx &= \frac{2}{3}\int \frac{1}{\sec x-1}\cdot\left(\frac{\sec x+1}{\sec x+1}\right)dx\\ &= \frac{2}{3}\int \frac{\sec x+1}{\tan^2 x}\,dx\\ &= \frac{2}{3}\int \frac{\sec x}{\tan^2 x}\,dx + \frac{2}{3}\int \cot^2 x\,dx\\ &= \frac{2}{3}\int \frac{\cos x}{\sin^2 x}\,dx + \frac{2}{3}\int(\csc^2 x - 1)\,dx\\ &= \frac{2}{3}\left(-\frac{1}{\sin x}\right) - \frac{2}{3}\cot x - \frac{2}{3}x + C\\ &= -\frac{2}{3}[\csc x + \cot x + x] + C\end{aligned}$$

43. Let $u = 2t - 1$, $du = 2\,dt$.
$$\begin{aligned}\int \frac{-1}{\sqrt{1-(2t-1)^2}}\,dt &= -\frac{1}{2}\int \frac{2}{\sqrt{1-(2t-1)^2}}\,dt\\ &= -\frac{1}{2}\arcsin(2t-1) + C\end{aligned}$$

44. Let $u = \sqrt{3}x$, $du = \sqrt{3}\,dx$.
$$\begin{aligned}\int \frac{1}{4+3x^2}\,dx &= \frac{1}{\sqrt{3}}\int \frac{\sqrt{3}}{4+(\sqrt{3}x)^2}\,dx\\ &= \frac{1}{2\sqrt{3}}\arctan\left(\frac{\sqrt{3}x}{2}\right) + C\end{aligned}$$

45. Let $u = \cos\left(\dfrac{2}{t}\right)$, $du = \dfrac{2\sin(2/t)}{t^2}\,dt$.
$$\begin{aligned}\int \frac{\tan(2/t)}{t^2}\,dt &= \frac{1}{2}\int \frac{1}{\cos(2/t)}\left[\frac{2\sin(2/t)}{t^2}\right]dt\\ &= \frac{1}{2}\ln\left|\cos\left(\frac{2}{t}\right)\right| + C\end{aligned}$$

46. Let $u = \dfrac{1}{t}$, $du = \dfrac{-1}{t^2}\,dt$.
$$\int \frac{e^{1/t}}{t^2}\,dt = -\int e^{1/t}\left(\frac{-1}{t^2}\right)dt = -e^{1/t} + C$$

47. $\displaystyle\int \frac{3}{\sqrt{6x - x^2}}\,dx = 3\int \frac{1}{\sqrt{9 - (x-3)^2}}\,dx = 3\arcsin\left(\frac{x-3}{3}\right) + C$

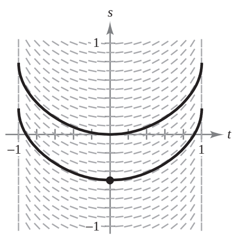
48. $\displaystyle\int \frac{1}{(x-1)\sqrt{4x^2 - 8x + 3}}\,dx = \int \frac{2}{[2(x-1)]\sqrt{[2(x-1)]^2 - 1}}\,dx = \operatorname{arcsec}|2(x-1)| + C$

49. $\displaystyle\int \frac{4}{4x^2 + 4x + 65}\,dx = \int \frac{1}{[x + (1/2)]^2 + 16}\,dx = \frac{1}{4}\arctan\left[\frac{x + (1/2)}{4}\right] + C = \frac{1}{4}\arctan\left(\frac{2x+1}{8}\right) + C$

50. $\displaystyle\int \frac{1}{\sqrt{1 - 4x - x^2}}\,dx = \int \frac{1}{\sqrt{5 - (x^2 + 4x + 4)}}\,dx = \int \frac{1}{\sqrt{5 - (x+2)^2}}\,dx = \arcsin\left(\frac{x+2}{\sqrt{5}}\right) + C, \quad \left(a = \sqrt{5}\right)$

51. $\displaystyle\frac{ds}{dt} = \frac{t}{\sqrt{1 - t^4}}, \quad \left(0, -\frac{1}{2}\right)$

(a)



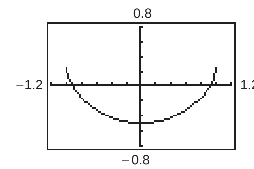
(b) $u = t^2$, $du = 2t\,dt$

$\displaystyle\int \frac{t}{\sqrt{1 - t^4}}\,dt = \frac{1}{2}\int \frac{2t}{\sqrt{1 - (t^2)^2}}\,dt$

$\displaystyle = \frac{1}{2}\arcsin t^2 + C$

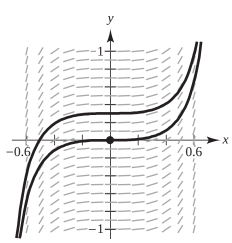
$\displaystyle\left(0, -\frac{1}{2}\right)\!: \ -\frac{1}{2} = \frac{1}{2}\arcsin 0 + C \implies C = -\frac{1}{2}$

$\displaystyle s = \frac{1}{2}\arcsin t^2 - \frac{1}{2}$



52. $\displaystyle\frac{dy}{dx} = \tan^2(2x), \quad (0, 0)$

(a)

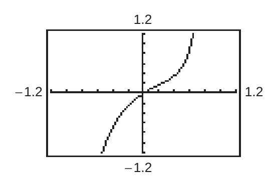


(b) $\displaystyle\int \tan^2(2x)\,dx = \int (\sec^2(2x) - 1)\,dx$

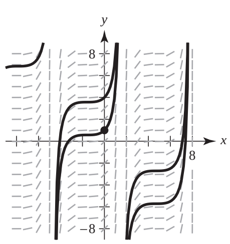
$\displaystyle = \frac{1}{2}\tan(2x) - x + C$

$(0, 0)$: $0 = C$

$\displaystyle y = \frac{1}{2}\tan(2x) - x$



53. (a)



(b) $\displaystyle y = \int (\sec x + \tan x)^2\,dx$

$\displaystyle = \int (\sec^2 x + 2\sec x\tan x + \tan^2 x)\,dx$

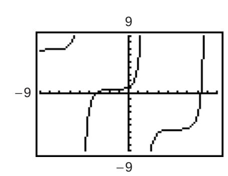
$\displaystyle = \int (\sec^2 x + 2\sec x\tan x + (\sec^2 x - 1))\,dx$

$\displaystyle = \int (2\sec^2 x + 2\sec x\tan x - 1)\,dx$

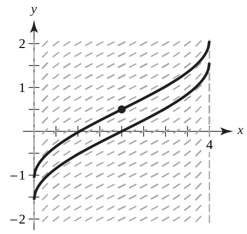
$= 2\tan x + 2\sec x - x + C$

At $(0, 1)$: $1 = 0 + 2 - 0 + C \implies C = -1$

$y = 2\tan x + 2\sec x - x - 1$



54. (a)



(b) $y = \int \frac{1}{\sqrt{4x - x^2}}\,dx$

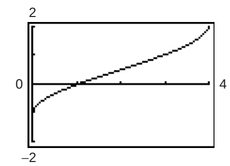
$= \int \frac{1}{\sqrt{4 - (x^2 - 4x + 4)}}\,dx$

$= \int \frac{1}{\sqrt{4 - (x - 2)^2}}\,dx$

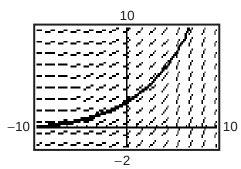
$= \arcsin\left(\frac{x - 2}{2}\right) + C$

At $\left(2, \frac{1}{2}\right)$: $\frac{1}{2} = \arcsin(0) + C \Rightarrow C = \frac{1}{2}$

$y = \arcsin\left(\frac{x - 2}{2}\right) + \frac{1}{2}$

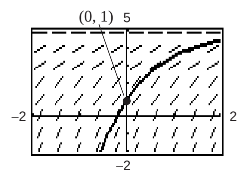


55.



$y = 3e^{0.2x}$

56.



$y = 5 - 4e^{-x}$

57. $y = \int (1 + e^x)^2\,dx$

$= \int (e^{2x} + 2e^x + 1)\,dx$

$= \frac{1}{2}e^{2x} + 2e^x + x + C$

58. $r = \int \frac{(1 + e^t)^2}{e^t}\,dt = \int \frac{1 + 2e^t + e^{2t}}{e^t}\,dt$

$= \int (e^{-t} + 2 + e^t)\,dt = -e^{-t} + 2t + e^t + C$

59. $\frac{dy}{dx} = \frac{\sec^2 x}{4 + \tan^2 x}$

Let $u = \tan x$, $du = \sec^2 x\,dx$.

$y = \int \frac{\sec^2 x}{4 + \tan^2 x}\,dx = \frac{1}{2}\arctan\left(\frac{\tan x}{2}\right) + C$

60. Let $u = 2x$, $du = 2\,dx$.

$y = \int \frac{1}{x\sqrt{4x^2 - 1}}\,dx = \int \frac{2}{2x\sqrt{(2x)^2 - 1}}\,dx$

$= \operatorname{arcsec}|2x| + C$

61. Let $u = 2x$, $du = 2\,dx$.

$\int_0^{\pi/4} \cos 2x\,dx = \frac{1}{2}\int_0^{\pi/4} \cos 2x(2)\,dx$

$= \left[\frac{1}{2}\sin 2x\right]_0^{\pi/4} = \frac{1}{2}$

62. Let $u = \sin t$, $du = \cos t\,dt$.

$\int_0^{\pi} \sin^2 t \cos t\,dt = \left[\frac{1}{3}\sin^3 t\right]_0^{\pi} = 0$

63. Let $u = -x^2$, $du = -2x\,dx$.

$\int_0^1 xe^{-x^2}\,dx = -\frac{1}{2}\int_0^1 e^{-x^2}(-2x)\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_0^1$

$= \frac{1}{2}(1 - e^{-1}) \approx 0.316$

64. Let $u = 1 - \ln x$, $du = \frac{-1}{x}\,dx$.

$\int_1^e \frac{1 - \ln x}{x}\,dx = -\int_1^e (1 - \ln x)\left(\frac{-1}{x}\right)dx$

$= \left[-\frac{1}{2}(1 - \ln x)^2\right]_1^e = \frac{1}{2}$

65. Let $u = x^2 + 9$, $du = 2x\,dx$.

$\int_0^4 \frac{2x}{\sqrt{x^2 + 9}}\,dx = \int_0^4 (x^2 + 9)^{-1/2}(2x)\,dx$

$= \left[2\sqrt{x^2 + 9}\right]_0^4 = 4$

66. $\int_1^2 \frac{x-2}{x}\,dx = \int_1^2 \left(1 - \frac{2}{x}\right) dx$

$= \left[x - 2\ln x\right]_1^2 = 1 - \ln 4 \approx -0.386$

67. Let $u = 3x$, $du = 3\,dx$.

$\int_0^{2/\sqrt{3}} \frac{1}{4+9x^2}\,dx = \frac{1}{3}\int_0^{2/\sqrt{3}} \frac{3}{4+(3x)^2}\,dx$

$= \left[\frac{1}{6}\arctan\left(\frac{3x}{2}\right)\right]_0^{2/\sqrt{3}}$

$= \frac{\pi}{18} \approx 0.175$

68. $\int_0^4 \frac{1}{\sqrt{25-x^2}}\,dx = \left[\arcsin\frac{x}{5}\right]_0^4 = \arcsin\frac{4}{5} \approx 0.927$

69. $A = \int_0^{5/2} (-2x+5)^{3/2}\,dx$

$= -\frac{1}{2}\int_0^{5/2} (5-2x)^{3/2}(-2)\,dx$

$= -\frac{1}{5}(5-2x)^{5/2}\Big]_0^{5/2}$

$= 0 + \frac{1}{5}(5)^{5/2} = 5^{3/2}$

$= 5\sqrt{5} \approx 11.1803$

70. $A = \int_0^2 x\sqrt{8-2x^2}\,dx$

$= -\frac{1}{4}\int_0^2 (8-2x^2)^{1/2}(-4x)\,dx$

$= -\frac{1}{6}(8-2x^2)^{3/2}\Big]_0^2$

$= 0 + \frac{1}{6}(8)^{3/2}$

$= \frac{8\sqrt{2}}{3} \approx 3.7712$

71. $A = \int_0^5 \frac{3x+2}{x^2+9}\,dx$

$= \int_0^5 \frac{3x}{x^2+9}\,dx + \int_0^5 \frac{2}{x^2+9}\,dx$

$= \left[\frac{3}{2}\ln|x^2+9| + \frac{2}{3}\arctan\left(\frac{x}{3}\right)\right]_0^5$

$= \frac{3}{2}\ln(34) + \frac{2}{3}\arctan\left(\frac{5}{3}\right) - \frac{3}{2}\ln 9$

$= \frac{3}{2}\ln\left(\frac{34}{9}\right) + \frac{2}{3}\arctan\left(\frac{5}{3}\right)$

≈ 2.6806

72. $A = \int_{-3}^3 \frac{3}{x^2+1}\,dx$

$= 2\int_0^3 \frac{3}{x^2+1}\,dx$

$= 6\arctan(x)\Big]_0^3$

$= 6\arctan(3)$

≈ 7.4943

73. $y^2 = x^2(1-x^2)$

$y = \pm\sqrt{x^2(1-x^2)}$

$A = 4\int_0^1 x\sqrt{1-x^2}\,dx$

$= -2\int_0^1 (1-x^2)^{1/2}(-2x)\,dx$

$= -\frac{4}{3}(1-x)^{3/2}\Big]_0^1$

$= -\frac{4}{3}(0-1) = \frac{4}{3}$

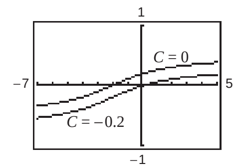
74. $A = \int_0^{\pi/2} \sin 2x\,dx$

$= -\frac{1}{2}\cos 2x\Big]_0^{\pi/2}$

$= -\frac{1}{2}(-1-1) = 1$

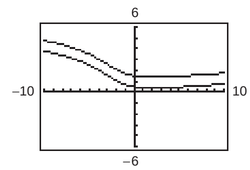
75. $\displaystyle\int \frac{1}{x^2 + 4x + 13}\,dx = \frac{1}{3}\arctan\left(\frac{x+2}{3}\right) + C$

The antiderivatives are vertical translations of each other.



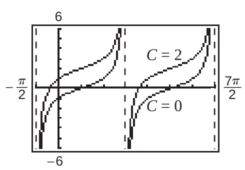
76. $\displaystyle\int \frac{x-2}{x^2 + 4x + 13}\,dx = \frac{1}{2}\ln(x^2 + 4x + 13) - \frac{4}{3}\arctan\left(\frac{x+2}{3}\right) + C$

The antiderivatives are vertical translations of each other.



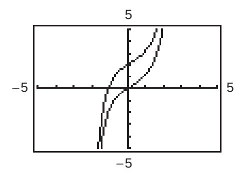
77. $\displaystyle\int \frac{1}{1 + \sin\theta}\,d\theta = \tan\theta - \sec\theta + C \quad \left(\text{or } \frac{-2}{1 + \tan(\theta/2)}\right)$

The antiderivatives are vertical translations of each other.



78. $\displaystyle\int \left(\frac{e^x + e^{-x}}{2}\right)^3 dx = \frac{1}{24}[e^{3x} + 9e^x - 9e^{-x} - e^{-3x}] + C$

The antiderivatives are vertical translations of each other.



79. Power Rule: $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$

$u = x^2 + 1, n = 3$

80. $\displaystyle\int \sec u \tan u\,du = \sec u + C$

81. Log Rule: $\displaystyle\int \frac{du}{u} = \ln|u| + C, \quad u = x^2 + 1$

82. Arctan Rule: $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$

83. They are equivalent because

$e^{x + C_1} = e^x \cdot e^{C_1} = Ce^x, C = e^{C_1}.$

84. They differ by a constant.

$\sec^2 x + C_1 = (\tan^2 x + 1) + C_1 = \tan^2 x + C$

85. $\sin x + \cos x = a\sin(x + b)$

$\sin x + \cos x = a\sin x\cos b + a\cos x\sin b$

$\sin x + \cos x = (a\cos b)\sin x + (a\sin b)\cos x$

Equate coefficients of like terms to obtain the following.

$1 = a\cos b \quad \text{and} \quad 1 = a\sin b$

Thus, $a = 1/\cos b$. Now, substitute for a in $1 = a\sin b$.

$$1 = \left(\frac{1}{\cos b}\right)\sin b$$

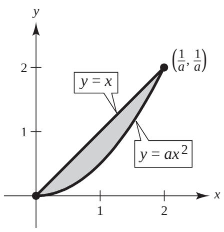
$$1 = \tan b \implies b = \frac{\pi}{4}$$

Since $b = \dfrac{\pi}{4}$, $a = \dfrac{1}{\cos(\pi/4)} = \sqrt{2}$. Thus, $\sin x + \cos x = \sqrt{2}\sin\left(x + \dfrac{\pi}{4}\right)$.

$$\int \frac{dx}{\sin x + \cos x} = \int \frac{dx}{\sqrt{2}\sin(x + (\pi/4))} = \frac{1}{\sqrt{2}}\int \csc\left(x + \frac{\pi}{4}\right)dx = -\frac{1}{\sqrt{2}}\ln\left|\csc\left(x + \frac{\pi}{4}\right) + \cot\left(x + \frac{\pi}{4}\right)\right| + C$$

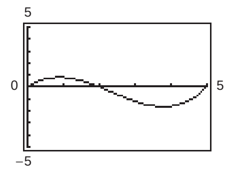
86. $\int_0^{1/a} (x - ax^2)\,dx = \left[\frac{1}{2}x^2 - \frac{a}{3}x^3\right]_0^{1/a} = \frac{1}{6a^2}$

Let $\frac{1}{6a^2} = \frac{2}{3}$, $12a^2 = 3$, $a = \frac{1}{2}$.



87. $f(x) = \frac{1}{5}(x^3 - 7x^2 + 10x)$

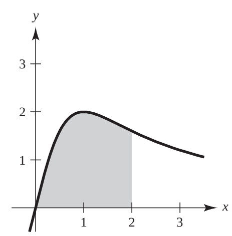
$\int_0^5 f(x)\,dx < 0$ because more area is below the x-axis than above.



88. No. When $u = x^2$, it does not follow that $x = \sqrt{u}$ since x is negative on $[-1, 0)$.

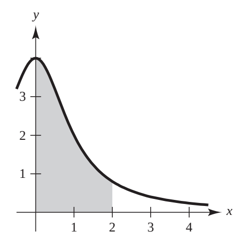
89. $\int_0^2 \frac{4x}{x^2 + 1}\,dx \approx 3$

Matches (a).

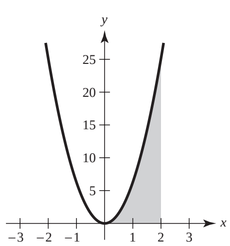


90. $\int_0^2 \frac{4}{x^2 + 1}\,dx \approx 4$

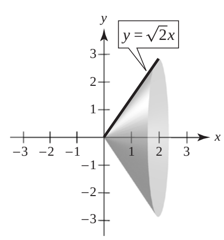
Matches (d).



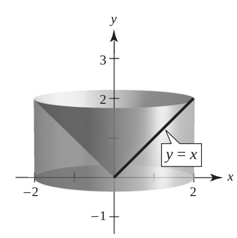
91. (a) $y = 2\pi x^2, \quad 0 \le x \le 2$



(b) $y = \sqrt{2}x, \quad 0 \le x \le 2$

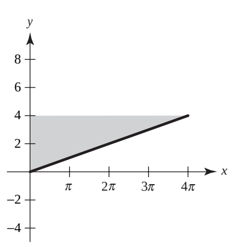


(c) $y = x, \quad 0 \le x \le 2$



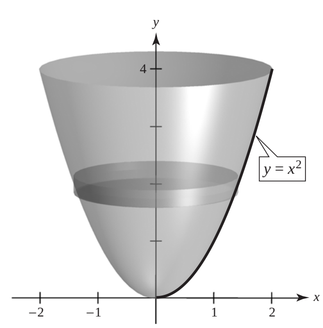
92. (a) $x = \pi y, \quad 0 \le y \le 4$

$y = \frac{1}{\pi}x, \quad 0 \le x \le 4\pi$



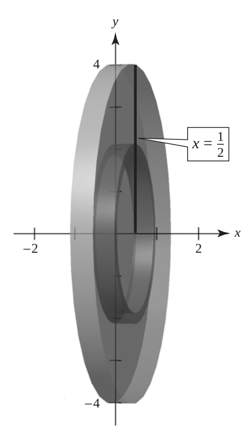
(b) $x = \sqrt{y}, \quad 0 \le y \le 4$

$y = x^2, \quad 0 \le x \le 2$



(c) $x = \frac{1}{2}, \quad 0 \le y \le 4$

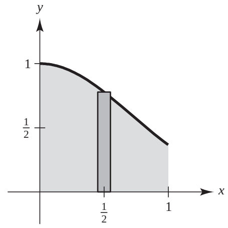
$2\pi\int_0^4 y\left(\frac{1}{2}\right) dy$



93. (a) **Shell Method:**

Let $u = -x^2$, $du = -2x\,dx$.

$$V = 2\pi\int_0^1 xe^{-x^2}\,dx$$
$$= -\pi\int_0^1 e^{-x^2}(-2x)\,dx$$
$$= \left[-\pi e^{-x^2}\right]_0^1$$
$$= \pi(1 - e^{-1}) \approx 1.986$$

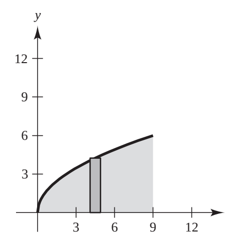


(b) **Shell Method:**

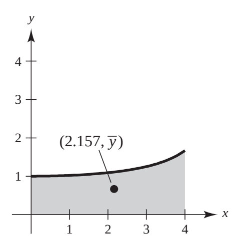
$$V = 2\pi\int_0^b xe^{-x^2}\,dx$$
$$= \left[-\pi e^{-x^2}\right]_0^b$$
$$= \pi(1 - e^{-b^2}) = \frac{4}{3}$$
$$e^{-b^2} = \frac{3\pi - 4}{3\pi}$$
$$b = \sqrt{\ln\left(\frac{3\pi}{3\pi - 4}\right)} \approx 0.743$$

94.
$$y = f(x) = \ln(\sin x)$$
$$f'(x) = \frac{\cos x}{\sin x}$$
$$s = \int_{\pi/4}^{\pi/2}\sqrt{1 + \frac{\cos^2 x}{\sin^2 x}}\,dx = \int_{\pi/4}^{\pi/2}\sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}}\,dx$$
$$= \int_{\pi/4}^{\pi/2}\frac{1}{\sin x}\,dx = \int_{\pi/4}^{\pi/2}\csc x\,dx$$
$$= -\ln\left|\csc x + \cot x\right|\Big]_{\pi/4}^{\pi/2}$$
$$= -\ln(1) + \ln\left(\sqrt{2} + 1\right)$$
$$= \ln\left(\sqrt{2} + 1\right) \approx 0.8814$$

95.
$$y = 2\sqrt{x}$$
$$y' = \frac{1}{\sqrt{x}}$$
$$1 + (y')^2 = 1 + \frac{1}{x} = \frac{x + 1}{x}$$
$$S = 2\pi\int_0^9 2\sqrt{x}\sqrt{\frac{x+1}{x}}\,dx$$
$$= 2\pi\int_0^9 2\sqrt{x + 1}\,dx$$
$$= \left[4\pi\left(\frac{2}{3}\right)(x + 1)^{3/2}\right]_0^9$$
$$= \frac{8\pi}{3}\left(10\sqrt{10} - 1\right)$$
$$\approx 256.545$$



96.
$$A = \int_0^4 \frac{5}{\sqrt{25 - x^2}}\,dx = \left[5\arcsin\frac{x}{5}\right]_0^4 = 5\arcsin\frac{4}{5}$$
$$\bar{x} = \frac{1}{A}\int_0^4 x\left(\frac{5}{\sqrt{25 - x^2}}\right)dx$$
$$= \frac{1}{5\arcsin(4/5)}\left(-\frac{5}{2}\right)\int_0^4 (25 - x^2)^{-1/2}(-2x)\,dx$$
$$= \frac{1}{5\arcsin(4/5)}(-5)\left[(25 - x^2)^{1/2}\right]_0^4$$
$$= -\frac{1}{\arcsin(4/5)}[3 - 5] = \frac{2}{\arcsin(4/5)} \approx 2.157$$



97. Average value
$$= \frac{1}{b - a}\int_a^b f(x)\,dx$$
$$= \frac{1}{3 - (-3)}\int_{-3}^3 \frac{1}{1 + x^2}\,dx$$
$$= \frac{1}{6}\arctan(x)\Big]_{-3}^3$$
$$= \frac{1}{6}[\arctan(3) - \arctan(-3)]$$
$$= \frac{1}{3}\arctan(3) \approx 0.4163$$

98. Average value
$$= \frac{1}{b - a}\int_a^b f(x)\,dx$$
$$= \frac{1}{(\pi/n) - 0}\int_0^{\pi/n}\sin(nx)\,dx$$
$$= \frac{n}{\pi}\left[\frac{-1}{n}\cos(nx)\right]_0^{\pi/n}$$
$$= -\frac{1}{\pi}[\cos(\pi) - \cos(0)]$$
$$= \frac{2}{\pi}$$

99.
$$y = \tan(\pi x)$$
$$y' = \pi \sec^2(\pi x)$$
$$1 + (y')^2 = 1 + \pi^2 \sec^4(\pi x)$$
$$s = \int_0^{1/4} \sqrt{1 + \pi^2 \sec^4(\pi x)}\, dx \approx 1.0320$$

100.
$$y = x^{2/3}$$
$$y' = \frac{2}{3x^{1/3}}$$
$$1 + (y')^2 = 1 + \frac{4}{9x^{2/3}}$$
$$s = \int_1^8 \sqrt{1 + \frac{4}{9x^{2/3}}}\, dx \approx 7.6337$$

101. (a) $$\int \cos^3 x\, dx = \int (1 - \sin^2 x) \cos x\, dx = \sin x - \frac{\sin^3 x}{3} + C$$

(b) $$\int \cos^5 x\, dx = \int (1 - \sin^2 x)^2 \cos x\, dx = \int (1 - 2\sin^2 x + \sin^4 x) \cos x\, dx = \sin x - \frac{2}{3}\sin^3 x + \frac{\sin^5 x}{5} + C$$

(c) $$\int \cos^7 x\, dx = \int (1 - \sin^2 x)^3 \cos x\, dx = \int (1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x) \cos x\, dx = \sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C$$

(d) $$\int \cos^{15} x\, dx = \int (1 - \cos^2 x)^7 \cos x\, dx$$

You would expand $(1 - \cos^2 x)^7$.

102. (a) $$\int \tan^3 x\, dx = \int (\sec^2 x - 1) \tan x\, dx = \int \sec^2 x \tan x\, dx - \int \tan x\, dx = \frac{\tan^2 x}{2} - \int \tan x\, dx$$
$$\int \tan^3 x\, dx = \frac{\tan^2 x}{2} + \ln|\cos x| + C$$

(b) $$\int \tan^5 x\, dx = \int (\sec^2 x - 1) \tan^3 x\, dx = \frac{\tan^4 x}{4} - \int \tan^3 x\, dx$$

(c) $$\int \tan^{2k+1} x\, dx = \int (\sec^2 x - 1) \tan^{2k-1} x\, dx = \frac{\tan^{2k} x}{2k} - \int \tan^{2k-1} x\, dx$$

(d) You would use these formulas recursively.

103. Let $f(x) = \frac{1}{2}\left(x\sqrt{x^2 + 1} + \ln\left|x + \sqrt{x^2 + 1}\right|\right) + C.$

$$f'(x) = \frac{1}{2}\left(x\frac{1}{2}(x^2 + 1)^{-1/2}(2x) + \sqrt{x^2 + 1} + \frac{1}{x + \sqrt{x^2 + 1}}\left(1 + \frac{1}{2}(x^2 + 1)^{-1/2}(2x)\right)\right)$$
$$= \frac{1}{2}\left(\frac{x^2}{\sqrt{x^2 + 1}} + \sqrt{x^2 + 1} + \frac{1}{x + \sqrt{x^2 + 1}}\left(1 + \frac{x}{\sqrt{x^2 + 1}}\right)\right)$$
$$= \frac{1}{2}\left(\frac{x^2 + (x^2 + 1)}{\sqrt{x^2 + 1}} + \frac{1}{x + \sqrt{x^2 + 1}}\left(\frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}}\right)\right)$$
$$= \frac{1}{2}\left(\frac{2x^2 + 1}{\sqrt{x^2 + 1}} + \frac{1}{\sqrt{x^2 + 1}}\right) = \frac{1}{2}\left(\frac{2(x^2 + 1)}{\sqrt{x^2 + 1}}\right) = \sqrt{x^2 + 1}$$

Thus, $\displaystyle\int \sqrt{x^2 + 1}\, dx = \frac{1}{2}\left(x\sqrt{x^2 + 1} + \ln\left|x + \sqrt{x^2 + 1}\right|\right) + C.$

—CONTINUED—

103. —CONTINUED—

Let $g(x) = \frac{1}{2}\left(x\sqrt{x^2+1} + \operatorname{arcsinh}(x)\right)$.

$$g'(x) = \frac{1}{2}\left(x\frac{1}{2}(x^2+1)^{-1/2}(2x) + \sqrt{x^2+1} + \frac{1}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{2}\left(\frac{x^2}{\sqrt{x^2+1}} + \sqrt{x^2+1} + \frac{1}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{2}\left(\frac{x^2+(x^2+1)+1}{\sqrt{x^2+1}}\right)$$

$$= \frac{1}{2}\left(\frac{2(x^2+1)}{\sqrt{x^2+1}}\right) = \sqrt{x^2+1}$$

Thus, $\int \sqrt{x^2+1}\,dx = \frac{1}{2}\left(x\sqrt{x^2+1} + \operatorname{arcsinh}(x)\right) + C$.

104. Let $I = \int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}\,dx$.

I is defined and continuous on $[2, 4]$. Note the symmetry: as x goes from 2 to 4, $9 - x$ goes from 7 to 5 and $x + 3$ goes from 5 to 7. So, let $y = 6 - x$, $dy = -dx$.

$$I = \int_4^2 \frac{\sqrt{\ln(3+y)}}{\sqrt{\ln(3+y)} + \sqrt{\ln(9-y)}}(-dy) = \int_2^4 \frac{\sqrt{\ln(3+y)}}{\sqrt{\ln(3+y)} + \sqrt{\ln(9-y)}}\,dy$$

Adding:

$$2I = \int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}\,dx + \int_2^4 \frac{\sqrt{\ln(3+x)}}{\sqrt{\ln(3+x)} + \sqrt{\ln(9-x)}}\,dx = \int_2^4 dx = 2 \implies I = 1$$

You can easily check this result numerically.

Section 8.2 Integration by Parts

1. $\frac{d}{dx}[\sin x - x\cos x] = \cos x - (-x\sin x + \cos x) = x\sin x$

Matches (b)

2. $\frac{d}{dx}[x^2\sin x + 2x\cos x - 2\sin x] = x^2\cos x + 2x\sin x - 2x\sin x + 2\cos x - 2\cos x = x^2\cos x$

Matches (d)

3. $\frac{d}{dx}[x^2e^x - 2xe^x + 2e^x] = x^2e^x + 2xe^x - 2xe^x - 2e^x + 2e^x$

$= x^2e^x$

Matches (c)

4. $\frac{d}{dx}[-x + x\ln x] = -1 + x\left(\frac{1}{x}\right) + \ln x = \ln x$

Matches (a)

5. $\int xe^{2x}\,dx$

$u = x,\ dv = e^{2x}\,dx$

6. $\int x^2e^{2x}\,dx$

$u = x^2,\ dv = e^{2x}\,dx$

7. $\int (\ln x)^2\,dx$

$u = (\ln x)^2,\ dv = dx$

8. $\int \ln 3x\,dx$

$u = \ln 3x,\ dv = dx$

9. $\int x\sec^2 x\,dx$

$u = x,\ dv = \sec^2 x\,dx$

10. $\int x^2\cos x\,dx$

$u = x^2,\ dv = \cos x\,dx$

11. $dv = e^{-2x}\,dx \implies v = \int e^{-2x}\,dx = -\frac{1}{2}e^{-2x}$

$u = x \implies du = dx$

$$\int xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}\,dx$$
$$= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$$
$$= \frac{-1}{4e^{2x}}(2x+1) + C$$

12. $dv = e^{-x}\,dx \implies v = \int e^{-x}\,dx = -e^{-x}$

$u = x \implies du = dx$

$$2\int \frac{x}{e^x}\,dx = 2\int xe^{-x}\,dx$$
$$= 2\left[-xe^{-x} - \int -e^{-x}\,dx\right]$$
$$= 2[-xe^{-x} - e^{-x}] + C$$
$$= -2xe^{-x} - 2e^{-x} + C$$

13. Use integration by parts three times.

(1) $dv = e^x\,dx \implies v = \int e^x\,dx = e^x$ (2) $dv = e^x\,dx \implies v = \int e^x\,dx = e^x$ (3) $dv = e^x\,dx \implies v = \int e^x\,dx = e^x$

$u = x^3 \implies du = 3x^2\,dx$ $u = x^2 \implies du = 2x\,dx$ $u = x \implies du = dx$

$$\int x^3e^x\,dx = x^3e^x - 3\int x^2e^x\,dx = x^3e^x - 3x^2e^x + 6\int xe^x\,dx$$
$$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C = e^x(x^3 - 3x^2 + 6x - 6) + C$$

14. $\int \frac{e^{1/t}}{t^2}\,dt = -\int e^{1/t}\left(\frac{-1}{t^2}\right)dt = -e^{1/t} + C$

15. $\int x^2e^{x^3}\,dx = \frac{1}{3}\int e^{x^3}(3x^2)\,dx = \frac{1}{3}e^{x^3} + C$

16. $dv = x^4\,dx \implies v = \frac{x^5}{5}$

$u = \ln x \implies du = \frac{1}{x}\,dx$

$$\int x^4\ln x\,dx = \frac{x^5}{5}\ln x - \int \frac{x^5}{5}\left(\frac{1}{x}\right)dx$$
$$= \frac{x^5}{5}\ln x - \frac{1}{5}\int x^4\,dx$$
$$= \frac{x^5}{5}\ln x - \frac{1}{25}x^5 + C$$
$$= \frac{x^5}{25}(5\ln x - 1) + C$$

17. $dv = t\,dt \implies v = \int t\,dt = \frac{t^2}{2}$

$u = \ln(t+1) \implies du = \frac{1}{t+1}\,dt$

$$\int t\ln(t+1)\,dt = \frac{t^2}{2}\ln(t+1) - \frac{1}{2}\int \frac{t^2}{t+1}\,dt$$
$$= \frac{t^2}{2}\ln(t+1) - \frac{1}{2}\int\left(t - 1 + \frac{1}{t+1}\right)dt$$
$$= \frac{t^2}{2}\ln(t+1) - \frac{1}{2}\left[\frac{t^2}{2} - t + \ln(t+1)\right] + C$$
$$= \frac{1}{4}[2(t^2-1)\ln|t+1| - t^2 + 2t] + C$$

18. Let $u = \ln x,\ du = \frac{1}{x}\,dx$.

$$\int \frac{1}{x(\ln x)^3}\,dx = \int (\ln x)^{-3}\left(\frac{1}{x}\right)dx = \frac{-1}{2(\ln x)^2} + C$$

19. Let $u = \ln x,\ du = \frac{1}{x}\,dx$.

$$\int \frac{(\ln x)^2}{x}\,dx = \int (\ln x)^2\left(\frac{1}{x}\right)dx = \frac{(\ln x)^3}{3} + C$$

20. $dv = \dfrac{1}{x^2}\,dx \implies v = \displaystyle\int \frac{1}{x^2}\,dx = -\frac{1}{x}$

$u = \ln x \implies du = \dfrac{1}{x}\,dx$

$$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + C$$

21. $dv = \dfrac{1}{(2x+1)^2}\,dx \implies v = \displaystyle\int (2x+1)^{-2}\,dx = -\frac{1}{2(2x+1)}$

$u = xe^{2x} \implies du = (2xe^{2x} + e^{2x})\,dx = e^{2x}(2x+1)\,dx$

$$\int \frac{xe^{2x}}{(2x+1)^2}\,dx = -\frac{xe^{2x}}{2(2x+1)} + \int \frac{e^{2x}}{2}\,dx = \frac{-xe^{2x}}{2(2x+1)} + \frac{e^{2x}}{4} + C = \frac{e^{2x}}{4(2x+1)} + C$$

22. $dv = \dfrac{x}{(x^2+1)^2}\,dx \implies v = \displaystyle\int (x^2+1)^{-2}\,x\,dx = -\frac{1}{2(x^2+1)}$

$u = x^2e^{x^2} \implies du = (2x^3e^{x^2} + 2xe^{x^2})\,dx = 2xe^{x^2}(x^2+1)\,dx$

$$\int \frac{x^3e^{x^2}}{(x^2+1)^2}\,dx = -\frac{x^2e^{x^2}}{2(x^2+1)} + \int xe^{x^2}\,dx = -\frac{x^2e^{x^2}}{2(x^2+1)} + \frac{e^{x^2}}{2} + C = \frac{e^{x^2}}{2(x^2+1)} + C$$

23. Use integration by parts twice.

(1) $dv = e^x\,dx \implies v = \displaystyle\int e^x\,dx = e^x$

$u = x^2 \implies du = 2x\,dx$

(2) $dv = e^x\,dx \implies v = \displaystyle\int e^x\,dx = e^x$

$u = x \implies du = dx$

$$\int (x^2-1)e^x\,dx = \int x^2e^x\,dx - \int e^x\,dx = x^2e^x - 2\int xe^x\,dx - e^x$$

$$= x^2e^x - 2\left[xe^x - \int e^x\,dx\right] - e^x = x^2e^x - 2xe^x + e^x + C = (x-1)^2e^x + C$$

24. $dv = \dfrac{1}{x^2}\,dx \implies v = \displaystyle\int \frac{1}{x^2}\,dx = -\frac{1}{x}$

$u = \ln 2x \implies du = \dfrac{1}{x}\,dx$

$$\int \frac{\ln(2x)}{x^2}\,dx = -\frac{\ln(2x)}{x} + \int \frac{1}{x^2}\,dx = -\frac{\ln(2x)}{x} - \frac{1}{x} + C = -\frac{\ln(2x)+1}{x} + C$$

25. $dv = \sqrt{x-1}\,dx \implies v = \displaystyle\int (x-1)^{1/2}\,dx = \frac{2}{3}(x-1)^{3/2}$

$u = x \implies du = dx$

$$\int x\sqrt{x-1}\,dx = \frac{2}{3}x(x-1)^{3/2} - \frac{2}{3}\int (x-1)^{3/2}\,dx = \frac{2}{3}x(x-1)^{3/2} - \frac{4}{15}(x-1)^{5/2} + C = \frac{2(x-1)^{3/2}}{15}(3x+2) + C$$

26. $dv = \dfrac{1}{\sqrt{2+3x}}\,dx \implies v = \displaystyle\int (2+3x)^{-1/2}\,dx = \frac{2}{3}\sqrt{2+3x}$

$u = x \implies du = dx$

$$\int \frac{x}{\sqrt{2+3x}}\,dx = \frac{2x\sqrt{2+3x}}{3} - \frac{2}{3}\int \sqrt{2+3x}\,dx$$

$$= \frac{2x\sqrt{2+3x}}{3} - \frac{4}{27}(2+3x)^{3/2} + C = \frac{2\sqrt{2+3x}}{27}[9x - 2(2+3x)] + C = \frac{2\sqrt{2+3x}}{27}(3x-4) + C$$

27. $dv = \cos x\,dx \implies v = \int \cos x\,dx = \sin x$

$u = x \implies du = dx$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$$

28. $dv = \sin x\,dx \implies v = -\cos x$

$u = x \implies du = dx$

$$\int x\sin dx = -x\cos x - \int -\cos x\,dx = -x\cos x + \sin x + C$$

29. Use integration by parts three times.

(1) $u = x^3,\ du = 3x^2\,dx,\ dv = \sin x\,dx,\ v = -\cos x$

$$\int x^3 \sin dx = -x^3\cos x + 3\int x^2\cos x\,dx$$

(2) $u = x^2,\ du = 2x\,dx,\ dv = \cos x\,dx,\ v = \sin x$

$$\int x^3\sin x\,dx = -x^3\cos x + 3\left[x^2\sin x - 2\int x\sin x\,dx\right] = -x^3\cos x + 3x^2\sin x - 6\int x\sin x\,dx$$

(3) $u = x,\ du = dx,\ dv = \sin x\,dx,\ v = -\cos x$

$$\int x^3\sin x\,dx = -x^3\cos x + 3x^2\sin x - 6\left[-x\cos x + \int \cos x\,dx\right] = -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$$

30. Use integration by parts twice.

(1) $u = x^2,\ du = 2x\,dx,\ dv = \cos x\,dx,\ v = \sin x$

$$\int x^2\cos x\,dx = x^2\sin x - 2\int x\sin x\,dx$$

(2) $u = x,\ du = dx,\ dv = \sin x\,dx,\ v = -\cos x$

$$\int x^2\cos x\,dx = x^2\sin x - 2\left[-x\cos x + \int \cos x\,dx\right] = x^2\sin x + 2x\cos x - 2\sin x + C$$

31. $u = t,\ du = dt,\ dv = \csc t\cot dt,\ v = -\csc t$

$$\int t\csc t\cot t\,dt = -t\csc t + \int \csc t\,dt = -t\csc t - \ln\left|\csc t + \cot t\right| + C$$

32. $dv = \sec\theta\tan\theta\,d\theta \implies v = \int \sec\theta\tan\theta\,d\theta = \sec\theta$

$u = \theta \implies du = d\theta$

$$\int \theta\sec\theta\tan\theta\,d\theta = \theta\sec\theta - \int \sec\theta\,d\theta = \theta\sec\theta - \ln\left|\sec\theta + \tan\theta\right| + C$$

33. $dv = dx \implies v = \int dx = x$

$u = \arctan x \implies du = \dfrac{1}{1+x^2}\,dx$

$$\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}\,dx = x\arctan x - \frac{1}{2}\ln(1+x^2) + C$$

34. $dv = dx \implies v = \int dx = x$

$u = \arccos x \implies du = -\dfrac{1}{\sqrt{1-x^2}}\,dx$

$$4\int \arccos x\,dx = 4\left[x\arccos x + \int \frac{x}{\sqrt{1-x^2}}\,dx\right] = 4\left[x\arccos x - \sqrt{1-x^2}\right] + C$$

35. Use integration by parts twice.

(1) $dv = e^{2x}\,dx \implies v = \int e^{2x}\,dx = \frac{1}{2}e^{2x}$

$u = \sin x \implies du = \cos x\,dx$

(2) $dv = e^{2x}\,dx \implies v = \int e^{2x}\,dx = \frac{1}{2}e^{2x}$

$u = \cos x \implies du = -\sin x\,dx$

—CONTINUED—

35. —CONTINUED—

$$\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\int e^{2x}\cos x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\left(\frac{1}{2}e^{2x}\cos x + \frac{1}{2}\int e^{2x}\sin x\,dx\right)$$

$$\frac{5}{4}\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x$$

$$\int e^{2x}\sin x\,dx = \frac{1}{5}e^{2x}(2\sin x - \cos x) + C$$

36. Use integration by parts twice.

(1) $dv = e^x\,dx \implies v = \int e^x\,dx = e^x$

$u = \cos 2x \implies du = -2\sin 2x\,dx$

(2) $dv = e^x\,dx \implies v = \int e^x\,dx = e^x$

$u = \sin 2x \implies du = 2\cos 2x\,dx$

$$\int e^x\cos 2x\,dx = e^x\cos 2x + 2\int e^x\sin 2x\,dx = e^x\cos 2x + 2\left(e^x\sin 2x - 2\int e^x\cos 2x\,dx\right)$$

$$5\int e^x\cos 2x\,dx = e^x\cos 2x + 2e^x\sin 2x$$

$$\int e^x\cos 2x\,dx = \frac{e^x}{5}(\cos 2x + 2\sin 2x) + C$$

37. $y' = xe^{x^2}$

$$y = \int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C$$

38. $dv = dx \implies v = x$

$u = \ln x \implies du = \frac{1}{x}\,dx$

$y' = \ln x$

$$y = \int \ln x\,dx = x\ln x - \int x\left(\frac{1}{x}\right)dx$$

$$= x\ln x - x + C = x(-1 + \ln x) + C$$

39. Use integration by parts twice.

(1) $dv = \frac{1}{\sqrt{2+3t}}\,dt \implies v = \int (2+3t)^{-1/2}\,dt = \frac{2}{3}\sqrt{2+3t}$

$u = t^2 \implies du = 2t\,dt$

(2) $dv = \sqrt{2+3t}\,dt \implies v = \int (2+3t)^{1/2}\,dt = \frac{2}{9}(2+3t)^{3/2}$

$u = t \implies du = dt$

$$y = \int \frac{t^2}{\sqrt{2+3t}}\,dt = \frac{2t^2\sqrt{2+3t}}{3} - \frac{4}{3}\int t\sqrt{2+3t}\,dt$$

$$= \frac{2t^2\sqrt{2+3t}}{3} - \frac{4}{3}\left[\frac{2t}{9}(2+3t)^{3/2} - \frac{2}{9}\int (2+3t)^{3/2}\,dt\right]$$

$$= \frac{2t^2\sqrt{2+3t}}{3} - \frac{8t}{27}(2+3t)^{3/2} + \frac{16}{405}(2+3t)^{5/2} + C$$

$$= \frac{2\sqrt{2+3t}}{405}(27t^2 - 24t + 32) + C$$

40. Use integration by parts twice.

(1) $dv = \sqrt{x-1}\,dx \implies v = \int (x-1)^{1/2}\,dx = \frac{2}{3}(x-1)^{3/2}$

$u = x^2 \implies du = 2x\,dx$

(2) $dv = (x-1)^{3/2}\,dx \implies v = \int (x-1)^{3/2}\,dx = \frac{2}{5}(x-1)^{5/2}$

$u = x \implies du = dx$

$$y = \int x^2\sqrt{x-1}\,dx$$

$$= \frac{2}{3}x^2(x-1)^{3/2} - \frac{4}{3}\int x(x-1)^{3/2}\,dx = \frac{2}{3}x^2(x-1)^{3/2} - \frac{4}{3}\left[\frac{2}{5}x(x-1)^{5/2} - \frac{2}{5}\int (x-1)^{5/2}\,dx\right]$$

$$= \frac{2}{3}x^2(x-1)^{3/2} - \frac{8}{15}x(x-1)^{5/2} + \frac{16}{105}(x-1)^{7/2} + C = \frac{2(x-1)^{3/2}}{105}(15x^2 + 12x + 8) + C$$

41. $(\cos y)y' = 2x$

$$\int \cos y\,dy = \int 2x\,dx$$

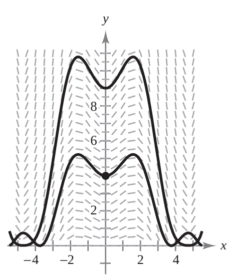
$$\sin y = x^2 + C$$

42. $dv = dx \implies v = \int dx = x$

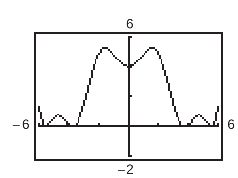
$u = \arctan\frac{x}{2} \implies du = \frac{1}{1 + (x/2)^2}\left(\frac{1}{2}\right)dx = \frac{2}{4 + x^2}\,dx$

$$y = \int \arctan\frac{x}{2}\,dx = x\arctan\frac{x}{2} - \int \frac{2x}{4 + x^2}\,dx = x\arctan\frac{x}{2} - \ln(4 + x^2) + C$$

43. (a)



(b) $\frac{dy}{dx} = x\sqrt{y}\cos x, \quad (0, 4)$



$$\int \frac{dy}{\sqrt{y}} = \int x\cos x\,dx$$

$$\int y^{-1/2}\,dy = \int x\cos x\,dx \qquad (u = x,\ du = dx,\ dv = \cos x\,dx,\ v = \sin x)$$

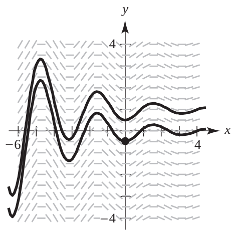
$$2y^{1/2} = x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x + C$$

$(0, 4)$: $2(4)^{1/2} = 0 + 1 + C \implies C = 3$

$2\sqrt{y} = x\sin x + \cos x + 3$

44. (a)



(b) $\dfrac{dy}{dx} = e^{-x/3}\sin 2x, \quad \left(0, -\dfrac{18}{37}\right)$

$$y = \int e^{-x/3}\sin 2x\, dx$$

Use integration by parts twice.

(1) $u = \sin 2x, \ du = 2\cos 2x$

$dv = e^{-x/3}\,dx, v = -3e^{-x/3}$

$$\int e^{-x/3}\sin 2x\,dx = -3e^{-x/3}\sin 2x + \int 6e^{-x/3}\cos 2x\,dx$$

(2) $u = \cos 2x, \ du = -2\sin 2x$

$dv = e^{-x/3}\,dx, v = -3e^{-x/3}$

$$\int e^{-x/3}\sin 2x\,dx = -3e^{-x/3}\sin 2x + 6\left[-3e^{-x/3}\cos 2x - \int 6e^{-x/3}\sin 2x\,dx\right] + C$$

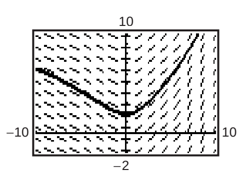
$$37\int e^{-x/3}\sin 2x\,dx = -3e^{-x/3}\sin 2x - 18e^{-x/3}\cos 2x + C$$

$$y = \int e^{-x/3}\sin 2x\,dx = \frac{1}{37}\left[-3e^{-x/3}\sin 2x - 18e^{-x/3}\cos 2x\right] + C$$

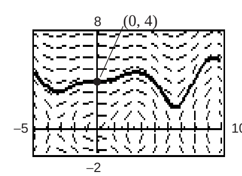
$$\left(0, \frac{-18}{37}\right): \ \frac{-18}{37} = \frac{1}{37}[0 - 18] + C \implies C = 0$$

$$y = \frac{-1}{37}\left[3e^{-x/3}\sin 2x + 18e^{-x/3}\cos 2x\right]$$

45. $\dfrac{dy}{dx} = \dfrac{x}{y}e^{x/8}, y(0) = 2$



46. $\dfrac{dy}{dx} = \dfrac{x}{y}\sin x, y(0) = 4$



47. $u = x, du = dx, dv = e^{-x/2}\,dx, v = -2e^{-x/2}$

$$\int xe^{-x/2}\,dx = -2xe^{-x/2} + \int 2e^{-x/2}\,dx = -2xe^{-x/2} - 4e^{-x/2} + C$$

Thus, $\displaystyle\int_0^4 xe^{-x/2}\,dx = \Big[-2xe^{-x/2} - 4e^{-x/2}\Big]_0^4$

$$= -8e^{-2} - 4e^{-2} + 4$$

$$= -12e^{-2} + 4 \approx 2.376.$$

48. See Exercise 3.

$$\int_0^1 x^2e^x\,dx = \Big[x^2e^x - 2xe^x + 2e^x\Big]_0^1 = e - 2 \approx 0.718$$

49. See Exercise 27.

$$\int_0^{\pi/2} x\cos x\,dx = \Big[x\sin x + \cos x\Big]_0^{\pi/2} = \frac{\pi}{2} - 1$$

50. $dv = \sin 2x\,dx \implies v = \int \sin 2x\,dx = -\frac{1}{2}\cos 2x$

$u = x \implies du = dx$

$$\int x\sin 2x\,dx = \frac{-1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\,dx$$

$$= \frac{-1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C$$

$$= \frac{1}{4}(\sin 2x - 2x\cos 2x) + C$$

Thus, $\int_0^{\pi} x\sin 2x\,dx = \left[\frac{1}{4}(\sin 2x - 2x\cos 2x)\right]_0^{\pi} = -\frac{\pi}{2}.$

51. $u = \arccos x,\ du = -\frac{1}{\sqrt{1-x^2}}\,dx,\ dv = dx,\ v = x$

$$\int \arccos x\,dx = x\arccos x + \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= x\arccos x - \sqrt{1-x^2} + C$$

Thus, $\int_0^{1/2} \arccos x = \left[x\arccos x - \sqrt{1-x^2}\right]_0^{1/2}$

$$= \frac{1}{2}\arccos\left(\frac{1}{2}\right) - \sqrt{\frac{3}{4}} + 1$$

$$= \frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1 \approx 0.658.$$

52. $dv = x\,dx \implies v = \int x\,dx = \frac{x^2}{2}$

$u = \arcsin x^2 \implies du = \frac{2x}{\sqrt{1-x^4}}\,dx$

$$\int x\arcsin x^2\,dx = \frac{x^2}{2}\arcsin x^2 - \int \frac{x^3}{\sqrt{1-x^4}}\,dx$$

$$= \frac{x^2}{2}\arcsin x^2 + \frac{1}{4}(2)(1-x^4)^{1/2} + C$$

$$= \frac{1}{2}\left[x^2\arcsin x^2 + \sqrt{1-x^4}\right] + C$$

Thus, $\int_0^1 x\arcsin x^2\,dx = \frac{1}{2}\left[x^2\arcsin x^2 + \sqrt{1-x^4}\right]_0^1 = \frac{1}{4}(\pi - 2).$

53. Use integration by parts twice.

(1) $dv = e^x\,dx \implies v = \int e^x\,dx = e^x$

$u = \sin x \implies du = \cos x\,dx$

(2) $dv = e^x\,dx \implies v = \int e^x\,dx = e^x$

$u = \cos x \implies du = -\sin x\,dx$

$$\int e^x\sin x\,dx = e^x\sin x - \int e^x\cos x\,dx = e^x\sin x - e^x\cos x - \int e^x\sin x\,dx$$

$$2\int e^x\sin x\,dx = e^x(\sin x - \cos x)$$

$$\int e^x\sin x\,dx = \frac{e^x}{2}(\sin x - \cos x) + C$$

Thus, $\int_0^1 e^x\sin x\,dx = \left[\frac{e^x}{2}(\sin x - \cos x)\right]_0^1 = \frac{e}{2}(\sin 1 - \cos 1) + \frac{1}{2} = \frac{e(\sin 1 - \cos 1) + 1}{2} \approx 0.909.$

54. Use integration by parts twice.

(1) $dv = e^{-x}, v = -e^{-x}, u = \cos x, du = -\sin x\, dx$

$$\int e^{-x} \cos x\, dx = -e^{-x}\cos x - \int e^{-x} \sin x\, dx$$

(2) $dv = e^{-x}\, dx, v = -e^{-x}, u = \sin x, du = \cos x\, dx$

$$\int e^{-x} \cos x\, dx = -e^{-x}\cos x - \left[-e^{-x}\sin x + \int e^{-x}\cos x\, dx\right] \Rightarrow 2\int e^{-x}\cos x\, dx = e^{-x}\sin x - e^{-x}\cos x$$

Thus, $\displaystyle\int_0^2 e^{-x}\cos x\, dx = \left[\frac{e^{-x}\sin x - e^{-x}\cos x}{2}\right]_0^2 = \frac{-e^{-2}}{2}[\sin 2 - \cos 2] + \frac{1}{2}.$

55. $dv = x^2\, dx, v = \dfrac{x^3}{3}, u = \ln x, du = \dfrac{1}{x}\, dx$

$$\int x^2 \ln x\, dx = \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\left(\frac{1}{x}\right) dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\, dx$$

Hence, $\displaystyle\int_1^2 x^2 \ln x\, dx = \left[\frac{x^3}{3}\ln x - \frac{1}{9}x^3\right]_1^2$

$$= \frac{8}{3}\ln 2 - \frac{8}{9} + \frac{1}{9}$$

$$= \frac{8}{3}\ln 2 - \frac{7}{9} \approx 1.071.$$

56. $dv = dx \quad \Rightarrow \quad v = \displaystyle\int dx = x$

$$u = \ln(1 + x^2) \Rightarrow du = \frac{2x}{1 + x^2}\, dx$$

$$\int \ln(1 + x^2)\, dx = x\ln(1 + x^2) - \int \frac{2x^2}{1 + x^2}\, dx$$

$$= x\ln(1 + x^2) - 2\int\left[1 - \frac{1}{1 + x^2}\right] dx$$

$$= x\ln(1 + x^2) - 2x + 2\arctan x + C$$

Thus,

$$\int_0^1 \ln(1 + x^2)\, dx = \left[x\ln(1 + x^2) - 2x + 2\arctan x\right]_0^1$$

$$= \ln 2 - 2 + \frac{\pi}{2}.$$

57. $dv = x\, dx, v = \dfrac{x^2}{2}, u = \operatorname{arcsec} x, du = \dfrac{1}{x\sqrt{x^2 - 1}}\, dx$

$$\int x \operatorname{arcsec} x\, dx = \frac{x^2}{2}\operatorname{arcsec} x - \int \frac{x^2/2}{x\sqrt{x^2 - 1}}\, dx$$

$$= \frac{x^2}{2}\operatorname{arcsec} x - \frac{1}{4}\int \frac{2x}{\sqrt{x^2 - 1}}\, dx$$

$$= \frac{x^2}{2}\operatorname{arcsec} x - \frac{1}{2}\sqrt{x^2 - 1} + C$$

Hence,

$$\int_2^4 x \operatorname{arcsec} x\, dx = \left[\frac{x^2}{2}\operatorname{arcsec} x - \frac{1}{2}\sqrt{x^2 - 1}\right]_2^4$$

$$= \left(8 \operatorname{arcsec} 4 - \frac{\sqrt{15}}{2}\right) - \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

$$= 8 \operatorname{arcsec} 4 - \frac{\sqrt{15}}{2} + \frac{\sqrt{3}}{2} - \frac{2\pi}{3}$$

$$\approx 7.380.$$

58. $u = x, du = dx, dv = \sec^2 x\, dx, v = \tan x$

$$\int x\sec^2 x\, dx = x\tan x - \int \tan x\, dx$$

Hence,

$$\int_0^{\pi/4} x\sec^2 x\, dx = \left[x\tan x + \ln|\cos x|\right]_0^{\pi/4}$$

$$= \left(\frac{\pi}{4} + \ln\frac{\sqrt{2}}{2}\right) - 0$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2.$$

59. $\int x^2e^{2x}\,dx = x^2\left(\frac{1}{2}e^{2x}\right) - (2x)\left(\frac{1}{4}e^{2x}\right) + 2\left(\frac{1}{8}e^{2x}\right) + C$

$= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$

$= \frac{1}{4}e^{2x}(2x^2 - 2x + 1) + C$

Alternate signs	u and its derivatives	v' and its antiderivatives
$+$	x^2	e^{2x}
$-$	$2x$	$\frac{1}{2}e^{2x}$
$+$	2	$\frac{1}{4}e^{2x}$
$-$	0	$\frac{1}{8}e^{2x}$

60. $\int x^3e^{-2x}\,dx = x^3\left(-\frac{1}{2}e^{-2x}\right) - 3x^2\left(\frac{1}{4}e^{-2x}\right) + 6x\left(-\frac{1}{8}e^{-2x}\right) - 6\left(\frac{1}{16}e^{-2x}\right) + C$

$= -\frac{1}{8}e^{-2x}(4x^3 + 6x^2 + 6x + 3) + C$

Alternate signs	u and its derivatives	v' and its antiderivatives
$+$	x^3	e^{-2x}
$-$	$3x^2$	$-\frac{1}{2}e^{-2x}$
$+$	$6x$	$\frac{1}{4}e^{-2x}$
$-$	6	$-\frac{1}{8}e^{-2x}$
$+$	0	$\frac{1}{16}e^{-2x}$

61. $\int x^3 \sin x\,dx = x^3(-\cos x) - 3x^2(-\sin x) + 6x\cos x - 6\sin x + C$

$= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$

$= (3x^2 - 6)\sin x - (x^3 - 6x)\cos x + C$

Alternate signs	u and its derivatives	v' and its antiderivatives
$+$	x^3	$\sin x$
$-$	$3x^2$	$-\cos x$
$+$	$6x$	$-\sin x$
$-$	6	$\cos x$
$+$	0	$\sin x$

62. $\int x^3\cos 2x\,dx = x^3\left(\frac{1}{2}\sin 2x\right) - 3x^2\left(-\frac{1}{4}\cos 2x\right) + 6x\left(-\frac{1}{8}\sin 2x\right) - 6\left(\frac{1}{16}\cos 2x\right) + C$

$= \frac{1}{2}x^3\sin 2x + \frac{3}{4}x^2\cos 2x - \frac{3}{4}x\sin 2x - \frac{3}{8}\cos 2x + C$

$= \frac{1}{8}[4x^3\sin 2x + 6x^2\cos 2x - 6x\sin 2x - 3\cos 2x] + C$

Alternate signs	u and its derivatives	v' and its antiderivatives
$+$	x^3	$\cos 2x$
$-$	$3x^2$	$\frac{1}{2}\sin 2x$
$+$	$6x$	$-\frac{1}{4}\cos 2x$
$-$	6	$-\frac{1}{8}\sin 2x$
$+$	0	$\frac{1}{16}\cos 2x$

63. $\int x\sec^2 x\,dx = x\tan x + \ln|\cos x| + C$

Alternate signs	u and its derivatives	v' and its antiderivatives
$+$	x	$\sec^2 x$
$-$	1	$\tan x$
$+$	0	$-\ln\lvert\cos x\rvert$

64. $\int x^2(x-2)^{3/2}\,dx = \frac{2}{5}x^2(x-2)^{5/2} - \frac{8}{35}x(x-2)^{7/2} + \frac{16}{315}(x-2)^{9/2} + C$

$= \frac{2}{315}(x-2)^{5/2}(35x^2 + 40x + 32) + C$

Alternate signs	u and its derivatives	v' and its antiderivatives
$+$	x^2	$(x-2)^{3/2}$
$-$	$2x$	$\frac{2}{5}(x-2)^{5/2}$
$+$	2	$\frac{4}{35}(x-2)^{7/2}$
$-$	0	$\frac{8}{315}(x-2)^{9/2}$

65. $u = \sqrt{x} \Rightarrow u^2 = x \Rightarrow 2u\,du = dx$

$\int \sin\sqrt{x}\,dx = \int \sin u(2u\,du) = 2\int u \sin u\,du$

Integration by parts: $w = u$, $dw = du$, $dv = \sin u\,du$, $v = -\cos u$

$2\int u\sin u\,du = 2\left(-u\cos u + \int \cos u\,du\right)$

$= 2(-u\cos u + \sin u) + C$

$= 2\left(-\sqrt{x}\cos\sqrt{x} + \sin\sqrt{x}\right) + C$

66. $u = x^2$, $du = 2x\,dx$

$\int 2x^3\cos(x^2)\,dx = \int x^2\cos(x^2)(2x)\,dx = \int u\cos u\,du$

Integration by parts: $w = u$, $dw = du$, $dv = \cos u\,du$, $v = \sin u$

$\int u\cos u\,du = u\sin u - \int \sin u\,du$

$= u\sin u + \cos u + C$

$= x^2\sin(x^2) + \cos(x^2) + C$

67. Let $u = 4 - x$, $du = -dx$, $x = 4 - u$.

$\int_0^4 x\sqrt{4-x}\,dx = \int_4^0 (4-u)u^{1/2}(-du)$

$= \int_0^4 (4u^{1/2} - u^{3/2})\,du$

$= \left[\frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_0^4$

$= \frac{8}{3}(8) - \frac{2}{5}(32) = \frac{128}{15}$

68. Let $u = \sqrt{2x}$, $u^2 = 2x$, $2u\,du = 2\,dx$.

$\int_0^2 e^{\sqrt{2x}}\,dx = \int_0^2 e^u(u\,du)$

$= \left[ue^u - e^u\right]_0^2$ (Integration by parts)

$= (2e^2 - e^2) - (0 - 1)$

$= e^2 + 1$

69. Let $w = \ln x$, $dw = \frac{1}{x}\,dx$, $x = e^w$, $dx = e^w\,dw$.

$\int \cos(\ln x)\,dx = \int \cos w(e^w\,dw)$

Now use integration by parts twice.

$\int \cos w e^w\,dw = \cos w e^w + \int \sin w e^w\,dw$ $\quad [u = \cos w, dv = e^w\,dw]$

$= \cos w e^w + \left[\sin w e^w - \int \cos w e^w\,dw\right]$ $\quad [u = \sin w, dv = e^w\,dw]$

$2\int \cos w e^w\,dw = \cos w e^w + \sin w e^w$

$\int \cos w e^w\,dw = \frac{1}{2}e^w[\cos w + \sin w] + C$

$\int \cos(\ln x)\,dx = \frac{1}{2}x[\cos(\ln x) + \sin(\ln x)] + C$

70. Let $w = 1 + x^2$, $dw = 2x\,dx$, $x^2 = w - 1$, $x = \sqrt{w-1}$.

$$\int \ln(x^2+1)\,dx = \int \ln(w)\frac{dw}{2\sqrt{w-1}}$$

Integration by parts: $u = \ln w$, $du = \frac{1}{w}\,dw$, $dv = \frac{1}{2\sqrt{w-1}}\,dw$, $v = \sqrt{w-1}$

$$\int \ln(x^2+1)\,dx = \ln(w)\sqrt{w-1} - \int \frac{\sqrt{w-1}}{2}\,dw$$

Substitution: $z = \sqrt{w-1}$, $z^2 = w-1$, $2z\,dz = dw$

$$\int \ln(x^2+1)\,dx = \ln(w)\sqrt{w-1} - \int \frac{z}{z^2+1}(2z\,dz)$$

$$= \ln(w)\sqrt{w-1} - 2\int\left(1 - \frac{1}{z^2+1}\right)dz$$

$$= \ln(w)\sqrt{w-1} - 2z + 2\arctan(z) + C$$

$$= \ln(1+x^2)x - 2x + 2\arctan(x) + C$$

71. Integration by parts is based on the Product Rule.

72. Answers will vary.

73. No
Substitution

74. Yes
$u = \ln x$, $dv = x\,dx$

75. Yes
$u = x^2$, $dv = e^{2x}\,dx$

76. No
Substitution

77. Yes. Let $u = x$ and

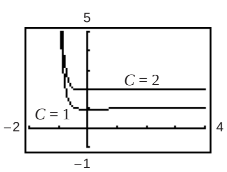
$$du = \frac{1}{\sqrt{x+1}}\,dx.$$

(Substitution also works. Let $u = \sqrt{x+1}$.)

78. No
Substitution

79. (a) $\int t^3 e^{-4t}\,dt = \frac{-e^{-4t}}{128}(32t^3 + 24t^2 + 12t + 3) + C$

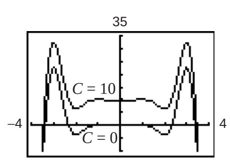
(b)



(c) The graphs are vertical translations of each other.

80. (a) $\int \alpha^4 \sin(\pi\alpha)\,d\alpha = \frac{1}{\pi^5}[-(\alpha\pi)^4\cos\pi\alpha + 4(\alpha\pi)^3\sin\pi\alpha + 12(\alpha\pi)^2\cos\pi\alpha - 24(\alpha\pi)\sin\pi\alpha - 24\cos\pi\alpha] + C$

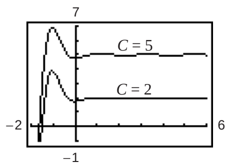
(b)



(c) The graphs are vertical translations of each other.

81. (a) $\displaystyle\int e^{-2x}\sin 3x\,dx = \frac{e^{-2x}}{13}[-2\sin 3x - 3\cos 3x] + C$

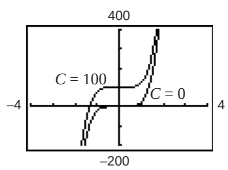
$$\int_0^{\pi/2} e^{-2x}\sin 3x\,dx = \frac{1}{13}[2e^{-\pi} + 3] \approx 0.2374$$

(b) 

(c) The graphs are vertical translations of each other.

82. (a) $\displaystyle\int x^4(25 - x^2)^{3/2}\,dx = \frac{1{,}171{,}875\arcsin|x/5|}{128} - \frac{x(2x^2 + 25)(25 - x^2)^{5/2}}{16} + \frac{625x(25 - x^2)^{3/2}}{64} + \frac{46{,}875x\sqrt{25 - x^2}}{128} + C$

$$\int_0^5 x^4(25 - x^2)^{2/3}\,dx = \frac{1{,}171{,}875}{256}\pi \approx 14{,}381.0699$$

(b) 

(c) The graphs are vertical translations of each other.

83. (a) $dv = \sqrt{2x - 3}\,dx \implies v = \displaystyle\int (2x - 3)^{1/2}\,dx = \frac{1}{3}(2x - 3)^{3/2}$

$u = 2x \implies du = 2\,dx$

$$\int 2x\sqrt{2x - 3}\,dx = \frac{2}{3}x(2x - 3)^{3/2} - \frac{2}{3}\int (2x - 3)^{3/2}\,dx$$

$$= \frac{2}{3}x(2x - 3)^{3/2} - \frac{2}{15}(2x - 3)^{5/2} + C$$

$$= \frac{2}{15}(2x - 3)^{3/2}(3x + 3) + C = \frac{2}{5}(2x - 3)^{3/2}(x + 1) + C$$

(b) $u = 2x - 3 \implies x = \dfrac{u + 3}{2}$ and $dx = \dfrac{1}{2}\,du$

$$\int 2x\sqrt{2x - 3}\,dx = \int 2\left(\frac{u + 3}{2}\right)u^{1/2}\left(\frac{1}{2}\right)du = \frac{1}{2}\int (u^{3/2} + 3u^{1/2})\,du = \frac{1}{2}\left[\frac{2}{5}u^{5/2} + 2u^{3/2}\right] + C$$

$$= \frac{1}{5}u^{3/2}(u + 5) + C = \frac{1}{5}(2x - 3)^{3/2}[(2x - 3) + 5] + C = \frac{2}{5}(2x - 3)^{3/2}(x + 1) + C$$

84. (a) $dv = \sqrt{4+x}\,dx \implies v = \int (4+x)^{1/2}\,dx = \frac{2}{3}(4+x)^{3/2}$

$u = x \implies du = dx$

$$\int x\sqrt{4+x}\,dx = \frac{2}{3}x(4+x)^{3/2} - \frac{2}{3}\int (4+x)^{3/2}\,dx$$

$$= \frac{2}{3}x(4+x)^{3/2} - \frac{4}{15}(4+x)^{5/2} + C = \frac{2}{15}(4+x)^{3/2}(3x-8) + C$$

(b) $u = 4 + x \implies x = u - 4$ and $dx = du$

$$\int x\sqrt{4+x}\,dx = \int (u-4)u^{1/2}\,du = \int (u^{3/2} - 4u^{1/2})\,du$$

$$= \frac{2}{5}u^{5/2} - \frac{8}{3}u^{3/2} + C = \frac{2}{15}u^{3/2}(3u-20) + C$$

$$= \frac{2}{15}(4+x)^{3/2}[3(4+x) - 20] + C = \frac{2}{15}(4+x)^{3/2}(3x-8) + C$$

85. (a) $dv = \frac{x}{\sqrt{4+x^2}}\,dx \implies v = \int (4+x^2)^{-1/2}x\,dx = \sqrt{4+x^2}$

$u = x^2 \implies du = 2x\,dx$

$$\int \frac{x^3}{\sqrt{4+x^2}}\,dx = x^2\sqrt{4+x^2} - 2\int x\sqrt{4+x^2}\,dx$$

$$= x^2\sqrt{4+x^2} - \frac{2}{3}(4+x^2)^{3/2} + C = \frac{1}{3}\sqrt{4+x^2}\,(x^2-8) + C$$

(b) $u = 4 + x^2 \implies x^2 = u - 4$ and $2x\,dx = du \implies x\,dx = \frac{1}{2}\,du$

$$\int \frac{x^3}{\sqrt{4+x^2}}\,dx = \int \frac{x^2}{\sqrt{4+x^2}}\,x\,dx = \int \frac{u-4}{\sqrt{u}}\,\frac{1}{2}\,du$$

$$= \frac{1}{2}\int (u^{1/2} - 4u^{-1/2})\,du = \frac{1}{2}\left(\frac{2}{3}u^{3/2} - 8u^{1/2}\right) + C$$

$$= \frac{1}{3}u^{1/2}(u-12) + C = \frac{1}{3}\sqrt{4+x^2}\,[(4+x^2) - 12] + C = \frac{1}{3}\sqrt{4+x^2}\,(x^2-8) + C$$

86. (a) $dv = \sqrt{4-x}\,dx \implies v = \int (4-x)^{1/2}\,dx$

$$= -\frac{2}{3}(4-x)^{3/2}$$

$u = x \implies du = dx$

$$\int x\sqrt{4-x}\,dx = -\frac{2}{3}x(4-x)^{3/2} + \frac{2}{3}\int (4-x)^{3/2}\,dx$$

$$= -\frac{2}{3}x(4-x)^{3/2} - \frac{4}{15}(4-x)^{5/2} + C$$

$$= -\frac{2}{15}(4-x)^{3/2}[5x + 2(4-x)] + C$$

$$= -\frac{2}{15}(4-x)^{3/2}(3x+8) + C$$

(b) $u = 4 - x \implies x = 4 - u$ and $dx = -du$

$$\int x\sqrt{4-x}\,dx = -\int (4-u)\sqrt{u}\,du$$

$$= -\int (4u^{1/2} - u^{3/2})\,du$$

$$= -\frac{8}{3}u^{3/2} + \frac{2}{5}u^{5/2} + C$$

$$= -\frac{2}{15}u^{3/2}(20 - 3u) + C$$

$$= -\frac{2}{15}(4-x)^{3/2}[20 - 3(4-x)] + C$$

$$= -\frac{2}{15}(4-x)^{3/2}(3x+8) + C$$

87. $n = 0$: $\int \ln x\,dx = x(\ln x - 1) + C$

$n = 1$: $\int x \ln x\,dx = \frac{x^2}{4}(2\ln x - 1) + C$

$n = 2$: $\int x^2 \ln x\,dx = \frac{x^3}{9}(3\ln x - 1) + C$

$n = 3$: $\int x^3 \ln x\,dx = \frac{x^4}{16}(4\ln x - 1) + C$

$n = 4$: $\int x^4 \ln x\,dx = \frac{x^5}{25}(5\ln x - 1) + C$

In general, $\int x^n \ln x\,dx = \frac{x^{n+1}}{(n+1)^2}[(n+1)\ln x - 1] + C$.

88. $n = 0$: $\int e^x\,dx = e^x + C$

$n = 1$: $\int xe^x\,dx = xe^x - e^x + C = xe^x - \int e^x\,dx$

$n = 2$: $\int x^2e^x\,dx = x^2e^x - 2xe^x + 2e^x + C = x^2e^x - 2\int xe^x\,dx$

$n = 3$: $\int x^3e^x\,dx = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C = x^3e^x - 3\int x^2e^x\,dx$

$n = 4$: $\int x^4e^x\,dx = x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x + C = x^4e^x - 4\int x^3e^x\,dx$

In general, $\int x^ne^x\,dx = x^ne^x - n\int x^{n-1}e^x\,dx$.

89. $dv = \sin x\,dx \implies v = -\cos x$

$u = x^n \implies du = nx^{n-1}\,dx$

$$\int x^n \sin x\,dx = -x^n\cos x + n\int x^{n-1}\cos x\,dx$$

90. $dv = \cos x\,dx \implies v = \sin x$

$u = x^n \implies du = nx^{n-1}\,dx$

$$\int x^n \cos x\,dx = x^n\sin x - n\int x^{n-1}\sin x\,dx$$

91. $dv = x^n\,dx \implies v = \frac{x^{n+1}}{n+1}$

$u = \ln x \implies du = \frac{1}{x}\,dx$

$$\int x^n \ln x\,dx = \frac{x^{n+1}}{n+1}\ln x - \int \frac{x^n}{n+1}\,dx$$

$$= \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + C$$

$$= \frac{x^{n+1}}{(n+1)^2}[(n+1)\ln x - 1] + C$$

92. $dv = e^{ax}\,dx \implies v = \frac{1}{a}e^{ax}$

$u = x^n \implies du = nx^{n-1}\,dx$

$$\int x^ne^{ax}\,dx = \frac{x^ne^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$$

93. Use integration by parts twice.

(1) $dv = e^{ax}\,dx \implies v = \frac{1}{a}e^{ax}$ (2) $dv = e^{ax}\,dx \implies v = \frac{1}{a}e^{ax}$

$u = \sin bx \implies du = b\cos bx\,dx$ $u = \cos bx \implies du = -b\sin bx\,dx$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}\sin bx}{a} - \frac{b}{a}\int e^{ax}\cos bx\,dx$$

$$= \frac{e^{ax}\sin bx}{a} - \frac{b}{a}\left[\frac{e^{ax}\cos bx}{a} + \frac{b}{a}\int e^{ax}\sin bx\,dx\right] = \frac{e^{ax}\sin bx}{a} - \frac{b}{a^2}e^{ax}\cos bx - \frac{b^2}{a^2}\int e^{ax}\sin bx\,dx$$

Therefore, $\left(1 + \frac{b^2}{a^2}\right)\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2}$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2 + b^2} + C.$$

94. Use integration by parts twice.

(1) $dv = e^{ax}\,dx \implies v = \frac{1}{a}e^{ax}$ (2) $dv = e^{ax}\,dx \implies v = \frac{1}{a}e^{ax}$

$u = \cos bx \implies du = -b\sin bx$ $u = \sin bx \implies du = b\cos bx$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}\cos bx}{a} + \frac{b}{a}\int e^{ax}\sin bx\,dx = \frac{e^{ax}\cos bx}{a} + \frac{b}{a}\left[\frac{e^{ax}\sin bx}{a} - \frac{b}{a}\int e^{ax}\cos bx\,dx\right]$$

$$= \frac{e^{ax}\cos bx}{a} + \frac{be^{ax}\sin bx}{a^2} - \frac{b^2}{a^2}\int e^{ax}\cos bx\,dx$$

Therefore, $\left(1 + \frac{b^2}{a^2}\right)\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2}$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2 + b^2} + C.$$

95. $n = 3$, (Use formula in Exercise 91.)

$$\int x^3\ln x\,dx = \frac{x^4}{16}[4\ln x - 1] + C$$

96. $n = 2$, (Use formula in Exercise 90.)

$\int x^2\cos x\,dx = x^2\sin x - 2\int x\sin x\,dx$, (Use formula in Exercise 83.) $(n = 1)$

$$= x^2\sin x - 2\left[-x\cos x + \int\cos x\,dx\right] = x^2\sin x + 2x\cos x - 2\sin x + C$$

97. $a = 2, b = 3$, (Use formula in Exercise 94.)

$$\int e^{2x}\cos 3x\,dx = \frac{e^{2x}(2\cos 3x + 3\sin 3x)}{13} + C$$

98. $n = 3, a = 2,$ (Use formula in Exercise 92 three times.)

$$\int x^3e^{2x}\,dx = \frac{x^3e^{2x}}{2} - \frac{3}{2}\int x^2e^{2x}\,dx, \quad (n = 3, a = 2)$$

$$= \frac{x^3e^{2x}}{2} - \frac{3}{2}\left[\frac{x^2e^{2x}}{2} - \int xe^{2x}\,dx\right], \quad (n = 2, a = 2)$$

$$= \frac{x^3e^{2x}}{2} - \frac{3x^2e^{2x}}{4} + \frac{3}{2}\left[\frac{xe^{2x}}{2} - \frac{1}{2}\int e^{2x}\,dx\right]$$

$$= \frac{x^3e^{2x}}{2} - \frac{3x^2e^{2x}}{4} + \frac{3xe^{2x}}{4} - \frac{3e^{2x}}{8} + C, \quad (n = 1, a = 2)$$

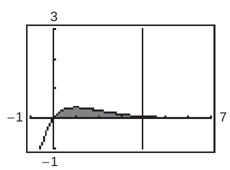
$$= \frac{e^{2x}}{8}(4x^3 - 6x^2 + 6x - 3) + C$$

99. $dv = e^{-x}\,dx \implies v = -e^{-x}$

$u = x \implies du = dx$

$$A = \int_0^4 xe^{-x}\,dx = \Big[-xe^{-x}\Big]_0^4 + \int_0^4 e^{-x}\,dx = \frac{-4}{e^4} - \Big[e^{-x}\Big]_0^4$$

$$= 1 - \frac{5}{e^4} \approx 0.908$$



100. $dv = e^{-x/3}\,dx \implies v = -3e^{-x/3}$

$u = x \implies du = dx$

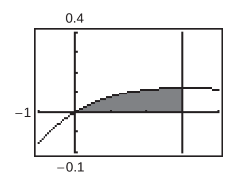
$$A = \frac{1}{9}\int_0^3 xe^{-x/3}\,dx$$

$$= \frac{1}{9}\left(\Big[-3xe^{-x/3}\Big]_0^3 + 3\int_0^3 e^{-x/3}\,dx\right)$$

$$= \frac{1}{9}\left(\frac{-9}{e} - \Big[9e^{-x/3}\Big]_0^3\right)$$

$$= -\frac{1}{e} - \frac{1}{e} + 1$$

$$= 1 - \frac{2}{e} \approx 0.264$$



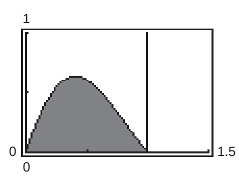
101. $$A = \int_0^1 e^{-x}\sin(\pi x)\,dx$$

$$= \left[\frac{e^{-x}(-\sin \pi x - \pi \cos \pi x)}{1 + \pi^2}\right]_0^1$$

$$= \frac{1}{1 + \pi^2}\left(\frac{\pi}{e} + \pi\right)$$

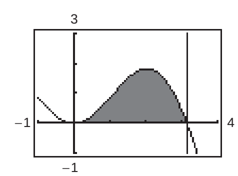
$$= \frac{\pi}{1 + \pi^2}\left(\frac{1}{e} + 1\right)$$

≈ 0.395 (See Exercise 93.)



102. $$A = \int_0^{\pi} x \sin x\,dx = \Big[-x\cos x + \sin x\Big]_0^{\pi}$$

$= \pi$ (See Exercise 89.)



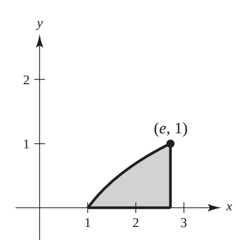
103. (a) $A = \int_1^e \ln x\,dx = \Big[-x + x\ln x\Big]_1^e = 1$ (See Exercise 4.)

(b) $R(x) = \ln x, r(x) = 0$

$$V = \pi\int_1^e (\ln x)^2\,dx$$

$$= \pi\Big[x(\ln x)^2 - 2x\ln x + 2x\Big]_1^e$$ (Use integration by parts twice, see Exercise 7.)

$$= \pi(e - 2) \approx 2.257$$



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103. —CONTINUED—

(c) $p(x) = x, h(x) = \ln x$

$$V = 2\pi\int_1^e x \ln x\,dx = 2\pi\left[\frac{x^2}{4}(-1 + 2\ln x)\right]_1^e$$

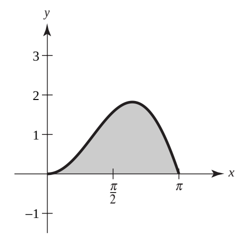
$$= \frac{(e^2+1)\pi}{2} \approx 13.177 \quad \text{(See Exercise 91.)}$$

(d) $$\bar{x} = \frac{\int_1^e x \ln x\,dx}{1} = \frac{e^2+1}{4} \approx 2.097$$

$$\bar{y} = \frac{\frac{1}{2}\int_1^e (\ln x)^2\,dx}{1} = \frac{e-2}{2} \approx 0.359$$

$$(\bar{x}, \bar{y}) = \left(\frac{e^2+1}{4}, \frac{e-2}{2}\right) \approx (2.097, 0.359)$$

104. $y = x \sin x, \quad 0 \le x \le \pi$



(a) $$V = \int_0^\pi \pi[x \sin x]^2\,dx = \pi\int_0^\pi x^2 \sin^2 x\,dx$$

Let $u = x^2, du = 2x\,dx, dv = \sin^2 x\,dx = \frac{1-\cos 2x}{2}dx, v = \frac{1}{2}x - \frac{\sin 2x}{4}$.

$$\int x^2 \sin^2 x\,dx = x^2\left[\frac{1}{2}x - \frac{\sin 2x}{4}\right] - \int\left(\frac{1}{2}x - \frac{\sin 2x}{4}\right)(2x\,dx)$$

$$= \frac{1}{2}x^3 - \frac{x^2\sin 2x}{4} - \int\left(x^2 - \frac{x\sin 2x}{2}\right)dx$$

$$= \frac{1}{2}x^3 - \frac{x^2\sin 2x}{4} - \frac{x^3}{3} + \int\frac{x\sin 2x}{2}\,dx$$

$$= \frac{1}{6}x^3 - \frac{1}{4}x^2\sin 2x + \frac{1}{8}(\sin 2x - 2x\cos 2x) + C \quad \text{(Integration by Parts)}$$

$$V = \pi\int_0^\pi x^2\sin^2 x\,dx = \pi\left[\frac{1}{6}x^3 - \frac{1}{4}x^2\sin 2x + \frac{1}{8}(\sin 2x - 2x\cos 2x)\right]_0^\pi = \frac{1}{6}\pi^4 - \frac{1}{4}\pi^2$$

(b) $$V = \int_0^\pi 2\pi x(x\sin x)\,dx = 2\pi\left[2\cos x + 2x\sin x - x^2\cos x\right]_0^\pi = 2\pi[\pi^2 - 4] = 2\pi^3 - 8\pi$$

(c) $$m = \int_0^\pi x\sin(x)\,dx = \left[\sin x - x\cos x\right]_0^\pi = \pi$$

$$M_x = \int_0^\pi \frac{1}{2}(x\sin x)^2\,dx$$

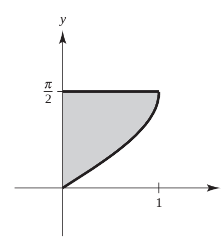
$$= \frac{1}{2}\left[\frac{1}{6}\pi^3 - \frac{1}{4}\pi\right] \quad \text{(See part (a).)}$$

$$= \frac{1}{12}\pi^3 - \frac{1}{8}\pi$$

$$M_y = \int_0^\pi x(x\sin x)\,dx = \pi^2 - 4 \quad \text{(See part (b).)}$$

$$\bar{x} = \frac{M_y}{m} = \frac{\pi^2-4}{\pi} \approx 1.8684, \quad \bar{y} = \frac{M_x}{m} = \frac{(1/12)\pi^3 - (1/8)\pi}{\pi} = \frac{1}{2}\pi^2 - \frac{1}{8} \approx 0.6975$$

105. In Example 6, we showed that the centroid of an equivalent region was $(1, \pi/8)$. By symmetry, the centroid of this region is $(\pi/8, 1)$. You can also solve this problem directly.



$$A = \int_0^1\left(\frac{\pi}{2} - \arcsin x\right)dx = \left[\frac{\pi}{2}x - x\arcsin x - \sqrt{1-x^2}\right]_0^1 \quad \text{(Example 3)}$$

$$= \left(\frac{\pi}{2} - \frac{\pi}{2} - 0\right) - (-1) = 1$$

$$\bar{x} = \frac{M_y}{A} = \int_0^1 x\left[\frac{\pi}{2} - \arcsin x\right]dx = \frac{\pi}{8}, \quad \bar{y} = \frac{M_x}{A} = \int_0^1 \frac{(\pi/2) + \arcsin x}{2}\left[\frac{\pi}{2} - \arcsin x\right]dx = 1$$

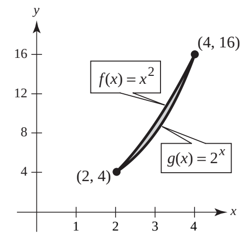
106. $f(x) = x^2,\ g(x) = 2^x$

$f(2) = g(2) = 4,\ f(4) = g(4) = 16$

$$m = \int_2^4 (x^2 - 2^x)\,dx = \left[\frac{x^3}{3} - \frac{1}{\ln 2}2^x\right]_2^4$$

$$= \left(\frac{64}{3} - \frac{16}{\ln 2}\right) - \left(\frac{8}{3} - \frac{4}{\ln 2}\right)$$

$$= \frac{56}{3} - \frac{12}{\ln 2} \approx 1.3543$$



$$M_x = \int_2^4 \frac{1}{2}(x^2 + 2^x)(x^2 - 2^x)\,dx$$

$$= \frac{1}{2}\int_2^4 (x^4 - 2^{2x})\,dx$$

$$= \frac{1}{2}\left[\frac{x^5}{5} - \frac{2^{2x}}{2\ln 2}\right]_2^4$$

$$= \frac{1}{2}\left[\left(\frac{1024}{5} - \frac{128}{\ln 2}\right) - \left(\frac{32}{5} - \frac{8}{\ln 2}\right)\right]$$

$$= \frac{496}{5} - \frac{60}{\ln 2} \approx 12.6383$$

$$M_y = \int_2^4 x[x^2 - 2^x]\,dx$$

$$= -\frac{56}{\ln 2} + \frac{12}{(\ln 2)^2} \approx 4.1855$$

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) \approx (3.0905, 9.3318)$$

107. $$\text{Average value} = \frac{1}{\pi}\int_0^{\pi} e^{-4t}(\cos 2t + 5\sin 2t)\,dt$$

$$= \frac{1}{\pi}\left[e^{-4t}\left(\frac{-4\cos 2t + 2\sin 2t}{20}\right) + 5e^{-4t}\left(\frac{-4\sin 2t - 2\cos 2t}{20}\right)\right]_0^{\pi}$$ (From Exercises 93 and 94)

$$= \frac{7}{10\pi}(1 - e^{-4\pi}) \approx 0.223$$

108. (a) $$\text{Average} = \int_1^2 (1.6t\ln t + 1)\,dt = \left[0.8t^2\ln t - 0.4t^2 + t\right]_1^2 = 3.2(\ln 2) - 0.2 \approx 2.018$$

(b) $$\text{Average} = \int_3^4 (1.6t\ln t + 1)\,dt = \left[0.8t^2\ln t - 0.4t^2 + t\right]_3^4 = 12.8(\ln 4) - 7.2(\ln 3) - 1.8 \approx 8.035$$

109. $c(t) = 100{,}000 + 4000t,\ r = 5\%,\ t_1 = 10$

$$P = \int_0^{10} (100{,}000 + 4000t)e^{-0.05t}\,dt$$

$$= 4000\int_0^{10} (25 + t)e^{-0.05t}\,dt$$

Let $u = 25 + t,\ dv = e^{-0.05t}dt,\ du = dt,\ v = -\frac{100}{5}e^{-0.05t}$.

$$P = 4000\left\{\left[(25 + t)\left(-\frac{100}{5}e^{-0.05t}\right)\right]_0^{10} + \frac{100}{5}\int_0^{10} e^{-0.05t}\,dt\right\}$$

$$= 4000\left\{\left[(25 + t)\left(-\frac{100}{5}e^{-0.05t}\right)\right]_0^{10} - \left[\frac{10{,}000}{25}e^{-0.05t}\right]_0^{10}\right\}$$

$\approx \$931{,}265$

110. $c(t) = 30{,}000 + 500t,\ r = 7\%,\ t_1 = 5$

$$P\int_0^5 (30{,}000 + 500t)e^{-0.07t}\,dt = 500\int_0^5 (60 + t)e^{-0.07t}\,dt$$

Let $u = 60 + t,\ dv = e^{-0.07t}\,dt,\ du = dt,\ v = -\frac{100}{7}e^{-0.07t}$.

$$P = 500\left\{\left[(60 + t)\left(-\frac{100}{7}e^{-0.07t}\right)\right]_0^5 + \frac{100}{7}\int_0^5 e^{-0.07t}\,dt\right\}$$

$$= 500\left\{\left[(60 + t)\left(-\frac{100}{7}e^{-0.07t}\right)\right]_0^5 - \left[\frac{10{,}000}{49}e^{-0.07t}\right]_0^5\right\}$$

$\approx \$131{,}528.68$

111. $$\int_{-\pi}^{\pi} x \sin nx\,dx = \left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_{-\pi}^{\pi}$$
$$= -\frac{\pi}{n}\cos \pi n - \frac{\pi}{n}\cos(-\pi n)$$
$$= -\frac{2\pi}{n}\cos \pi n$$
$$= \begin{cases} -(2\pi/n), & \text{if } n \text{ is even} \\ (2\pi/n), & \text{if } n \text{ is odd} \end{cases}$$

112. $$\int_{-\pi}^{\pi} x^2 \cos nx\,dx = \left[\frac{x^2}{n}\sin nx + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx\right]_{-\pi}^{\pi}$$
$$= \frac{2\pi}{n^2}\cos n\pi + \frac{2\pi}{n^2}\cos(-n\pi)$$
$$= \frac{4\pi}{n^2}\cos n\pi$$
$$= \begin{cases} (4\pi/n^2), & \text{if } n \text{ is even} \\ -(4\pi/n^2), & \text{if } n \text{ is odd} \end{cases}$$
$$= \frac{(-1)^n 4\pi}{n^2}$$

113. Let $u = x$, $dv = \sin\left(\frac{n\pi}{2}x\right)dx$, $du = dx$, $v = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)$.

$$I_1 = \int_0^1 x\sin\left(\frac{n\pi}{2}x\right)dx = \left[\frac{-2x}{n\pi}\cos\left(\frac{n\pi}{2}x\right)\right]_0^1 + \frac{2}{n\pi}\int_0^1 \cos\left(\frac{n\pi}{2}x\right)dx$$
$$= -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \left[\left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}x\right)\right]_0^1$$
$$= -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right)$$

Let $u = (-x + 2)$, $dv = \sin\left(\frac{n\pi}{2}x\right)dx$, $du = -dx$, $v = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)$.

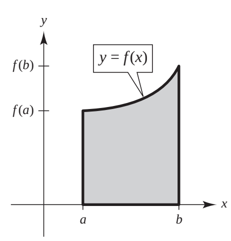
$$I_2 = \int_1^2 (-x+2)\sin\left(\frac{n\pi}{2}x\right)dx = \left[\frac{-2(-x+2)}{n\pi}\cos\left(\frac{n\pi}{2}x\right)\right]_1^2 - \frac{2}{n\pi}\int_1^2 \cos\left(\frac{n\pi}{2}x\right)dx$$
$$= \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \left[\left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}x\right)\right]_1^2$$
$$= \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right)$$

$$h(I_1 + I_2) = b_n = h\left[\left(\frac{2}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right)\right] = \frac{8h}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$

114. For any integrable function, $\int f(x)\,dx = C + \int f(x)\,dx$, but this cannot be used to imply that $C = 0$.

115. **Shell Method:**

$$V = 2\pi\int_a^b x\,f(x)\,dx$$
$$dv = x\,dx \implies v = \frac{x^2}{2}$$
$$u = f(x) \implies du = f'(x)\,dx$$
$$V = 2\pi\left[\frac{x^2}{2}f(x) - \int\frac{x^2}{2}f'(x)\,dx\right]_a^b$$
$$= \pi\left[(b^2f(b) - a^2f(a)) - \int_a^b x^2f'(x)\,dx\right]$$



Disk Method:

$$V = \pi\int_0^{f(a)}(b^2 - a^2)\,dy + \pi\int_{f(a)}^{f(b)}\left[b^2 - [f^{-1}(y)]^2\right]dy$$
$$= \pi(b^2 - a^2)f(a) + \pi b^2(f(b) - f(a)) - \pi\int_{f(a)}^{f(b)}[f^{-1}(y)]^2\,dy$$
$$= \pi\left[(b^2f(b) - a^2f(a)) - \int_{f(a)}^{f(b)}[f^{-1}(y)]^2\,dy\right]$$

Since $x = f^{-1}(y)$, we have $f(x) = y$ and $f'(x)\,dx = dy$. When $y = f(a)$, $x = a$. When $y = f(b)$, $x = b$. Thus,

$$\int_{f(a)}^{f(b)}[f^{-1}(y)]^2\,dy = \int_a^b x^2f'(x)\,dx$$

and the volumes are the same.

116. $f'(x) = xe^{-x}$

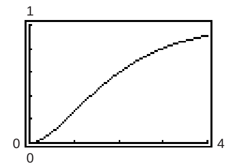
(a) $f(x) = \int xe^{-x}\,dx = -xe^{-x} - e^{-x} + C$

(Parts: $u = x$, $dv = e^{-x}\,dx$)

$f(0) = 0 = -1 + C \implies C = 1$

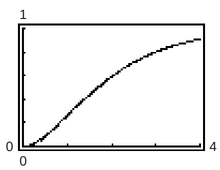
$f(x) = -xe^{-x} - e^{-x} + 1$

(b)



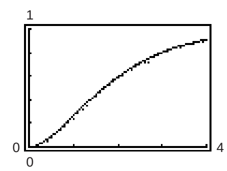
(c) You obtain the points:

n	x_n	y_n
0	0	0
1	0.05	0
2	0.10	2.378×10^{-3}
3	0.15	0.0069
4	0.20	0.0134
⋮	⋮	⋮
80	4.0	0.9064



(d) You obtain the points:

n	x_n	y_n
0	0	0
1	0.1	0
2	0.2	0.0090484
3	0.3	0.025423
4	0.4	0.047648
⋮	⋮	⋮
40	4.0	0.9039



(e) The result in part (c) is better because h is smaller.

117. $f'(x) = 3x\sin(2x)$, $f(0) = 0$

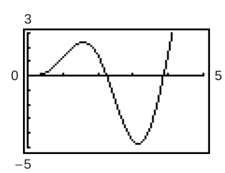
(a) $f(x) = \int 3x\sin 2x\,dx = -\frac{3}{4}(2x\cos 2x - \sin 2x) + C$

(Parts: $u = 3x$, $dv = \sin 2x\,dx$)

$f(0) = 0 = -\frac{3}{4}(0) + C \implies C = 0$

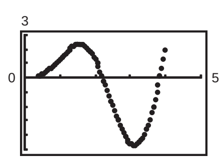
$f(x) = -\frac{3}{4}(2x\cos 2x - \sin 2x)$

(b)



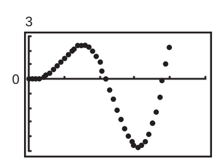
(c) Using $h = 0.05$, you obtain the points:

n	x_n	y_n
0	0	0
1	0.05	0.05
2	0.10	7.4875×10^{-4}
3	0.15	0.0037
4	0.20	0.0104
⋮	⋮	⋮
80	4.0	1.3181



(d) Using $h = 0.1$, you obtain the points:

n	x_n	y_n
0	0	0
1	0.1	0
2	0.2	0.0060
3	0.3	0.0293
4	0.4	0.0801
⋮	⋮	⋮
40	4.0	1.0210



118. $f'(x) = \cos\sqrt{x},\ f(0) = 1$

(a) Let $w = \sqrt{x}$, $w^2 = x$, $2w\,dw = dx$.

$$\int \cos\sqrt{x}\,dx = \int \cos w(2w\,dw)$$

Now use parts: $u = 2w$, $dv = \cos w\,dw$.

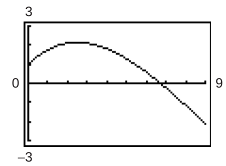
$$\int \cos\sqrt{x}\,dx = 2w\sin w + 2\cos w + C$$

$$= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$$

$f(0) = 1 = 2 + C \implies C = -1$

$f(x) = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} - 1$

(b)



(c) Using $h = 0.05$, you obtain the points:

n	x_n	y_n
0	0	1
1	0.05	1.05
2	0.1	1.0988
3	0.15	1.1463
4	0.2	1.1926
⋮	⋮	⋮
80	4.0	1.8404

(d) Using $h = 0.1$, you obtain the points:

n	x_n	y_n
0	0	1
1	0.1	1.1
2	0.2	1.1950
3	0.3	1.2852
4	0.4	1.3706
⋮	⋮	⋮
80	4.0	1.8759

119. On $\left[0, \dfrac{\pi}{2}\right]$, $\sin x \le 1 \implies x\sin x \le x \implies \displaystyle\int_0^{\pi/2} x\sin x\,dx \le \int_0^{\pi/2} x\,dx$.

120. (a) $A = \displaystyle\int_0^{\pi} x\sin x\,dx = \Big[\sin x - x\cos x\Big]_0^{\pi} = \pi$

(b) $\displaystyle\int_{\pi}^{2\pi} x\sin x\,dx = \Big[\sin x - x\cos x\Big]_{\pi}^{2\pi} = -2\pi - \pi = -3\pi$

$A = 3\pi$

(c) $\displaystyle\int_{2\pi}^{3\pi} x\sin x\,dx = \Big[\sin x - x\cos x\Big]_{2\pi}^{3\pi} = 3\pi + 2\pi = 5\pi$

$A = 5\pi$

The area between $y = x\sin x$ and $y = 0$ on $[n\pi, (n+1)\pi]$ is $(2n+1)\pi$:

$$\int_{n\pi}^{(n+1)\pi} x\sin x\,dx = \Big[\sin x - x\cos x\Big]_{n\pi}^{(n+1)\pi} = \pm(n+1)\pi \pm n\pi = \pm(2n+1)\pi$$

$A = |\pm(2n+1)\pi| = (2n+1)\pi$

Section 8.3 Trigonometric Integrals

1. $y = \sec x$

$y' = \sec x \tan x = \sin x \sec^2 x$

$\int \sin x \sec^2 x\,dx = \sec x + C$

Matches (c)

2. $y = \cos x + \sec x$

$y' = -\sin x + \sec x \tan x$

$= -\sin x + \sin x \sec^2 x$

$= -\sin x(1 - \sec^2 x)$

$= \sin x \tan^2 x$

$\int \sin x \tan^2 x\,dx = \cos x + \sec x + C$

Matches (a)

3. $y = x - \tan x + \frac{1}{3}\tan^3 x$

$y' = 1 - \sec^2 x + \tan^2 x(\sec^2 x)$

$= -\tan^2 x + \tan^2 x(1 + \tan^2 x)$

$= \tan^4 x$

$\int \tan^4 x\,dx = x - \tan x + \frac{1}{3}\tan^3 x + C$

Matches (d)

4.
$$y = 3x + 2\sin x \cos^3 x + 3\sin x \cos x$$
$$y' = 3 + 2\cos^4 x - 6\sin^2 x \cos^2 x + 3\cos^2 x - 3\sin^2 x$$
$$= 3 + 2\cos^4 x - 6\cos^2 x(1 - \cos^2 x) + 3\cos^2 x - 3(1 - \cos^2 x) = 8\cos^4 x$$

$\int 8\cos^4 x\,dx = 3x + 2\sin x \cos^3 x + 3\sin x \cos x + C$

Matches (b)

5. Let $u = \cos x$, $du = -\sin x\,dx$.

$$\int \cos^3 x \sin x\,dx = -\int \cos^3 x(-\sin x)\,dx = -\frac{1}{4}\cos^4 x + C$$

6.
$$\int \cos^3 x \sin^4 x\,dx = \int \cos x(1 - \sin^2 x)\sin^4 x\,dx = \int (\sin^4 x - \sin^6 x)\cos x\,dx = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$$

7. Let $u = \sin 2x$, $du = 2\cos 2x\,dx$.

$$\int \sin^5 2x \cos 2x\,dx = \frac{1}{2}\int \sin^5 2x(2\cos 2x)\,dx = \frac{1}{12}\sin^6 2x + C$$

8. Let $u = \cos x$, $du = -\sin x\,dx$.

$$\int \sin^3 x\,dx = \int \sin x(1 - \cos^2 x)\,dx = \int \cos^2 x(-\sin x)\,dx + \int \sin x\,dx = \frac{1}{3}\cos^3 x - \cos x + C$$

9. Let $u = \cos x$, $du = -\sin x\,dx$.

$$\int \sin^5 x \cos^2 x\,dx = \int \sin x(1 - \cos^2 x)^2 \cos^2 x\,dx$$
$$= -\int (\cos^2 x - 2\cos^4 x + \cos^6 x)(-\sin x)\,dx = \frac{-1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C$$

10. Let $u = \sin\frac{x}{3}$, $du = \frac{1}{3}\cos\frac{x}{3}\,dx$.

$$\begin{aligned}\int \cos^3\frac{x}{3}\,dx &= \int\left(\cos\frac{x}{3}\right)\left(1-\sin^2\frac{x}{3}\right)dx\\ &= 3\int\left(1-\sin^2\frac{x}{3}\right)\left(\frac{1}{3}\cos\frac{x}{3}\right)dx\\ &= 3\left(\sin\frac{x}{3}-\frac{1}{3}\sin^3\frac{x}{3}\right)+C\\ &= 3\sin\frac{x}{3}-\sin^3\frac{x}{3}+C\end{aligned}$$

11.
$$\begin{aligned}\int\cos^3\theta\sqrt{\sin\theta}\,d\theta &= \int\cos\theta(1-\sin^2\theta)(\sin\theta)^{1/2}\,d\theta\\ &= \int\left[(\sin\theta)^{1/2}-(\sin\theta)^{5/2}\right]\cos\theta\,d\theta\\ &= \frac{2}{3}(\sin\theta)^{3/2}-\frac{2}{7}(\sin\theta)^{7/2}+C\end{aligned}$$

12.
$$\begin{aligned}\int\frac{\sin^5 t}{\sqrt{\cos t}}\,dt &= \int\sin t(1-\cos^2 t)^2(\cos t)^{-1/2}\,dt\\ &= \int\sin t(1-2\cos^2 t+\cos^4 t)(\cos t)^{-1/2}\,dt\\ &= \int\left[(\cos t)^{-1/2}-2(\cos t)^{3/2}+(\cos t)^{7/2}\right]\sin t\,dt = -2(\cos t)^{1/2}+\frac{4}{5}(\cos t)^{5/2}-\frac{2}{9}(\cos t)^{9/2}+C\end{aligned}$$

13.
$$\begin{aligned}\int\cos^2 3x\,dx &= \int\frac{1+\cos 6x}{2}\,dx\\ &= \frac{1}{2}\left(x+\frac{1}{6}\sin 6x\right)+C\\ &= \frac{1}{12}(6x+\sin 6x)+C\end{aligned}$$

14.
$$\begin{aligned}\int\sin^2 2x\,dx &= \int\frac{1-\cos 4x}{2}\,dx = \frac{1}{2}\left(x-\frac{1}{4}\sin 4x\right)+C\\ &= \frac{1}{8}(4x-\sin 4x)+C\end{aligned}$$

15.
$$\begin{aligned}\int\sin^2\alpha\cdot\cos^2\alpha\,d\alpha &= \int\frac{1-\cos 2\alpha}{2}\cdot\frac{1+\cos 2\alpha}{2}\,d\alpha\\ &= \frac{1}{4}\int(1-\cos^2 2\alpha)\,d\alpha\\ &= \frac{1}{4}\int\left(1-\frac{1+\cos 4\alpha}{2}\right)d\alpha\\ &= \frac{1}{8}\int(1-\cos 4\alpha)\,d\alpha\\ &= \frac{1}{8}\left[\alpha-\frac{1}{4}\sin 4\alpha\right]+C\\ &= \frac{1}{32}[4\alpha-\sin 4\alpha]+C\end{aligned}$$

16.
$$\begin{aligned}\int\sin^4 2\theta\,d\theta &= \int\frac{1-\cos 4\theta}{2}\cdot\frac{1-\cos 4\theta}{2}\,d\theta\\ &= \frac{1}{4}\int(1-2\cos 4\theta+\cos^2 4\theta)\,d\theta\\ &= \frac{1}{4}\int\left(1-2\cos 4\theta+\frac{1+\cos 8\theta}{2}\right)d\theta\\ &= \frac{1}{4}\int\left(\frac{3}{2}-2\cos 4\theta+\frac{1}{2}\cos 8\theta\right)d\theta\\ &= \frac{1}{4}\left[\frac{3}{2}\theta-\frac{1}{2}\sin 4\theta+\frac{1}{16}\sin 8\theta\right]+C\\ &= \frac{3}{8}\theta-\frac{1}{8}\sin 4\theta+\frac{1}{64}\sin 8\theta+C\end{aligned}$$

17. Integration by parts:

$$dv = \sin^2 x\,dx = \frac{1-\cos 2x}{2} \implies v = \frac{x}{2}-\frac{\sin 2x}{4} = \frac{1}{4}(2x-\sin 2x)$$

$$u = x \implies du = dx$$

$$\begin{aligned}\int x\sin^2 x\,dx &= \frac{1}{4}x(2x-\sin 2x)-\frac{1}{4}\int(2x-\sin 2x)\,dx\\ &= \frac{1}{4}x(2x-\sin 2x)-\frac{1}{4}\left(x^2+\frac{1}{2}\cos 2x\right)+C = \frac{1}{8}(2x^2-2x\sin 2x-\cos 2x)+C\end{aligned}$$

18. Use integration by parts twice.

$dv = \sin^2 x\,dx = \dfrac{1-\cos 2x}{2} \implies v = \dfrac{x}{2} - \dfrac{\sin 2x}{4} = \dfrac{1}{4}(2x - \sin 2x)$

$u = x^2 \implies du = 2x\,dx$

$dv = \sin 2x\,dx \implies v = -\dfrac{1}{2}\cos 2x$

$u = x \implies du = dx$

$$\begin{aligned}\int x^2 \sin^2 x\,dx &= \frac{1}{4}x^2(2x - \sin 2x) - \frac{1}{2}\int (2x^2 - x\sin 2x)\,dx\\ &= \frac{1}{2}x^3 - \frac{1}{4}x^2\sin 2x - \frac{1}{3}x^3 + \frac{1}{2}\int x\sin 2x\,dx\\ &= \frac{1}{6}x^3 - \frac{1}{4}x^2\sin 2x + \frac{1}{2}\left[-\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\,dx\right]\\ &= \frac{1}{6}x^3 - \frac{1}{4}x^2\sin 2x - \frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x + C\\ &= \frac{1}{24}(4x^3 - 6x^2\sin 2x - 6x\cos 2x + 3\sin 2x) + C\end{aligned}$$

19. $\displaystyle\int_0^{\pi/2} \cos^3 x\,dx = \frac{2}{3}, \quad (n = 3)$

20. $\displaystyle\int_0^{\pi/2} \cos^5 x\,dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right) = \frac{8}{15}, \quad (n = 5)$

21. $\displaystyle\int_0^{\pi/2} \cos^7 x\,dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right) = \frac{16}{35}, \quad (n = 7)$

22. $\displaystyle\int_0^{\pi/2} \sin^2 x\,dx = \left(\frac{1}{2}\right)\frac{\pi}{2} = \frac{\pi}{4}, \quad (n = 2)$

23. $\displaystyle\int_0^{\pi/2} \sin^6 x\,dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\frac{\pi}{2} = \frac{5\pi}{32}, \quad (n = 6)$

24. $\displaystyle\int_0^{\pi/2} \sin^7 x\,dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right) = \frac{16}{35}, \quad (n = 7)$

25. $\displaystyle\int \sec(3x)\,dx = \frac{1}{3}\ln\left|\sec 3x + \tan 3x\right| + C$

26. $\displaystyle\int \sec^2(2x - 1)\,dx = \frac{1}{2}\tan(2x - 1) + C$

27.
$$\begin{aligned}\int \sec^4 5x\,dx &= \int (1 + \tan^2 5x)\sec^2 5x\,dx\\ &= \frac{1}{5}\left(\tan 5x + \frac{\tan^3 5x}{3}\right) + C\\ &= \frac{\tan 5x}{15}(3 + \tan^2 5x) + C\end{aligned}$$

28.
$$\begin{aligned}\int \sec^6 3x\,dx &= \int (1 + \tan^2 3x)^2 \sec^2 3x\,dx\\ &= \int (1 + 2\tan^2 3x + \tan^4 3x)\sec^2 3x\,dx\\ &= \frac{1}{3}\tan 3x + \frac{2}{9}\tan^3 3x + \frac{1}{15}\tan^5 3x + C\end{aligned}$$

29. $dv = \sec^2 \pi x\,dx \implies v = \dfrac{1}{\pi}\tan \pi x$

$u = \sec \pi x \implies du = \pi \sec \pi x \tan \pi x\,dx$

$$\int \sec^3 \pi x\,dx = \frac{1}{\pi}\sec \pi x\tan \pi x - \int \sec \pi x\tan^2 \pi x\,dx = \frac{1}{\pi}\sec \pi x\tan \pi x - \int \sec \pi x(\sec^2 \pi x - 1)\,dx$$

$$2\int \sec^3 \pi x\,dx = \frac{1}{\pi}\left(\sec \pi x\tan \pi x + \ln\left|\sec \pi x + \tan \pi x\right|\right) + C_1$$

$$\int \sec^3 \pi x\,dx = \frac{1}{2\pi}\left(\sec \pi x\tan \pi x + \ln\left|\sec \pi x + \tan \pi x\right|\right) + C$$

30. $\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C$

31. $\int \tan^5 \frac{x}{4}\,dx = \int \left(\sec^2 \frac{x}{4} - 1\right) \tan^3 \frac{x}{4}\,dx$

$= \int \tan^3 \frac{x}{4} \sec^2 \frac{x}{4}\,dx - \int \tan^3 \frac{x}{4}\,dx$

$= \tan^4 \frac{x}{4} - \int \left(\sec^2 \frac{x}{4} - 1\right) \tan \frac{x}{4}\,dx$

$= \tan^4 \frac{x}{4} - 2\tan^2 \frac{x}{4} - 4\ln\left|\cos \frac{x}{4}\right| + C$

32. $\int \tan^3 \frac{\pi x}{2} \sec^2 \frac{\pi x}{2}\,dx = \frac{1}{2\pi} \tan^4 \frac{\pi x}{2} + C$

33. $u = \tan x,\ du = \sec^2 x\,dx$

$\int \sec^2 x \tan x\,dx = \frac{1}{2}\tan^2 x + C$

[or, $u = \sec x,\ du = \sec x \tan x\,dx,$

$\int \sec^2 x \tan x\,dx = \frac{1}{2}\sec^2 x + C.$]

34. Let $u = \sec 2t,\ du = 2\sec 2t \tan 2t.$

$\int \tan^3 2t \cdot \sec^3 2t\,dt = \int (\sec^2 2t - 1)\sec^3 2t \cdot \tan 2t\,dt = \int (\sec^4 2t - \sec^2 2t)(\sec 2t \tan 2t)\,dt = \frac{\sec^5 2t}{10} - \frac{\sec^3 2t}{6} + C$

35. $\int \tan^2 x \sec^2 x\,dx = \frac{\tan^3 x}{3} + C$

36. $\int \tan^5 2x \sec^2 2x\,dx = \frac{1}{12}\tan^6 2x + C$

37. $\int \sec^6 4x \tan 4x\,dx = \frac{1}{4}\int \sec^5 4x(4\sec 4x \tan 4x)\,dx$

$= \frac{\sec^6 4x}{24} + C$

38. $\int \sec^2 \frac{x}{2} \tan \frac{x}{2}\,dx = 2\int \sec \frac{x}{2}\left(\frac{1}{2}\sec \frac{x}{2} \tan \frac{x}{2}\right)dx$

$= \sec^2 \frac{x}{2} + C$ or

$\int \sec^2 \frac{x}{2} \tan \frac{x}{2}\,dx = 2\int \tan \frac{x}{2}\left(\frac{1}{2}\sec^2 \frac{x}{2}\right)dx = \tan^2 \frac{x}{2} + C$

39. Let $u = \sec x,\ du = \sec x \tan x\,dx.$

$\int \sec^3 x \tan x\,dx = \int \sec^2 x(\sec x \tan x)\,dx$

$= \frac{1}{3}\sec^3 x + C$

40. $\int \tan^3 3x\,dx = \int (\sec^2 3x - 1)\tan 3x\,dx$

$= \frac{1}{3}\int \tan 3x(3\sec^2 3x)\,dx + \frac{1}{3}\int \frac{-3\sin 3x}{\cos 3x}\,dx$

$= \frac{1}{6}\tan^2 3x + \frac{1}{3}\ln|\cos 3x| + C$

41. $\int \frac{\tan^2 x}{\sec x}\,dx = \int \frac{(\sec^2 x - 1)}{\sec x}\,dx$

$= \int (\sec x - \cos x)\,dx$

$= \ln|\sec x + \tan x| - \sin x + C$

42. $\int \frac{\tan^2 x}{\sec^5 x} = \int \frac{\sin^2 x}{\cos^2 x} \cdot \cos^5 x\,dx$

$= \int \sin^2 x \cdot \cos^3 x\,dx$

$= \int \sin^2 x(1 - \sin^2 x)\cos x\,dx$

$= \int (\sin^2 x - \sin^4 x)\cos x\,dx$

$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$

43. $r = \int \sin^4(\pi\theta)\,d\theta = \frac{1}{4}\int[1 - \cos(2\pi\theta)]^2\,d\theta$

$= \frac{1}{4}\int[1 - 2\cos(2\pi\theta) + \cos^2(2\pi\theta)]\,d\theta$

$= \frac{1}{4}\int\left[1 - 2\cos(2\pi\theta) + \frac{1 + \cos(4\pi\theta)}{2}\right]d\theta$

$= \frac{1}{4}\left[\theta - \frac{1}{\pi}\sin(2\pi\theta) + \frac{\theta}{2} + \frac{1}{8\pi}\sin(4\pi\theta)\right] + C$

$= \frac{1}{32\pi}[12\pi\theta - 8\sin(2\pi\theta) + \sin(4\pi\theta)] + C$

44. $s = \int \sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}\,d\alpha$

$= \int\left(\frac{1 - \cos\alpha}{2}\right)\left(\frac{1 + \cos\alpha}{2}\right)d\alpha = \int\frac{1 - \cos^2\alpha}{4}\,d\alpha$

$= \frac{1}{4}\int\sin^2\alpha\,d\alpha = \frac{1}{8}\int(1 - \cos 2\alpha)\,d\alpha$

$= \frac{1}{8}\left[\theta - \frac{\sin 2\alpha}{2}\right] + C$

$= \frac{1}{16}(2\alpha - \sin 2\alpha) + C$

45. $y = \int\tan^3 3x\sec 3x\,dx$

$= \int(\sec^2 3x - 1)\sec 3x\tan 3x\,dx$

$= \frac{1}{3}\int\sec^2 3x(3\sec 3x\tan 3x)\,dx - \frac{1}{3}\int 3\sec 3x\tan 3x\,dx$

$= \frac{1}{9}\sec^3 3x - \frac{1}{3}\sec 3x + C$

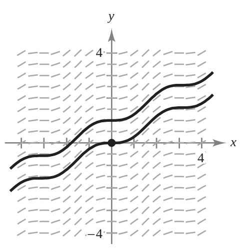
46. $y = \int\sqrt{\tan x}\sec^4 x\,dx$

$= \int\tan^{1/2}x(\tan^2 x + 1)\sec^2 x\,dx$

$= \int(\tan^{5/2}x + \tan^{1/2}x)\sec^2 x\,dx$

$= \frac{2}{7}\tan^{7/2}x + \frac{2}{3}\tan^{3/2}x + C$

47. (a)



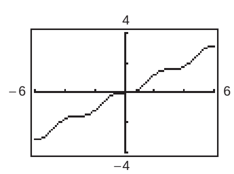
(b) $\frac{dy}{dx} = \sin^2 x, \quad (0, 0)$

$y = \int\sin^2 x\,dx$

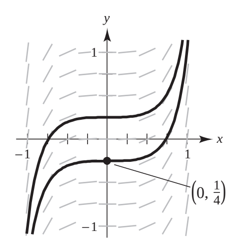
$= \int\frac{1 - \cos 2x}{2}\,dx$

$= \frac{1}{2}x - \frac{\sin 2x}{4} + C$

$(0, 0)$: $0 = C, y = \frac{1}{2}x - \frac{\sin 2x}{4}$



48. (a)



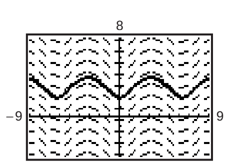
(b) $\frac{dy}{dx} = \sec^2 x\tan^2 x, \quad \left(0, -\frac{1}{4}\right)$

$y = \int\sec^2 x\tan^2 x\,dx \quad u = \tan x, du = \sec^2 x\,dx$

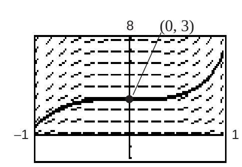
$y = \frac{\tan^3 x}{3} + C$

$\left(0, -\frac{1}{4}\right)$: $-\frac{1}{4} = C \Rightarrow y = \frac{1}{3}\tan^3 x - \frac{1}{4}$

49. $\frac{dy}{dx} = \frac{3\sin x}{y}, y(0) = 2$



50. $\frac{dy}{dx} = 3\sqrt{y}\tan^2 x, y(0) = 3$



51. $\int \sin 3x \cos 2x \, dx = \frac{1}{2}\int (\sin 5x + \sin x)\, dx$

$= \frac{-1}{2}\left(\frac{1}{5}\cos 5x + \cos x\right) + C$

$= \frac{-1}{10}(\cos 5x + 5\cos x) + C$

52. $\int \cos 4\theta \cos(-3\theta)\, d\theta = \int \cos 4\theta \cos 3\theta \, d\theta$

$= \frac{1}{2}\int (\cos 7\theta + \cos\theta)\, d\theta$

$= \frac{\sin 7\theta}{14} + \frac{\sin\theta}{2} + C$

53. $\int \sin\theta \sin 3\theta \, d\theta = \frac{1}{2}\int (\cos 2\theta - \cos 4\theta)\, d\theta$

$= \frac{1}{2}\left(\frac{1}{2}\sin 2\theta - \frac{1}{4}\sin 4\theta\right) + C$

$= \frac{1}{8}(2\sin 2\theta - \sin 4\theta) + C$

54. $\int \sin(-4x)\cos 3x \, dx = -\int \sin 4x \cos 3x \, dx$

$= -\frac{1}{2}\int (\sin x + \sin 7x)\, dx$

$= -\frac{1}{2}\left[-\cos x - \frac{1}{7}\cos 7x\right] + C$

$= \frac{1}{14}[7\cos x + \cos 7x] + C$

55. $\int \cot^3 2x \, dx = \int (\csc^2 2x - 1)\cot 2x \, dx$

$= -\frac{1}{2}\int \cot 2x(-2\csc^2 2x)\, dx - \frac{1}{2}\int \frac{2\cos 2x}{\sin 2x}\, dx$

$= -\frac{1}{4}\cot^2 2x - \frac{1}{2}\ln|\sin 2x| + C$

$= \frac{1}{4}(\ln|\csc^2 2x| - \cot^2 2x) + C$

56. Let $u = \tan\frac{x}{2}$, $du = \frac{1}{2}\sec^2\frac{x}{2}\, dx$.

$\int \tan^4\frac{x}{2}\sec^4\frac{x}{2}\, dx = \int \tan^4\frac{x}{2}\left(\tan^2\frac{x}{2} + 1\right)\sec^2\frac{x}{2}\, dx$

$= 2\int\left(\tan^6\frac{x}{2} + \tan^4\frac{x}{2}\right)\left(\frac{1}{2}\sec^2\frac{x}{2}\right) dx$

$= \frac{2}{7}\tan^7\frac{x}{2} + \frac{2}{5}\tan^5\frac{x}{2} + C$

57. Let $u = \cot\theta$, $du = -\csc^2\theta\, d\theta$.

$\int \csc^4\theta\, d\theta = \int \csc^2\theta(1 + \cot^2\theta)\, d\theta$

$= \int \csc^2\theta\, d\theta + \int \csc^2\theta\cot^2\theta\, d\theta$

$= -\cot\theta - \frac{1}{3}\cot^3\theta + C$

58. $u = \cot 3x$, $du = -3\csc^2 3x\, dx$

$\int \csc^2 3x \cot 3x\, dx = -\frac{1}{3}\int \cot 3x(-3\csc^2 3x)\, dx$

$= -\frac{1}{6}\cot^2 3x + C$

59. $\int \frac{\cot^2 t}{\csc t}\, dt = \int \frac{\csc^2 t - 1}{\csc t}\, dt$

$= \int (\csc t - \sin t)\, dt$

$= \ln|\csc t - \cot t| + \cos t + C$

60. $\int \frac{\cot^3 t}{\csc t}\, dt = \int \frac{\cos^3 t}{\sin^2 t}\, dt = \int \frac{(1 - \sin^2 t)\cos t}{\sin^2 t}\, dt$

$= \int \frac{\cos t}{\sin^2 t}\, dt - \int \cos t\, dt$

$= \frac{-1}{\sin t} - \sin t + C = -\csc t - \sin t + C$

61. $\int \frac{1}{\sec x \tan x}\, dx = \int \frac{\cos^2 x}{\sin x}\, dx = \int \frac{1 - \sin^2 x}{\sin x}\, dx$

$= \int (\csc x - \sin x)\, dx$

$= \ln|\csc x - \cot x| + \cos x + C$

62. $\int \frac{\sin^2 x - \cos^2 x}{\cos x}\, dx = \int \frac{1 - 2\cos^2 x}{\cos x}\, dx$

$= \int (\sec x - 2\cos x)\, dx$

$= \ln|\sec x + \tan x| - 2\sin x + C$

63. $\int (\tan^4 t - \sec^4 t)\,dt = \int (\tan^2 t + \sec^2 t)(\tan^2 t - \sec^2 t)\,dt, \qquad (\tan^2 t - \sec^2 t = -1)$

$$= -\int (\tan^2 t + \sec^2 t)\,dt = -\int (2\sec^2 t - 1)\,dt = -2\tan t + t + C$$

64. $\int \frac{1-\sec t}{\cos t - 1}\,dt = \int \frac{\cos t - 1}{(\cos t - 1)\cos t}\,dt$

$$= \int \sec t\,dt = \ln|\sec t + \tan t| + C$$

65. $\int_{-\pi}^{\pi} \sin^2 x\,dx = 2\int_0^{\pi} \frac{1-\cos 2x}{2}\,dx$

$$= \left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \pi$$

66. $\int_0^{\pi/3} \tan^2 x\,dx = \int_0^{\pi/3} (\sec^2 x - 1)\,dx$

$$= \left[\tan x - x\right]_0^{\pi/3} = \sqrt{3} - \frac{\pi}{3}$$

67. $\int_0^{\pi/4} \tan^3 x\,dx = \int_0^{\pi/4} (\sec^2 x - 1)\tan x\,dx$

$$= \int_0^{\pi/4} \sec^2 x \tan x\,dx - \int_0^{\pi/4} \frac{\sin x}{\cos x}\,dx$$

$$= \left[\frac{1}{2}\tan^2 x + \ln|\cos x|\right]_0^{\pi/4}$$

$$= \frac{1}{2}(1 - \ln 2)$$

68. Let $u = \tan t$, $du = \sec^2 t\,dt$.

$$\int_0^{\pi/4} \sec^2 t\sqrt{\tan t}\,dt = \left[\frac{2}{3}\tan^{3/2} t\right]_0^{\pi/4} = \frac{2}{3}$$

69. Let $u = 1 + \sin t$, $du = \cos t\,dt$.

$$\int_0^{\pi/2} \frac{\cos t}{1 + \sin t}\,dt = \left[\ln|1 + \sin t|\right]_0^{\pi/2} = \ln 2$$

70. $\int_{-\pi}^{\pi} \sin 3\theta \cos\theta\,d\theta = \frac{1}{2}\int_{-\pi}^{\pi} (\sin 4\theta + \sin 2\theta)\,d\theta$

$$= -\frac{1}{2}\left[\frac{1}{4}\cos 4\theta + \frac{1}{2}\cos 2\theta\right]_{-\pi}^{\pi} = 0$$

71. Let $u = \sin x$, $du = \cos x\,dx$.

$$\int_{-\pi/2}^{\pi/2} \cos^3 x\,dx = 2\int_0^{\pi/2} (1 - \sin^2 x)\cos x\,dx$$

$$= 2\left[\sin x - \frac{1}{3}\sin^3 x\right]_0^{\pi/2} = \frac{4}{3}$$

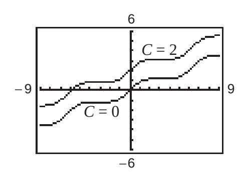
72. $\int_{-\pi/2}^{\pi/2} (\sin^2 x + 1)\,dx = \int_{-\pi/2}^{\pi/2} \left(\frac{1-\cos 2x}{2} + 1\right)dx$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{3}{2} - \frac{1}{2}\cos 2x\right)dx$$

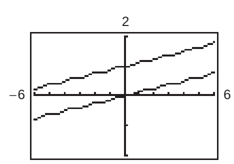
$$= \left[\frac{3}{2}x - \frac{1}{4}\sin 2x\right]_{-\pi/2}^{\pi/2} = \frac{3\pi}{2}$$

73. $\int \cos^4 \frac{x}{2}\,dx = \frac{1}{16}[6x + 8\sin x + \sin 2x] + C$

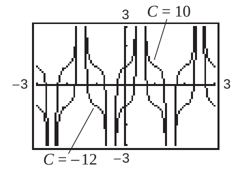
$$= \frac{1}{8}\left[4\sin\frac{x}{2}\cos^3\frac{x}{2} + 6\sin\frac{x}{2}\cos\frac{x}{2} + 3x\right] + C$$



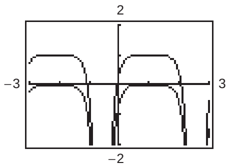
74. $\int \sin^2 x \cos^2 x\,dx = \frac{1}{32}[4x - \sin 4x] + C$



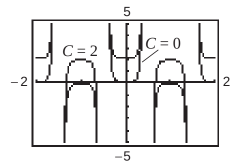
75. $\int \sec^5 \pi x \, dx = \frac{1}{4\pi}\left\{\sec^3 \pi x \tan \pi x + \frac{3}{2}\left[\sec \pi x \tan \pi x + \ln\left|\sec \pi x + \tan \pi x\right|\right]\right\} + C$



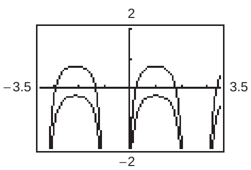
76. $\int \tan^3(1 - x) \, dx = -\frac{\tan^2(1 - x)}{2} - \ln\left|\cos(1 - x)\right| + C$



77. $\int \sec^5 \pi x \tan \pi x \, dx = \frac{1}{5\pi}\sec^5 \pi x + C$



78. $\int \sec^4(1 - x) \tan(1 - x) \, dx = -\frac{\sec^4(1 - x)}{4} + C$



79. $\int_0^{\pi/4} \sin 2\theta \sin 3\theta \, d\theta = \frac{1}{2}\left[\sin \theta - \frac{1}{5}\sin 5\theta\right]_0^{\pi/4} = \frac{3\sqrt{2}}{10}$

80. $\int_0^{\pi/2} (1 - \cos \theta)^2 \, d\theta = \left[\frac{3}{2}\theta - 2\sin \theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/2}$

$$= \frac{3\pi}{4} - 2$$

81. $\int_0^{\pi/2} \sin^4 x \, dx = \frac{1}{4}\left[\frac{3x}{2} - \sin 2x + \frac{1}{8}\sin 4x\right]_0^{\pi/2}$

$$= \frac{3\pi}{16}$$

82. $\int_0^{\pi/2} \sin^6 x \, dx = \frac{1}{8}\left[\frac{5x}{2} - 2\sin 2x + \frac{3}{8}\sin 4x + \frac{1}{6}\sin^3 2x\right]_0^{\pi/2} = \frac{5\pi}{32}$

83. (a) Save one sine factor and convert the remaining sine factors to cosine. Then expand and integrate.

(b) Save one cosine factor and convert the remaining cosine factors to sine. Then expand and integrate.

(c) Make repeated use of the power reducing formula to convert the integrand to odd powers of the cosine.

84. See guidelines on page 537.

85. (a) Let $u = \tan 3x$, $du = 3\sec^2 3x \, dx$.

$$\int \sec^4 3x \tan^3 3x \, dx = \int \sec^2 3x \tan^3 3x \sec^2 3x \, dx = \frac{1}{3}\int (\tan^2 3x + 1)\tan^3 3x(3\sec^2 3x) \, dx$$

$$= \frac{1}{3}\int (\tan^5 3x + \tan^3 3x)(3\sec^2 3x) \, dx = \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + C_1$$

Or let $u = \sec 3x$, $du = 3\sec 3x \tan 3x \, dx$.

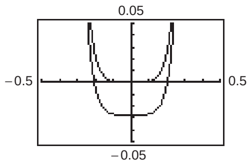
$$\int \sec^4 3x \tan^3 3x \, dx = \int \sec^3 3x \tan^2 3x \sec 3x \tan 3x \, dx$$

$$= \frac{1}{3}\int \sec^3 3x(\sec^2 3x - 1)(3\sec 3x \tan 3x) \, dx = \frac{\sec^6 3x}{18} - \frac{\sec^4 3x}{12} + C$$

—CONTINUED—

85. —CONTINUED—

(b)



(c) $\dfrac{\sec^6 3x}{18} - \dfrac{\sec^4 3x}{12} + C = \dfrac{(1 + \tan^2 3x)^3}{18} - \dfrac{(1 + \tan^2 3x)^2}{12} + C$

$$= \frac{1}{18}\tan^6 3x + \frac{1}{6}\tan^4 3x + \frac{1}{6}\tan^2 3x + \frac{1}{18} - \frac{1}{12}\tan^4 3x - \frac{1}{6}\tan^2 3x - \frac{1}{12} + C$$

$$= \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + \left(\frac{1}{18} - \frac{1}{12}\right) + C$$

$$= \frac{\tan^6 3x}{18} + \frac{\tan^4 3x}{12} + C_2$$

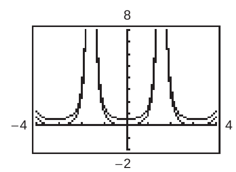
86. (a) Let $u = \tan x$, $du = \sec^2 x\, dx$.

$$\int \sec^2 x \tan x\, dx = \frac{1}{2}\tan^2 x + C_1$$

Or let $u = \sec x$, $du = \sec x \tan x\, dx$.

$$\int \sec x(\sec x \tan x)\, dx = \frac{1}{2}\sec^2 x + C$$

(b)



(c) $\dfrac{1}{2}\sec^2 x + C = \dfrac{1}{2}(\tan^2 x + 1) + C = \dfrac{1}{2}\tan^2 x + \left(\dfrac{1}{2} + C\right) = \dfrac{1}{2}\tan^2 x + C_2$

87. $A = \displaystyle\int_0^{\pi/2} (\sin x - \sin^3 x)\, dx$

$$= \int_0^{\pi/2} \sin x\, dx - \int_0^{\pi/2} \sin^3 x\, dx$$

$$= \Big[-\cos x\Big]_0^{\pi/2} - \frac{2}{3} \qquad \text{(Wallis's Formula)}$$

$$= 1 - \frac{2}{3} = \frac{1}{3}$$

88. $A = \displaystyle\int_0^1 \sin^2(\pi x)\, dx$

$$= \int_0^1 \frac{1 - \cos(2\pi x)}{2}\, dx$$

$$= \left[\frac{1}{2}x - \frac{\sin 2\pi x}{4\pi}\right]_0^1$$

$$= \frac{1}{2}$$

89. $A = \displaystyle\int_{-\pi/4}^{\pi/4} [\cos^2 x - \sin^2 x]\, dx$

$$= \int_{-\pi/4}^{\pi/4} \cos 2x\, dx$$

$$= \frac{\sin 2x}{2}\Bigg]_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

90. $A = \displaystyle\int_{-\pi/2}^{\pi/4} [\cos^2 x - \sin x \cos x]\, dx$

$$= \int_{-\pi/2}^{\pi/4} \left[\frac{1 + \cos 2x}{2} - \sin x \cos x\right] dx$$

$$= \left[\frac{1}{2}x + \frac{\sin 2x}{4} - \frac{\sin^2 x}{2}\right]_{-\pi/2}^{\pi/4}$$

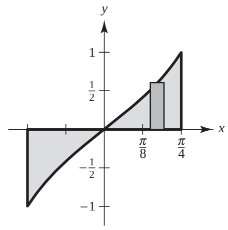
$$= \left(\frac{\pi}{8} + \frac{1}{4} - \frac{1}{4}\right) - \left(-\frac{\pi}{4} - \frac{1}{2}\right)$$

$$= \frac{3\pi}{8} + \frac{1}{2}$$

91. Disks

$R(x) = \tan x,\ r(x) = 0$

$$V = 2\pi\int_0^{\pi/4} \tan^2 x\, dx$$
$$= 2\pi\int_0^{\pi/4} (\sec^2 x - 1)\, dx$$
$$= 2\pi\Big[\tan x - x\Big]_0^{\pi/4}$$
$$= 2\pi\left(1 - \frac{\pi}{4}\right) \approx 1.348$$



92. $V = \pi\int_0^{\pi/2} \left[\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)\right] dx$

$$= \pi\int_0^{\pi/2} \cos x\, dx$$
$$= \pi\Big[\sin x\Big]_0^{\pi/2} = \pi$$

93. (a) $V = \pi\int_0^{\pi} \sin^2 x\, dx = \frac{\pi}{2}\int_0^{\pi}(1 - \cos 2x)\, dx = \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{\pi^2}{2}$

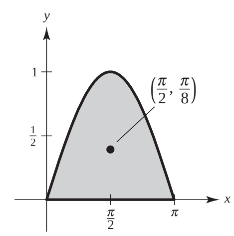
(b) $A = \int_0^{\pi} \sin x\, dx = \Big[-\cos x\Big]_0^{\pi} = 1 + 1 = 2$

Let $u = x$, $dv = \sin x\, dx$, $du = dx$, $v = -\cos x$.

$$\bar{x} = \frac{1}{A}\int_0^{\pi} x\sin x\, dx = \frac{1}{2}\left[\Big[-x\cos x\Big]_0^{\pi} + \int_0^{\pi}\cos x\, dx\right] = \frac{1}{2}\Big[-x\cos x + \sin x\Big]_0^{\pi} = \frac{\pi}{2}$$

$$\bar{y} = \frac{1}{2A}\int_0^{\pi}\sin^2 x\, dx = \frac{1}{8}\int_0^{\pi}(1 - \cos 2x)\, dx = \frac{1}{8}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{\pi}{8}$$

$$(\bar{x}, \bar{y}) = \left(\frac{\pi}{2}, \frac{\pi}{8}\right)$$



94. (a) $V = \pi\int_0^{\pi/2} \cos^2 x\, dx = \frac{\pi}{2}\int_0^{\pi/2}(1 + \cos 2x)\, dx = \frac{\pi}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi/2} = \frac{\pi^2}{4}$

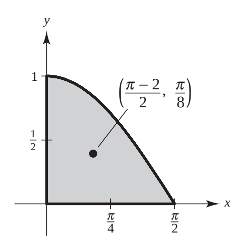
(b) $A = \int_0^{\pi/2} \cos x\, dx = \Big[\sin x\Big]_0^{\pi/2} = 1$

Let $u = x$, $dv = \cos x\, dx$, $du = dx$, $v = \sin x$.

$$\bar{x} = \int_0^{\pi/2} x\cos x\, dx = \Big[x\sin x\Big]_0^{\pi/2} - \int_0^{\pi/2}\sin x\, dx = \Big[x\sin x + \cos x\Big]_0^{\pi/2} = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2}$$

$$\bar{y} = \frac{1}{2}\int_0^{\pi/2}\cos^2 x\, dx = \frac{1}{4}\int_0^{\pi/2}(1 + \cos 2x)\, dx = \frac{1}{4}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi/2} = \frac{\pi}{8}$$

$$(\bar{x}, \bar{y}) = \left(\frac{\pi - 2}{2}, \frac{\pi}{8}\right)$$



95. $dv = \sin x\, dx \implies v = -\cos x$

$u = \sin^{n-1} x \implies du = (n - 1)\sin^{n-2} x\cos x\, dx$

$$\int \sin^n x\, dx = -\sin^{n-1} x\cos x + (n - 1)\int \sin^{n-2} x\cos^2 x\, dx = -\sin^{n-1} x\cos x + (n - 1)\int \sin^{n-2} x(1 - \sin^2 x)\, dx$$

$$= -\sin^{n-1} x\cos x + (n - 1)\int \sin^{n-2} x\, dx - (n - 1)\int \sin^n x\, dx$$

Therefore, $n\int \sin^n x\, dx = -\sin^{n-1} x\cos x + (n - 1)\int \sin^{n-2} x\, dx$

$$\int \sin^n x\, dx = \frac{-\sin^{n-1} x\cos x}{n} + \frac{n - 1}{n}\int \sin^{n-2} x\, dx.$$

96. $dv = \cos x\,dx \implies v = \sin x$

$u = \cos^{n-1} x \implies du = -(n-1)\cos^{n-2} x \sin x\,dx$

$$\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x\,dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)\,dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$$

Therefore, $n\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx$

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$$

97. Let $u = \sin^{n-1} x$, $du = (n-1)\sin^{n-2} x \cos x\,dx$, $dv = \cos^m x \sin x\,dx$, $v = \dfrac{-\cos^{m+1} x}{m+1}$.

$$\int \cos^m x \sin^n x\,dx = \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^{n-2} x \cos^{m+2} x\,dx$$

$$= \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^{n-2} x \cos^m x(1-\sin^2 x)\,dx$$

$$= \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^{n-2} x \cos^m x\,dx - \frac{n-1}{m+1}\int \sin^n x \cos^m x\,dx$$

$$\frac{m+n}{m+1}\int \cos^m x \sin^n x\,dx = \frac{-\sin^{n-1} x \cos^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \sin^{n-2} x \cos^m x\,dx$$

$$\int \cos^m x \sin^n x\,dx = \frac{-\cos^{m+1} x \sin^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \cos^m x \sin^{n-2} x\,dx$$

98. Let $u = \sec^{n-2} x$, $du = (n-2)\sec^{n-2} x \tan x\,dx$, $dv = \sec^2 x\,dx$, $v = \tan x$.

$$\int \sec^n x\,dx = \sec^{n-2} x \tan x - \int (n-2)\sec^{n-2} x \tan^2 x\,dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)\,dx$$

$$= \sec^{n-2} x \tan x - (n-2)\left[\int \sec^n x\,dx - \int \sec^{n-2} x\,dx\right]$$

$$(n-1)\int \sec^n x\,dx = \sec^{n-2} x \tan x + (n-2)\int \sec^{n-2} x\,dx$$

$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$$

99. $\displaystyle\int \sin^5 x\,dx = -\frac{\sin^4 x \cos x}{5} + \frac{4}{5}\int \sin^3 x\,dx$

$$= -\frac{\sin^4 x \cos x}{5} + \frac{4}{5}\left[-\frac{\sin^2 x \cos x}{3} + \frac{2}{3}\int \sin x\,dx\right]$$

$$= -\frac{1}{5}\sin^4 x \cos x - \frac{4}{15}\sin^2 x \cos x - \frac{8}{15}\cos x + C$$

$$= -\frac{\cos x}{15}[3\sin^4 x + 4\sin^2 x + 8] + C$$

100. $\displaystyle\int \cos^4 x\,dx = \frac{\cos^3 x \sin x}{4} + \frac{3}{4}\int \cos^2 x\,dx$

$\displaystyle = \frac{\cos^3 x \sin x}{4} + \frac{3}{4}\left[\frac{\cos x \sin x}{2} + \frac{1}{2}\int dx\right]$

$\displaystyle = \frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3}{8}x + C$

$\displaystyle = \frac{1}{8}[2\cos^3 x \sin x + 3\cos x \sin x + 3x] + C$

101. $\displaystyle\int \sec^4 \frac{2\pi x}{5}\,dx = \frac{5}{2\pi}\int \sec^4\left(\frac{2\pi x}{5}\right)\frac{2\pi}{5}\,dx$

$\displaystyle = \frac{5}{2\pi}\left[\frac{1}{3}\sec^2\left(\frac{2\pi x}{5}\right)\tan\left(\frac{2\pi x}{5}\right) + \frac{2}{3}\int \sec^2\left(\frac{2\pi x}{5}\right)\frac{2\pi}{5}\,dx\right]$

$\displaystyle = \frac{5}{6\pi}\left[\sec^2\left(\frac{2\pi x}{5}\right)\tan\left(\frac{2\pi x}{5}\right) + 2\tan\left(\frac{2\pi x}{5}\right)\right] + C$

$\displaystyle = \frac{5}{6\pi}\tan\left(\frac{2\pi x}{5}\right)\left[\sec^2\left(\frac{2\pi x}{5}\right) + 2\right] + C$

102. $\displaystyle\int \sin^4 x \cos^2 x\,dx = -\frac{\cos^3 x \sin^3 x}{6} + \frac{1}{2}\int \cos^2 x \sin^2 x\,dx$

$\displaystyle = -\frac{\cos^3 x \sin^3 x}{6} + \frac{1}{2}\left[-\frac{\cos^3 x \sin x}{4} + \frac{1}{4}\int \cos^2 x\,dx\right]$

$\displaystyle = -\frac{1}{6}\cos^3 x \sin^3 x - \frac{1}{8}\cos^3 x \sin x + \frac{1}{8}\left[\frac{\cos x \sin x}{2} + \frac{x}{2}\right] + C$

$\displaystyle = -\frac{1}{48}[8\cos^3 x \sin^3 x + 6\cos^3 x \sin x - 3\cos x \sin x - 3x] + C$

103. $\displaystyle f(t) = a_0 + a_1 \cos\frac{\pi t}{6} + b_1 \sin\frac{\pi t}{6}$

$\displaystyle a_0 = \frac{1}{12}\int_0^{12} f(t)\,dt,\;\; a_1 = \frac{1}{6}\int_0^{12} f(t)\cos\frac{\pi t}{6}\,dt,\;\; b_1 = \frac{1}{6}\int_0^{12} f(t)\sin\frac{\pi t}{6}\,dt$

(a) $\displaystyle a_0 \approx \frac{1}{12}\cdot\frac{(12-0)}{3(12)}[33.5 + 4(35.4) + 2(44.7) + 4(55.6) + 2(67.4) + 4(76.2) + 2(80.4) + 4(79.0) + 2(72.0)$

$+ 4(61.0) + 2(49.3) + 4(38.6) + 33.5]$

≈ 57.72

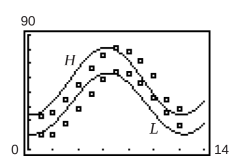
$a_1 \approx -23.36$

$b_1 \approx -2.75$ (Answers will vary.)

$\displaystyle H(t) \approx 57.72 - 23.36\cos\left(\frac{\pi t}{6}\right) - 2.75\sin\left(\frac{\pi t}{6}\right)$

(b) $\displaystyle L(t) \approx 42.04 - 20.91\cos\left(\frac{\pi t}{6}\right) - 4.33\sin\left(\frac{\pi t}{6}\right)$

(c)



Temperature difference is greatest in the summer ($t \approx 4.9$ or end of May).

104. (a) n is odd and $n \geq 3$.

$$\int_0^{\pi/2} \cos^n x\,dx = \left[\frac{\cos^{n-1} x \sin x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x\,dx$$

$$= \frac{n-1}{n}\left[\left[\frac{\cos^{n-3} x \sin x}{n-2}\right]_0^{\pi/2} + \frac{n-3}{n-2}\int_0^{\pi/2} \cos^{n-4} x\,dx\right]$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\left[\left[\frac{\cos^{n-5} x \sin x}{n-4}\right]_0^{\pi/2} + \frac{n-5}{n-4}\int_0^{\pi/2} \cos^{n-6} x\,dx\right]$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\int_0^{\pi/2} \cos^{n-6} x\,dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\int_0^{\pi/2} \cos x\,dx$$

$$= \left[\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots(\sin x)\right]_0^{\pi/2}$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots 1 \qquad \text{(Reverse the order.)}$$

$$= (1)\left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right)$$

$$= \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right)$$

(b) n is even and $n \geq 2$.

$$\int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\int_0^{\pi/2} \cos^2 x\,dx \qquad \text{(From part (a))}$$

$$= \left[\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\left(\frac{x}{2}+\frac{1}{4}\sin 2x\right)\right]_0^{\pi/2}$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{\pi}{4} \qquad \text{(Reverse the order.)}$$

$$= \left(\frac{\pi}{2}\cdot\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)$$

$$= \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$$

105. $$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi} = 0, \quad (m \neq n)$$

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m-n)x - \cos(m+n)x]\,dx$$

$$= \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0, \quad (m \neq n)$$

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} [\sin(m+n)x + \sin(m-n)x]\,dx$$

$$= -\frac{1}{2}\left[\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n}\right]_{-\pi}^{\pi}, \quad (m \neq n)$$

$$= -\frac{1}{2}\left[\left(\frac{\cos(m+n)\pi}{m+n} + \frac{\cos(m-n)\pi}{m-n}\right) - \left(\frac{\cos(m+n)(-\pi)}{m+n} + \frac{\cos(m-n)(-\pi)}{m-n}\right)\right]$$

$$= 0, \quad \text{since } \cos(-\theta) = \cos\theta.$$

$$\int_{-\pi}^{\pi} \sin(mx)\cos(mx)\,dx = \frac{1}{m}\left[\frac{\sin^2(mx)}{2}\right]_{-\pi}^{\pi} = 0$$

106. $f(x) = \sum_{i=1}^{N} a_i \sin(ix)$

(a)

$$f(x)\sin(nx) = \left[\sum_{i=1}^{N} a_i \sin(ix)\right]\sin(nx)$$

$$\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \int_{-\pi}^{\pi}\left[\sum_{i=1}^{N} a_i \sin(ix)\right]\sin(nx)\,dx$$

$$= \int_{-\pi}^{\pi} a_n \sin^2(nx)\,dx \qquad \text{(by Exercise 106)}$$

$$= \int_{-\pi}^{\pi} a_n \frac{1-\cos(2nx)}{2}\,dx$$

$$= \left[\frac{a_n}{2}\left(x - \frac{\sin(2nx)}{2n}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{a_n}{2}(\pi + \pi) = a_n\pi$$

Hence, $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$

(b) $f(x) = x$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\,dx = 2$$

$$a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin 2x\,dx = -1$$

$$a_3 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin 3x\,dx = \frac{2}{3}$$

Section 8.4 Trigonometric Substitution

1. $$\frac{d}{dx}\left[4\ln\left|\frac{\sqrt{x^2+16}-4}{x}\right| + \sqrt{x^2+16} + C\right] = \frac{d}{dx}\left[4\ln\left|\sqrt{x^2+16}-4\right| - 4\ln|x| + \sqrt{x^2+16} + C\right]$$

$$= 4\left[\frac{x/\sqrt{x^2+16}}{\sqrt{x^2+16}-4}\right] - \frac{4}{x} + \frac{x}{\sqrt{x^2+16}}$$

$$= \frac{4x}{\sqrt{x^2+16}\left(\sqrt{x^2+16}-4\right)} - \frac{4}{x} + \frac{x}{\sqrt{x^2+16}}$$

$$= \frac{4x^2 - 4\sqrt{x^2+16}\left(\sqrt{x^2+16}-4\right) + x^2\left(\sqrt{x^2+16}-4\right)}{x\sqrt{x^2+16}\left(\sqrt{x^2+16}-4\right)}$$

$$= \frac{4x^2 - 4(x^2+16) + 16\sqrt{x^2+16} + x^2\sqrt{x^2+16} - 4x^2}{x\sqrt{x^2+16}\left(\sqrt{x^2+16}-4\right)}$$

$$= \frac{\sqrt{x^2+16}(x^2+16) - 4(x^2+16)}{x\sqrt{x^2+16}\left(\sqrt{x^2+16}-4\right)}$$

$$= \frac{(x^2+16)\left(\sqrt{x^2+16}-4\right)}{x\sqrt{x^2+16}\left(\right.}$$

Indefinite integral: $\displaystyle\int \frac{\sqrt{x^2+16}}{x}\,dx$, matches (b).

2. $$\frac{d}{dx}\left[8\ln\left|\sqrt{x^2-16}+x\right| + \frac{1}{2}x\sqrt{x^2-16} + C\right] = 8\left[\frac{\left(x/\sqrt{x^2-16}\right)+1}{\sqrt{x^2-16}+x}\right] + \frac{1}{2}x\left(\frac{x}{\sqrt{x^2-16}}\right) + \frac{1}{2}\sqrt{x^2-16}$$

$$= \frac{8\left(x+\sqrt{x^2-16}\right)}{\sqrt{x^2-16}\left(\sqrt{x^2-16}+x\right)} + \frac{x^2}{2\sqrt{x^2+16}} + \frac{\sqrt{x^2-16}}{2}$$

$$= \frac{16 + x^2 + x^2 - 16}{2\sqrt{x^2-16}}$$

$$= \frac{x^2}{\sqrt{x^2-16}}$$

Indefinite integral: $\displaystyle\int \frac{x^2}{\sqrt{x^2-16}}$, matches (d).

3. $$\frac{d}{dx}\left[8\arcsin\frac{x}{4} - \frac{x\sqrt{16-x^2}}{2} + C\right] = 8\frac{1/4}{\sqrt{1-(x/4)^2}} - \frac{x(1/2)(16-x^2)^{-1/2}(-2x) + \sqrt{16-x^2}}{2}$$
$$= \frac{8}{\sqrt{16-x^2}} + \frac{x^2}{2\sqrt{16-x^2}} - \frac{\sqrt{16-x^2}}{2}$$
$$= \frac{16}{2\sqrt{16-x^2}} + \frac{x^2}{2\sqrt{16-x^2}} - \frac{(16-x^2)}{2\sqrt{16-x^2}} = \frac{x^2}{\sqrt{16-x^2}}$$

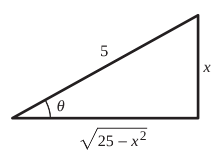
Matches (a)

4. $$\frac{d}{dx}\left[8\arcsin\frac{x-3}{4} + \frac{(x-3)\sqrt{7+6x-x^2}}{2} + C\right] = 8\left[\frac{1}{\sqrt{1-[(x-3)/4]^2}}\cdot\frac{1}{4}\right] + \frac{1}{2}(x-3)\frac{3-x}{\sqrt{7+6x-x^2}} + \frac{1}{2}\sqrt{7+6x-x^2}$$
$$= \frac{8}{\sqrt{16-(x-3)^2}} - \frac{(x-3)^2}{2\sqrt{16-(x-3)^2}} + \frac{\sqrt{16-(x-3)^2}}{2}$$
$$= \frac{16-(x^2-6x+9)+16-(x^2-6x+9)}{2\sqrt{16-(x-3)^2}}$$
$$= \frac{2[16-(x-3)^2]}{2\sqrt{16-(x-3)^2}}$$
$$= \sqrt{16-(x-3)^2}$$
$$= \sqrt{7+6x-x^2}$$

Indefinite integral: $\int \sqrt{7+6x-x^2}\,dx$, matches (c).

5. Let $x = 5\sin\theta$, $dx = 5\cos\theta\,d\theta$, $\sqrt{25-x^2} = 5\cos\theta$.

$$\int \frac{1}{(25-x^2)^{3/2}}\,dx = \int \frac{5\cos\theta}{(5\cos\theta)^3}\,d\theta$$
$$= \frac{1}{25}\int \sec^2\theta\,d\theta$$
$$= \frac{1}{25}\tan\theta + C$$
$$= \frac{x}{25\sqrt{25-x^2}} + C$$



6. Same substitution as in Exercise 5

$$\int \frac{10}{x^2\sqrt{25-x^2}}\,dx = 10\int \frac{5\cos\theta\,d\theta}{(25\sin^2\theta)(5\cos\theta)} = \frac{2}{5}\int \csc^2\theta\,d\theta = -\frac{2}{5}\cot\theta + C = \frac{-2\sqrt{25-x^2}}{5x} + C$$

7. Same substitution as in Exercise 5

$$\int \frac{\sqrt{25-x^2}}{x}\,dx = \int \frac{25\cos^2\theta\,d\theta}{5\sin\theta} = 5\int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta = 5\int (\csc\theta - \sin\theta)\,d\theta$$
$$= 5[\ln|\csc\theta - \cot\theta| + \cos\theta] + C = 5\ln\left|\frac{5-\sqrt{25-x^2}}{x}\right| + \sqrt{25-x^2} + C$$

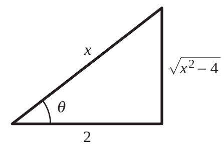
8. Same substitution as in Exercise 5

$$\int \frac{x^2}{\sqrt{25-x^2}}\,dx = \int \frac{25\sin^2\theta}{5\cos\theta}(5\cos\theta)\,d\theta = \frac{25}{2}\int (1-\cos 2\theta)\,d\theta$$

$$= \frac{25}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C = \frac{25}{2}(\theta - \sin\theta\cos\theta) + C$$

$$= \frac{25}{2}\left[\arcsin\left(\frac{x}{5}\right) - \left(\frac{x}{5}\right)\left(\frac{\sqrt{25-x^2}}{5}\right)\right] + C = \frac{1}{2}\left[25\arcsin\left(\frac{x}{5}\right) - x\sqrt{25-x^2}\right] + C$$

9. Let $x = 2\sec\theta$, $dx = 2\sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-4} = 2\tan\theta$.



$$\int \frac{1}{\sqrt{x^2-4}}\,dx = \int \frac{2\sec\theta\tan\theta\,d\theta}{2\tan\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C_1$$

$$= \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C_1$$

$$= \ln\left|x + \sqrt{x^2-4}\right| - \ln 2 + C_1 = \ln\left|x + \sqrt{x^2-4}\right| + C$$

10. Same substitution as in Exercise 9

$$\int \frac{\sqrt{x^2-4}}{x}\,dx = \int \frac{2\tan\theta}{2\sec\theta}(2\sec\theta\tan\theta)\,d\theta = 2\int \tan^2\theta\,d\theta = 2\int(\sec^2\theta - 1)\,d\theta$$

$$= 2(\tan\theta - \theta) + C = 2\left[\frac{\sqrt{x^2-4}}{2} - \operatorname{arcsec}\left(\frac{x}{2}\right)\right] + C = \sqrt{x^2-4} - 2\operatorname{arcsec}\left(\frac{x}{2}\right) + C$$

11. Same substitution as in Exercise 9

$$\int x^3\sqrt{x^2-4}\,dx = \int (8\sec^3\theta)(2\tan\theta)(2\sec\theta\tan\theta)\,d\theta = 32\int \tan^2\theta\sec^4\theta\,d\theta$$

$$= 32\int \tan^2\theta(1+\tan^2\theta)\sec^2\theta\,d\theta = 32\left(\frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5}\right) + C$$

$$= \frac{32}{15}\tan^3\theta[5 + 3\tan^2\theta] + C = \frac{32}{15}\frac{(x^2-4)^{3/2}}{8}\left[5 + 3\frac{(x^2-4)}{4}\right] + C$$

$$= \frac{1}{15}(x^2-4)^{3/2}[20 + 3(x^2-4)] + C = \frac{1}{15}(x^2-4)^{3/2}(3x^2+8) + C$$

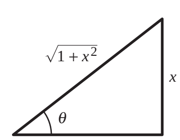
12. Same substitution as in Exercise 9

$$\int \frac{x^3}{\sqrt{x^2-4}}\,dx = \int \frac{8\sec^3\theta}{2\tan\theta}(2\sec\theta\tan\theta)\,d\theta = 8\int \sec^4\theta\,d\theta$$

$$= 8\int (1+\tan^2\theta)\sec^2\theta\,d\theta = 8\left(\tan\theta + \frac{\tan^3\theta}{3}\right) + C = \frac{8}{3}\tan\theta(3+\tan^2\theta) + C$$

$$= \frac{8}{3}\left(\frac{\sqrt{x^2-4}}{2}\right)\left(3 + \frac{x^2-4}{4}\right) + C = \frac{1}{3}\sqrt{x^2-4}\,(12 + x^2 - 4) + C = \frac{1}{3}\sqrt{x^2-4}\,(x^2+8) + C$$

13. Let $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$, $\sqrt{1+x^2} = \sec\theta$.



$$\int x\sqrt{1+x^2}\,dx = \int \tan\theta(\sec\theta)\sec^2\theta\,d\theta = \frac{\sec^3\theta}{3} + C = \frac{1}{3}(1+x^2)^{3/2} + C$$

Note: This integral could have been evaluated with the Power Rule.

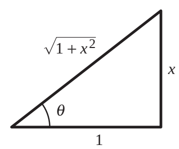
14. Same substitution as in Exercise 13

$$\int \frac{9x^3}{\sqrt{1+x^2}}\,dx = 9\int \frac{\tan^3\theta}{\sec\theta}\sec^2\theta\,d\theta = 9\int(\sec^2\theta - 1)\sec\theta\tan\theta\,d\theta = 9\left[\frac{\sec^3\theta}{3} - \sec\theta\right] + C$$

$$= 3\sec\theta(\sec^2\theta - 3) + C = 3\sqrt{1+x^2}\left[(1+x^2) - 3\right] + C = 3\sqrt{1+x^2}(x^2-2) + C$$

15. Same substitution as in Exercise 13

$$\int \frac{1}{(1+x^2)^2}\,dx = \int \frac{1}{\left(\sqrt{1+x^2}\right)^4}\,dx = \int \frac{\sec^2\theta\,d\theta}{\sec^4\theta}$$



$$= \int \cos^2\theta\,d\theta = \frac{1}{2}\int(1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]$$

$$= \frac{1}{2}[\theta + \sin\theta\cos\theta] + C$$

$$= \frac{1}{2}\left[\arctan x + \left(\frac{x}{\sqrt{1+x^2}}\right)\left(\frac{1}{\sqrt{1+x^2}}\right)\right] + C$$

$$= \frac{1}{2}\left[\arctan x + \frac{x}{1+x^2}\right] + C$$

16. Same substitution as in Exercise 13

$$\int \frac{x^2}{(1+x^2)^2}\,dx = \int \frac{x^2}{\left(\sqrt{1+x^2}\right)^4}\,dx = \int \frac{\tan^2\theta\sec^2\theta\,d\theta}{\sec^4\theta} = \int \sin^2\theta\,d\theta$$

$$= \frac{1}{2}\int(1-\cos 2\theta)\,d\theta = \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] = \frac{1}{2}[\theta - \sin\theta\cos\theta] + C$$

$$= \frac{1}{2}\left[\arctan x - \left(\frac{x}{\sqrt{1+x^2}}\right)\left(\frac{1}{\sqrt{1+x^2}}\right)\right] + C = \frac{1}{2}\left[\arctan x - \frac{x}{1+x^2}\right] + C$$

17. Let $u = 3x$, $a = 2$, and $du = 3\,dx$.

$$\int \sqrt{4+9x^2}\,dx = \frac{1}{3}\int \sqrt{(2)^2 + (3x)^2}\,3\,dx$$

$$= \frac{1}{3}\left(\frac{1}{2}\right)\left(3x\sqrt{4+9x^2} + 4\ln\left|3x + \sqrt{4+9x^2}\right|\right) + C$$

$$= \frac{1}{2}x\sqrt{4+9x^2} + \frac{2}{3}\ln\left|3x + \sqrt{4+9x^2}\right| + C$$

18. Let $u = x$, $a = 1$, and $du = dx$.

$$\int \sqrt{1+x^2}\,dx = \frac{1}{2}\left(x\sqrt{1+x^2} + \ln\left|x + \sqrt{1+x^2}\right|\right) + C$$

19. $\displaystyle\int \sqrt{25-4x^2}\,dx = \int 2\sqrt{\frac{25}{4} - x^2}\,dx, \quad a = \frac{5}{2}$

$$= 2\frac{1}{2}\left[\frac{25}{4}\arcsin\left(\frac{2x}{5}\right) + x\sqrt{\frac{25}{4} - x^2}\right] + C$$

$$= \frac{25}{4}\arcsin\left(\frac{2x}{5}\right) + \frac{x}{2}\sqrt{25-4x^2} + C$$

20. $\int \sqrt{2x^2 - 1}\, dx = \int \sqrt{(\sqrt{2}x)^2 - 1}\, dx, \quad u = \sqrt{2}x, du = \sqrt{2}\, dx$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{2}\right)\left[\sqrt{2}x\sqrt{2x^2 - 1} - \ln\left|\sqrt{2}x + \sqrt{2x^2 - 1}\right|\right] + C$$

$$= \frac{x}{2}\sqrt{2x^2 - 1} - \frac{\sqrt{2}}{4}\ln\left|\sqrt{2}x + \sqrt{2x^2 - 1}\right| + C$$

21. $\int \frac{x}{\sqrt{x^2 + 9}}\, dx = \frac{1}{2}\int (x^2 + 9)^{-1/2}(2x)\, dx$

$$= \sqrt{x^2 + 9} + C$$

(Power Rule)

22. $\int \frac{x}{\sqrt{9 - x^2}}\, dx = -\frac{1}{2}\int (9 - x^2)^{-1/2}(-2x)\, dx$

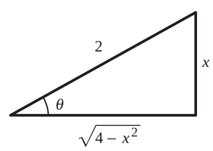
$$= -(9 - x^2)^{1/2} + C$$

(Power Rule)

23. $\int \frac{1}{\sqrt{16 - x^2}}\, dx = \arcsin\left(\frac{x}{4}\right) + C$

24. $\int \frac{1}{\sqrt{25 - x^2}}\, dx = \arcsin \frac{x}{5} + C$

25. Let $x = 2 \sin \theta$, $dx = 2 \cos \theta\, d\theta$, $\sqrt{4 - x^2} = 2 \cos \theta$.



$$\int \sqrt{16 - 4x^2}\, dx = 2\int \sqrt{4 - x^2}\, dx$$

$$= 2\int 2 \cos \theta(2 \cos \theta\, d\theta)$$

$$= 8\int \cos^2 \theta\, d\theta$$

$$= 4\int (1 + \cos 2\theta)\, d\theta$$

$$= 4\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$$

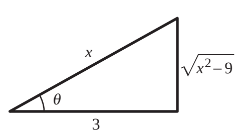
$$= 4\theta + 4 \sin \theta \cos \theta + C$$

$$= 4 \arcsin\left(\frac{x}{2}\right) + x\sqrt{4 - x^2} + C$$

26. Let $u = 16 - 4x^2$, $du = -8x\, dx$.

$$\int x\sqrt{16 - 4x^2}\, dx = -\frac{1}{8}\int (16 - 4x^2)^{1/2}(-8x)\, dx = \left[-\frac{1}{12}(16 - 4x^2)^{3/2}\right] + C = -\frac{2}{3}(4 - x^2)^{3/2} + C$$

27. Let $x = 3 \sec \theta$, $dx = 3 \sec \theta \tan \theta\, d\theta$, $\sqrt{x^2 - 9} = 3 \tan \theta$.



$$\int \frac{1}{\sqrt{x^2 - 9}}\, dx = \int \frac{3 \sec \theta \tan \theta\, d\theta}{3 \tan \theta}$$

$$= \int \sec \theta\, d\theta$$

$$= \ln|\sec \theta + \tan \theta| + C_1$$

$$= \ln\left|\frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3}\right| + C_1$$

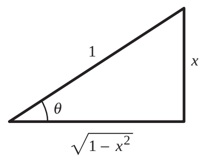
$$= \ln\left|x + \sqrt{x^2 - 9}\right| + C$$

28. Let $u = 1 - t^2$, $du = -2t\,dt$.

$$\int \frac{t}{(1-t^2)^{3/2}}\,dt = -\frac{1}{2}\int (1-t^2)^{-3/2}(-2t)\,dt = \frac{1}{\sqrt{1-t^2}} + C$$

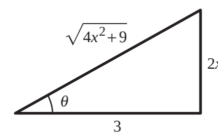
29. Let $x = \sin\theta$, $dx = \cos\theta\,d\theta$, $\sqrt{1-x^2} = \cos\theta$.

$$\begin{aligned}\int \frac{\sqrt{1-x^2}}{x^4}\,dx &= \int \frac{\cos\theta(\cos\theta\,d\theta)}{\sin^4\theta}\\ &= \int \cot^2\theta\csc^2\theta\,d\theta\\ &= -\frac{1}{3}\cot^3\theta + C\\ &= \frac{-(1-x^2)^{3/2}}{3x^3} + C\end{aligned}$$



30. Let $2x = 3\tan\theta$, $dx = \frac{3}{2}\sec^2\theta\,d\theta$, $\sqrt{4x^2+9} = 3\sec\theta$.

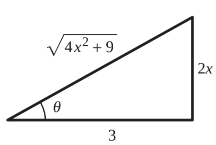
$$\begin{aligned}\int \frac{\sqrt{4x^2+9}}{x^4}\,dx &= \int \frac{3\sec\theta[(3/2)\sec^2\theta\,d\theta]}{(3/2)^4\tan^4\theta}\\ &= \frac{8}{9}\int \frac{\cos\theta}{\sin^4\theta}\,d\theta\\ &= \frac{-8}{27\sin^3\theta} + C\\ &= -\frac{8}{27}\csc^3\theta + C\\ &= \frac{-(4x^2+9)^{3/2}}{27x^3} + C\end{aligned}$$



31. Same substitution as in Exercise 30

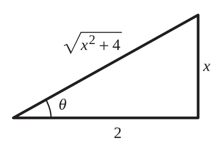
$x = \frac{3}{2}\tan\theta$, $dx = \frac{3}{2}\sec^2\theta\,d\theta$

$$\begin{aligned}\int \frac{1}{x\sqrt{4x^2+9}}\,dx &= \int \frac{(3/2)\sec^2\theta\,d\theta}{(3/2)\tan\theta\,3\sec\theta}\\ &= \frac{1}{3}\int \csc\theta\,d\theta\\ &= -\frac{1}{3}\ln\left|\csc\theta + \cot\theta\right| + C\\ &= -\frac{1}{3}\ln\left|\frac{\sqrt{4x^2+9}+3}{2x}\right| + C\end{aligned}$$



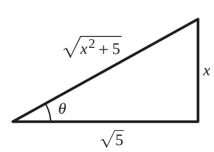
32. Let $2x = 4\tan\theta$, $dx = 2\sec^2\theta\,d\theta$, $\sqrt{4x^2+16} = 4\sec\theta$.

$$\begin{aligned}\int \frac{1}{x\sqrt{4x^2+16}}\,dx &= \int \frac{2\sec^2\theta\,d\theta}{2\tan\theta(4\sec\theta)}\\ &= \frac{1}{4}\int \frac{\sec\theta}{\tan\theta}\,d\theta = \frac{1}{4}\int \csc\theta\,d\theta\\ &= -\frac{1}{4}\ln\left|\csc\theta + \cot\theta\right| + C\\ &= -\frac{1}{4}\ln\left|\frac{\sqrt{x^2+4}+2}{x}\right| + C\end{aligned}$$



33. Let $x = \sqrt{5}\tan\theta$, $dx = \sqrt{5}\sec^2\theta\,d\theta$, $x^2 + 5 = 5\sec^2\theta$.

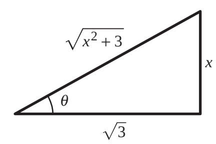
$$\begin{aligned}\int \frac{-5x}{(x^2+5)^{3/2}}\,dx &= \int \frac{-5\sqrt{5}\tan\theta}{(5\sec^2\theta)^{3/2}}\sqrt{5}\sec^2\theta\,d\theta\\ &= -\sqrt{5}\int \frac{\tan\theta}{\sec\theta}\,d\theta\\ &= -\sqrt{5}\int \sin\theta\,d\theta\\ &= \sqrt{5}\cos\theta + C\\ &= \sqrt{5}\,\frac{\sqrt{5}}{\sqrt{x^2+5}} + C\\ &= \frac{5}{\sqrt{x^2+5}} + C\end{aligned}$$



34. Let $x = \sqrt{3}\tan\theta$, $dx = \sqrt{3}\sec^2\theta\,d\theta$, $x^2 + 3 = 3\sec^2\theta$.

$$\int \frac{1}{(x^2+3)^{3/2}}\,dx = \int \frac{\sqrt{3}\sec^2\theta\,d\theta}{3\sqrt{3}\sec^3\theta}$$

$$= \frac{1}{3}\int \cos\theta\,d\theta = \frac{1}{3}\sin\theta + C = \frac{x}{3\sqrt{x^2+3}} + C$$



35. Let $u = 1 + e^{2x}$, $du = 2e^{2x}\,dx$.

$$\int e^{2x}\sqrt{1+e^{2x}}\,dx = \frac{1}{2}\int (1+e^{2x})^{1/2}(2e^{2x})\,dx = \frac{1}{3}(1+e^{2x})^{3/2} + C$$

36. Let $u = x^2 + 2x + 2$, $du = (2x+2)\,dx$.

$$\int (x+1)\sqrt{x^2+2x+2}\,dx = \frac{1}{2}\int (x^2+2x+2)^{1/2}(2x+2)\,dx = \frac{1}{3}(x^2+2x+2)^{3/2} + C$$

37. Let $e^x = \sin\theta$, $e^x\,dx = \cos\theta\,d\theta$, $\sqrt{1-e^{2x}} = \cos\theta$.

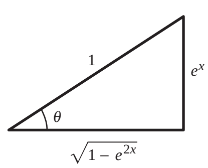
$$\int e^x\sqrt{1-e^{2x}}\,dx = \int \cos^2\theta\,d\theta$$

$$= \frac{1}{2}\int (1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]$$

$$= \frac{1}{2}(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{1}{2}\left(\arcsin e^x + e^x\sqrt{1-e^{2x}}\right) + C$$



38. Let $\sqrt{x} = \sin\theta$, $x = \sin^2\theta$, $dx = 2\sin\theta\cos\theta\,d\theta$, $\sqrt{1-x} = \cos\theta$.

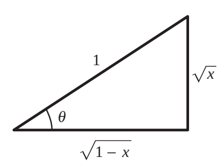
$$\int \frac{\sqrt{1-x}}{\sqrt{x}}\,dx = \int \frac{\cos\theta(2\sin\theta\cos\theta\,d\theta)}{\sin\theta}$$

$$= 2\int \cos^2\theta\,d\theta$$

$$= \int (1+\cos 2\theta)\,d\theta$$

$$= (\theta + \sin\theta\cos\theta) + C$$

$$= \arcsin\sqrt{x} + \sqrt{x}\sqrt{1-x} + C$$



39. Let $x = \sqrt{2}\tan\theta$, $dx = \sqrt{2}\sec^2\theta\,d\theta$, $x^2 + 2 = 2\sec^2\theta$.

$$\int \frac{1}{4+4x^2+x^4}\,dx = \int \frac{1}{(x^2+2)^2}\,dx = \int \frac{\sqrt{2}\sec^2\theta\,d\theta}{4\sec^4\theta}$$

$$= \frac{\sqrt{2}}{4}\int \cos^2\theta\,d\theta$$

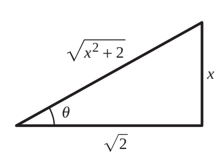
$$= \frac{\sqrt{2}}{4}\left(\frac{1}{2}\right)\int (1+\cos 2\theta)\,d\theta$$

$$= \frac{\sqrt{2}}{8}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

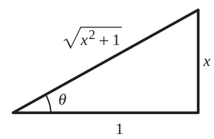
$$= \frac{\sqrt{2}}{8}(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{\sqrt{2}}{8}\left(\arctan\frac{x}{\sqrt{2}} + \frac{x}{\sqrt{x^2+2}}\cdot\frac{\sqrt{2}}{\sqrt{x^2+2}}\right)$$

$$= \frac{1}{4}\left[\frac{x}{x^2+2} + \frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}}\right] + C$$

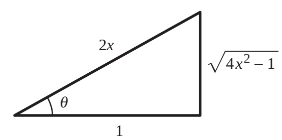


40. Let $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$, $x^2 + 1 = \sec^2\theta$.



$$\int \frac{x^3 + x + 1}{x^4 + 2x^2 + 1}\,dx = \frac{1}{4}\int \frac{4x^3 + 4x}{x^4 + 2x^2 + 1}\,dx + \int \frac{1}{(x^2+1)^2}\,dx$$
$$= \frac{1}{4}\ln(x^4 + 2x^2 + 1) + \int \frac{\sec^2\theta\,d\theta}{\sec^4\theta}$$
$$= \frac{1}{2}\ln(x^2+1) + \frac{1}{2}\int (1 + \cos 2\theta)\,d\theta$$
$$= \frac{1}{2}\ln(x^2+1) + \frac{1}{2}(\theta + \sin\theta\cos\theta) + C$$
$$= \frac{1}{2}\left[\ln(x^2+1) + \arctan x + \frac{x}{x^2+1}\right] + C$$

41. Use integration by parts. Since $x > \frac{1}{2}$,



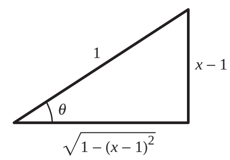
$$u = \operatorname{arcsec} 2x \implies du = \frac{1}{x\sqrt{4x^2-1}}\,dx,\; dv = dx \implies v = x$$
$$\int \operatorname{arcsec} 2x\,dx = x \operatorname{arcsec} 2x - \int \frac{1}{\sqrt{4x^2-1}}\,dx$$
$$2x = \sec\theta,\; dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta,\; \sqrt{4x^2-1} = \tan\theta$$
$$\int \operatorname{arcsec} 2x\,dx = x\operatorname{arcsec} 2x - \int \frac{(1/2)\sec\theta\tan\theta\,d\theta}{\tan\theta}$$
$$= x\operatorname{arcsec} 2x - \frac{1}{2}\int \sec\theta\,d\theta$$
$$= x\operatorname{arcsec} 2x - \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C$$
$$= x\operatorname{arcsec} 2x - \frac{1}{2}\ln\left|2x + \sqrt{4x^2-1}\right| + C.$$

42. $u = \arcsin x \implies du = \dfrac{1}{\sqrt{1-x^2}}\,dx,\; dv = x\,dx \implies v = \dfrac{x^2}{2}$

$$\int x\arcsin x\,dx = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx$$
$$x = \sin\theta,\; dx = \cos\theta\,d\theta,\; \sqrt{1-x^2} = \cos\theta$$
$$\int x\arcsin x\,dx = \frac{x^2}{2}\arcsin x = \frac{1}{2}\int \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{4}\int (1 - \cos 2\theta)\,d\theta$$
$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\left[\theta - \frac{1}{2}\sin 2\theta\right] + C = \frac{x^2}{2}\arcsin x - \frac{1}{4}[\theta - \sin\theta\cos\theta] + C$$
$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\left[\arcsin x - x\sqrt{1-x^2}\right] + C = \frac{1}{4}\left[(2x^2-1)\arcsin x + x\sqrt{1-x^2}\right] + C$$

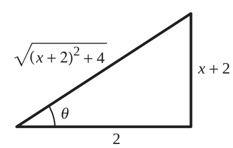
43. $\displaystyle\int \frac{1}{\sqrt{4x - x^2}}\,dx = \int \frac{1}{\sqrt{4 - (x-2)^2}}\,dx = \arcsin\left(\frac{x-2}{2}\right) + C$

44. Let $x - 1 = \sin\theta$, $dx = \cos\theta\, d\theta$, $\sqrt{1-(x-1)^2} = \sqrt{2x - x^2} = \cos\theta$.



$$\begin{aligned}
\int \frac{x^2}{\sqrt{2x-x^2}}\,dx &= \int \frac{x^2}{\sqrt{1-(x-1)^2}}\,dx \\
&= \int \frac{(1+\sin\theta)^2(\cos\theta\, d\theta)}{\cos\theta} \\
&= \int (1 + 2\sin\theta + \sin^2\theta)\, d\theta \\
&= \int \left(\frac{3}{2} + 2\sin\theta - \frac{1}{2}\cos 2\theta\right) d\theta \\
&= \frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta + C \\
&= \frac{3}{2}\theta - 2\cos\theta - \frac{1}{2}\sin\theta\cos\theta + C \\
&= \frac{3}{2}\arcsin(x-1) - 2\sqrt{2x-x^2} - \frac{1}{2}(x-1)\sqrt{2x-x^2} + C \\
&= \frac{3}{2}\arcsin(x-1) - \frac{1}{2}\sqrt{2x-x^2}(x+3) + C
\end{aligned}$$

45. Let $x + 2 = 2\tan\theta$, $dx = 2\sec^2\theta\, d\theta$, $\sqrt{(x+2)^2+4} = 2\sec\theta$.



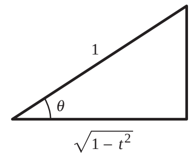
$$\begin{aligned}
\int \frac{x}{\sqrt{x^2+4x+8}}\,dx = \int \frac{x}{\sqrt{(x+2)^2+4}}\,dx &= \int \frac{(2\tan\theta - 2)(2\sec^2\theta)\, d\theta}{2\sec\theta} \\
&= 2\int (\tan\theta - 1)(\sec\theta)\, d\theta \\
&= 2\left[\sec\theta - \ln\left|\sec\theta + \tan\theta\right|\right] + C_1 \\
&= 2\left[\frac{\sqrt{(x+2)^2+4}}{2} - \ln\left|\frac{\sqrt{(x+2)^2+4}}{2} + \frac{x+2}{2}\right|\right] + C_1 \\
&= \sqrt{x^2+4x+8} - 2\left[\ln\left|\sqrt{x^2+4x+8} + (x+2)\right| - \ln 2\right] + C_1 \\
&= \sqrt{x^2+4x+8} - 2\ln\left|\sqrt{x^2+4x+8} + (x+2)\right| + C
\end{aligned}$$

46. Let $x - 3 = 2\sec\theta$, $dx = 2\sec\theta\tan\theta\, d\theta$, $\sqrt{(x-3)^2-4} = 2\tan\theta$.

$$\begin{aligned}
\int \frac{x}{\sqrt{x^2-6x+5}}\,dx &= \int \frac{x}{\sqrt{(x-3)^2-4}}\,dx \\
&= \int \frac{(2\sec\theta + 3)}{2\tan\theta}(2\sec\theta\tan\theta)\, d\theta \\
&= \int (2\sec^2\theta + 3\sec\theta)\, d\theta \\
&= 2\tan\theta + 3\ln\left|\sec\theta + \tan\theta\right| + C_1 \\
&= 2\left(\frac{\sqrt{(x-3)^3-4}}{2}\right) + 3\ln\left|\frac{x-3}{2} + \frac{\sqrt{(x-3)^2-4}}{2}\right| + C_1 \\
&= \sqrt{x^2-6x+5} + 3\ln\left|(x-3) + \sqrt{x^2-6x+5}\right| + C
\end{aligned}$$

47. Let $t = \sin\theta$, $dt = \cos\theta\,d\theta$, $1 - t^2 = \cos^2\theta$.

(a) $$\int \frac{t^2}{(1-t^2)^{3/2}}\,dt = \int \frac{\sin^2\theta\cos\theta\,d\theta}{\cos^3\theta}$$
$$= \int \tan^2\theta\,d\theta = \int (\sec^2\theta - 1)\,d\theta$$
$$= \tan\theta - \theta + C$$
$$= \frac{t}{\sqrt{1-t^2}} - \arcsin t + C$$



Thus, $$\int_0^{\sqrt{3}/2} \frac{t^2}{(1-t^2)^{3/2}}\,dt = \left[\frac{t}{\sqrt{1-t^2}} - \arcsin t\right]_0^{\sqrt{3}/2} = \frac{\sqrt{3}/2}{\sqrt{1/4}} - \arcsin\frac{\sqrt{3}}{2} = \sqrt{3} - \frac{\pi}{3} \approx 0.685.$$

(b) When $t = 0$, $\theta = 0$. When $t = \sqrt{3}/2$, $\theta = \pi/3$. Thus,
$$\int_0^{\sqrt{3}/2} \frac{t^2}{(1-t^2)^{3/2}}\,dt = \left[\tan\theta - \theta\right]_0^{\pi/3} = \sqrt{3} - \frac{\pi}{3} \approx 0.685.$$

48. Same substitution as in Exercise 47

(a) $$\int \frac{1}{(1-t^2)^{5/2}}\,dt = \int \frac{\cos\theta\,d\theta}{\cos^5\theta} = \int \sec^4\theta\,d\theta = \int (\tan^2\theta + 1)\sec^2\theta\,d\theta$$
$$= \frac{1}{3}\tan^3\theta + \tan\theta + C = \frac{1}{3}\left(\frac{t}{\sqrt{1-t^2}}\right)^3 + \frac{t}{\sqrt{1-t^2}} + C$$

Thus, $$\int_0^{\sqrt{3}/2} \frac{1}{(1-t^2)^{5/2}}\,dt = \left[\frac{t^3}{3(1-t^2)^{3/2}} + \frac{t}{\sqrt{1-t^2}}\right]_0^{\sqrt{3}/2}$$
$$= \frac{3\sqrt{3}/8}{3(1/4)^{3/2}} + \frac{\sqrt{3}/2}{\sqrt{1/4}} = \sqrt{3} + \sqrt{3} = 2\sqrt{3} \approx 3.464.$$

(b) When $t = 0$, $\theta = 0$. When $t = \sqrt{3}/2$, $\theta = \pi/3$. Thus,
$$\int_0^{\sqrt{3}/2} \frac{1}{(1-t^2)^{5/2}}\,dt = \left[\frac{1}{3}\tan^3\theta + \tan\theta\right]_0^{\pi/3} = \frac{1}{3}\left(\sqrt{3}\right)^3 + \sqrt{3} = 2\sqrt{3} \approx 3.464.$$

49. (a) Let $x = 3\tan\theta$, $dx = 3\sec^2\theta\,d\theta$, $\sqrt{x^2+9} = 3\sec\theta$.
$$\int \frac{x^3}{\sqrt{x^2+9}}\,dx = \int \frac{(27\tan^3\theta)(3\sec^2\theta\,d\theta)}{3\sec\theta}$$
$$= 27\int (\sec^2\theta - 1)\sec\theta\tan\theta\,d\theta$$
$$= 27\left[\frac{1}{3}\sec^3\theta - \sec\theta\right] + C = 9[\sec^3\theta - 3\sec\theta] + C$$
$$= 9\left[\left(\frac{\sqrt{x^2+9}}{3}\right)^3 - 3\left(\frac{\sqrt{x^2+9}}{3}\right)\right] + C = \frac{1}{3}(x^2+9)^{3/2} - 9\sqrt{x^2+9} + C$$

Thus, $$\int_0^3 \frac{x^3}{\sqrt{x^2+9}}\,dx = \left[\frac{1}{3}(x^2+9)^{3/2} - 9\sqrt{x^2+9}\right]_0^3$$
$$= \left(\frac{1}{3}\left(54\sqrt{2}\right) - 27\sqrt{2}\right) - (9 - 27)$$
$$= 18 - 9\sqrt{2} = 9\left(2 - \sqrt{2}\right) \approx 5.272.$$

(b) When $x = 0$, $\theta = 0$. When $x = 3$, $\theta = \pi/4$. Thus,
$$\int_0^3 \frac{x^3}{\sqrt{x^2+9}}\,dx = 9\left[\sec^3\theta - 3\sec\theta\right]_0^{\pi/4} = 9\left(2\sqrt{2} - 3\sqrt{2}\right) - 9(1 - 3) = 9\left(2 - \sqrt{2}\right) \approx 5.272.$$

50. (a) Let $5x = 3\sin\theta$, $dx = \frac{3}{5}\cos\theta\, d\theta$, $\sqrt{9-25x^2} = 3\cos\theta$.

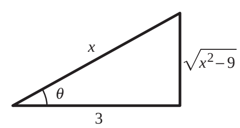
$$\int \sqrt{9-25x^2}\,dx = \int (3\cos\theta)\frac{3}{5}\cos\theta\,d\theta$$
$$= \frac{9}{5}\int \frac{1+\cos 2\theta}{2}\,d\theta$$
$$= \frac{9}{10}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$$
$$= \frac{9}{10}[\theta + \sin\theta\cos\theta] + C$$
$$= \frac{9}{10}\left[\arcsin\frac{5x}{3} + \frac{5x}{3}\cdot\frac{\sqrt{9-25x^2}}{3}\right] + C$$

Thus, $\displaystyle\int_0^{3/5} \sqrt{9-25x^2}\,dx = \frac{9}{10}\arcsin\frac{5x}{3} + \frac{5x\sqrt{9-25x^2}}{9}\Bigg]_0^{3/5} = \frac{9}{10}\left[\frac{\pi}{2}\right] = \frac{9\pi}{20}$.

(b) When $x = 0$, $\theta = 0$. When $x = \frac{3}{5}$, $\theta = \frac{\pi}{2}$.

Thus, $\displaystyle\int_0^{3/5} \sqrt{9-25x^2}\,dx = \left[\frac{9}{10}(\theta + \sin\theta\cos\theta)\right]_0^{\pi/2} = \frac{9}{10}\left(\frac{\pi}{2}\right) = \frac{9\pi}{20}$.

51. (a) Let $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-9} = 3\tan\theta$.



$$\int \frac{x^2}{\sqrt{x^2-9}}\,dx = \int \frac{9\sec^2\theta}{3\tan\theta}3\sec\theta\tan\theta\,d\theta$$
$$= 9\int \sec^3\theta\,d\theta$$
$$= 9\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int\sec\theta\,d\theta\right] \quad \text{(8.3 Exercise 98 or Example 5, Section 8.2)}$$
$$= \frac{9}{2}[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|]$$
$$= \frac{9}{2}\left[\frac{x}{3}\cdot\frac{\sqrt{x^2-9}}{3} + \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right|\right]$$

Hence,

$$\int_4^6 \frac{x^2}{\sqrt{x^2-9}}\,dx = \frac{9}{2}\left[\frac{x\sqrt{x^2-9}}{9} + \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right|\right]_4^6$$
$$= \frac{9}{2}\left[\left(\frac{6\sqrt{27}}{9} + \ln\left|2 + \frac{\sqrt{27}}{3}\right|\right) - \left(\frac{4\sqrt{7}}{9} + \ln\left|\frac{4}{3} + \frac{\sqrt{7}}{3}\right|\right)\right]$$
$$= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2}\left(\ln\left(\frac{6+\sqrt{27}}{3}\right) - \ln\left(\frac{4+\sqrt{7}}{3}\right)\right)$$
$$= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2}\ln\left(\frac{6+3\sqrt{3}}{4+\sqrt{7}}\right)$$
$$= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2}\ln\left(\frac{(4-\sqrt{7})(2+\sqrt{3})}{3}\right) \approx 12.644.$$

—CONTINUED—

51. —CONTINUED—

(b) When $x = 4$, $\theta = \text{arcsec}\left(\frac{4}{3}\right)$. When $x = 6$, $\theta = \text{arcsec}(2) = \frac{\pi}{3}$.

$$\int_4^6 \frac{x^2}{\sqrt{x^2-9}}\,dx = \frac{9}{2}\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_{\text{arcsec}(4/3)}^{\pi/3}$$

$$= \frac{9}{2}\big[2\cdot\sqrt{3} + \ln|2+\sqrt{3}|\big] - \frac{9}{2}\left[\frac{4}{3}\frac{\sqrt{7}}{3} + \ln\left|\frac{4}{3} + \frac{\sqrt{7}}{3}\right|\right]$$

$$= 9\sqrt{3} - 2\sqrt{7} + \frac{9}{2}\ln\left(\frac{6+3\sqrt{3}}{4+\sqrt{7}}\right) \approx 12.644$$

52. (a) Let $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\,d\theta$,

$\sqrt{x^2-9} = 3\tan\theta.$

$$\int \frac{\sqrt{x^2-9}}{x^2}\,dx = \int \frac{3\tan\theta}{9\sec^2\theta}3\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{\tan^2\theta}{\sec\theta}\,d\theta = \int \frac{\sin^2\theta}{\cos\theta}\,d\theta$$

$$= \int \frac{1-\cos^2\theta}{\cos\theta}\,d\theta$$

$$= \int (\sec\theta - \cos\theta)\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| - \sin\theta + C$$

$$= \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| - \frac{\sqrt{x^2-9}}{x} + C$$

Hence,

$$\int_3^6 \frac{\sqrt{x^2-9}}{x^2}\,dx = \left[\ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| - \frac{\sqrt{x^2-9}}{x}\right]_3^6$$

$$= \ln|2+\sqrt{3}| - \frac{\sqrt{3}}{2}.$$

(b) When $x = 3$, $\theta = 0$; when $x = 6$, $\theta = \frac{\pi}{3}$. Hence,

$$\int_3^6 \frac{\sqrt{x^2-9}}{x^2}\,dx = \Big[\ln|\sec\theta + \tan\theta| - \sin\theta\Big]_0^{\pi/3}$$

$$= \ln|2+\sqrt{3}| - \frac{\sqrt{3}}{2}.$$

53. $x\frac{dy}{dx} = \sqrt{x^2-9},\ x \ge 3,\ y(3) = 1$

$$y = \int \frac{\sqrt{x^2-9}}{x}\,dx$$

Let $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-9} = 3\tan\theta$.

$$y = \int \frac{3\tan\theta}{3\sec\theta}3\sec\theta\tan\theta\,d\theta = 3\int \tan^2\theta\,d\theta$$

$$= 3\int(\sec^2\theta - 1)\,d\theta = 3[\tan\theta - \theta] + C$$

$$= 3\left[\frac{\sqrt{x^2-9}}{3} - \arctan\left(\frac{\sqrt{x^2-9}}{3}\right)\right] + C$$

$$= \sqrt{x^2-9} - 3\arctan\left(\frac{\sqrt{x^2-9}}{3}\right) + C$$

$y(3) = 1$: $1 = 0 - 3(0) + C \implies C = 1$

$$y = \sqrt{x^2-9} - 3\arctan\left(\frac{\sqrt{x^2-9}}{3}\right) + 1$$

54. $\sqrt{x^2+4}\,\frac{dy}{dx} = 1,\ x \ge -2,\ y(0) = 4$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+4}}$$

$$y = \int \frac{1}{\sqrt{x^2+4}}\,dx$$

Let $x = 2\tan\theta$, $x^2 + 4 = 4\sec^2\theta$, $dx = 2\sec^2\theta\,d\theta$.

$$y = \int \frac{1}{2\sec\theta}2\sec^2\theta\,d\theta = \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C$$

$$= \ln\left|\sqrt{x^2+4} + x\right| + C_1$$

$y(0) = 4 \implies 4 = \ln|2| + C_1 \implies C_1 = 4 - \ln 2$

$$y = \ln\left|\sqrt{x^2+4} + x\right| + 4 - \ln 2$$

55. $\displaystyle\int \frac{x^2}{\sqrt{x^2+10x+9}}\,dx = \frac{1}{2}\sqrt{x^2+10x+9}\,(x-15) + 33\ln\left|(x+5)+\sqrt{x^2+10x+9}\right| + C$

56. $\displaystyle\int (x^2+2x+11)^{3/2}\,dx = \frac{1}{4}(x+1)(x^2+2x+26)\sqrt{x^2+2x+11} + \frac{75}{2}\ln\left|\sqrt{x^2+2x+11}+(x+1)\right| + C$

57. $\displaystyle\int \frac{x^2}{\sqrt{x^2-1}}\,dx = \frac{1}{2}\left(x\sqrt{x^2-1}+\ln\left|x+\sqrt{x^2-1}\right|\right) + C$

58. $\displaystyle\int x^2\sqrt{x^2-4}\,dx = \frac{1}{4}x^3\sqrt{x^2-4} - \frac{1}{2}x\sqrt{x^2-4} - 2\ln\left|x+\sqrt{x^2-4}\right| + C$

59. (a) $u = a\sin\theta$

(b) $u = a\tan\theta$

(c) $u = a\sec\theta$

60. (a) Substitution: $u = x^2+1,\ du = 2x\,dx$

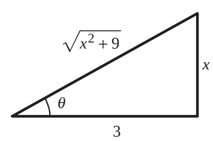
(b) Trigonometric substitution: $x = \sec\theta$

61. (a) $u = x^2+9,\ du = 2x\,dx$

$$\int \frac{x}{x^2+9}\,dx = \frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+9) + C$$

(b) Let $x = 3\tan\theta,\ x^2+9 = 9\sec^2\theta,\ dx = 3\sec^2\theta\,d\theta.$

$$\begin{aligned}\int \frac{x}{x^2+9}\,dx &= \int\frac{3\tan\theta}{9\sec^2\theta}3\sec^2\theta\,d\theta = \int\tan\theta\,d\theta\\ &= -\ln|\cos\theta| + C_1\\ &= -\ln\left|\frac{3}{\sqrt{x^2+9}}\right| + C_1\\ &= -\ln 3 + \ln\sqrt{x^2+9} + C_1\\ &= \frac{1}{2}\ln(x^2+9) + C_2\end{aligned}$$



The answers are equivalent.

62. (a) $\displaystyle\int\frac{x^2}{x^2+9}\,dx = \int\frac{x^2+9-9}{x^2+9}\,dx = \int\left(1-\frac{9}{x^2+9}\right)dx = x - 3\arctan\left(\frac{x}{3}\right) + C$

(b) Let $x = 3\tan\theta,\ x^2+9 = 9\sec^2\theta,\ dx = 3\sec^2\theta\,d\theta.$

$$\begin{aligned}\int\frac{x^2}{x^2+9}\,dx &= \int\frac{9\tan^2\theta}{9\sec^2\theta}3\sec^2\theta\,d\theta\\ &= 3\int\tan^2\theta\,d\theta = 3\int(\sec^2\theta-1)\,d\theta\\ &= 3\tan\theta - 3\theta + C_1\\ &= x - 3\arctan\left(\frac{x}{3}\right) + C_1\end{aligned}$$

The answers are equivalent.

63. True

$$\int\frac{dx}{\sqrt{1-x^2}} = \int\frac{\cos\theta\,d\theta}{\cos\theta} = \int d\theta$$

64. False

$$\int\frac{\sqrt{x^2-1}}{x}\,dx = \int\frac{\tan\theta}{\sec\theta}(\sec\theta\tan\theta\,d\theta) = \int\tan^2\theta\,d\theta$$

65. False

$$\int_0^{\sqrt{3}} \frac{dx}{\left(\sqrt{1+x^2}\right)^3} = \int_0^{\pi/3} \frac{\sec^2\theta\, d\theta}{\sec^3\theta} = \int_0^{\pi/3} \cos\theta\, d\theta$$

66. True

$$\int_{-1}^1 x^2\sqrt{1-x^2}\,dx = 2\int_0^1 x^2\sqrt{1-x^2}\,dx = 2\int_0^{\pi/2} (\sin^2\theta)(\cos\theta)(\cos\theta\, d\theta) = 2\int_0^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta$$

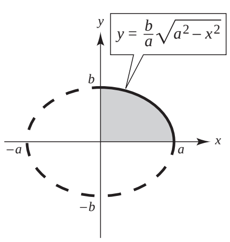
67. $A = 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$

$$= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$= \left[\frac{4b}{a}\left(\frac{1}{2}\right)\left(a^2\arcsin\frac{x}{a} + x\sqrt{a^2-x^2}\right)\right]_0^a$$

$$= \frac{2b}{a}\left(a^2\left(\frac{\pi}{2}\right)\right) = \pi ab$$

Note: See Theorem 8.2 for $\int\sqrt{a^2-x^2}\,dx$.



68. $x^2 + y^2 = a^2$

$$x = \pm\sqrt{a^2-y^2}$$

$$A = 2\int_h^a \sqrt{a^2-y^2}\,dy$$

$$= \left[a^2\arcsin\left(\frac{y}{a}\right) + y\sqrt{a^2-y^2}\right]_h^a \qquad \text{(Theorem 8.2)}$$

$$= \left(a^2\frac{\pi}{2}\right) - \left(a^2\arcsin\left(\frac{h}{a}\right) + h\sqrt{a^2-h^2}\right)$$

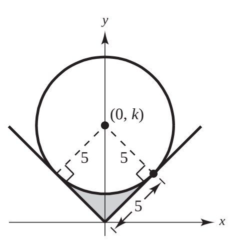
$$= \frac{a^2\pi}{2} - a^2\arcsin\left(\frac{h}{a}\right) - h\sqrt{a^2-h^2}$$

69. (a) $x^2 + (y-k)^2 = 25$

Radius of circle $= 5$

$k^2 = 5^2 + 5^2 = 50$

$k = 5\sqrt{2}$



(b) $\text{Area} = \text{square} - \frac{1}{4}(\text{circle})$

$$= 25 - \frac{1}{4}\pi(5)^2 = 25\left(1 - \frac{\pi}{4}\right)$$

(c) $\text{Area} = r^2 - \frac{1}{4}\pi r^2 = r^2\left(1 - \frac{\pi}{4}\right)$

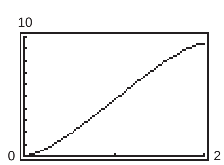
70. (a) Place the center of the circle at $(0, 1)$; $x^2 + (y-1)^2 = 1$. The depth d satisfies $0 \le d \le 2$. The volume is

$$V = 3\cdot 2\int_0^d \sqrt{1-(y-1)^2}\,dy = 6\cdot\frac{1}{2}\left[\arcsin(y-1) + (y-1)\sqrt{1-(y-1)^2}\right]_0^d \qquad \text{(Theorem 8.2 (1))}$$

$$= 3\left[\arcsin(d-1) + (d-1)\sqrt{1-(d-1)^2} - \arcsin(-1)\right]$$

$$= \frac{3\pi}{2} + 3\arcsin(d-1) + 3(d-1)\sqrt{2d-d^2}.$$

(b)



—CONTINUED—

(c) The full tank holds $3\pi \approx 9.4248$ cubic meters. The horizontal lines

$$y = \frac{3\pi}{4},\ y = \frac{3\pi}{2},\ y = \frac{9\pi}{4}$$

intersect the curve at $d = 0.596, 1.0, 1.404$. The dipstick would have these markings on it.

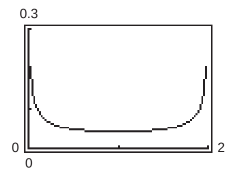
70. —CONTINUED—

(d) $V = 6\int_0^d \sqrt{1-(y-1)^2}\,dy$

$$\frac{dV}{dt} = \frac{dV}{dd}\cdot\frac{dd}{dt} = 6\sqrt{1-(d-1)^2}\cdot d'(t)$$

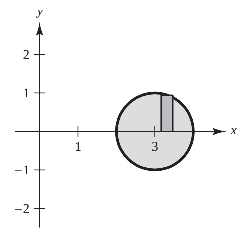
$$= \frac{1}{4} \Rightarrow d'(t) = \frac{1}{24\sqrt{1-(d-1)^2}}$$

(e)



The minimum occurs at $d = 1$, which is the widest part of the tank.

71. Let $x - 3 = \sin\theta$, $dx = \cos\theta\,d\theta$, $\sqrt{1-(x-3)^2} = \cos\theta$.



Shell Method:

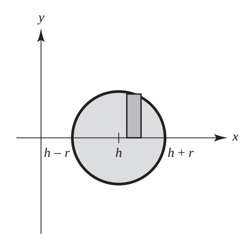
$$V = 4\pi\int_2^4 x\sqrt{1-(x-3)^2}\,dx$$

$$= 4\pi\int_{-\pi/2}^{\pi/2}(3+\sin\theta)\cos^2\theta\,d\theta$$

$$= 4\pi\left[\frac{3}{2}\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta + \int_{-\pi/2}^{\pi/2}\cos^2\theta\sin\theta\,d\theta\right]$$

$$= 4\pi\left[\frac{3}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) - \frac{1}{3}\cos^3\theta\right]_{-\pi/2}^{\pi/2} = 6\pi^2$$

72. Let $x - h = r\sin\theta$, $dx = r\cos\theta\,d\theta$, $\sqrt{r^2-(x-h)^2} = r\cos\theta$.



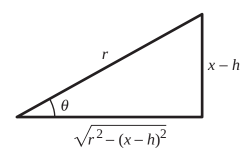
Shell Method:

$$V = 4\pi\int_{h-r}^{h+r} x\sqrt{r^2-(x-h)^2}\,dx$$

$$= 4\pi\int_{-\pi/2}^{\pi/2}(h + r\sin\theta)r\cos\theta(r\cos\theta)\,d\theta = 4\pi r^2\int_{-\pi/2}^{\pi/2}(h+r\sin\theta)\cos^2\theta\,d\theta$$

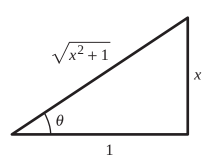
$$= 4\pi r^2\left[\frac{h}{2}\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta + r\int_{-\pi/2}^{\pi/2}\sin\theta\cos^2\theta\,d\theta\right]$$

$$= 2\pi r^2 h\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\pi/2}^{\pi/2} - \left[4\pi r^3\left(\frac{\cos^3\theta}{3}\right)\right]_{-\pi/2}^{\pi/2} = 2\pi^2 r^2 h$$



73. $y = \ln x$, $y' = \dfrac{1}{x}$, $1 + (y')^2 = 1 + \dfrac{1}{x^2} = \dfrac{x^2+1}{x^2}$

Let $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$, $\sqrt{x^2+1} = \sec\theta$.



$$s = \int_1^5 \sqrt{\frac{x^2+1}{x^2}}\,dx = \int_1^5 \frac{\sqrt{x^2+1}}{x}\,dx$$

$$= \int_a^b \frac{\sec\theta}{\tan\theta}\sec^2\theta\,d\theta = \int_a^b \frac{\sec\theta}{\tan\theta}(1+\tan^2\theta)\,d\theta$$

$$= \int_a^b (\csc\theta + \sec\theta\tan\theta)\,d\theta = \Big[-\ln|\csc\theta + \cot\theta| + \sec\theta\Big]_a^b$$

$$= \left[-\ln\left|\frac{\sqrt{x^2+1}}{x} + \frac{1}{x}\right| + \sqrt{x^2+1}\right]_1^5$$

$$= \left[-\ln\left(\frac{\sqrt{26}+1}{5}\right) + \sqrt{26}\right] - \left[-\ln\left(\sqrt{2}+1\right) + \sqrt{2}\right]$$

$$= \ln\left[\frac{5\left(\sqrt{2}+1\right)}{\sqrt{26}+1}\right] + \sqrt{26} - \sqrt{2} \approx 4.367 \text{ or } \ln\left[\frac{\sqrt{26}-1}{5\left(\sqrt{2}-1\right)}\right] + \sqrt{26} - \sqrt{2}$$

74. $y = \frac{1}{2}x^2, y' = x, 1 + (y')^2 = 1 + x^2$

$$s = \int_0^4 \sqrt{1 + x^2}\,dx = \left[\frac{1}{2}\left(x\sqrt{x^2+1} + \ln\left|x + \sqrt{x^2+1}\right|\right)\right]_0^4 \quad \text{(Theorem 8.2)}$$

$$= \frac{1}{2}\left[4\sqrt{17} + \ln\left(4 + \sqrt{17}\right)\right] \approx 9.2936$$

75. Length of one arch of sine curve: $y = \sin x, y' = \cos x$

$$L_1 = \int_0^{\pi} \sqrt{1 + \cos^2 x}\,dx$$

Length of one arch of cosine curve: $y = \cos x, y' = -\sin x$

$$L_2 = \int_{-\pi/2}^{\pi/2} \sqrt{1 + \sin^2 x}\,dx$$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{1 + \cos^2\left(x - \frac{\pi}{2}\right)}\,dx, \qquad u = x - \frac{\pi}{2}, du = dx$$

$$= \int_{-\pi}^{0} \sqrt{1 + \cos^2 u}\,du$$

$$= \int_0^{\pi} \sqrt{1 + \cos^2 u}\,du = L_1$$

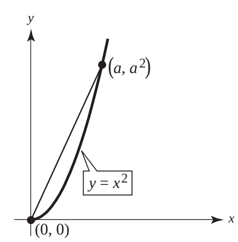
76. (a) Along line: $d_1 = \sqrt{a^2 + a^4} = a\sqrt{1 + a^2}$

Along parabola: $y = x^2, y' = 2x$

$$d_2 = \int_0^a \sqrt{1 + 4x^2}\,dx$$

$$= \frac{1}{4}\left[2x\sqrt{4x^2+1} + \ln\left|2x + \sqrt{4x^2+1}\right|\right]_0^a \quad \text{(Theorem 8.2)}$$

$$= \frac{1}{4}\left[2a\sqrt{4a^2+1} + \ln\left(2a + \sqrt{4a^2+1}\right)\right]$$

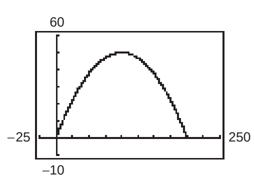


(b) For $a = 1$, $d_1 = \sqrt{2}$ and $d_2 = \frac{\sqrt{5}}{2} + \frac{1}{4}\ln\left(2 + \sqrt{5}\right) \approx 1.4789$.

For $a = 10$, $d_1 = 10\sqrt{101} \approx 100.4988$, $d_2 \approx 101.0473$.

(c) As a increases, $d_2 - d_1 \to 0$.

77. (a)



(b) $y = 0$ for $x = 200$ (range)

(c) $y = x - 0.005x^2, y' = 1 - 0.01x, 1 + (y')^2 = 1 + (1 - 0.01x)^2$

Let $u = 1 - 0.01x$, $du = -0.01\,dx$, $a = 1$. (See Theorem 8.2.)

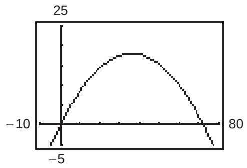
$$s = \int_0^{200} \sqrt{1 + (1 - 0.01x)^2}\,dx = -100\int_0^{200} \sqrt{(1 - 0.01x)^2 + 1}\,(-0.01)\,dx$$

$$= -50\left[(1 - 0.01x)\sqrt{(1 - 0.01x)^2 + 1} + \ln\left|(1 - 0.01x) + \sqrt{(1 - 0.01x)^2 + 1}\right|\right]_0^{200}$$

$$= -50\left[\left(-\sqrt{2} + \ln\left|-1 + \sqrt{2}\right|\right) - \left(\sqrt{2} + \ln\left|1 + \sqrt{2}\right|\right)\right]$$

$$= 100\sqrt{2} + 50\ln\left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1}\right) \approx 229.559$$

78. (a)



(b) $y = 0$ for $x = 72$

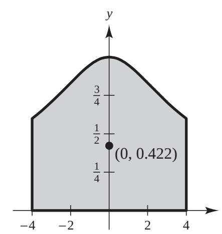
(c) $y = x - \frac{x^2}{72}, y' = 1 - \frac{x}{36}, 1 + (y')^2 = 1 + \left(1 - \frac{x}{36}\right)^2$

$$s = \int_0^{72} \sqrt{1 + \left(1 - \frac{x}{36}\right)^2}\, dx = -36\int_0^{72} \sqrt{1 + \left(1 - \frac{x}{36}\right)^2}\left(-\frac{1}{36}\right)dx$$

$$= -\frac{36}{2}\left[\left(1 - \frac{x}{36}\right)\sqrt{1 + \left(1 - \frac{x}{36}\right)^2} + \ln\left|\left(1 - \frac{x}{36}\right) + \sqrt{1 + \left(1 - \frac{x}{36}\right)^2}\right|\right]_0^{72}$$

$$= -18\left[\left(-\sqrt{2} + \ln\left|-1 + \sqrt{2}\right|\right) - \left(\sqrt{2} + \ln\left|1 + \sqrt{2}\right|\right)\right] = 36\sqrt{2} + 18\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) \approx 82.641$$

79. Let $x = 3\tan\theta$, $dx = 3\sec^2\theta\, d\theta$, $\sqrt{x^2+9} = 3\sec\theta$.



$$A = 2\int_0^4 \frac{3}{\sqrt{x^2+9}}\, dx = 6\int_0^4 \frac{dx}{\sqrt{x^2+9}} = 6\int_a^b \frac{3\sec^2\theta\, d\theta}{3\sec\theta}$$

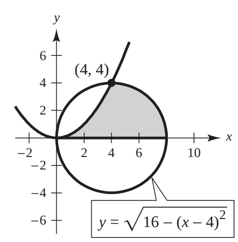
$$= 6\int_a^b \sec\theta\, d\theta = \Big[6\ln\left|\sec\theta + \tan\theta\right|\Big]_a^b = \left[6\ln\left|\frac{\sqrt{x^2+9}+x}{3}\right|\right]_0^4 = 6\ln 3$$

$\bar{x} = 0$ (by symmetry)

$$\bar{y} = \frac{1}{2}\left(\frac{1}{A}\right)\int_{-4}^4 \left(\frac{3}{\sqrt{x^2+9}}\right)^2 dx = \frac{9}{12\ln 3}\int_{-4}^4 \frac{1}{x^2+9}\, dx = \frac{3}{4\ln 3}\left[\frac{1}{3}\arctan\frac{x}{3}\right]_{-4}^4 = \frac{2}{4\ln 3}\arctan\frac{4}{3} \approx 0.422$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{1}{2\ln 3}\arctan\frac{4}{3}\right) \approx (0, 0.422)$$

80. First find where the curves intersect.



$$y^2 = 16 - (x-4)^2 = \frac{1}{16}x^4$$

$$16^2 - 16(x-4)^2 = x^4$$

$$16^2 - 16x^2 + 128x - 16^2 = x^4$$

$$x^4 + 16x^2 - 128x = 0$$

$$x(x-4)(x^2+4x+32) \implies x = 0, 4$$

$$A = \int_0^4 \frac{1}{4}x^2\, dx + \frac{1}{4}\pi(4)^2 = \frac{1}{12}x^3\Big]_0^4 + 4\pi = \frac{16}{3} + 4\pi$$

$$M_y = \int_0^4 x\left[\frac{1}{4}x^2\right]dx + \int_4^8 x\sqrt{16-(x-4)^2}\, dx$$

$$= \frac{x^4}{16}\Big]_0^4 + \int_4^8 (x-4)\sqrt{16-(x-4)^2}\, dx + \int_4^8 4\sqrt{16-(x-4)^2}\, dx$$

$$= 16 + \left[\frac{-1}{3}(16-(x-4)^2)^{3/2}\right]_4^8 + 2\left[16\arcsin\frac{x-4}{4} + (x-4)\sqrt{16-(x-4)^2}\right]_4^8$$

$$= 16 + \frac{1}{3}16^{3/2} + 2\left[16\left(\frac{\pi}{2}\right)\right] = 16 + \frac{64}{3} + 16\pi = \frac{112}{3} + 16\pi$$

—CONTINUED—

80. —CONTINUED—

$$M_x = \int_0^4 \frac{1}{2}\left(\frac{1}{4}x^2\right)^2 dx + \int_4^8 \frac{1}{2}(16 - (x-4)^2)\,dx$$

$$= \left[\frac{1}{32}\cdot\frac{x^5}{5}\right]_0^4 + \left[8x - \frac{(x-4)^3}{6}\right]_4^8$$

$$= \frac{32}{5} + \left(64 - \frac{64}{6}\right) - 32 = \frac{416}{15}$$

$$\bar{x} = \frac{M_y}{A} = \frac{112/3 + 16\pi}{16/3 + 4\pi} = \frac{112 + 48\pi}{16 + 12\pi} = \frac{28 + 12\pi}{4 + 3\pi} \approx 4.89$$

$$\bar{y} = \frac{M_x}{A} = \frac{416/15}{(16/3) + 4\pi} = \frac{104}{5(4 + 3\pi)} \approx 1.55$$

$(\bar{x}, \bar{y}) \approx (4.89, 1.55)$

81. $y = x^2,\ y' = 2x,\ 1 + (y') = 1 + 4x^2$

$2x = \tan\theta,\ dx = \frac{1}{2}\sec^2\theta\,d\theta,\ \sqrt{1+4x^2} = \sec\theta$

(For $\int\sec^5\theta\,d\theta$ and $\int\sec^3\theta\,d\theta$, see Exercise 98 in Section 8.3.)

$$S = 2\pi\int_0^{\sqrt{2}} x^2\sqrt{1+4x^2}\,dx = 2\pi\int_a^b\left(\frac{\tan\theta}{2}\right)^2(\sec\theta)\left(\frac{1}{2}\sec^2\theta\right)d\theta$$

$$= \frac{\pi}{4}\int_a^b \sec^3\theta\tan^2\theta\,d\theta = \frac{\pi}{4}\left[\int_a^b \sec^5\theta\,d\theta - \int_a^b \sec^3\theta\,d\theta\right]$$

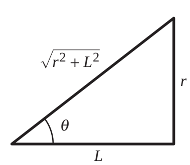
$$= \frac{\pi}{4}\left\{\frac{1}{4}\left[\sec^3\theta\tan\theta + \frac{3}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|)\right] - \frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|)\right\}\Bigg]_a^b$$

$$= \frac{\pi}{4}\left[\frac{1}{4}[(1+4x^2)^{3/2}(2x)] - \frac{1}{8}\left[(1+4x^2)^{1/2}(2x) + \ln\left|\sqrt{1+4x^2} + 2x\right|\right]\right]_0^{\sqrt{2}}$$

$$= \frac{\pi}{4}\left[\frac{54\sqrt{2}}{4} - \frac{6\sqrt{2}}{8} - \frac{1}{8}\ln\left(3 + 2\sqrt{2}\right)\right]$$

$$= \frac{\pi}{4}\left(\frac{51\sqrt{2}}{4} - \frac{\ln\left(3+2\sqrt{2}\right)}{8}\right) = \frac{\pi}{32}\left[102\sqrt{2} - \ln\left(3 + 2\sqrt{2}\right)\right] \approx 13.989$$

82. Let $r = L\tan\theta,\ dr = L\sec^2\theta\,d\theta,\ r^2 + L^2 = L^2\sec^2\theta.$



$$\frac{1}{R}\int_0^R \frac{2mL}{(r^2+L^2)^{3/2}}\,dr = \frac{2mL}{R}\int_a^b \frac{L\sec^2\theta\,d\theta}{L^3\sec^3\theta}$$

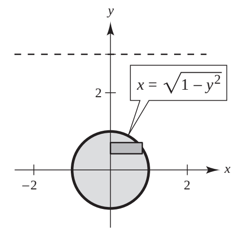
$$= \frac{2m}{RL}\int_a^b \cos\theta\,d\theta$$

$$= \left[\frac{2m}{RL}\sin\theta\right]_a^b$$

$$= \left[\frac{2m}{RL}\frac{r}{\sqrt{r^2+L^2}}\right]_0^R$$

$$= \frac{2m}{L\sqrt{R^2+L^2}}$$

83. (a) Area of representative rectangle: $2\sqrt{1-y^2}\,\Delta y$



Force: $2(62.4)(3-y)\sqrt{1-y^2}\,\Delta y$

$$F = 124.8\int_{-1}^{1}(3-y)\sqrt{1-y^2}\,dy$$

$$= 124.8\left[3\int_{-1}^{1}\sqrt{1-y^2}\,dy - \int_{-1}^{1}y\sqrt{1-y^2}\,dy\right]$$

$$= 124.8\left[\frac{3}{2}\left(\arcsin y + y\sqrt{1-y^2}\right) + \frac{1}{2}\left(\frac{2}{3}\right)(1-y^2)^{3/2}\right]_{-1}^{1}$$

$$= (62.4)3[\arcsin 1 - \arcsin(-1)] = 187.2\pi \text{ lb}$$

(b) $$F = 124.8\int_{-1}^{1}(d-y)\sqrt{1-y^2}\,dy = 124.8d\int_{-1}^{1}\sqrt{1-y^2}\,dy - 124.8\int_{-1}^{1}y\sqrt{1-y^2}\,dy$$

$$= 124.8\left(\frac{d}{2}\right)\left[\arcsin y + y\sqrt{1-y^2}\right]_{-1}^{1} - 124.8(0) = 62.4\pi d \text{ lb}$$

84. (a) $$F_{\text{inside}} = 48\int_{-1}^{0.8}(0.8-y)(2)\sqrt{1-y^2}\,dy$$

$$= 96\left[0.8\int_{-1}^{0.8}\sqrt{1-y^2}dy - \int_{-1}^{0.8}y\sqrt{1-y^2}\,dy\right]$$

$$= 96\left[\frac{0.8}{2}\left(\arcsin y + y\sqrt{1-y^2}\right) + \frac{1}{3}(1-y^2)^{3/2}\right]_{-1}^{0.8} \approx 96(1.263) \approx 121.3 \text{ lbs}$$

(b) $$F_{\text{outside}} = 64\int_{-1}^{0.4}(0.4-y)(2)\sqrt{1-y^2}\,dy$$

$$= 128\left[0.4\int_{-1}^{0.4}\sqrt{1-y^2}\,dy - \int_{-1}^{0.4}y\sqrt{1-y^2}\,dy\right] = 128\left[\frac{0.4}{2}\left(\arcsin y + y\sqrt{1-y^2}\right) + \frac{1}{3}(1-y^2)^{3/2}\right]_{-1}^{0.4} \approx 92.98$$

85. Let $u = a\sin\theta$, $du = a\cos\theta\,d\theta$, $\sqrt{a^2-u^2} = a\cos\theta$.

$$\int\sqrt{a^2-u^2}\,du = \int a^2\cos^2\theta\,d\theta = a^2\int\frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \frac{a^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C = \frac{a^2}{2}(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{a^2}{2}\left[\arcsin\frac{u}{a} + \left(\frac{u}{a}\right)\left(\frac{\sqrt{a^2+u^2}}{a}\right)\right] + C = \frac{1}{2}\left[a^2\arcsin\frac{u}{a} + u\sqrt{a^2-u^2}\right] + C$$

Let $u = a\sec\theta$, $du = a\sec\theta\tan\theta\,d\theta$, $\sqrt{u^2-a^2} = a\tan\theta$.

$$\int\sqrt{u^2-a^2}\,du = \int a\tan\theta(a\sec\theta\tan\theta)\,d\theta = a^2\int\tan^2\theta\sec\theta\,d\theta$$

$$= a^2\int(\sec^2\theta - 1)\sec\theta\,d\theta = a^2\int(\sec^3\theta - \sec\theta)\,d\theta$$

$$= a^2\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int\sec\theta\,d\theta\right] - a^2\int\sec\theta\,d\theta = a^2\left[\frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta|\right]$$

$$= \frac{a^2}{2}\left[\frac{u}{a}\cdot\frac{\sqrt{u^2-a^2}}{a} - \ln\left|\frac{u}{a} + \frac{\sqrt{u^2-a^2}}{a}\right|\right] + C_1$$

$$= \frac{1}{2}\left[u\sqrt{u^2-a^2} - a^2\ln\left|u + \sqrt{u^2-a^2}\right|\right] + C$$

—CONTINUED—

85. —CONTINUED—

Let $u = a\tan\theta$, $du = a\sec^2\theta\, d\theta$, $\sqrt{u^2 + a^2} = a\sec\theta$.

$$\int \sqrt{u^2 + a^2}\, du = \int (a\sec\theta)(a\sec^2\theta)\, d\theta$$

$$= a^2 \int \sec^3\theta\, d\theta = a^2\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right|\right] + C_1$$

$$= \frac{a^2}{2}\left[\frac{\sqrt{u^2 + a^2}}{a}\cdot\frac{u}{a} + \ln\left|\frac{\sqrt{u^2 + a^2}}{a} + \frac{u}{a}\right|\right] + C_1 = \frac{1}{2}\left[u\sqrt{u^2 + a^2} + a^2\ln\left|u + \sqrt{u^2 + a^2}\right|\right] + C$$

86. $y = \sin x$ on $[0, 2]$

$y' = \cos x$

$$s_1 = 2\int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx \qquad (\approx 3.820197789)$$

Ellipse: $x^2 + 2y^2 = 2$

Upper half: $y = \sqrt{1 - \frac{1}{2}x^2}, \quad -\sqrt{2} \le x \le \sqrt{2}$

$$y' = \frac{-x}{2\sqrt{1 - (1/2)x^2}}$$

$$s_2 = 2\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{1 + \frac{x^2}{4(1 - (1/2)x^2)}}\, dx = 2\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{1 + \frac{x^2}{4 - 2x^2}}\, dx$$

Let $x = \sqrt{2}\sin\theta$, $dx = \sqrt{2}\cos\theta\, d\theta$, $x^2 = 2\sin^2\theta$, $4 - 2x^2 = 4 - 4\sin^2\theta = 4\cos^2\theta$.

$$s_2 = 2\int_{-\pi/2}^{\pi/2} \sqrt{1 + \frac{2\sin^2\theta}{4\cos^2\theta}}\sqrt{2}\cos\theta\, d\theta$$

$$= 2\int_{-\pi/2}^{\pi/2} \frac{\sqrt{4\cos^2\theta + 2\sin^2\theta}}{2\cos\theta}\sqrt{2}\cos\theta\, d\theta$$

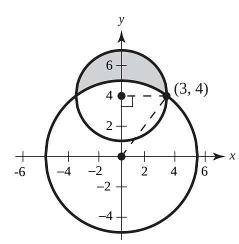
$$= 2\int_{-\pi/2}^{\pi/2} \frac{\sqrt{2 + 2\cos^2\theta}}{\sqrt{2}}\, d\theta$$

$$= 2\int_{-\pi/2}^{\pi/2} \sqrt{1 + \cos^2\theta}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{1 + \cos^2\theta}\, d\theta = s_1$$

87. Large circle: $x^2 + y^2 = 25$

$$y = \sqrt{25 - x^2}, \quad \text{upper half}$$



From the right triangle, the center of the small circle is $(0, 4)$.

$x^2 + (y - 4)^2 = 9$

$$y = 4 + \sqrt{9 - x^2}, \quad \text{upper half}$$

$$A = 2\int_0^3 \left[\left(4 + \sqrt{9 - x^2}\right) - \sqrt{25 - x^2}\right] dx$$

$$= 2\left[4x + \frac{1}{2}\left[9\arcsin\left(\frac{x}{3}\right) + x\sqrt{9 - x^2}\right] - \frac{1}{2}\left[25\arcsin\left(\frac{x}{5}\right) + x\sqrt{25 - x^2}\right]\right]_0^3$$

$$= 2\left[12 + \frac{9}{2}\arcsin(1) - \frac{25}{2}\arcsin\frac{3}{5} - 6\right]$$

$$= 12 + \frac{9\pi}{2} - 25\arcsin\frac{3}{5} \approx 10.050$$

Section 8.5 Partial Fractions

1. $\frac{5}{x^2-10x} = \frac{5}{x(x-10)} = \frac{A}{x} + \frac{B}{x-10}$

2. $\frac{4x^2+3}{(x-5)^3} = \frac{A}{x-5} + \frac{B}{(x-5)^2} + \frac{C}{(x-5)^3}$

3. $\frac{2x-3}{x^3+10x} = \frac{2x-3}{x(x^2+10)} = \frac{A}{x} + \frac{Bx+C}{x^2+10}$

4. $\frac{x-2}{x^2+4x+3} = \frac{x-2}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$

5. $\frac{16}{x(x-10)} = \frac{A}{x} + \frac{B}{x-10}$

6. $\frac{2x-1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$

7. $\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$

$$1 = A(x-1) + B(x+1)$$

When $x = -1$, $1 = -2A$, $A = -\frac{1}{2}$.

When $x = 1$, $1 = 2B$, $B = \frac{1}{2}$.

$$\int \frac{1}{x^2-1}\,dx = -\frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{1}{2}\int \frac{1}{x-1}\,dx$$
$$= -\frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C$$
$$= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$$

8. $\frac{1}{4x^2-9} = \frac{1}{(2x-3)(2x+3)} = \frac{A}{2x-3} + \frac{B}{2x+3}$

$$1 = A(2x+3) + B(2x-3)$$

When $x = \frac{3}{2}$, $1 = 6A$, $A = \frac{1}{6}$.

When $x = -\frac{3}{2}$, $1 = -6B$, $B = -\frac{1}{6}$.

$$\int \frac{1}{4x^2-9}\,dx = \frac{1}{6}\left[\int \frac{1}{2x-3}\,dx - \int \frac{1}{2x+3}\,dx\right]$$
$$= \frac{1}{12}[\ln|2x-3| - \ln|2x+3|] + C$$
$$= \frac{1}{12}\ln\left|\frac{2x-3}{2x+3}\right| + C$$

9. $\frac{3}{x^2+x-2} = \frac{3}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$

$$3 = A(x+2) + B(x-1)$$

When $x = 1$, $3 = 3A$, $A = 1$.

When $x = -2$, $3 = -3B$, $B = -1$.

$$\int \frac{3}{x^2+x-2}\,dx = \int \frac{1}{x-1}\,dx - \int \frac{1}{x+2}\,dx$$
$$= \ln|x-1| - \ln|x+2| + C$$
$$= \ln\left|\frac{x-1}{x+2}\right| + C$$

10. $\int \frac{x+1}{x^2+4x+3}\,dx = \int \frac{(x+1)}{(x+1)(x+3)}\,dx$

$$= \int \frac{1}{x+3}\,dx = \ln|x+3| + C$$

11. $\frac{5-x}{2x^2+x-1} = \frac{5-x}{(2x-1)(x+1)} = \frac{A}{2x-1} + \frac{B}{x+1}$

$$5 - x = A(x+1) + B(2x-1)$$

When $x = \frac{1}{2}$, $\frac{9}{2} = \frac{3}{2}A$, $A = 3$.

When $x = -1$, $6 = -3B$, $B = -2$.

$$\int \frac{5-x}{2x^2+x-1}\,dx = 3\int \frac{1}{2x-1}\,dx - 2\int \frac{1}{x+1}\,dx$$
$$= \frac{3}{2}\ln|2x-1| - 2\ln|x+1| + C$$

12. $\frac{5x^2-12x-12}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}$

$$5x^2 - 12x - 12 = A(x^2-4) + Bx(x+2) + Cx(x-2)$$

When $x = 0$, $-12 = -4A \implies A = 3$. When $x = 2$, $-16 = 8B \implies B = -2$. When $x = -2$, $32 = 8C \implies C = 4$.

$$\int \frac{5x^2-12x-12}{x^3-4x}\,dx$$
$$= \int \frac{3}{x}\,dx + \int \frac{-2}{x-2}\,dx + \int \frac{4}{x+2}\,dx$$
$$= 3\ln|x| - 2\ln|x-2| + 4\ln|x+2| + C$$

13. $\dfrac{x^2+12x+12}{x(x+2)(x-2)} = \dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{x-2}$

$x^2 + 12x + 12 = A(x+2)(x-2) + Bx(x-2) + Cx(x+2)$

When $x = 0$, $12 = -4A$, $A = -3$. When $x = -2$, $-8 = 8B$, $B = -1$. When $x = 2$, $40 = 8C$, $C = 5$.

$$\int \frac{x^2+12x+12}{x^3-4x}\,dx = 5\int \frac{1}{x-2}\,dx - \int \frac{1}{x+2}\,dx - 3\int \frac{1}{x}\,dx$$
$$= 5\ln|x-2| - \ln|x+2| - 3\ln|x| + C$$

14. $\dfrac{x^3-x+3}{x^2+x-2} = x - 1 + \dfrac{2x+1}{(x+2)(x-1)} = x - 1 + \dfrac{A}{x+2} + \dfrac{B}{x-1}$

$2x + 1 = A(x-1) + B(x+2)$

When $x = -2$, $-3 = -3A$, $A = 1$. When $x = 1$, $3 = 3B$, $B = 1$.

$$\int \frac{x^3-x+3}{x^2+x-2}\,dx = \int \left[x - 1 + \frac{1}{x+2} + \frac{1}{x-1}\right] dx$$
$$= \frac{x^2}{2} - x + \ln|x+2| + \ln|x-1| + C = \frac{x^2}{2} - x + \ln|x^2+x-2| + C$$

15. $\dfrac{2x^3-4x^2-15x+5}{x^2-2x-8} = 2x + \dfrac{x+5}{(x-4)(x+2)} = 2x + \dfrac{A}{x-4} + \dfrac{B}{x+2}$

$x + 5 = A(x+2) + B(x-4)$

When $x = 4$, $9 = 6A$, $A = \frac{3}{2}$. When $x = -2$, $3 = -6B$, $B = -\frac{1}{2}$.

$$\int \frac{2x^3-4x^2-15x+5}{x^2-2x-8}\,dx = \int \left[2x + \frac{3/2}{x-4} - \frac{1/2}{x+2}\right] dx$$
$$= x^2 + \frac{3}{2}\ln|x-4| - \frac{1}{2}\ln|x+2| + C$$

16. $\dfrac{x+2}{x(x-4)} = \dfrac{A}{x-4} + \dfrac{B}{x}$

$x + 2 = Ax + B(x-4)$

When $x = 4$, $6 = 4A$, $A = \frac{3}{2}$.

When $x = 0$, $2 = -4B$, $B = -\frac{1}{2}$.

$$\int \frac{x+2}{x^2-4x}\,dx = \int \left[\frac{3/2}{x-4} - \frac{1/2}{x}\right] dx$$
$$= \frac{3}{2}\ln|x-4| - \frac{1}{2}\ln|x| + C$$

17. $\dfrac{4x^2+2x-1}{x^2(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1}$

$4x^2 + 2x - 1 = Ax(x+1) + B(x+1) + Cx^2$

When $x = 0$, $B = -1$. When $x = -1$, $C = 1$. When $x = 1$, $A = 3$.

$$\int \frac{4x^2+2x-1}{x^3+x^2}\,dx = \int \left[\frac{3}{x} - \frac{1}{x^2} + \frac{1}{x+1}\right] dx$$
$$= 3\ln|x| + \frac{1}{x} + \ln|x+1| + C$$
$$= \frac{1}{x} + \ln|x^4 + x^3| + C$$

18. $\dfrac{2x-3}{(x-1)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2}$

$2x - 3 = A(x-1) + B$

When $x = 1$, $B = -1$. When $x = 0$, $A = 2$.

$$\int \frac{2x-3}{(x-1)^2}\,dx = \int \left[\frac{2}{x-1} - \frac{1}{(x-1)^2}\right] dx = 2\ln|x-1| + \frac{1}{x-1} + C$$

19. $\dfrac{x^2+3x-4}{x^3-4x^2+4x} = \dfrac{x^2+3x-4}{x(x-2)^2} = \dfrac{A}{x} + \dfrac{B}{(x-2)} + \dfrac{C}{(x-2)^2}$

$x^2+3x-4 = A(x-2)^2 + Bx(x-2) + Cx$

When $x = 0$, $-4 = 4A \implies A = -1$. When $x = 2$, $6 = 2C \implies C = 3$. When $x = 1$, $0 = -1 - B + 3 \implies B = 2$.

$$\int \frac{x^2+3x-4}{x^3-4x^2+4x}\,dx = \int \frac{-1}{x}\,dx + \int \frac{2}{(x-2)}\,dx + \int \frac{3}{(x-2)^2}\,dx$$

$$= -\ln|x| + 2\ln|x-2| - \frac{3}{(x-2)} + C$$

20. $\dfrac{4x^2}{x^3+x^2-x-1} = \dfrac{4x^2}{x^2(x+1)-(x+1)} = \dfrac{4x^2}{(x^2-1)(x+1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

$4x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$

When $x = -1$, $4 = -2C \implies C = -2$. When $x = 1$, $4 = 4A \implies A = 1$. When $x = 0$, $0 = 1 - B + 2 \implies B = 3$.

$$\int \frac{4x^2}{x^3+x^2-x-1}\,dx = \int \frac{1}{x-1}\,dx + \int \frac{3}{x+1}\,dx - \int \frac{2}{(x+1)^2}\,dx$$

$$= \ln|x-1| + 3\ln|x+1| + \frac{2}{(x+1)} + C$$

21. $\dfrac{x^2-1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

$x^2 - 1 = A(x^2+1) + (Bx+C)x$

When $x = 0$, $A = -1$. When $x = 1$, $0 = -2 + B + C$. When $x = -1$, $0 = -2 + B - C$.
Solving these equations we have $A = -1$, $B = 2$, $C = 0$.

$$\int \frac{x^2-1}{x^3+x}\,dx = -\int \frac{1}{x}\,dx + \int \frac{2x}{x^2+1}\,dx$$

$$= -\ln|x| + \ln|x^2+1| + C$$

$$= \ln\left|\frac{x^2+1}{x}\right| + C$$

22. $\dfrac{6x}{x^3-8} = \dfrac{6x}{(x-2)(x^2+2x+4)} = \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+2x+4}$

$6x = A(x^2+2x+4) + (Bx+C)(x-2)$

When $x = 2$, $12 = 12A \implies A = 1$. When $x = 0$, $0 = 4 - 2C \implies C = 2$.
When $x = 1$, $6 = 7 + (B+2)(-1) \implies B = -1$.

$$\int \frac{6x}{x^3-8}\,dx = \int \frac{1}{x-2}\,dx + \int \frac{-x+2}{x^2+2x+4}\,dx$$

$$= \int \frac{1}{x-2}\,dx + \int \frac{-x-1}{x^2+2x+4}\,dx + \int \frac{3}{(x^2+2x+1)+3}\,dx$$

$$= \ln|x-2| - \frac{1}{2}\ln|x^2+2x+4| + \frac{3}{\sqrt{3}}\arctan\left(\frac{x+1}{\sqrt{3}}\right) + C$$

$$= \ln|x-2| - \frac{1}{2}\ln|x^2+2x+4| + \sqrt{3}\arctan\left(\frac{\sqrt{3}(x+1)}{3}\right) + C$$

23. $\dfrac{x^2}{x^4 - 2x^2 - 8} = \dfrac{A}{x-2} + \dfrac{B}{x+2} + \dfrac{Cx+D}{x^2+2}$

$$x^2 = A(x+2)(x^2+2) + B(x-2)(x^2+2) + (Cx+D)(x+2)(x-2)$$

When $x = 2$, $4 = 24A$. When $x = -2$, $4 = -24B$. When $x = 0$, $0 = 4A - 4B - 4D$, and when $x = 1$, $1 = 9A - 3B - 3C - 3D$. Solving these equations we have $A = \frac{1}{6}$, $B = -\frac{1}{6}$, $C = 0$, $D = \frac{1}{3}$.

$$\int \frac{x^2}{x^4 - 2x^2 - 8}\,dx = \frac{1}{6}\left[\int \frac{1}{x-2}\,dx - \int \frac{1}{x+2}\,dx + 2\int \frac{1}{x^2+2}\,dx\right]$$

$$= \frac{1}{6}\left[\ln\left|\frac{x-2}{x+2}\right| + \sqrt{2}\arctan\frac{x}{\sqrt{2}}\right] + C$$

24. $\dfrac{x^2 - x + 9}{(x^2+9)^2} = \dfrac{Ax+B}{x^2+9} + \dfrac{Cx+D}{(x^2+9)^2}$

$$x^2 - x + 9 = (Ax+B)(x^2+9) + Cx + D$$

$$= Ax^3 + Bx^2 + (9A + C)x + (9B + D)$$

By equating coefficients of like terms, we have $A = 0$, $B = 1$, $D = 0$, and $C = -1$.

$$\int \frac{x^2 - x - 9}{(x^2+9)^2} = \int \frac{1}{x^2+9}\,dx - \int \frac{x}{(x^2+9)^2}\,dx = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + \frac{1}{2(x^2+9)} + C$$

25. $\dfrac{x}{(2x-1)(2x+1)(4x^2+1)} = \dfrac{A}{2x-1} + \dfrac{B}{2x+1} + \dfrac{Cx+D}{4x^2+1}$

$$x = A(2x+1)(4x^2+1) + B(2x-1)(4x^2+1) + (Cx+D)(2x-1)(2x+1)$$

When $x = \frac{1}{2}$, $\frac{1}{2} = 4A$. When $x = -\frac{1}{2}$, $-\frac{1}{2} = -4B$. When $x = 0$, $0 = A - B - D$, and when $x = 1$, $1 = 15A + 5B + 3C + 3D$. Solving these equations we have $A = \frac{1}{8}$, $B = \frac{1}{8}$, $C = -\frac{1}{2}$, $D = 0$.

$$\int \frac{x}{16x^4 - 1}\,dx = \frac{1}{8}\left[\int \frac{1}{2x-1}\,dx + \int \frac{1}{2x+1}\,dx - 4\int \frac{x}{4x^2+1}\,dx\right] = \frac{1}{16}\ln\left|\frac{4x^2-1}{4x^2+1}\right| + C$$

26. $\dfrac{x^2 - 4x + 7}{(x+1)(x^2 - 2x + 3)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2 - 2x + 3}$

$$x^2 - 4x + 7 = A(x^2 - 2x + 3) + (Bx + C)(x + 1)$$

When $x = -1$, $12 = 6A$. When $x = 0$, $7 = 3A + C$. When $x = 1$, $4 = 2A + 2B + 2C$.
Solving these equations we have $A = 2$, $B = -1$, $C = 1$.

$$\int \frac{x^2 - 4x + 7}{x^3 - x^2 + x + 3}\,dx = 2\int \frac{1}{x+1}\,dx + \int \frac{-x+1}{x^2 - 2x + 3}\,dx$$

$$= 2\ln|x+1| - \frac{1}{2}\ln|x^2 - 2x + 3| + C$$

27. $\dfrac{x^2 + 5}{(x+1)(x^2 - 2x + 3)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2 - 2x + 3}$

$$x^2 + 5 = A(x^2 - 2x + 3) + (Bx + C)(x + 1)$$

$$= (A + B)x^2 + (-2A + B + C)x + (3A + C)$$

When $x = -1$, $A = 1$. By equating coefficients of like terms, we have $A + B = 1$, $-2A + B + C = 0$, $3A + C = 5$. Solving these equations we have $A = 1$, $B = 0$, $C = 2$.

$$\int \frac{x^2 + 5}{x^3 - x^2 + x + 3}\,dx = \int \frac{1}{x+1}\,dx + 2\int \frac{1}{(x-1)^2 + 2}\,dx$$

$$= \ln|x+1| + \sqrt{2}\arctan\left(\frac{x-1}{\sqrt{2}}\right) + C$$

28. $\dfrac{x^2 + x + 3}{(x^2 + 3)^2} = \dfrac{Ax + B}{x^2 + 3} + \dfrac{Cx + D}{(x^2 + 3)^2}$

$$x^2 + x + 3 = (Ax + B)(x^2 + 3) + Cx + D$$
$$= Ax^3 + Bx^2 + (3A + C)x + (3B + D)$$

By equating coefficients of like terms, we have $A = 0$, $B = 1$, $3A + C = 1$, $3B + D = 3$. Solving these equations we have $A = 0$, $B = 1$, $C = 1$, $D = 0$.

$$\int \frac{x^2 + x + 3}{x^4 + 6x^2 + 9}\,dx = \int \left[\frac{1}{x^2 + 3} + \frac{x}{(x^2 + 3)^2}\right] dx$$
$$= \frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}} - \frac{1}{2(x^2 + 3)} + C$$

29. $\dfrac{3}{(2x + 1)(x + 2)} = \dfrac{A}{2x + 1} + \dfrac{B}{x + 2}$

$$3 = A(x + 2) + B(2x + 1)$$

When $x = -\frac{1}{2}$, $A = 2$. When $x = -2$, $B = -1$.

$$\int_0^1 \frac{3}{2x^2 + 5x + 2}\,dx = \int_0^1 \frac{2}{2x + 1}\,dx - \int_0^1 \frac{1}{x + 2}\,dx$$
$$= \Big[\ln|2x + 1| - \ln|x + 2|\Big]_0^1$$
$$= \ln 2$$

30. $\dfrac{x - 1}{x^2(x + 1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x + 1}$

$$x - 1 = Ax(x + 1) + B(x + 1) + Cx^2$$

When $x = 0$, $B = -1$. When $x = -1$, $C = -2$. When $x = 1$, $0 = 2A + 2B + C$. Solving these equations we have $A = 2$, $B = -1$, $C = -2$.

$$\int_1^5 \frac{x - 1}{x^2(x + 1)}\,dx = 2\int_1^5 \frac{1}{x}\,dx - \int_1^5 \frac{1}{x^2}\,dx - 2\int_1^5 \frac{1}{x + 1}\,dx$$
$$= \left[2 \ln|x| + \frac{1}{x} - 2 \ln|x + 1|\right]_1^5$$
$$= \left[2 \ln\left|\frac{x}{x + 1}\right| + \frac{1}{x}\right]_1^5$$
$$= 2 \ln \frac{5}{3} - \frac{4}{5}$$

31. $\dfrac{x + 1}{x(x^2 + 1)} = \dfrac{A}{x} + \dfrac{Bx + C}{x^2 + 1}$

$$x + 1 = A(x^2 + 1) + (Bx + C)x$$

When $x = 0$, $A = 1$. When $x = 1$, $2 = 2A + B + C$. When $x = -1$, $0 = 2A + B - C$. Solving these equations we have $A = 1$, $B = -1$, $C = 1$.

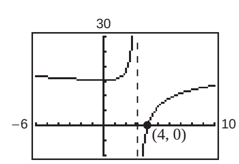
$$\int_1^2 \frac{x + 1}{x(x^2 + 1)}\,dx = \int_1^2 \frac{1}{x}\,dx - \int_1^2 \frac{x}{x^2 + 1}\,dx + \int_1^2 \frac{1}{x^2 + 1}\,dx$$
$$= \left[\ln|x| - \frac{1}{2} \ln(x^2 + 1) + \arctan x\right]_1^2$$
$$= \frac{1}{2} \ln \frac{8}{5} - \frac{\pi}{4} + \arctan 2$$
$$\approx 0.557$$

32. $\displaystyle\int_0^1 \frac{x^2 - x}{x^2 + x + 1}\,dx = \int_0^1 dx - \int_0^1 \frac{2x + 1}{x^2 + x + 1}\,dx$

$$= \Big[x - \ln|x^2 + x + 1|\Big]_0^1$$
$$= 1 - \ln 3$$

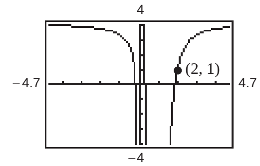
33. $\displaystyle\int \frac{3x\,dx}{x^2 - 6x + 9} = 3 \ln|x - 3| - \frac{9}{x - 3} + C$

$(4, 0)$: $\quad 3 \ln|4 - 3| - \dfrac{9}{4 - 3} + C = 0 \Longrightarrow C = 9$



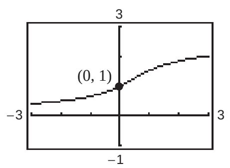
34. $\int \frac{6x^2+1}{x^2(x-1)^3}\,dx = 3\ln\left|\frac{x-1}{x}\right| + \frac{1}{x} + \frac{2}{x-1} - \frac{7}{2(x-1)^2} + C$

$(2, 1)$: $3\ln\left|\frac{1}{2}\right| + \frac{1}{2} + \frac{2}{1} - \frac{7}{2} + C = 1 \Rightarrow C = 2 - 3\ln\frac{1}{2}$



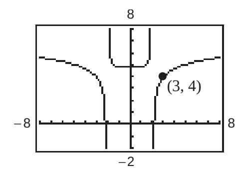
35. $\int \frac{x^2+x+2}{(x^2+2)^2}\,dx = \frac{\sqrt{2}}{2}\arctan\frac{x}{\sqrt{2}} - \frac{1}{2(x^2+2)} + C$

$(0, 1)$: $0 - \frac{1}{4} + C = 1 \Rightarrow C = \frac{5}{4}$



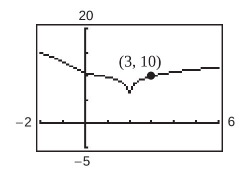
36. $\int \frac{x^3}{(x^2-4)^2}\,dx = \frac{1}{2}\ln|x^2-4| - \frac{2}{x^2-4} + C$

$(3, 4)$: $\frac{1}{2}\ln 5 - \frac{2}{5} + C = 4 \Rightarrow C = \frac{22}{5} - \frac{1}{2}\ln 5$



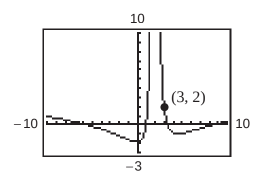
37. $\int \frac{2x^2-2x+3}{x^3-x^2-x-2}\,dx = \ln|x-2| + \frac{1}{2}\ln|x^2+x+1| - \sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$

$(3, 10)$: $0 + \frac{1}{2}\ln 13 - \sqrt{3}\arctan\frac{7}{\sqrt{3}} + C = 10 \Rightarrow C = 10 - \frac{1}{2}\ln 13 + \sqrt{3}\arctan\frac{7}{\sqrt{3}}$



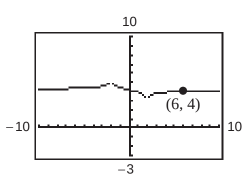
38. $\int \frac{x(2x-9)}{x^3-6x^2+12x-8}\,dx = 2\ln|x-2| + \frac{1}{x-2} + \frac{5}{(x-2)^2} + C$

$(3, 2)$: $0 + 1 + 5 + C = 2 \Rightarrow C = -4$



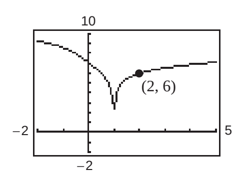
39. $\int \frac{1}{x^2-4}\,dx = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$

$(6, 4)$: $\frac{1}{4}\ln\left|\frac{4}{8}\right| + C = 4 \Rightarrow C = 4 - \frac{1}{4}\ln\frac{1}{2} = 4 + \frac{1}{4}\ln 2$



40. $\int \frac{x^2-x+2}{x^3-x^2+x-1}\,dx = -\arctan x + \ln|x-1| + C$

$(2, 6)$: $-\arctan 2 + 0 + C = 6 \Rightarrow C = 6 + \arctan 2$



41. Let $u = \cos x$ $du = -\sin x\,dx$.

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$$

$$1 = A(u-1) + Bu$$

When $u = 0$, $A = -1$. When $u = 1$, $B = 1$, $u = \cos x$, $du = -\sin x\,dx$.

$$\int \frac{\sin x}{\cos x(\cos x - 1)}\,dx = -\int \frac{1}{u(u-1)}\,du$$

$$= \int \frac{1}{u}\,du - \int \frac{1}{u-1}\,du = \ln|u| - \ln|u-1| + C = \ln\left|\frac{u}{u-1}\right| + C = \ln\left|\frac{\cos x}{\cos x - 1}\right| + C$$

42. Let $u = \cos x$, $du = \sin x\,dx$.

$$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$$

$$1 = A(u+1) + Bu$$

When $u = 0$, $A = 1$. When $u = -1$, $B = -1$, $u = \cos x$, $du = -\sin dx$.

$$\begin{aligned}\int \frac{\sin x}{\cos x + \cos^2 x}\,dx &= -\int \frac{1}{u(u+1)}\,du\\ &= \int \frac{1}{u+1}\,du - \int \frac{1}{u}\,du\\ &= \ln|u+1| - \ln|u| + C\\ &= \ln\left|\frac{u+1}{u}\right| + C\\ &= \ln\left|\frac{\cos x + 1}{\cos x}\right| + C\\ &= \ln|1 + \sec x| + C\end{aligned}$$

43. $$\begin{aligned}\int \frac{3\cos x}{\sin^2 x + \sin x - 2}\,dx &= 3\int \frac{1}{u^2 + u - 2}\,du\\ &= \ln\left|\frac{u-1}{u+2}\right| + C\\ &= \ln\left|\frac{-1 + \sin x}{2 + \sin x}\right| + C\end{aligned}$$

(From Exercise 9 with $u = \sin x$, $du = \cos x\,dx$)

44. $\dfrac{1}{u(u+1)} = \dfrac{A}{u} + \dfrac{B}{u+1}$, $u = \tan x$, $du = \sec^2 x\,dx$

$$1 = A(u+1) + Bu$$

When $u = 0$, $A = 1$.
When $u = -1$, $1 = -B \Rightarrow B = -1$.

$$\begin{aligned}\int \frac{\sec^2 x\,dx}{\tan x(\tan x + 1)} &= \int \frac{1}{u(u+1)}\,du\\ &= \int\left(\frac{1}{u} - \frac{1}{u+1}\right)du\\ &= \ln|u| - \ln|u+1| + C\\ &= \ln\left|\frac{u}{u+1}\right| + C\\ &= \ln\left|\frac{\tan x}{\tan x + 1}\right| + C\end{aligned}$$

45. Let $u = e^x$, $du = e^x\,dx$.

$$\frac{1}{(u-1)(u+4)} = \frac{A}{u-1} + \frac{B}{u+4}$$

$$1 = A(u+4) + B(u-1)$$

When $u = 1$, $A = \frac{1}{5}$. When $u = -4$, $B = -\frac{1}{5}$, $u = e^x$, $du = e^x\,dx$.

$$\begin{aligned}\int \frac{e^x}{(e^x - 1)(e^x + 4)}\,dx &= \int \frac{1}{(u-1)(u+4)}\,du\\ &= \frac{1}{5}\left(\int \frac{1}{u-1}\,du - \int \frac{1}{u+4}\,du\right)\\ &= \frac{1}{5}\ln\left|\frac{u-1}{u+4}\right| + C\\ &= \frac{1}{5}\ln\left|\frac{e^x - 1}{e^x + 4}\right| + C\end{aligned}$$

46. Let $u = e^x$, $du = e^x\,dx$.

$$\frac{1}{(u^2+1)(u-1)} = \frac{A}{u-1} + \frac{Bu + C}{u^2 + 1}$$

$$1 = A(u^2 + 1) + (Bu + C)(u - 1)$$

When $u = 1$, $A = \frac{1}{2}$. When $u = 0$, $1 = A - C$. When $u = -1$, $1 = 2A + 2B - 2C$.
Solving these equations we have $A = \frac{1}{2}$, $B = -\frac{1}{2}$, $C = -\frac{1}{2}$, $u = e^x$, $du = e^x\,dx$.

$$\begin{aligned}\int \frac{e^x}{(e^{2x} + 1)(e^x - 1)}\,dx &= \int \frac{1}{(u^2+1)(u-1)}\,du\\ &= \frac{1}{2}\left(\int \frac{1}{u-1}\,du - \int \frac{u+1}{u^2+1}\,du\right)\\ &= \frac{1}{2}\left(\ln|u-1| - \frac{1}{2}\ln|u^2+1| - \arctan u\right) + C\\ &= \frac{1}{4}\left(2\ln|e^x - 1| - \ln|e^{2x} + 1| - 2\arctan e^x\right) + C\end{aligned}$$

47. $\dfrac{1}{x(a+bx)} = \dfrac{A}{x} + \dfrac{B}{a+bx}$

$$1 = A(a+bx) + Bx$$

When $x = 0$, $1 = aA \Rightarrow A = 1/a$.
When $x = -a/b$, $1 = -(a/b)B \Rightarrow B = -b/a$.

$$\int \frac{1}{x(a+bx)}\,dx = \frac{1}{a}\int\left(\frac{1}{x} - \frac{b}{a+bx}\right)dx = \frac{1}{a}\left(\ln|x| - \ln|a+bx|\right) + C = \frac{1}{a}\ln\left|\frac{x}{a+bx}\right| + C$$

48. $\dfrac{1}{a^2-x^2} = \dfrac{A}{a-x} + \dfrac{B}{a+x}$

$$1 = A(a+x) + B(a-x)$$

When $x = a$, $A = 1/2a$.
When $x = -a$, $B = 1/2a$.

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\int\left(\frac{1}{a-x} + \frac{1}{a+x}\right)dx = \frac{1}{2a}\left(-\ln|a-x| + \ln|a+x|\right) + C = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

49. $\dfrac{x}{(a+bx)^2} = \dfrac{A}{a+bx} + \dfrac{B}{(a+bx)^2}$

$$x = A(a+bx) + B$$

When $x = -a/b$, $B = -a/b$.
When $x = 0$, $0 = aA + B \Rightarrow A = 1/b$.

$$\int \frac{x}{(a+bx)^2}\,dx = \int\left(\frac{1/b}{a+bx} + \frac{-a/b}{(a+bx)^2}\right)dx = \frac{1}{b}\int\frac{1}{a+bx}\,dx - \frac{a}{b}\int\frac{1}{(a+bx)^2}\,dx = \frac{1}{b^2}\ln|a+bx| + \frac{a}{b^2}\left(\frac{1}{a+bx}\right) + C = \frac{1}{b^2}\left(\frac{a}{a+bx} + \ln|a+bx|\right) + C$$

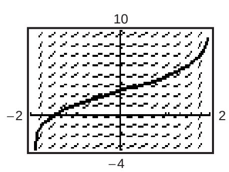
50. $\dfrac{1}{x^2(a+bx)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{a+bx}$

$$1 = Ax(a+bx) + B(a+bx) + Cx^2$$

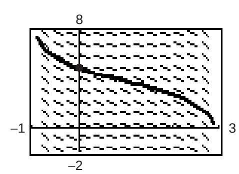
When $x = 0$, $1 = Ba \Rightarrow B = 1/a$. When $x = -a/b$, $1 = C(a^2/b^2) \Rightarrow C = b^2/a^2$. When $x = 1$, $1 = (a+b)A + (a+b)B + C \Rightarrow A = -b/a^2$.

$$\int \frac{1}{x^2(a+bx)}\,dx = \int\left(\frac{-b/a^2}{x} + \frac{1/a}{x^2} + \frac{b^2/a^2}{a+bx}\right)dx = -\frac{b}{a^2}\ln|x| - \frac{1}{ax} + \frac{b}{a^2}\ln|a+bx| + C = -\frac{1}{ax} + \frac{b}{a^2}\ln\left|\frac{a+bx}{x}\right| + C = -\frac{1}{ax} - \frac{b}{a^2}\ln\left|\frac{x}{a+bx}\right| + C$$

51. $\dfrac{dy}{dx} = \dfrac{6}{4-x^2}$, $y(0) = 3$



52. $\dfrac{dy}{dx} = \dfrac{4}{(x^2-2x-3)}$, $y(0) = 5$



53. Dividing x^3 by $x - 5$

54. (a) $\dfrac{N(x)}{D(x)} = \dfrac{A_1}{px+q} + \dfrac{A_2}{(px+q)^2} + \cdots + \dfrac{A_m}{(px+q)^m}$

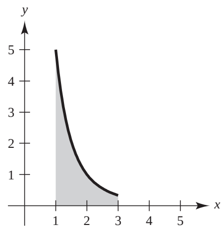
(b) $\dfrac{N(x)}{D(x)} = \dfrac{A_1 + B_1x}{(ax^2+bx+c)} + \cdots + \dfrac{A_n + B_nx}{(ax^2+bx+c)^n}$

55. (a) Substitution: $u = x^2 + 2x - 8$

(b) Partial fractions

(c) Trigonometric substitution (tan) or inverse tangent rule

56. $A = \int_1^3 \frac{10}{x(x^2+1)}\,dx \approx 3$, matches (c).



57. $A = \int_0^1 \frac{12}{x^2+5x+6}\,dx$

$$\frac{12}{x^2+5x+6} = \frac{12}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$12 = A(x+3) + B(x+2)$$

Let $x = -3$: $12 = B(-1) \implies B = -12$

Let $x = -2$: $12 = A(1) \implies A = 12$

$$A = \int_0^1 \left(\frac{12}{x+2} - \frac{12}{x+3}\right) dx$$

$$= \Big[12\ln|x+2| - 12\ln|x+3|\Big]_0^1$$

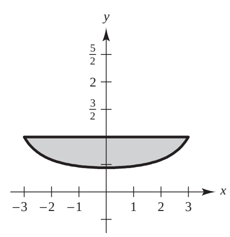
$$= 12[\ln 3 - \ln 4 - \ln 2 + \ln 3]$$

$$= 12\ln\left(\frac{9}{8}\right) \approx 1.4134$$

58. $A = 2\int_0^3 \left(1 - \frac{7}{16-x^2}\right) dx = 2\int_0^3 dx - 14\int_0^3 \frac{1}{16-x^2}\,dx$

$$= \left[2x - \frac{14}{8}\ln\left|\frac{4+x}{4-x}\right|\right]_0^3 \quad \text{(From Exercise 48)}$$

$$= 6 - \frac{7}{4}\ln 7 \approx 2.595$$



59. Average cost $= \frac{1}{80-75}\int_{75}^{80} \frac{124p}{(10+p)(100-p)}\,dp$

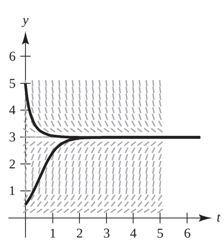
$$= \frac{1}{5}\int_{75}^{80}\left(\frac{-124}{(10+p)11} + \frac{1240}{(100-p)11}\right) dp$$

$$= \frac{1}{5}\left[\frac{-124}{11}\ln(10+p) - \frac{1240}{11}\ln(100-p)\right]_{75}^{80}$$

$$\approx \frac{1}{5}(24.51) = 4.9$$

Approximately \$490,000

60. (a)



(b) The slope is negative because the function is decreasing.

(c) For $y > 0$, $\lim_{t\to\infty} y(t) = 3$.

(d) $\frac{dy}{y(L-y)} = \frac{A}{y} + \frac{B}{L-y}$

$$1 = A(L-y) + By \implies A = \frac{1}{L},\ B = \frac{1}{L}$$

$$\int \frac{dy}{y(L-y)} = \int k\,dt$$

$$\frac{1}{L}\left[\int \frac{1}{y}dy + \int \frac{1}{L-y}dy\right] = \int k\,dt$$

$$\frac{1}{L}\big[\ln|y| - \ln|L-y|\big] = kt + C_1$$

$$\ln\left|\frac{y}{L-y}\right| = kLt + LC_1$$

$$C_2 e^{kLt} = \frac{y}{L-y}$$

When $t = 0$, $\frac{y_0}{L-y_0} = C_2 \implies \frac{y}{L-y} = \frac{y_0}{L-y_0}e^{kLt}$.

Solving for y, you obtain $y = \frac{y_0 L}{y_0 + (L-y_0)e^{-kLt}}$.

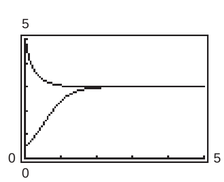
(e) $k = 1, L = 3$

(i) $y(0) = 5$: $y = \frac{15}{5 - 2e^{-3t}}$

(ii) $y(0) = \frac{1}{2}$:

$$y = \frac{3/2}{(1/2) + (5/2)e^{-3t}}$$

$$= \frac{3}{1 + 5e^{-3t}}$$



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60. —CONTINUED—

(f) $\dfrac{dy}{dt} = ky(L - y)$

$$\frac{d^2y}{dt^2} = k\left[y\left(\frac{-dy}{dt}\right) + (L - y)\frac{dy}{dt}\right] = 0$$

$$\Rightarrow y\frac{dy}{dt} = (L - y)\frac{dy}{dt}$$

$$\Rightarrow \quad y = \frac{L}{2}$$

From the first derivative test, this is a maximum.

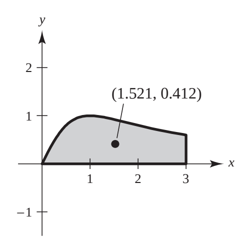
61. $V = \pi\int_0^3 \left(\frac{2x}{x^2 + 1}\right)^2 dx = 4\pi\int_0^3 \frac{x^2}{(x^2 + 1)^2}\,dx$

$$= 4\pi\int_0^3 \left(\frac{1}{x^2 + 1} - \frac{1}{(x^2 + 1)^2}\right) dx \qquad \text{(partial fractions)}$$

$$= 4\pi\left[\arctan x - \frac{1}{2}\left(\arctan x + \frac{x}{x^2 + 1}\right)\right]_0^3 \qquad \text{(trigonometric substitution)}$$

$$= 2\pi\left[\arctan x - \frac{x}{x^2 + 1}\right]_0^3 = 2\pi\left[\arctan 3 - \frac{3}{10}\right] \approx 5.963$$

$$A = \int_0^3 \frac{2x}{x^2 + 1}\,dx = \left[\ln(x^2 + 1)\right]_0^3 = \ln 10$$



$$\bar{x} = \frac{1}{A}\int_0^3 \frac{2x^2}{x^2 + 1}\,dx = \frac{1}{\ln 10}\int_0^3 \left(2 - \frac{2}{x^2 + 1}\right) dx$$

$$= \frac{1}{\ln 10}\left[2x - 2\arctan x\right]_0^3 = \frac{2}{\ln 10}[3 - \arctan 3] \approx 1.521$$

$$\bar{y} = \frac{1}{A}\left(\frac{1}{2}\right)\int_0^3 \left(\frac{2x}{x^2 + 1}\right)^2 dx = \frac{2}{\ln 10}\int_0^3 \frac{x^2}{(x^2 + 1)^2}\,dx$$

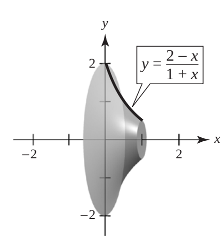
$$= \frac{2}{\ln 10}\int_0^3 \left(\frac{1}{x^2 + 1} - \frac{1}{(x^2 + 1)^2}\right) dx \qquad \text{(partial fractions)}$$

$$= \frac{2}{\ln 10}\left[\arctan x - \frac{1}{2}\left(\arctan x + \frac{x}{x^2 + 1}\right)\right]_0^3 \qquad \text{(trigonometric substitution)}$$

$$= \frac{2}{\ln 10}\left[\frac{1}{2}\arctan x - \frac{x}{2(x^2 + 1)}\right]_0^3 = \frac{1}{\ln 10}\left[\arctan x - \frac{x}{x^2 + 1}\right]_0^3 = \frac{1}{\ln 10}\left[\arctan 3 - \frac{3}{10}\right] \approx 0.412$$

$(\bar{x}, \bar{y}) \approx (1.521, 0.412)$

62. $y^2 = \dfrac{(2 - x)^2}{(1 + x)^2}$, $[0, 1]$



$$V = \int_0^1 \pi\frac{(2 - x)^2}{(1 + x)^2}\,dx$$

$$= \pi\left[\int_0^1 \frac{4}{(1 + x)^2}\,dx - \int_0^1 \frac{4x}{(1 + x)^2}\,dx + \int_0^1 \frac{x^2}{(1 + x)^2}\,dx\right]$$

$$= \pi\left[2 - (4\ln 2 - 2) + \frac{3}{2} - 2\ln 2\right]$$

$$= \pi\left[\frac{11}{2} - 6\ln 2\right] = \frac{\pi}{2}[11 - 12\ln 2]$$

63. $$\frac{1}{(x+1)(n-x)} = \frac{A}{x+1} + \frac{B}{n-x}, A = B = \frac{1}{n+1}$$

$$\frac{1}{n+1}\int\left(\frac{1}{x+1} + \frac{1}{n-x}\right)dx = kt + C$$

$$\frac{1}{n+1}\ln\left|\frac{x+1}{n-x}\right| = kt + C$$

When $t = 0, x = 0, C = \frac{1}{n+1}\ln\frac{1}{n}$.

$$\frac{1}{n+1}\ln\left|\frac{x+1}{n-x}\right| = kt + \frac{1}{n+1}\ln\frac{1}{n}$$

$$\frac{1}{n+1}\left[\ln\left|\frac{x+1}{n-x}\right| - \ln\frac{1}{n}\right] = kt$$

$$\ln\frac{nx+n}{n-x} = (n+1)kt$$

$$\frac{nx+n}{n-x} = e^{(n+1)kt}$$

$$x = \frac{n[e^{(n+1)kt} - 1]}{n + e^{(n+1)kt}}$$

Note: $\lim_{t\to\infty} x = n$

64. (a) $$\frac{1}{(y_0 - x)(z_0 - x)} = \frac{A}{y_0 - x} + \frac{B}{z_0 - x},$$

$$A = \frac{1}{z_0 - y_0}, B = -\frac{1}{z_0 - y_0}, \qquad \text{(Assume } y_0 \neq z_0.)$$

$$\frac{1}{z_0 - y_0}\int\left(\frac{1}{y_0 - x} - \frac{1}{z_0 - x}\right)dx = kt + C$$

$$\frac{1}{z_0 - y_0}\ln\left|\frac{z_0 - x}{y_0 - x}\right| = kt + C, \text{ when } t = 0, x = 0$$

$$C = \frac{1}{z_0 - y_0}\ln\frac{z_0}{y_0}$$

$$\frac{1}{z_0 - y_0}\left[\ln\left|\frac{z_0 - x}{y_0 - x}\right| - \ln\left(\frac{z_0}{y_0}\right)\right] = kt$$

$$\ln\left[\frac{y_0(z_0 - x)}{z_0(y_0 - x)}\right] = (z_0 - y_0)kt$$

$$\frac{y_0(z_0 - x)}{z_0(y_0 - x)} = e^{(z_0 - y_0)kt}$$

$$x = \frac{y_0 z_0[e^{(z_0 - y_0)kt} - 1]}{z_0 e^{(z_0 - y_0)kt} - y_0}$$

(b) (1) If $y_0 < z_0$, $\lim_{t\to\infty} x = y_0$.

(2) If $y_0 > z_0$, $\lim_{t\to\infty} x = z_0$.

(3) If $y_0 = z_0$, then the original equation is:

$$\int\frac{1}{(y_0 - x)^2}\,dx = \int k\,dt$$

$$(y_0 - x)^{-1} = kt + C_1$$

$$x = 0 \text{ when } t = 0 \implies \frac{1}{y_0} = C_1$$

$$\frac{1}{y_0 - x} = kt + \frac{1}{y_0} = \frac{kty_0 + 1}{y_0}$$

$$y_0 - x = \frac{y_0}{kty_0 + 1}$$

$$x = y_0 - \frac{y_0}{kty_0 + 1}$$

As $t \to \infty$, $x \to y_0 = x_0$.

65. $\dfrac{x}{1+x^4} = \dfrac{Ax+B}{x^2+\sqrt{2}x+1} + \dfrac{Cx+D}{x^2-\sqrt{2}x+1}$

$$x = (Ax+B)(x^2-\sqrt{2}x+1) + (Cx+D)(x^2+\sqrt{2}x+1)$$
$$= (A+C)x^3 + (B+D-\sqrt{2}A+\sqrt{2}C)x^2 + (A+C-\sqrt{2}B+\sqrt{2}D)x + (B+D)$$

$0 = A + C \implies C = -A$

$0 = B + D - \sqrt{2}A + \sqrt{2}C \qquad -2\sqrt{2}A = 0 \implies A = 0$ and $C = 0$

$1 = A + C - \sqrt{2}B + \sqrt{2}D \qquad -2\sqrt{2}B = 1 \implies B = -\dfrac{\sqrt{2}}{4}$ and $D = \dfrac{\sqrt{2}}{4}$

$0 = B + D \implies D = -B$

Thus,

$$\int_0^1 \frac{x}{1+x^4}\,dx = \int_0^1 \left[\frac{-\sqrt{2}/4}{x^2+\sqrt{2}x+1} + \frac{\sqrt{2}/4}{x^2-\sqrt{2}x+1}\right]dx$$
$$= \frac{\sqrt{2}}{4}\int_0^1 \left[\frac{-1}{\left[x+(\sqrt{2}/2)\right]^2+(1/2)} + \frac{1}{\left[x-(\sqrt{2}/2)\right]^2+(1/2)}\right]dx$$
$$= \frac{\sqrt{2}}{4}\cdot\frac{1}{1/\sqrt{2}}\left[-\arctan\left(\frac{x+(\sqrt{2}/2)}{1/\sqrt{2}}\right) + \arctan\left(\frac{x-(\sqrt{2}/2)}{1/\sqrt{2}}\right)\right]_0^1$$
$$= \frac{1}{2}\left[-\arctan(\sqrt{2}x+1) + \arctan(\sqrt{2}x-1)\right]_0^1$$
$$= \frac{1}{2}\left[\left(-\arctan(\sqrt{2}+1) + \arctan(\sqrt{2}-1)\right) - (-\arctan 1 + \arctan(-1))\right]$$
$$= \frac{1}{2}\left[\arctan(\sqrt{2}-1) - \arctan(\sqrt{2}+1) + \frac{\pi}{4} + \frac{\pi}{4}\right].$$

Since $\arctan x - \arctan y = \arctan[(x-y)/(1+xy)]$, we have:

$$\int_0^1 \frac{x}{1+x^4}\,dx = \frac{1}{2}\left[\arctan\left(\frac{(\sqrt{2}-1)-(\sqrt{2}+1)}{1+(\sqrt{2}-1)(\sqrt{2}+1)}\right) + \frac{\pi}{2}\right] = \frac{1}{2}\left[\arctan\left(\frac{-2}{2}\right) + \frac{\pi}{2}\right] = \frac{1}{2}\left[-\frac{\pi}{4} + \frac{\pi}{2}\right] = \frac{\pi}{8}$$

66. The partial fraction decomposition is:

$$\frac{x^4(1-x)^4}{1+x^2} = x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}$$

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \left[\frac{x^7}{7} - \frac{2x^6}{3} + x^5 - \frac{4}{3}x^3 + 4x - 4\arctan x\right]_0^1$$
$$= \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - 4\left(\frac{\pi}{4}\right)$$
$$= \frac{22}{7} - \pi$$

Note: You can easily verify this calculation with a graphing utility.

Section 8.6 Integration by Tables and Other Integration Techniques

1. By Formula 6: $\displaystyle\int \frac{x^2}{1+x}\,dx = -\frac{x}{2}(2-x) + \ln|1+x| + C$

2. By Formula 13: $(b = 2, a = -5)$

$$\frac{2}{3}\int \frac{1}{x^2(2x-5)^2}\,dx = \frac{2}{3}\left(\frac{-1}{25}\right)\left[\frac{-5+4x}{x(-5+2x)} + \frac{4}{-5}\ln\left|\frac{x}{2x-5}\right|\right] + C$$

$$= \frac{8}{375}\ln\left|\frac{x}{2x-5}\right| - \frac{2}{75}\,\frac{(4x-5)}{x(2x-5)} + C$$

3. By Formula 26: $\displaystyle\int e^x\sqrt{1+e^{2x}}\,dx = \frac{1}{2}\left[e^x\sqrt{e^{2x}+1} + \ln\left|e^x + \sqrt{e^{2x}+1}\right|\right] + C$

$u = e^x,\ du = e^x\,dx$

4. By Formula 29: $(a = 3)$

$$\frac{1}{3}\int \frac{\sqrt{x^2-9}}{x}\,dx = \frac{1}{3}\sqrt{x^2-9} - \operatorname{arcsec}\frac{|x|}{3} + C$$

5. By Formula 44: $\displaystyle\int \frac{1}{x^2\sqrt{1-x^2}}\,dx = -\frac{\sqrt{1-x^2}}{x} + C$

6. By Formula 41: $\displaystyle\int \frac{x}{\sqrt{9-x^4}}\,dx = \frac{1}{2}\int \frac{2x}{\sqrt{3^2-(x^2)^2}}\,dx$

$$= \frac{1}{2}\arcsin\frac{x^2}{3} + C$$

7. By Formulas 50 and 48: $\displaystyle\int \sin^4(2x)\,dx = \frac{1}{2}\int \sin^4(2x)(2)\,dx$

$$= \frac{1}{2}\left[\frac{-\sin^3(2x)\cos(2x)}{4} + \frac{3}{4}\int \sin^2(2x)(2)\,dx\right]$$

$$= \frac{1}{2}\left[\frac{-\sin^3(2x)\cos(2x)}{4} + \frac{3}{8}(2x - \sin 2x\cos 2x)\right] + C$$

$$= \frac{1}{16}(6x - 3\sin 2x\cos 2x - 2\sin^3 2x\cos 2x) + C$$

8. By Formulas 51 and 47: $\displaystyle\int \frac{\cos^3\sqrt{x}}{\sqrt{x}}\,dx = 2\int \cos^3\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right)dx$

$$= 2\left[\frac{\cos^2\sqrt{x}\sin\sqrt{x}}{3} + \frac{2}{3}\int \cos\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right)dx\right] = \frac{2}{3}\sin\sqrt{x}\left(\cos^2\sqrt{x} + 2\right) + C$$

$u = \sqrt{x},\ du = \dfrac{1}{2\sqrt{x}}\,dx$

9. By Formula 57: $\displaystyle\int \frac{1}{\sqrt{x}\left(1-\cos\sqrt{x}\right)}\,dx = 2\int \frac{1}{1-\cos\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right)dx$

$$= -2\left(\cot\sqrt{x} + \csc\sqrt{x}\right) + C$$

$u = \sqrt{x},\ du = \dfrac{1}{2\sqrt{x}}\,dx$

10. By Formula 71:

$$\int \frac{1}{1 - \tan 5x}\,dx = \frac{1}{5}\int \frac{1}{1 - \tan 5x}(5)\,dx$$
$$= \frac{1}{5}\left(\frac{1}{2}\right)\left(u - \ln\left|\cos u - \sin u\right|\right) + C$$
$$= \frac{1}{10}\left(5x - \ln\left|\cos 5x - \sin 5x\right|\right) + C$$

$u = 5x,\ du = 5\,dx$

11. By Formula 84:

$$\int \frac{1}{1 + e^{2x}}\,dx = x - \frac{1}{2}\ln(1 + e^{2x}) + C$$

12. By Formula 85: $\left(a = -\frac{1}{2}, b = 2\right)$

$$\int e^{-x/2}\sin 2x\,dx = \frac{e^{-x/2}}{(1/4) + 4}\left(-\frac{1}{2}\sin 2x - 2\cos 2x\right) + C$$
$$= \frac{4}{17}e^{-x/2}\left(-\frac{1}{2}\sin 2x - 2\cos 2x\right) + C$$

13. By Formula 89:

$$\int x^3 \ln x\,dx = \frac{x^4}{16}\left(4\ln|x| - 1\right) + C$$

14. By Formulas 90 and 91: $\int (\ln x)^3\,dx = x(\ln x)^3 - 3\int (\ln x)^2\,dx$

$$= x(\ln x)^3 - 3x\left[2 - 2\ln x + (\ln x)^2\right] + C$$
$$= x\left[(\ln x)^3 - 3(\ln x)^2 + 6\ln x - 6\right] + C$$

15. (a) By Formulas 83 and 82: $\int x^2 e^x\,dx = x^2 e^x - 2\int xe^x\,dx$

$$= x^2 e^x - 2\left[(x - 1)e^x + C_1\right]$$
$$= x^2 e^x - 2xe^x + 2e^x + C$$

(b) Integration by parts: $u = x^2,\ du = 2x\,dx,\ dv = e^x\,dx,\ v = e^x$

$$\int x^2 e^x\,dx = x^2 e^x - \int 2xe^x\,dx$$

Parts again: $u = 2x,\ du = 2\,dx,\ dv = e^x\,dx,\ v = e^x$

$$\int x^2 e^x\,dx = x^2 e^x - \left[2xe^x - \int 2e^x\,dx\right] = x^2 e^x - 2xe^x + 2e^x + C$$

16. (a) By Formula 89: $\int x^4 \ln x\,dx = \frac{x^5}{5^2}\left[-1 + (4 + 1)\ln x\right] + C = \frac{-x^5}{25} + \frac{1}{5}x^5 \ln x + C$

(b) Integration by parts: $u = \ln x,\ du = \frac{1}{x}\,dx,\ dv = x^4\,dx,\ v = \frac{x^5}{5}$

$$\int x^4 \ln x\,dx = \frac{x^5}{5}\ln x - \int \frac{x^5}{5}\frac{1}{x}\,dx = \frac{x^5}{5}\ln x - \frac{x^5}{25} + C$$

17. (a) By Formula: 12, $a = b = 1$, $u = x$, and

$$\int \frac{1}{x^2(x+1)}\,dx = \frac{-1}{1}\left(\frac{1}{x} + \frac{1}{1}\ln\left|\frac{x}{1+x}\right|\right) + C$$

$$= \frac{-1}{x} - \ln\left|\frac{x}{1+x}\right| + C$$

$$= \frac{-1}{x} + \ln\left|\frac{x+1}{x}\right| + C$$

(b) Partial fractions:

$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$1 = Ax(x+1) + B(x+1) + Cx^2$$

$x = 0$: $1 = B$

$x = -1$: $1 = C$

$x = 1$: $1 = 2A + 2 + 1 \Rightarrow A = -1$

$$\int \frac{1}{x^2(x+1)}\,dx = \int\left[\frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right]dx$$

$$= -\ln|x| - \frac{1}{x} + \ln|x+1| + C$$

$$= -\frac{1}{x} - \ln\left|\frac{x}{x+1}\right| + C$$

18. (a) By Formula 24: $a = \sqrt{75}$, $x = u$, and

$$\int \frac{1}{x^2 - 75}\,dx = \frac{1}{2\sqrt{75}}\ln\left|\frac{x - \sqrt{75}}{x + \sqrt{75}}\right| + C$$

$$= \frac{\sqrt{3}}{30}\ln\left|\frac{x - \sqrt{75}}{x + \sqrt{75}}\right| + C$$

(b) Partial fractions:

$$\frac{1}{x^2 - 75} = \frac{A}{x - \sqrt{75}} + \frac{B}{x + \sqrt{75}}$$

$$1 = A\left(x + \sqrt{75}\right) + B\left(x - \sqrt{75}\right)$$

$x = \sqrt{75}$: $1 = 2A\sqrt{75} \Rightarrow A = \dfrac{1}{2\sqrt{75}} = \dfrac{1}{10\sqrt{3}} = \dfrac{\sqrt{3}}{30}$

$x = -\sqrt{75}$: $1 = -2B\sqrt{75} \Rightarrow B = -\dfrac{\sqrt{3}}{30}$

$$\int \frac{1}{x^2 - 75}\,dx = \int\left[\frac{\sqrt{3}/30}{x - \sqrt{75}} - \frac{\sqrt{3}/30}{x + \sqrt{75}}\right]dx$$

$$= \frac{\sqrt{3}}{30}\ln\left|\frac{x - \sqrt{75}}{x + \sqrt{75}}\right| + C$$

19. By Formula 79: $\displaystyle\int x \operatorname{arcsec}(x^2+1)\,dx = \frac{1}{2}\int \operatorname{arcsec}(x^2+1)(2x)\,dx$

$$= \frac{1}{2}\left[(x^2+1)\operatorname{arcsec}(x^2+1) - \ln\left((x^2+1) + \sqrt{x^4 + 2x^2}\right)\right] + C$$

$u = x^2 + 1$, $du = 2x\,dx$

20. By Formula 79: $\displaystyle\int \operatorname{arcsec} 2x\,dx = \frac{1}{2}\left[2x \operatorname{arcsec} 2x - \ln\left|2x + \sqrt{4x^2 - 1}\right|\right] + C$

$u = 2x$, $du = 2\,dx$

21. By Formula 35: $\displaystyle\int \frac{1}{x^2\sqrt{x^2 - 4}}\,dx = \frac{\sqrt{x^2 - 4}}{4x} + C$

22. By Formula 14: $\displaystyle\int \frac{1}{x^2 + 2x + 2}\,dx = \frac{2}{\sqrt{4}}\arctan\left(\frac{2x+2}{2}\right) + C = \arctan(x+1) + C$

23. By Formula 4: $\displaystyle\int \frac{2x}{(1-3x)^2}\,dx = 2\int \frac{x}{(1-3x)^2}\,dx = \frac{2}{9}\left(\ln|1 - 3x| + \frac{1}{1-3x}\right) + C$

24. By Formula 56:

$$\int \frac{\theta^2}{1 - \sin \theta^3}\, d\theta = \frac{1}{3}\int \frac{1}{1 - \sin \theta^3} 3\theta^2\, d\theta$$

$$= \frac{1}{3}(\tan \theta^3 + \sec \theta^3) + C$$

25. By Formula 76:

$$\int e^x \arccos e^x\, dx = e^x \arccos e^x - \sqrt{1 - e^{2x}} + C$$

$u = e^x$, $du = e^x\, dx$

26. By Formula 71:

$$\int \frac{e^x}{1 - \tan e^x}\, dx = \frac{1}{2}\left(e^x - \ln\left|\cos e^x - \sin e^x\right|\right) + C$$

$u = e^x$, $du = e^x\, dx$

27. By Formula 73:

$$\int \frac{x}{1 - \sec x^2}\, dx = \frac{1}{2}\int \frac{2x}{1 - \sec x^2}\, dx$$

$$= \frac{1}{2}(x^2 + \cot x^2 + \csc x^2) + C$$

28. By Formula 23: $$\int \frac{1}{t[1 + (\ln t)^2]}\, dt = \int \frac{1}{1 + (\ln t)^2}\left(\frac{1}{t}\right) dt = \arctan(\ln t) + C$$

$u = \ln t$, $du = \frac{1}{t}\, dt$

29. By Formula 14: $$\int \frac{\cos \theta}{3 + 2\sin \theta + \sin^2 \theta}\, d\theta = \frac{\sqrt{2}}{2} \arctan\left(\frac{1 + \sin \theta}{\sqrt{2}}\right) + C \qquad (b^2 = 4 < 12 = 4ac)$$

$u = \sin \theta$, $du = \cos \theta\, d\theta$

30. By Formula 27: $$\int x^2\sqrt{2 + (3x)^2}\, dx = \frac{1}{27}\int (3x)^2\sqrt{\left(\sqrt{2}\right)^2 + (3x)^2}\, 3\, dx$$

$$= \frac{1}{8(27)}\left[3x(18x^2 + 2)\sqrt{2 + 9x^2} - 4\ln\left|3x + \sqrt{2 + 9x^2}\right|\right] + C$$

31. By Formula 35: $$\int \frac{1}{x^2\sqrt{2 + 9x^2}}\, dx = 3\int \frac{3}{(3x)^2\sqrt{\left(\sqrt{2}\right)^2 + (3x)^2}}\, dx$$

$$= -\frac{3\sqrt{2 + 9x^2}}{6x} + C$$

$$= -\frac{\sqrt{2 + 9x^2}}{2x} + C$$

32. By Formula 77: $$\int \sqrt{x} \arctan(x^{3/2})\, dx = \frac{2}{3}\int \arctan(x^{3/2})\left(\frac{3}{2}\sqrt{x}\right) dx$$

$$= \frac{2}{3}\left[x^{3/2} \arctan(x^{3/2}) - \ln \sqrt{1 + x^3}\right] + C$$

33. By Formula 3: $$\int \frac{\ln x}{x(3 + 2\ln x)}\, dx = \frac{1}{4}\left(2\ln|x| - 3\ln\left|3 + 2\ln|x|\right|\right) + C$$

$u = \ln x$, $du = \frac{1}{x}\, dx$

34. By Formula 45: $\displaystyle\int \frac{e^x}{(1 - e^{2x})^{3/2}}\,dx = \frac{e^x}{\sqrt{1 - e^{2x}}} + C$

$u = e^x,\ du = e^x\,dx$

35. By Formulas 1, 25, and 33: $\displaystyle\int \frac{x}{(x^2 - 6x + 10)^2}\,dx = \frac{1}{2}\int \frac{2x - 6 + 6}{(x^2 - 6x + 10)^2}\,dx$

$$= \frac{1}{2}\int (x^2 - 6x + 10)^{-2}(2x - 6)\,dx + 3\int \frac{1}{[(x - 3)^2 + 1]^2}\,dx$$

$$= -\frac{1}{2(x^2 - 6x + 10)} + \frac{3}{2}\left[\frac{x - 3}{x^2 - 6x + 10} + \arctan(x - 3)\right] + C$$

$$= \frac{3x - 10}{2(x^2 - 6x + 10)} + \frac{3}{2}\arctan(x - 3) + C$$

36. By Formula 27:

$$\int (2x - 3)^2\sqrt{(2x - 3)^2 + 4}\,dx = \frac{1}{2}\int (2x - 3)^2\sqrt{(2x - 3)^2 + 4}(2)\,dx$$

$$= \frac{1}{8}(2x - 3)[(2x - 3)^2 + 2]\sqrt{(2x - 3)^2 + 4} - \ln\left|2x - 3 + \sqrt{(2x - 3)^2 + 4}\right| + C$$

$u = 2x - 3,\ du = 2\,dx$

37. By Formula 31: $\displaystyle\int \frac{x}{\sqrt{x^4 - 6x^2 + 5}}\,dx = \frac{1}{2}\int \frac{2x}{\sqrt{(x^2 - 3)^2 - 4}}\,dx$

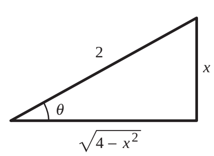
$$= \frac{1}{2}\ln\left|x^2 - 3 + \sqrt{x^4 - 6x^2 + 5}\right| + C$$

$u = x^2 - 3,\ du = 2x\,dx$

38. By Formula 31: $\displaystyle\int \frac{\cos x}{\sqrt{\sin^2 x + 1}}\,dx = \ln\left|\sin x + \sqrt{\sin^2 x + 1}\right| + C$

$u = \sin x,\ du = \cos x\,dx$

39. $\displaystyle\int \frac{x^3}{\sqrt{4 - x^2}}\,dx = \int \frac{8\sin^3\theta(2\cos\theta\,d\theta)}{2\cos\theta}$



$$= 8\int (1 - \cos^2\theta)\sin\theta\,d\theta$$

$$= 8\int [\sin\theta - \cos^2\theta(\sin\theta)]\,d\theta$$

$$= -8\cos\theta + \frac{8\cos^3\theta}{3} + C$$

$$= -8\frac{\sqrt{4 - x^2}}{2} + \frac{8}{3}\left(\frac{\sqrt{4 - x^2}}{2}\right)^3 + C$$

$$= \sqrt{4 - x^2}\left[-4 + \frac{1}{3}(4 - x^2)\right] + C$$

$$= \frac{-\sqrt{4 - x^2}}{3}(x^2 + 8) + C$$

$x = 2\sin\theta,\ dx = 2\cos\theta\,d\theta,\ \sqrt{4 - x^2} = 2\cos\theta$

40. $$\int \sqrt{\frac{3-x}{3+x}}\,dx = \int \frac{3-x}{\sqrt{9-x^2}}\,dx$$

$$= 3\int \frac{1}{\sqrt{9-x^2}}\,dx + \int \frac{-x}{\sqrt{9-x^2}}\,dx$$

$$= 3\arcsin\frac{x}{3} + \sqrt{9-x^2} + C$$

41. By Formula 8:

$$\int \frac{e^{3x}}{(1+e^x)^3}\,dx = \int \frac{(e^x)^2}{(1+e^x)^3}(e^x)\,dx$$

$$= \frac{2}{1+e^x} - \frac{1}{2(1+e^x)^2} + \ln|1+e^x| + C$$

$u = e^x,\ du = e^x\,dx$

42. By Formula 67:

$$\int \tan^3\theta\,d\theta = \frac{\tan^2\theta}{2} - \int \tan\theta\,d\theta$$

$$= \frac{\tan^2\theta}{2} + \ln|\cos x| + C$$

43. $$\int_0^1 xe^{x^2}\,dx$$

By Formula 81:

$$\int_0^1 xe^{x^2}\,dx = \frac{1}{2}e^{x^2}\Big]_0^1 = \frac{1}{2}(e-1)$$

44. By Formula 21:

$$\int_0^3 \frac{x}{\sqrt{1+x}}\,dx = \left[\frac{-2}{3}(2-x)\sqrt{1+x}\right]_0^3$$

$$= \frac{-2}{3}(-1)(2) + \frac{2}{3}(2) = \frac{8}{3}$$

45. By Formula 89:

$$\int_1^3 x^2\ln x\,dx = \left[\frac{x^3}{9}\left(-1+3\ln|x|\right)\right]_1^3$$

$$= 3(-1+3\ln 3) + \frac{1}{9} = 9\ln 3 - \frac{26}{9}$$

46. By Formula 52:

$$\int_0^\pi x\sin x\,dx = \Big[\sin x - x\cos x\Big]_0^\pi$$

$$= \pi$$

47. By Formula 23, and letting $u = \sin x$:

$$\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+\sin^2 x}\,dx = \Big[\arctan(\sin x)\Big]_{-\pi/2}^{\pi/2}$$

$$= \arctan(1) - \arctan(-1) = \frac{\pi}{2}$$

48. By Formula 7:

$$\int_2^4 \frac{x^2}{(3x-5)^2}\,dx = \left[\frac{1}{27}\left(3x - \frac{25}{3x-5} + 10\ln|3x-5|\right)\right]_2^4$$

$$= \frac{1}{27}\left[\left(12 - \frac{25}{7} + 10\ln 7\right) - (6-25)\right] = \frac{64}{63} + \frac{10}{27}\ln 7$$

49. By Formulas 54 and 55:

$$\int t^3\cos t\,dt = t^3\sin t - 3\int t^2\sin t\,dt$$

$$= t^3\sin t - 3\left[-t^2\cos t + 2\int t\cos t\,dt\right]$$

$$= t^3\sin t + 3t^2\cos t - 6\left[t\sin t - \int \sin t\,dt\right]$$

$$= t^3\sin t + 3t^2\cos t - 6t\sin t - 6\cos t + C$$

Thus,

$$\int_0^{\pi/2} t^3\cos t\,dt = \Big[t^3\sin t + 3t^2\cos t - 6t\sin t - 6\cos t\Big]_0^{\pi/2}$$

$$= \left(\frac{\pi^3}{8} - 3\pi\right) + 6 = \frac{\pi^3}{8} + 6 - 3\pi$$

50. By Formula 26:

$$\int_0^1 \sqrt{3+x^2}\,dx = \left[\frac{1}{2}\left(x\sqrt{x^2+3} + 3\ln\left|x + \sqrt{x^2+3}\right|\right)\right]_0^1 = \frac{1}{2}\left[(2) + 3\ln 3 - 3\ln\sqrt{3}\right] = 1 + \frac{3}{4}\ln 3$$

51. $\dfrac{u^2}{(a+bu)^2} = \dfrac{1}{b^2} - \dfrac{(2a/b)u + (a^2/b^2)}{(a+bu)^2} = \dfrac{1}{b^2} + \dfrac{A}{a+bu} + \dfrac{B}{(a+bu)^2}$

$$-\frac{2a}{b}u - \frac{a^2}{b^2} = A(a+bu) + B = (aA + B) + bAu$$

Equating the coefficients of like terms we have $aA + B = -a^2/b^2$ and $bA = -2a/b$. Solving these equations we have $A = -2a/b^2$ and $B = a^2/b^2$.

$$\int \frac{u^2}{(a+bu)^2}\,du = \frac{1}{b^2}\int du - \frac{2a}{b^2}\left(\frac{1}{b}\right)\int \frac{1}{a+bu}b\,du + \frac{a^2}{b^2}\left(\frac{1}{b}\right)\int \frac{1}{(a+bu)^2}b\,du = \frac{1}{b^2}u - \frac{2a}{b^3}\ln|a+bu| - \frac{a^2}{b^3}\left(\frac{1}{a+bu}\right) + C$$

$$= \frac{1}{b^3}\left(bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$$

52. Integration by parts: $w = u^n$, $dw = nu^{n-1}\,du$, $dv = \dfrac{du}{\sqrt{a+bu}}$, $v = \dfrac{2}{b}\sqrt{a+bu}$

$$\int \frac{u^n}{\sqrt{a+bu}}\,du = \frac{2u^n}{b}\sqrt{a+bu} - \frac{2n}{b}\int u^{n-1}\sqrt{a+bu}\,du$$

$$= \frac{2u^n}{b}\sqrt{a+bu} - \frac{2n}{b}\int u^{n-1}\sqrt{a+bu}\cdot\frac{\sqrt{a+bu}}{\sqrt{a+bu}}\,du$$

$$= \frac{2u^n}{b}\sqrt{a+bu} - \frac{2n}{b}\int \frac{au^{n-1} + bu^n}{\sqrt{a+bu}}\,du$$

$$= \frac{2u^n}{b}\sqrt{a+bu} - \frac{2na}{b}\int \frac{u^{n-1}}{\sqrt{a+bu}}\,du - 2n\int \frac{u^n}{\sqrt{a+bu}}\,du$$

Therefore, $(2n+1)\displaystyle\int \frac{u^n}{\sqrt{a+bu}}\,du = \frac{2}{b}\left[u^n\sqrt{a+bu} - na\int \frac{u^{n-1}}{\sqrt{a+bu}}\,du\right]$ and

$$\int \frac{u^n}{\sqrt{a+bu}} = \frac{2}{(2n+1)b}\left[u^n\sqrt{a+bu} - na\int \frac{u^{n-1}}{\sqrt{a+bu}}\,du\right].$$

53. When we have $u^2 + a^2$:

$$u = a\tan\theta$$

$$du = a\sec^2\theta\,d\theta$$

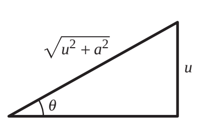
$$u^2 + a^2 = a^2\sec^2\theta$$

$$\int \frac{1}{(u^2+a^2)^{3/2}}\,du = \int \frac{a\sec^2\theta\,d\theta}{a^3\sec^3\theta}$$

$$= \frac{1}{a^2}\int \cos\theta\,d\theta$$

$$= \frac{1}{a^2}\sin\theta + C$$

$$= \frac{u}{a^2\sqrt{u^2+a^2}} + C$$



When we have $u^2 - a^2$:

$$u = a\sec\theta$$

$$du = a\sec\theta\tan\theta\,d\theta$$

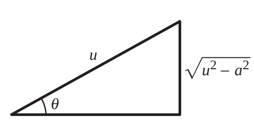
$$u^2 - a^2 = a^2\tan^2\theta$$

$$\int \frac{1}{(u^2-a^2)^{3/2}}\,du = \int \frac{a\sec\theta\tan\theta\,d\theta}{a^3\tan^3\theta}$$

$$= \frac{1}{a^2}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{a^2}\int \csc\theta\cot\theta\,d\theta$$

$$= -\frac{1}{a^2}\csc\theta + C$$

$$= \frac{-u}{a^2\sqrt{u^2-a^2}} + C$$



54. $\int u^n(\cos u)\,du = u^n \sin u - n\int u^{n-1}(\sin u)\,du$

$w = u^n,\ dv = \cos u\,du,\ dw = nu^{n-1}\,du,\ v = \sin u$

55. $\int(\arctan u)\,du = u \arctan u - \frac{1}{2}\int \frac{2u}{1+u^2}\,du$

$$= u \arctan u - \frac{1}{2}\ln(1+u^2) + C$$

$$= u \arctan u - \ln\sqrt{1+u^2} + C$$

$w = \arctan u,\ dv = du,\ dw = \frac{du}{1+u^2},\ v = u$

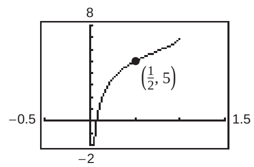
56. $\int(\ln u)^n\,du = u(\ln u)^n - \int n(\ln u)^{n-1}\left(\frac{1}{u}\right)u\,du = u(\ln u)^n - n\int(\ln u)^{n-1}\,du$

$w = (\ln u)^n,\ dv = du,\ dw = n(\ln u)^{n-1}\left(\frac{1}{u}\right)du,\ v = u$

57. $\int \frac{1}{x^{3/2}\sqrt{1-x}}\,dx = \frac{-2\sqrt{1-x}}{\sqrt{x}} + C$

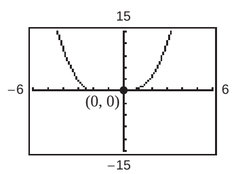
$\left(\frac{1}{2}, 5\right)$: $\frac{-2\sqrt{1/2}}{\sqrt{1/2}} + C = 5 \Rightarrow C = 7$

$$y = \frac{-2\sqrt{1-x}}{\sqrt{x}} + 7$$



58. $\int x\sqrt{x^2+2x}\,dx = \frac{1}{6}\left[2(x^2+2x)^{3/2} - 3(x+1)\sqrt{x^2+2x} + 3\ln\left|x+1+\sqrt{x^2+2x}\right|\right] + C$

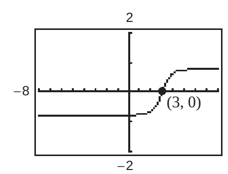
$(0, 0)$: $\frac{1}{6}[3\ln|1|] + C = 0 \Rightarrow C = 0$



59. $\int \frac{1}{(x^2-6x+10)^2}\,dx = \frac{1}{2}\left[\tan^{-1}(x-3) + \frac{x-3}{x^2-6x+10}\right] + C$

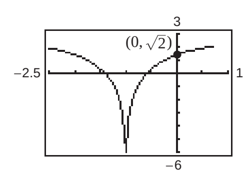
$(3, 0)$: $\frac{1}{2}\left[0 + \frac{0}{10}\right] + C = 0 \Rightarrow C = 0$

$$y = \frac{1}{2}\left[\tan^{-1}(x-3) + \frac{x-3}{x^2-6x+10}\right]$$



60. $\int \frac{\sqrt{2-2x-x^2}}{x+1}\,dx = \sqrt{2-2x-x^2} - \sqrt{3}\ln\left|\frac{\sqrt{3}+\sqrt{2-2x-x^2}}{x+1}\right| + C$

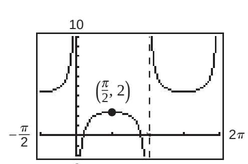
$(0, \sqrt{2})$: $\sqrt{2} - \sqrt{3}\ln(\sqrt{3}+\sqrt{2}) + C = \sqrt{2} \Rightarrow C = \sqrt{3}\ln(\sqrt{3}+\sqrt{2})$



61. $\int \frac{1}{\sin\theta\tan\theta}\,d\theta = -\csc\theta + C$

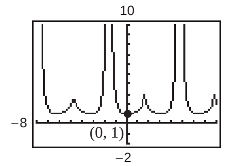
$\left(\frac{\pi}{4}, 2\right)$: $-\frac{2}{\sqrt{2}} + C = 2 \Rightarrow C = 2 + \sqrt{2}$

$$y = -\csc\theta + 2 + \sqrt{2}$$



62. $\displaystyle\int \frac{\sin\theta}{(\cos\theta)(1+\sin\theta)}\,d\theta = \frac{1}{2}\left[\frac{-\sin\theta}{1+\sin\theta} + \ln\left|\frac{1+\sin\theta}{\cos\theta}\right|\right] + C$

$(0, 1)$: $C = 1 \Rightarrow y = \dfrac{1}{2}\left[\dfrac{-\sin\theta}{1+\sin\theta} + \ln\left|\dfrac{1+\sin\theta}{\cos\theta}\right|\right] + 1$



63. $\displaystyle\int \frac{1}{2-3\sin\theta}\,d\theta = \int\left[\frac{\frac{2\,du}{1+u^2}}{2-3\left(\frac{2u}{1+u^2}\right)}\right], u = \tan\frac{\theta}{2}$

$$= \int \frac{2}{2(1+u^2)-6u}\,du$$

$$= \int \frac{1}{u^2-3u+1}\,du$$

$$= \int \frac{1}{\left(u-\frac{3}{2}\right)^2 - \frac{5}{4}}\,du$$

$$= \frac{1}{\sqrt{5}}\ln\left|\frac{\left(u-\frac{3}{2}\right)-\frac{\sqrt{5}}{2}}{\left(u-\frac{3}{2}\right)+\frac{\sqrt{5}}{2}}\right| + C$$

$$= \frac{1}{\sqrt{5}}\ln\left|\frac{2u-3-\sqrt{5}}{2u-3+\sqrt{5}}\right| + C$$

$$= \frac{1}{\sqrt{5}}\ln\left|\frac{2\tan\left(\frac{\theta}{2}\right)-3-\sqrt{5}}{2\tan\left(\frac{\theta}{2}\right)-3+\sqrt{5}}\right| + C$$

64. $\displaystyle\int \frac{\sin\theta}{1+\cos^2\theta}\,d\theta = -\int \frac{-\sin\theta}{1+(\cos\theta)^2}\,d\theta$

$$= -\arctan(\cos\theta) + C$$

65. $\displaystyle\int_0^{\pi/2} \frac{1}{1+\sin\theta+\cos\theta}\,d\theta = \int_0^1 \left[\frac{\frac{2\,du}{1+u^2}}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}}\right]$

$$= \int_0^1 \frac{1}{1+u}\,du$$

$$= \Big[\ln|1+u|\Big]_0^1$$

$$= \ln 2$$

$u = \tan\dfrac{\theta}{2}$

66. $\displaystyle\int_0^{\pi/2} \frac{1}{3-2\cos\theta}\,d\theta = \int_0^1 \left[\frac{\frac{2u}{1+u^2}}{3-\frac{2(1-u^2)}{1+u^2}}\right]$

$$= 2\int_0^1 \frac{1}{5u^2+1}\,du$$

$$= \left[\frac{2}{\sqrt{5}}\arctan\left(\sqrt{5}\,u\right)\right]_0^1$$

$$= \frac{2}{\sqrt{5}}\arctan\sqrt{5}$$

67. $\displaystyle\int \frac{\sin\theta}{3-2\cos\theta}\,d\theta = \frac{1}{2}\int \frac{2\sin\theta}{3-2\cos\theta}\,d\theta$

$$= \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln(3-2\cos\theta) + C$$

$u = 3 - 2\cos\theta$, $du = 2\sin\theta\,d\theta$

68. $\displaystyle\int \frac{\cos\theta}{1+\cos\theta}\,d\theta = \int \frac{\cos\theta(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}\,d\theta$

$$= \int \frac{\cos\theta - \cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \int (\csc\theta\cot\theta - \cot^2\theta)\,d\theta$$

$$= \int (\csc\theta\cot\theta - (\csc^2\theta - 1))\,d\theta$$

$$= -\csc\theta + \cot\theta + \theta + C$$

69. $\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}}\,d\theta = 2\int \cos\sqrt{\theta}\left(\frac{1}{2\sqrt{\theta}}\right)d\theta$

$= 2\sin\sqrt{\theta} + C$

$u = \sqrt{\theta},\ du = \frac{1}{2\sqrt{\theta}}\,d\theta$

70. $\int \frac{1}{\sec\theta - \tan\theta}\,d\theta = \int \frac{1}{(1/\cos\theta) - (\sin\theta/\cos\theta)}\,d\theta$

$= -\int \frac{-\cos\theta}{1 - \sin\theta}\,d\theta$

$= -\ln|1 - \sin\theta| + C$

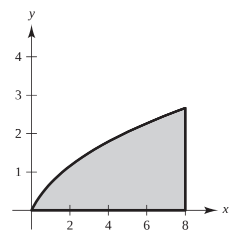
$u = 1 - \sin\theta,\ du = -\cos\theta\,d\theta$

71. $A = \int_0^8 \frac{x}{\sqrt{x+1}}\,dx$

$= \left[\frac{-2(2-x)}{3}\sqrt{x+1}\right]_0^8$

$= 12 - \left(-\frac{4}{3}\right)$

$= \frac{40}{3} \approx 13.333$ square units



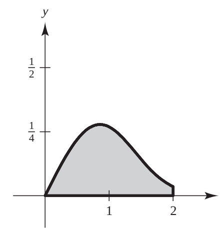
72. $A = \int_0^2 \frac{x}{1 + e^{x^2}}\,dx$

$= \frac{1}{2}\int_0^2 \frac{2x\,dx}{1 + e^{x^2}}$

$= \frac{1}{2}\left[x^2 - \ln(1 + e^{x^2})\right]_0^2$

$= \frac{1}{2}\left[4 - \ln(1 + e^4)\right] + \frac{1}{2}\ln 2$

≈ 0.337 square units



73. Arctangent Formula, Formula 23,

$\int \frac{1}{u^2 + 1}\,du,\ u = e^x$

74. Log Rule: $\int \frac{1}{u}\,du,\ u = e^x + 1$

75. Substitution: $u = x^2,\ du = 2x\,dx$

Then Formula 81.

76. Integration by parts

77. Cannot be integrated.

78. Formula 16 with $u = e^{2x}$

79. (a) $n = 1$: $u = \ln x,\ du = \frac{1}{x}\,dx,\ dv = x\,dx,\ v = \frac{x^2}{2}$

$\int x\ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\frac{1}{x}\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$

$n = 2$: $u = \ln x,\ du = \frac{1}{x}\,dx,\ dv = x^2\,dx,\ v = \frac{x^3}{3}$

$\int x^2\ln x\,dx = \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\frac{1}{x}\,dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$

$n = 3$: $u = \ln x,\ du = \frac{1}{x}\,dx,\ dv = x^3\,dx,\ v = \frac{x^4}{4}$

$\int x^3\ln x\,dx = \frac{x^4}{4}\ln x - \int \frac{x^4}{4}\frac{1}{x}\,dx = \frac{x^4}{4}\ln x - \frac{x^4}{16} + C$

(b) $\int x^n\ln x\,dx = \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + C$

80. A reduction formula reduces an integral to the sum of a function and a simpler integral. For example, see Formula 50, 54.

81. False. You might need to convert your integral using substitution or algebra.

82. True

83. $W = \int_0^5 2000xe^{-x}\,dx$

$= -2000\int_0^5 -xe^{-x}\,dx$

$= 2000\int_0^5 (-x)e^{-x}(-1)\,dx$

$= 2000\Big[(-x)e^{-x} - e^{-x}\Big]_0^5$

$= 2000\left(-\frac{6}{e^5} + 1\right)$

≈ 1919.145 ft · lbs

84. $W = \int_0^5 \frac{500x}{\sqrt{26 - x^2}}\,dx$

$= -250\int_0^5 (26 - x^2)^{-1/2}(-2x)\,dx$

$= \Big[-500\sqrt{26 - x^2}\Big]_0^5$

$= 500\left(\sqrt{26} - 1\right)$

≈ 2049.51 ft · lbs

85. $V = 20(2)\int_0^3 \frac{2}{\sqrt{1 + y^2}}\,dy$

$= \Big[80 \ln\left|y + \sqrt{1 + y^2}\right|\Big]_0^3$

$= 80 \ln\left(3 + \sqrt{10}\right)$

≈ 145.5 cubic feet

$W = 148\left(80 \ln\left(3 + \sqrt{10}\right)\right)$

$= 11{,}840 \ln\left(3 + \sqrt{10}\right)$

$\approx 21{,}530.4$ lb

By symmetry, $\bar{x} = 0$.

$$M = \rho(2)\int_0^3 \frac{2}{\sqrt{1 + y^2}}\,dy = \Big[4\rho \ln\left|y + \sqrt{1 + y^2}\right|\Big]_0^3 = 4\rho \ln\left(3 + \sqrt{10}\right)$$

$$M_x = 2\rho\int_0^3 \frac{2y}{\sqrt{1 + y^2}}\,dy = \Big[4\rho\sqrt{1 + y^2}\Big]_0^3 = 4\rho\left(\sqrt{10} - 1\right)$$

$$\bar{y} = \frac{M_x}{M} = \frac{4\rho\left(\sqrt{10} - 1\right)}{4\rho \ln\left(3 + \sqrt{10}\right)} \approx 1.19$$

Centroid: $(\bar{x}, \bar{y}) \approx (0, 1.19)$

86. $\frac{1}{2 - 0}\int_0^2 \frac{5000}{1 + e^{4.8 - 1.9t}}\,dt = \frac{2500}{-1.9}\int_0^2 \frac{-1.9\,dt}{1 + e^{4.8 - 1.9t}}$

$= -\frac{2500}{1.9}\Big[(4.8 - 1.9t) - \ln(1 + e^{4.8 - 1.9t})\Big]_0^2$

$= -\frac{2500}{1.9}[(1 - \ln(1 + e)) - (4.8 - \ln(1 + e^{4.8}))]$

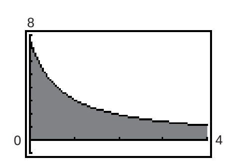
$= \frac{2500}{1.9}\left[3.8 + \ln\left(\frac{1 + e}{1 + e^{4.8}}\right)\right] \approx 401.4$

87. (a) $\int_0^4 \frac{k}{2 + 3x}\,dx = 10$

$$k = \frac{10}{\int_0^4 \frac{1}{2 + 3x}\,dx} = \frac{10}{\frac{1}{3}\ln 7} \approx \frac{10}{0.6486}$$

$= 15.417 \quad \left(= \frac{30}{\ln 7}\right)$

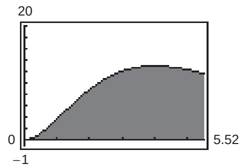
(b) $\int_0^4 \frac{15.417}{2 + 3x}\,dx$



88. (a) $\int_0^k 6x^2e^{-x/2}\,dx = 50$

By trial and error, $k = 5.51897$.

(b) $\int_0^{5.51897} 6x^2e^{-x/2}\,dx$



89. Let $I = \int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}}$.

For $x = \frac{\pi}{2} - u$, $dx = -du$, and

$$I = \int_{\pi/2}^{0} \frac{-du}{1 + (\tan(\pi/2 - u))^{\sqrt{2}}} = \int_0^{\pi/2} \frac{du}{1 + (\cot u)^{\sqrt{2}}} = \int_0^{\pi/2} \frac{(\tan u)^{\sqrt{2}}}{(\tan u)^{\sqrt{2}} + 1}\,du.$$

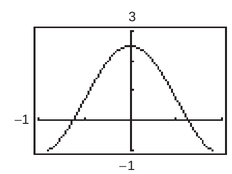
$$2I = \int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}} + \int_0^{\pi/2} \frac{(\tan x)^{\sqrt{2}}}{(\tan x)^{\sqrt{2}} + 1}\,dx = \int_0^{\pi/2} dx = \frac{\pi}{2}$$

Thus, $I = \frac{\pi}{4}$.

Section 8.7 Indeterminate Forms and L'Hôpital's Rule

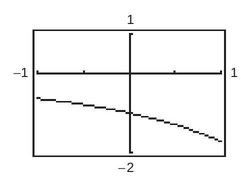
1. $\lim_{x\to 0} \frac{\sin 5x}{\sin 2x} \approx 2.5 \left(\text{exact: } \frac{5}{2}\right)$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	2.4132	2.4991	2.500	2.500	2.4991	2.4132



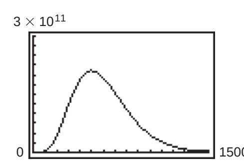
2. $\lim_{x\to 0} \frac{1 - e^x}{x} \approx -1$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	-0.9516	-0.9950	-0.9995	-1.00005	-1.005	-1.0517



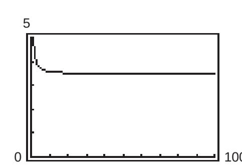
3. $\lim_{x\to\infty} x^5e^{-x/100} \approx 0$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$	0.9900	90,484	3.7×10^9	4.5×10^{10}	0	0



4. $\lim_{x\to\infty} \frac{6x}{\sqrt{3x^2 - 2x}} \approx 3.4641 \left(\text{exact: } \frac{6}{\sqrt{3}}\right)$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$	6	3.5857	3.4757	3.4653	3.4642	3.4641



5. (a) $\lim_{x\to 3} \frac{2(x-3)}{x^2-9} = \lim_{x\to 3} \frac{2(x-3)}{(x+3)(x-3)} = \lim_{x\to 3} \frac{2}{x+3} = \frac{1}{3}$

(b) $\lim_{x\to 3} \frac{2(x-3)}{x^2-9} = \lim_{x\to 3} \frac{(d/dx)[2(x-3)]}{(d/dx)[x^2-9]} = \lim_{x\to 3} \frac{2}{2x} = \frac{2}{6} = \frac{1}{3}$

6. (a) $\lim_{x\to -1} \frac{2x^2-x-3}{x+1} = \lim_{x\to -1} \frac{(2x-3)(x+1)}{x+1} = \lim_{x\to -1} (2x-3) = -5$

(b) $\lim_{x\to -1} \frac{2x^2-x-3}{x+1} = \lim_{x\to -1} \frac{(d/dx)[2x^2-x-3]}{(d/dx)[x+1]} = \lim_{x\to -1} \frac{4x-1}{1} = -5$

7. (a) $\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3} = \lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} = \lim_{x\to 3} \frac{(x+1)-4}{(x-3)\left[\sqrt{x+1}+2\right]} = \lim_{x\to 3} \frac{1}{\sqrt{x+1}+2} = \frac{1}{4}$

(b) $\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3} = \lim_{x\to 3} \frac{(d/dx)\left[\sqrt{x+1}-2\right]}{(d/dx)[x-3]} = \lim_{x\to 3} \frac{1/\left(2\sqrt{x+1}\right)}{1} = \frac{1}{4}$

8. (a) $\lim_{x\to 0} \frac{\sin 4x}{2x} = \lim_{x\to 0} 2\left(\frac{\sin 4x}{4x}\right) = 2(1) = 2$

(b) $\lim_{x\to 0} \frac{\sin 4x}{2x} = \lim_{x\to 0} \frac{(d/dx)[\sin 4x]}{(d/dx)[2x]} = \lim_{x\to 0} \frac{4\cos 4x}{2} = 2$

9. (a) $\lim_{x\to \infty} \frac{5x^2-3x+1}{3x^2-5} = \lim_{x\to \infty} \frac{5-(3/x)+(1/x^2)}{3-(5/x^2)} = \frac{5}{3}$

(b) $\lim_{x\to \infty} \frac{5x^2-3x+1}{3x^2-5} = \lim_{x\to \infty} \frac{(d/dx)[5x^2-3x+1]}{(d/dx)[3x^2-5]} = \lim_{x\to \infty} \frac{10x-3}{6x} = \lim_{x\to \infty} \frac{(d/dx)[10x-3]}{(d/dx)[6x]} = \lim_{x\to \infty} \frac{10}{6} = \frac{5}{3}$

10. (a) $\lim_{x\to \infty} \frac{2x+1}{4x^2+x} = \lim_{x\to \infty} \frac{(2/x)+(1/x^2)}{4+(1/x)} = \frac{0}{4} = 0$

(b) $\lim_{x\to \infty} \frac{2x+1}{4x^2+x} = \lim_{x\to \infty} \frac{(d/dx)[2x+1]}{(d/dx)[4x^2+x]} = \lim_{x\to \infty} \frac{2}{8x+1} = 0$

11. $\lim_{x\to 2} \frac{x^2-x-2}{x-2} = \lim_{x\to 2} \frac{2x-1}{1} = 3$

12. $\lim_{x\to -1} \frac{x^2-x-2}{x+1} = \lim_{x\to -1} \frac{2x-1}{1} = -3$

13. $\lim_{x\to 0} \frac{\sqrt{4-x^2}-2}{x} = \lim_{x\to 0} \frac{-x/\sqrt{4-x^2}}{1} = 0$

14. $\lim_{x\to 2^-} \frac{\sqrt{4-x^2}}{x-2} = \lim_{x\to 2^-} \frac{-x/\sqrt{4-x^2}}{1}$

$= \lim_{x\to 2^-} \frac{-x}{\sqrt{4-x^2}} = -\infty$

15. $\lim_{x\to 0} \frac{e^x-(1-x)}{x} = \lim_{x\to 0} \frac{e^x+1}{1} = 2$

16. $\lim_{x\to 1} \frac{\ln x^2}{x^2-1} = \lim_{x\to 1} \frac{2\ln x}{x^2-1}$

$= \lim_{x\to 1} \frac{2/x}{2x}$

$= \lim_{x\to 1} \frac{1}{x^2} = 1$

17. $\lim_{x\to 0^+} \frac{e^x-(1+x)}{x^3} = \lim_{x\to 0^+} \frac{e^x-1}{3x^2}$

$= \lim_{x\to 0^+} \frac{e^x}{6x} = \infty$

18. Case 1: $n = 1$

$$\lim_{x\to 0^+}\frac{e^x-(1+x)}{x} = \lim_{x\to 0^+}\frac{e^x-1}{1} = 0$$

Case 2: $n = 2$

$$\lim_{x\to 0^+}\frac{e^x-(1+x)}{x^2} = \lim_{x\to 0^+}\frac{e^x-1}{2x} = \lim_{x\to 0^+}\frac{e^x}{2} = \frac{1}{2}$$

Case 3: $n \ge 3$

$$\lim_{x\to 0^+}\frac{e^x-(1+x)}{x^n} = \lim_{x\to 0^+}\frac{e^x-1}{nx^{n-1}} = \lim_{x\to 0^+}\frac{e^x}{n(n-1)x^{n-2}} = \infty$$

19. $\displaystyle\lim_{x\to 0}\frac{\sin 2x}{\sin 3x} = \lim_{x\to 0}\frac{2\cos 2x}{3\cos 3x} = \frac{2}{3}$

20. $\displaystyle\lim_{x\to 0}\frac{\sin ax}{\sin bx} = \lim_{x\to 0}\frac{a\cos ax}{b\cos bx} = \frac{a}{b}$

21. $\displaystyle\lim_{x\to 0}\frac{\arcsin x}{x} = \lim_{x\to 0}\frac{1/\sqrt{1-x^2}}{1} = 1$

22. $\displaystyle\lim_{x\to 1}\frac{\arctan x-(\pi/4)}{x-1} = \lim_{x\to 1}\frac{1/(1+x^2)}{1} = \frac{1}{2}$

23. $\displaystyle\lim_{x\to\infty}\frac{3x^2-2x+1}{2x^2+3} = \lim_{x\to\infty}\frac{6x-2}{4x}$

$\displaystyle= \lim_{x\to\infty}\frac{6}{4} = \frac{3}{2}$

24. $\displaystyle\lim_{x\to\infty}\frac{x-1}{x^2+2x+3} = \lim_{x\to\infty}\frac{1}{2x+2} = 0$

25. $\displaystyle\lim_{x\to\infty}\frac{x^2+2x+3}{x-1} = \lim_{x\to\infty}\frac{2x+2}{1} = \infty$

26. $\displaystyle\lim_{x\to\infty}\frac{x^3}{x+1} = \lim_{x\to\infty}\frac{3x^2}{1} = \infty$

27. $\displaystyle\lim_{x\to\infty}\frac{x^3}{e^{x/2}} = \lim_{x\to\infty}\frac{3x^2}{(1/2)e^{x/2}}$

$\displaystyle= \lim_{x\to\infty}\frac{6x}{(1/4)e^{x/2}} = \lim_{x\to\infty}\frac{6}{(1/8)e^{x/2}} = 0$

28. $\displaystyle\lim_{x\to\infty}\frac{x^2}{e^x} = \lim_{x\to\infty}\frac{2x}{e^x} = \lim_{x\to\infty}\frac{2}{e^x} = 0$

29. $\displaystyle\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty}\frac{1}{\sqrt{1+(1/x^2)}} = 1$

Note: L'Hôpital's Rule does not work on this limit. See Exercise 79.

30. $\displaystyle\lim_{x\to\infty}\frac{x^2}{\sqrt{x^2+1}} = \lim_{x\to\infty}\frac{x}{\sqrt{1+(1/x)^2}} = \infty$

31. $\displaystyle\lim_{x\to\infty}\frac{\cos x}{x} = 0$ by Squeeze Theorem

$\displaystyle\left(\frac{\cos x}{x} \le \frac{1}{x}, \text{ for } x > 0\right)$

32. $\displaystyle\lim_{x\to\infty}\frac{\sin x}{x-\pi} = 0$

Note: Use the Squeeze Theorem for $x > \pi$.

$$-\frac{1}{x-\pi} \le \frac{\sin x}{x-\pi} \le \frac{1}{x-\pi}$$

33. $\displaystyle\lim_{x\to\infty}\frac{\ln x}{x^2} = \lim_{x\to\infty}\frac{1/x}{2x} = \lim_{x\to\infty}\frac{1}{2x^2} = 0$

34. $\displaystyle\lim_{x\to\infty}\frac{\ln x^4}{x^3} = \lim_{x\to\infty}\frac{4\ln x}{x^3} = \lim_{x\to\infty}\frac{4/x}{3x^2}$

$\displaystyle= \lim_{x\to\infty}\frac{4}{3x^3} = 0$

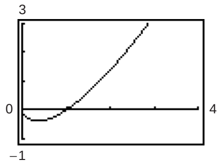
35. $\displaystyle\lim_{x\to\infty}\frac{e^x}{x^2} = \lim_{x\to\infty}\frac{e^x}{2x} = \lim_{x\to\infty}\frac{e^x}{2} = \infty$

36. $\displaystyle\lim_{x\to\infty}\frac{e^{x/2}}{x} = \lim_{x\to\infty}\frac{(1/2)e^{x/2}}{1} = \infty$

37. (a) $\lim_{x\to\infty} x \ln x$, not indeterminate

(b) $\lim_{x\to\infty} x \ln x = (\infty)(\infty) = \infty$

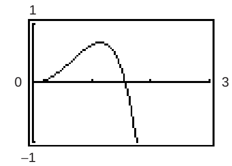
(c)



38. (a) $\lim_{x\to 0^+} x^3 \cot x = (0)(\infty)$

(b) $\lim_{x\to 0^+} x^3 \cot x = \lim_{x\to 0^+} \dfrac{x^3}{\tan x} = \lim_{x\to 0^+} \dfrac{3x^2}{\sec^2 x} = 0$

(c)



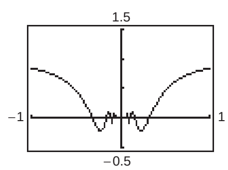
39. (a) $\lim_{x\to\infty} \left(x \sin \dfrac{1}{x}\right) = (\infty)(0)$

(b) $\lim_{x\to\infty} x \sin \dfrac{1}{x} = \lim_{x\to\infty} \dfrac{\sin(1/x)}{1/x}$

$$= \lim_{x\to\infty} \frac{(-1/x^2)\cos(1/x)}{-1/x^2}$$

$$= \lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = 1$$

(c)



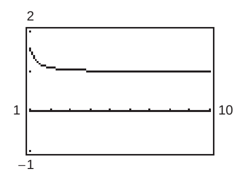
40. (a) $\lim_{x\to\infty} \left(x \tan \dfrac{1}{x}\right) = (\infty)(0)$

(b) $\lim_{x\to\infty} x \tan \dfrac{1}{x} = \lim_{x\to\infty} \dfrac{\tan(1/x)}{1/x}$

$$= \lim_{x\to\infty} \frac{-(1/x^2)\sec^2(1/x)}{-(1/x^2)}$$

$$= \lim_{x\to\infty} \sec^2\left(\frac{1}{x}\right) = 1$$

(c)



41. (a) $\lim_{x\to 0^+} x^{1/x} = 0^\infty = 0$, not indeterminate (See Exercise 106).

(b) Let $y = x^{1/x}$

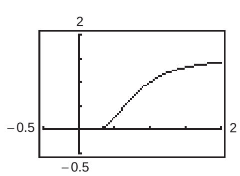
$$\ln y = \ln x^{1/x} = \frac{1}{x} \ln x.$$

Since $x \to 0^+$, $\dfrac{1}{x} \ln x \to (\infty)(-\infty) = -\infty$. Hence,

$$\ln y \to -\infty \implies y \to 0^+.$$

Therefore, $\lim_{x\to 0^+} x^{1/x} = 0$.

(c)



42. (a) $\lim_{x\to 0^+} (e^x + x)^{2/x} = 1^\infty$

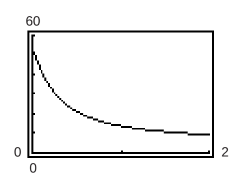
(b) Let $y = \lim_{x\to 0^+} (e^x + x)^{2/x}$.

$$\ln y = \lim_{x\to 0^+} \frac{2 \ln(e^x + x)}{x}$$

$$= \lim_{x\to 0^+} \frac{2(e^x + 1)/(e^x + x)}{1} = 4$$

Thus, $\ln y = 4 \implies y = e^4 \approx 54.598$.

(c)



43. (a) $\lim_{x\to\infty} x^{1/x} = \infty^0$

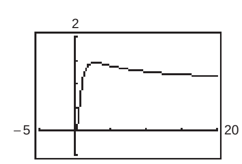
(b) Let $y = \lim_{x\to\infty} x^{1/x}$.

$$\ln y = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \left(\frac{1/x}{1}\right) = 0$$

Thus, $\ln y = 0 \implies y = e^0 = 1$. Therefore,

$$\lim_{x\to\infty} x^{1/x} = 1.$$

(c)



44. (a) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = 1^\infty$

(b) Let $y = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

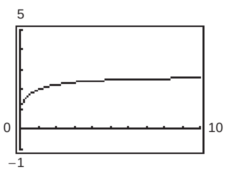
$$\ln y = \lim_{x\to\infty}\left[x\ln\left(1+\frac{1}{x}\right)\right] = \lim_{x\to\infty}\frac{\ln[1+(1/x)]}{1/x}$$

$$= \lim_{x\to\infty}\frac{\left[\frac{(-1/x^2)}{1+(1/x)}\right]}{(-1/x^2)} = \lim_{x\to\infty}\frac{1}{1+(1/x)} = 1$$

Thus, $\ln y = 1 \implies y = e^1 = e$. Therefore,

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$$

(c)



45. (a) $\lim_{x\to 0^+}(1+x)^{1/x} = 1^\infty$

(b) Let $y = \lim_{x\to 0^+}(1+x)^{1/x}$.

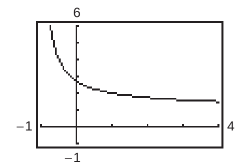
$$\ln y = \lim_{x\to 0^+}\frac{\ln(1+x)}{x}$$

$$= \lim_{x\to 0^+}\left(\frac{1/(1+x)}{1}\right) = 1$$

Thus, $\ln y = 1 \implies y = e^1 = e$.

Therefore, $\lim_{x\to 0^+}(1+x)^{1/x} = e$.

(c)



46. (a) $\lim_{x\to\infty}(1+x)^{1/x} = \infty^0$

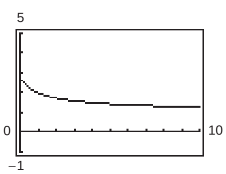
(b) Let $y = \lim_{x\to\infty}(1+x)^{1/x}$.

$$\ln y = \lim_{x\to\infty}\frac{\ln(1+x)}{x}$$

$$= \lim_{x\to\infty}\left(\frac{1/(1+x)}{1}\right) = 0$$

Thus, $\ln y = 0 \implies y = e^0 = 1$.
Therefore, $\lim_{x\to\infty}(1+x)^{1/x} = 1$.

(c)



47. (a) $\lim_{x\to 0^+}[3(x)^{x/2}] = 0^0$

(b) Let $y = \lim_{x\to 0^+}3(x)^{x/2}$.

$$\ln y = \lim_{x\to 0^+}\left[\ln 3 + \frac{x}{2}\ln x\right]$$

$$= \lim_{x\to 0^+}\left[\ln 3 + \frac{\ln x}{2/x}\right]$$

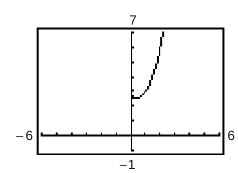
$$= \lim_{x\to 0^+}\ln 3 + \lim_{x\to 0^+}\frac{1/x}{-2/x^2}$$

$$= \lim_{x\to 0^+}\ln 3 - \lim_{x\to 0^+}\frac{x}{2}$$

$$= \ln 3$$

Hence, $\lim_{x\to 0^+}3(x)^{x/2} = 3$.

(c)



48. (a) $\lim_{x\to 4^+}[3(x-4)]^{x-4} = 0^0$

(b) Let $y = \lim_{x\to 4^+}[3(x-4)]^{x-4}$.

$$\ln y = \lim_{x\to 4^+}(x-4)\ln[3(x-4)]$$

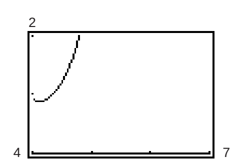
$$= \lim_{x\to 4^+}\frac{\ln[3(x-4)]}{1/(x-4)}$$

$$= \lim_{x\to 4^+}\frac{1/(x-4)}{-1/(x-4)^2}$$

$$= \lim_{x\to 4^+}[-(x-4)] = 0$$

Hence, $\lim_{x\to 4^+}[3(x-4)]^{x-4} = 1$.

(c)



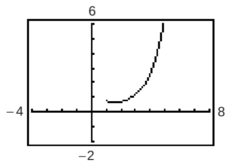
49. (a) $\lim_{x\to 1^+} (\ln x)^{x-1} = 0^0$

(b) Let $y = \lim_{x\to 1^+} (\ln x)^{x-1}$

$$= \lim_{x\to 1^+} (x-1)\ln x = 0.$$

Hence, $\lim_{x\to 1^+} (\ln x)^{x-1} = 1$.

(c)



50. (a) $\lim_{x\to 0^+} \left[\cos\left(\frac{\pi}{2} - x\right)\right]^x = 0^0$

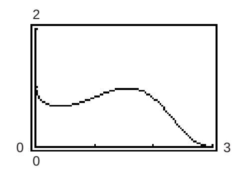
(b) Let $y = \lim_{x\to 0^+} \left[\cos\left(\frac{\pi}{2} - x\right)\right]^x$.

$$\ln y = \lim_{x\to 0^+} x \ln\left[\cos\left(\frac{\pi}{2} - x\right)\right]$$

$$= 0 \cdot 0 = 0$$

Hence, $\lim_{x\to 0^+} \left[\cos\left(\frac{\pi}{2} - x\right)\right]^x = 1$.

(c)



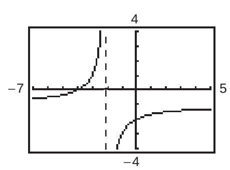
51. (a) $\lim_{x\to 2^+} \left(\frac{8}{x^2-4} - \frac{x}{x-2}\right) = \infty - \infty$

(b) $$\lim_{x\to 2^+} \left(\frac{8}{x^2-4} - \frac{x}{x-2}\right) = \lim_{x\to 2^+} \frac{8 - x(x+2)}{x^2-4}$$

$$= \lim_{x\to 2^+} \frac{(2-x)(4+x)}{(x+2)(x-2)}$$

$$= \lim_{x\to 2^+} \frac{-(x+4)}{x+2} = \frac{-3}{2}$$

(c)



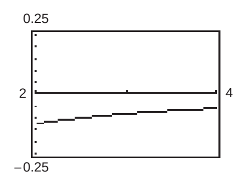
52. (a) $\lim_{x\to 2^+} \left(\frac{1}{x^2-4} - \frac{\sqrt{x-1}}{x^2-4}\right) = \infty - \infty$

(b) $$\lim_{x\to 2^+} \left(\frac{1}{x^2-4} - \frac{\sqrt{x-1}}{x^2-4}\right) = \lim_{x\to 2^+} \frac{1 - \sqrt{x-1}}{x^2-4}$$

$$= \lim_{x\to 2^+} \frac{-1/\left(2\sqrt{x-1}\right)}{2x}$$

$$= \lim_{x\to 2^+} \frac{-1}{4x\sqrt{x-1}} = \frac{-1}{8}$$

(c)

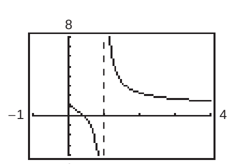


53. (a) $\lim_{x\to 1^+} \left(\frac{3}{\ln x} - \frac{2}{x-1}\right) = \infty - \infty$

(b) $$\lim_{x\to 1^+} \left(\frac{3}{\ln x} - \frac{2}{x-1}\right) = \lim_{x\to 1^+} \frac{3x - 3 - 2\ln x}{(x-1)\ln x}$$

$$= \lim_{x\to 1^+} \frac{3 - (2/x)}{[(x-1)/x] + \ln x} = \infty$$

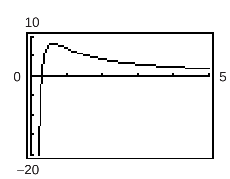
(c)



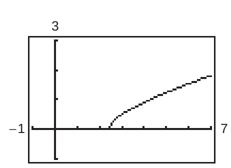
54. (a) $\lim_{x\to 0^+} \left(\frac{10}{x} - \frac{3}{x^2}\right) = \infty - \infty$

(b) $$\lim_{x\to 0^+} \left(\frac{10}{x} - \frac{3}{x^2}\right) = \lim_{x\to 0^+} \left(\frac{10x - 3}{x^2}\right) = -\infty$$

(c)



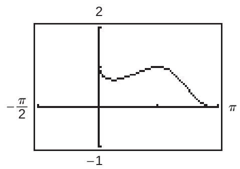
55. (a)



(b) $$\lim_{x\to 3} \frac{x-3}{\ln(2x-5)} = \lim_{x\to 3} \frac{1}{2/(2x-5)}$$

$$= \lim_{x\to 3} \frac{2x-5}{2} = \frac{1}{2}$$

56. (a)

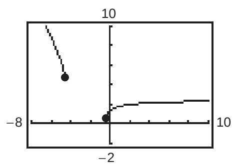


(b) Let $y = (\sin x)^x$, then $\ln y = x \ln(\sin x)$.

$$\lim_{x\to 0^+} \frac{\ln(\sin x)}{1/x} = \lim_{x\to 0^+} \frac{\cos x/\sin x}{-1/x^2} = \lim_{x\to 0^+} \frac{-x^2}{\tan x} = \lim_{x\to 0^+} \frac{-2x}{\sec^2 x} = 0$$

Therefore, since $\ln y = 0$, $y = 1$ and $\lim_{x\to 0^+} (\sin x)^x = 1$.

57. (a)



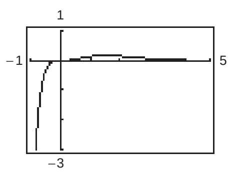
(b) $$\lim_{x\to\infty} \left(\sqrt{x^2+5x+2} - x\right) = \lim_{x\to\infty} \left(\sqrt{x^2+5x+2} - x\right)\frac{\left(\sqrt{x^2+5x+2}+x\right)}{\left(\sqrt{x^2+5x+2}+x\right)}$$

$$= \lim_{x\to\infty} \frac{(x^2+5x+2) - x^2}{\sqrt{x^2+5x+2}+x}$$

$$= \lim_{x\to\infty} \frac{5x+2}{\sqrt{x^2+5x+2}+x}$$

$$= \lim_{x\to\infty} \frac{5+(2/x)}{\sqrt{1+(5/x)+(2/x^2)}+1} = \frac{5}{2}$$

58. (a)



(b) $$\lim_{x\to\infty} \frac{x^3}{e^{2x}} = \lim_{x\to\infty} \frac{3x^2}{2e^{2x}} = \lim_{x\to\infty} \frac{6x}{4e^{2x}} = \lim_{x\to\infty} \frac{6}{8e^{2x}} = 0$$

59. $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, 1^\infty, 0^0, \infty - \infty, \infty^0$

60. See Theorem 8.4.

61. (a) Let $f(x) = x^2 - 25$ and $g(x) = x - 5$.

(b) Let $f(x) = (x-5)^2$ and $g(x) = x^2 - 25$.

(c) Let $f(x) = x^2 - 25$ and $g(x) = (x-5)^3$.

62. Let $f(x) = x + 25$ and $g(x) = x$.

63.

x	10	10^2	10^4	10^6	10^8	10^{10}
$\frac{(\ln x)^4}{x}$	2.811	4.498	0.720	0.036	0.001	0.000

64.

x	1	5	10	20	30	40	50	100
$\frac{e^x}{x^5}$	2.718	0.047	0.220	151.614	4.40×10^5	2.30×10^9	1.66×10^{13}	2.69×10^{33}

65. $$\lim_{x\to\infty} \frac{x^2}{e^{5x}} = \lim_{x\to\infty} \frac{2x}{5e^{5x}} = \lim_{x\to\infty} \frac{2}{25e^{5x}} = 0$$

66. $$\lim_{x\to\infty} \frac{x^3}{e^{2x}} = \lim_{x\to\infty} \frac{3x^2}{2e^{2x}} = \lim_{x\to\infty} \frac{6x}{4e^{2x}} = \lim_{x\to\infty} \frac{6}{8e^{2x}} = 0$$

67. $\lim_{x\to\infty} \frac{(\ln x)^3}{x} = \lim_{x\to\infty} \frac{3(\ln x)^2(1/x)}{1}$

$= \lim_{x\to\infty} \frac{3(\ln x)^2}{x}$

$= \lim_{x\to\infty} \frac{6(\ln x)(1/x)}{1}$

$= \lim_{x\to\infty} \frac{6(\ln x)}{x} = \lim_{x\to\infty} \frac{6}{x} = 0$

68. $\lim_{x\to\infty} \frac{(\ln x)^2}{x^3} = \lim_{x\to\infty} \frac{(2\ln x)/x}{3x^2}$

$= \lim_{x\to\infty} \frac{2\ln x}{3x^3}$

$= \lim_{x\to\infty} \frac{2/x}{9x^2} = \lim_{x\to\infty} \frac{2}{9x^3} = 0$

69. $\lim_{x\to\infty} \frac{(\ln x)^n}{x^m} = \lim_{x\to\infty} \frac{n(\ln x)^{n-1}/x}{mx^{m-1}}$

$= \lim_{x\to\infty} \frac{n(\ln x)^{n-1}}{mx^m}$

$= \lim_{x\to\infty} \frac{n(n-1)(\ln x)^{n-2}}{m^2x^m}$

$= \cdots = \lim_{x\to\infty} \frac{n!}{m^n x^m} = 0$

70. $\lim_{x\to\infty} \frac{x^m}{e^{nx}} = \lim_{x\to\infty} \frac{mx^{m-1}}{ne^{nx}}$

$= \lim_{x\to\infty} \frac{m(m-1)x^{m-2}}{n^2e^{nx}}$

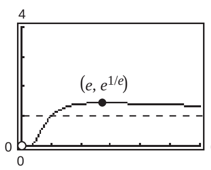
$= \cdots = \lim_{x\to\infty} \frac{m!}{n^m e^{nx}} = 0$

71. $y = x^{1/x}, x > 0$

Horizontal asymptote: $y = 1$ (See Exercise 43.)

$\ln y = \frac{1}{x}\ln x$

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\left(\frac{1}{x}\right) + (\ln x)\left(-\frac{1}{x^2}\right)$



$\frac{dy}{dx} = x^{1/x}\left(\frac{1}{x^2}\right)(1 - \ln x) = x^{(1/x)-2}(1 - \ln x) = 0$

Critical number: $x = e$

Intervals:	$(0, e)$	(e, ∞)
Sign of dy/dx:	$+$	$-$
$y = f(x)$:	Increasing	Decreasing

Relative maximum: $(e, e^{1/e})$

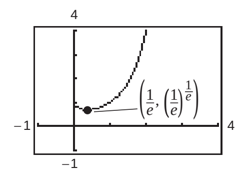
72. $y = x^x, x > 0$

$\lim_{x\to\infty} x^x = \infty$ and $\lim_{x\to 0^+} x^x = 1$

No horizontal asymptotes

$\ln y = x \ln x$

$\frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln x$



$\frac{dy}{dx} = x^x(1 + \ln x) = 0$

Critical number: $x = e^{-1}$

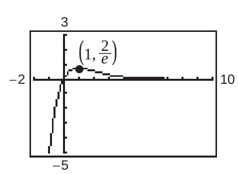
Intervals:	$(0, e^{-1})$	$(e^{-1}, 0)$
Sign of dy/dx:	$-$	$+$
$y = f(x)$:	Decreasing	Increasing

Relative minimum: $(e^{-1}, (e^{-1})^{e^{-1}}) = \left(\frac{1}{e}, \left(\frac{1}{e}\right)^{1/e}\right)$

73. $y = 2xe^{-x}$

$\lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$

Horizontal asymptote: $y = 0$



$\frac{dy}{dx} = 2x(-e^{-x}) + 2e^{-x}$

$= 2e^{-x}(1 - x) = 0$

Critical number: $x = 1$

Intervals:	$(-\infty, 1)$	$(1, \infty)$
Sign of dy/dx:	$+$	$-$
$y = f(x)$:	Increasing	Decreasing

Relative maximum: $\left(1, \frac{2}{e}\right)$

74. $y = \frac{\ln x}{x}$

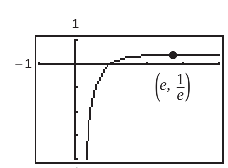
Horizontal asymptote: $y = 0$ (See Exercise 29.)

$\frac{dy}{dx} = \frac{x(1/x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2} = 0$

Critical number: $x = e$

Intervals:	$(0, e)$	(e, ∞)
Sign of dy/dx:	$+$	$-$
$y = f(x)$:	Increasing	Decreasing

Relative maximum: $\left(e, \frac{1}{e}\right)$



75. $\lim_{x \to 0} \frac{e^{2x} - 1}{e^x} = \frac{0}{1} = 0$

Limit is not of the form $0/0$ or ∞/∞.
L'Hôpital's Rule does not apply.

76. $\lim_{x \to \infty} \frac{\sin \pi x - 1}{x} = 0$ (Numerator is bounded)

Limit is not of the form $0/0$ or ∞/∞.
L'Hôpital's Rule does not apply.

77. $\lim_{x \to \infty} x \cos \frac{1}{x} = \infty(1) = \infty$

Limit is not of the form $0/0$ or ∞/∞.
L'Hôpital's Rule does not apply.

78. $\lim_{x \to \infty} \frac{e^{-x}}{1 + e^{-x}} = \frac{0}{1 + 0} = 0$

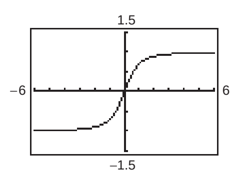
Limit is not of the form $0/0$ or ∞/∞.
L'Hôpital's Rule does not apply.

79. (a) Applying L'Hôpital's Rule twice results in the original limit, so L'Hôpital's Rule fails:

$$\lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x \to \infty} \frac{1}{x/\sqrt{x^2 + 1}}$$
$$= \lim_{x \to \infty} \frac{\sqrt{x^2 + 1}}{x}$$
$$= \lim_{x \to \infty} \frac{x/\sqrt{x^2 + 1}}{1}$$
$$= \lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}}$$

(b) $$\lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x \to \infty} \frac{x/x}{\sqrt{x^2 + 1}/x}$$
$$= \lim_{x \to \infty} \frac{1}{\sqrt{1 + 1/x^2}}$$
$$= \frac{1}{\sqrt{1 + 0}} = 1$$

(c)

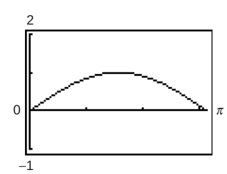


80. (a) $\lim_{x \to \pi/2^-} \frac{\tan x}{\sec x}$ is indeterminant: $\frac{\infty}{\infty}$

$$\lim_{x \to \pi/2^-} \frac{\tan x}{\sec x} = \lim_{x \to \pi/2^-} \frac{\sec^2 x}{\sec x \tan x}$$
$$= \lim_{x \to \pi/2^-} \frac{\sec x}{\tan x} \quad \left(\frac{\infty}{\infty}\right)$$
$$= \lim_{x \to \pi/2^-} \frac{\sec x \tan x}{\sec^2 x}$$
$$= \lim_{x \to \pi/2^-} \frac{\tan x}{\sec x}, \text{ the original problem!}$$

(b) $$\lim_{x \to \pi/2^-} \frac{\tan x}{\sec x} = \lim_{x \to \pi/2^-} \frac{\sin x}{\cos x}(\cos x)$$
$$= \lim_{x \to \pi/2^-} \sin x = 1$$

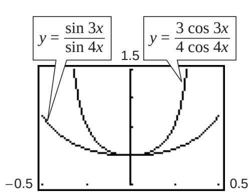
(c)



81. $f(x) = \sin(3x), g(x) = \sin(4x)$

$f'(x) = 3\cos(3x), g'(x) = 4\cos(4x)$

$y_1 = \frac{f(x)}{g(x)} = \frac{\sin 3x}{\sin 4x}, \quad y_2 = \frac{f'(x)}{g'(x)} = \frac{3\cos 3x}{4\cos 4x}$



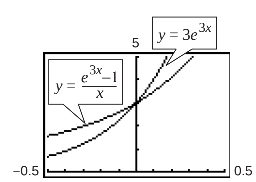
As $x \to 0$, $y_1 \to 0.75$ and $y_2 \to 0.75$
By L'Hôpital's Rule,

$$\lim_{x \to 0} \frac{\sin 3x}{\sin 4x} = \lim_{x \to 0} \frac{3\cos 3x}{4\cos 4x} = \frac{3}{4}$$

82. $f(x) = e^{3x} - 1, g(x) = x$

$f'(x) = 3e^{3x}, g'(x) = 1$

$y_1 = \frac{f(x)}{g(x)} = \frac{e^{3x} - 1}{x}, \quad y_2 = \frac{f'(x)}{g'(x)} = 3e^{3x}$



As $x \to 0$, $y_1 \to 3$ and $y_2 \to 3$
By L'Hôpital's Rule,

$$\lim_{x \to 0} \frac{e^{3x} - 1}{x} = \lim_{x \to 0} \frac{3e^{3x}}{1} = 3$$

83. $\displaystyle\lim_{k\to 0}\frac{32\left(1 - e^{-kt} + \frac{v_0 k e^{-kt}}{32}\right)}{k} = \lim_{k\to 0}\frac{32(1 - e^{-kt})}{k} + \lim_{k\to 0}(v_0 e^{-kt})$

$$= \lim_{k\to 0}\frac{32(0 + te^{-kt})}{1} + \lim_{k\to 0}\left(\frac{v_0}{e^{kt}}\right) = 32t + v_0$$

84. $A = P\left(1 + \frac{r}{n}\right)^{nt}$

$$\ln A = \ln P + nt\ln\left(1 + \frac{r}{n}\right) = \ln P + \frac{\ln\left(1 + \frac{r}{n}\right)}{\frac{1}{nt}}$$

$$\lim_{n\to\infty}\left[\frac{\ln\left(1 + \frac{r}{n}\right)}{\frac{1}{nt}}\right] = \lim_{n\to\infty}\left[\frac{-\frac{r}{n^2}\left(\frac{1}{1 + (r/n)}\right)}{-\left(\frac{1}{n^2 t}\right)}\right] = \lim_{n\to\infty}\left[rt\left(\frac{1}{1 + \frac{r}{n}}\right)\right] = rt$$

Since $\lim_{n\to\infty}\ln A = \ln P + rt$, we have $\lim_{n\to\infty} A = e^{(\ln P + rt)} = e^{\ln P}e^{rt} = Pe^{rt}$. Alternatively,

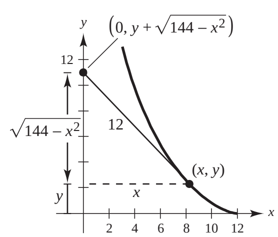
$$\lim_{n\to\infty} A = \lim_{n\to\infty} P\left(1 + \frac{r}{n}\right)^{nt} = \lim_{n\to\infty} P\left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} = Pe^{rt}.$$

85. Let N be a fixed value for n. Then

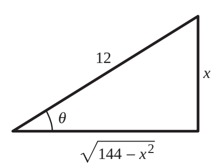
$$\lim_{x\to\infty}\frac{x^{N-1}}{e^x} = \lim_{x\to\infty}\frac{(N-1)x^{N-2}}{e^x} = \lim_{x\to\infty}\frac{(N-1)(N-2)x^{N-3}}{e^x} = \ldots = \lim_{x\to\infty}\left[\frac{(N-1)!}{e^x}\right] = 0. \qquad \text{(See Exercise 70.)}$$

86. (a) $\displaystyle m = \frac{dy}{dx} = \frac{y - \left(y + \sqrt{144 - x^2}\right)}{x - 0}$

$$= -\frac{\sqrt{144 - x^2}}{x}$$



(b) $\displaystyle y = -\int\frac{\sqrt{144 - x^2}}{x}\,dx$



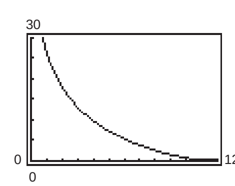
Let $x = 12\sin\theta$, $dx = 12\cos\theta\,d\theta$, $\sqrt{144 - x^2} = 12\cos\theta$.

$$y = -\int\frac{12\cos\theta}{12\sin\theta}12\cos\theta\,d\theta = -12\int\frac{1 - \sin^2\theta}{\sin\theta}\,d\theta$$

$$= -12\int(\csc\theta - \sin\theta)\,d\theta = -12\ln|\csc\theta - \cot\theta| - 12\cos\theta + C$$

$$= -12\ln\left|\frac{12}{x} - \frac{\sqrt{144 - x^2}}{x}\right| - 12\left(\frac{\sqrt{144 - x^2}}{12}\right) + C$$

$$= -12\ln\left|\frac{12 - \sqrt{144 - x^2}}{x}\right| - \sqrt{144 - x^2} + C$$



When $x = 12$, $y = 0 \Rightarrow C = 0$. Thus, $y = -12\ln\left(\frac{12 - \sqrt{144 - x^2}}{x}\right) - \sqrt{144 - x^2}$.

Note: $\frac{12 - \sqrt{144 - x^2}}{x} > 0$ for $0 < x \le 12$

—CONTINUED—

86. —CONTINUED—

(c) Vertical asymptote: $x = 0$

(d) $y + \sqrt{144 - x^2} = 12 \implies y = 12 - \sqrt{144 - x^2}$

Thus,

$$\begin{aligned}
12 - \sqrt{144 - x^2} &= -12 \ln\left(\frac{12 - \sqrt{144 - x^2}}{x}\right) - \sqrt{144 - x^2} \\
-1 &= \ln\left(\frac{12 - \sqrt{144 - x^2}}{x}\right) \\
xe^{-1} &= 12 - \sqrt{144 - x^2} \\
(xe^{-1} - 12)^2 &= \left(-\sqrt{144 - x^2}\right)^2 \\
x^2e^{-2} - 24xe^{-1} + 144 &= 144 - x^2 \\
x^2(e^{-2} + 1) - 24xe^{-1} &= 0 \\
x[x(e^{-2} + 1) - 24e^{-1}] &= 0 \\
x = 0 \text{ or } x &= \frac{24e^{-1}}{e^{-2} + 1} \approx 7.77665.
\end{aligned}$$

Therefore,

$$\begin{aligned}
s &= \int_{7.77665}^{12} \sqrt{1 + \left(-\frac{\sqrt{144 - x^2}}{x}\right)^2}\, dx = \int_{7.77665}^{12} \sqrt{\frac{x^2 + (144 - x^2)}{x^2}}\, dx \\
&= \int_{7.77665}^{12} \frac{12}{x}\, dx = \Big[12 \ln|x|\Big]_{7.77665}^{12} = 12(\ln 12 - \ln 7.77665) \approx 5.2 \text{ meters.}
\end{aligned}$$

87. $f(x) = x^3, g(x) = x^2 + 1, [0, 1]$

$$\begin{aligned}
\frac{f(b) - f(a)}{g(b) - g(a)} &= \frac{f'(c)}{g'(c)} \\
\frac{f(1) - f(0)}{g(1) - g(0)} &= \frac{3c^2}{2c} \\
\frac{1}{1} &= \frac{3c}{2} \\
c &= \frac{2}{3}
\end{aligned}$$

88. $f(x) = \frac{1}{x}, g(x) = x^2 - 4, [1, 2]$

$$\begin{aligned}
\frac{f(2) - f(1)}{g(2) - g(1)} &= \frac{f'(c)}{g'(c)} \\
\frac{-1/2}{3} &= \frac{-1/c^2}{2c} \\
-\frac{1}{6} &= -\frac{1}{2c^3} \\
2c^3 &= 6 \\
c &= \sqrt[3]{3}
\end{aligned}$$

89. $f(x) = \sin x, g(x) = \cos x, \left[0, \frac{\pi}{2}\right]$

$$\begin{aligned}
\frac{f(\pi/2) - f(0)}{g(\pi/2) - g(0)} &= \frac{f'(c)}{g'(c)} \\
\frac{1}{-1} &= \frac{\cos c}{-\sin c} \\
-1 &= -\cot c \\
c &= \frac{\pi}{4}
\end{aligned}$$

90. $f(x) = \ln x, g(x) = x^3, [1, 4]$

$$\begin{aligned}
\frac{f(4) - f(1)}{g(4) - g(1)} &= \frac{f'(c)}{g'(c)} \\
\frac{\ln 4}{63} &= \frac{1/c}{3c^2} = \frac{1}{3c^3} \\
3c^3 \ln 4 &= 63 \\
c^3 &= \frac{21}{\ln 4} \\
c &= \sqrt[3]{\frac{21}{\ln 4}} \approx 2.474
\end{aligned}$$

91. False. L'Hôpital's Rule does not apply since

$$\lim_{x \to 0} (x^2 + x + 1) \neq 0.$$

$$\lim_{x \to 0^+} \frac{x^2 + x + 1}{x} = \lim_{x \to 0^+} \left(x + 1 + \frac{1}{x}\right) = 1 + \infty = \infty$$

92. False. If $y = e^x/x^2$, then

$$y' = \frac{x^2 e^x - 2xe^x}{x^4} = \frac{xe^x(x - 2)}{x^4} = \frac{e^x(x - 2)}{x^3}.$$

93. True

94. False. Let $f(x) = x$ and $g(x) = x + 1$. Then

$$\lim_{x \to \infty} \frac{x}{x + 1} = 1, \text{ but } \lim_{x \to \infty} [x - (x + 1)] = -1.$$

95. Area of triangle: $\frac{1}{2}(2x)(1 - \cos x) = x - x \cos x$

Shaded area: Area of rectangle − Area under curve

$$2x(1 - \cos x) - 2\int_0^x (1 - \cos t)\, dt = 2x(1 - \cos x) - 2\Big[t - \sin t\Big]_0^x$$

$$= 2x(1 - \cos x) - 2(x - \sin x) = 2 \sin x - 2x \cos x$$

Ratio: $$\lim_{x \to 0} \frac{x - x \cos x}{2 \sin x - 2x \cos x} = \lim_{x \to 0} \frac{1 + x \sin x - \cos x}{2 \cos x + 2x \sin x - 2 \cos x}$$

$$= \lim_{x \to 0} \frac{1 + x \sin x - \cos x}{2x \sin x}$$

$$= \lim_{x \to 0} \frac{x \cos x + \sin x + \sin x}{2x \cos x + 2 \sin x}$$

$$= \lim_{x \to 0} \frac{x \cos x + 2 \sin x}{2x \cos x + 2 \sin x} \cdot \frac{1/\cos x}{1/\cos x}$$

$$= \lim_{x \to 0} \frac{x + 2 \tan x}{2x + 2 \tan x}$$

$$= \lim_{x \to 0} \frac{1 + 2 \sec^2 x}{2 + 2 \sec^2 x} = \frac{3}{4}$$

96. (a) $\sin \theta = BD$

$\cos \theta = DO \implies AD = 1 - \cos \theta$

$$\text{Area } \triangle ABD = \frac{1}{2}bh = \frac{1}{2}(1 - \cos \theta) \sin \theta = \frac{1}{2} \sin \theta - \frac{1}{2} \sin \theta \cos \theta$$

(b) Area of sector: $\frac{1}{2}\theta$

$$\text{Shaded area: } \frac{1}{2}\theta - \text{Area } \triangle OBD = \frac{1}{2}\theta - \frac{1}{2}(\cos \theta)(\sin \theta) = \frac{1}{2}\theta - \frac{1}{2} \sin \theta \cos \theta$$

(c) $$R = \frac{(1/2) \sin \theta - (1/2) \sin \theta \cos \theta}{(1/2)\theta - (1/2) \sin \theta \cos \theta} = \frac{\sin \theta - \sin \theta \cos \theta}{\theta - \sin \theta \cos \theta}$$

(d) $$\lim_{\theta \to 0} R = \lim_{\theta \to 0} \frac{\sin \theta - (1/2) \sin 2\theta}{\theta - (1/2) \sin 2\theta}$$

$$= \lim_{\theta \to 0} \frac{\cos \theta - \cos 2\theta}{1 - \cos 2\theta} = \lim_{\theta \to 0} \frac{-\sin \theta + 2 \sin 2\theta}{2 \sin 2\theta} = \lim_{\theta \to 0} \frac{-\cos \theta + 4 \cos 2\theta}{4 \cos 2\theta} = \frac{3}{4}$$

97. $\lim_{x\to 0} \frac{4x - 2\sin 2x}{2x^3} = \lim_{x\to 0} \frac{4 - 4\cos 2x}{6x^2}$

$= \lim_{x\to 0} \frac{8\sin 2x}{12x}$

$= \lim_{x\to 0} \frac{16\cos 2x}{12} = \frac{16}{12} = \frac{4}{3}$

Let $c = \frac{4}{3}$.

98. Let $y = (e^x + x)^{1/x}$.

$\ln y = \frac{1}{x}\ln(e^x + x) = \frac{\ln(e^x + x)}{x}$

$\lim_{x\to 0} \frac{\ln(e^x + x)}{x} = \lim_{x\to 0} \frac{e^x + 1}{e^x + x} = \frac{2}{1} = 2$

Hence, $\lim_{x\to 0}(e^x + x)^{1/x} = e^2$.

Let $c = e^2 \approx 7.389$.

99. $\lim_{x\to 0} \frac{a - \cos bx}{x^2} = 2$

Near $x = 0$, $\cos bx \approx 1$ and $x^2 \approx 0 \implies a = 1$.

Using L'Hôpital's Rule,

$\lim_{x\to 0} \frac{1 - \cos bx}{x^2} = \lim_{x\to 0} \frac{b\sin bx}{2x}$

$= \lim_{x\to 0} \frac{b^2\cos bx}{2} = 2.$

Hence, $b^2 = 4$ and $b = \pm 2$.

Answer: $a = 1, b = \pm 2$

100. We use mathematical induction.

For $n = 1$, $\lim_{x\to\infty} \frac{x^1}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$.

Assume that $\lim_{x\to\infty} \frac{x^k}{e^x} = 0$.

Then, $\lim_{x\to\infty} \frac{x^{k+1}}{e^x} = \lim_{x\to\infty} \frac{(k+1)x^k}{e^x}$

$= (k+1)\lim_{x\to 0} \frac{x^k}{e^x}$

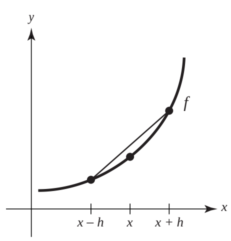
$= (k+1)(0) = 0.$

101. (a) $\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} = \lim_{h\to 0} \frac{f'(x+h)(1) - f'(x-h)(-1)}{2}$

$= \lim_{h\to 0}\left[\frac{f'(x+h) + f'(x-h)}{2}\right]$

$= \frac{f'(x) + f'(x)}{2} = f'(x)$

(b)



Graphically, the slope of the line joining $(x - h, f(x-h))$ and $(x + h, f(x+h))$ is approximately $f'(x)$. And, as $h \to 0$,

$\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} = f'(x)$.

102. $\lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = \lim_{h\to 0} \frac{f'(x+h)(1) + f'(x-h)(-1)}{2h}$

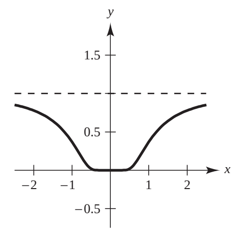
$= \lim_{h\to 0} \frac{f'(x+h) - f'(x-h)}{2h}$

$= \lim_{h\to 0} \frac{f''(x+h)(1) - f''(x-h)(-1)}{2}$

$= \lim_{h\to 0} \frac{f''(x+h) + f''(x-h)}{2}$

$= \frac{f''(x) + f''(x)}{2} = f''(x)$

103. $g(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$



$$g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{e^{-1/x^2}}{x}$$

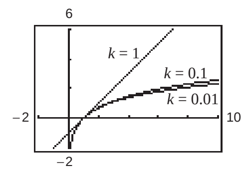
Let $y = \dfrac{e^{-1/x^2}}{x}$, then $\ln y = \ln\left(\dfrac{e^{-1/x^2}}{x}\right) = -\dfrac{1}{x^2} - \ln x = \dfrac{-1 - x^2 \ln x}{x^2}$. Since

$$\lim_{x \to 0} x^2 \ln x = \lim_{x \to 0} \frac{\ln x}{1/x^2} = \lim_{x \to 0} \frac{1/x}{-2/x^3} = \lim_{x \to 0} \left(-\frac{x^2}{2}\right) = 0$$

we have $\lim_{x \to 0} \left(\dfrac{-1 - x^2 \ln x}{x^2}\right) = -\infty$. Thus, $\lim_{x \to 0} y = e^{-\infty} = 0 \implies g'(0) = 0$.

Note: The graph appears to support this conclusion—the tangent line is horizontal at (0, 0).

104. $f(x) = \dfrac{x^k - 1}{k}$



$k = 1, \quad f(x) = x - 1$

$k = 0.1, \quad f(x) = \dfrac{x^{0.1} - 1}{0.1} = 10(x^{0.1} - 1)$

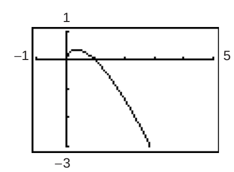
$k = 0.01, \quad f(x) = \dfrac{x^{0.01} - 1}{0.01} = 100(x^{0.01} - 1)$

$$\lim_{k \to 0^+} \frac{x^k - 1}{k} = \lim_{k \to 0^+} \frac{x^k(\ln x)}{1} = \ln x$$

105. (a) $\lim_{x \to 0^+} (-x \ln x)$ is the form $0 \cdot \infty$.

(b) $\lim_{x \to 0^+} \dfrac{-\ln x}{1/x} = \lim_{x \to 0^+} \dfrac{-1/x}{-1/x^2} = \lim_{x \to 0^+} (x) = 0$

(c)



106. $\lim_{x \to a} f(x)^{g(x)}$

$y = f(x)^{g(x)}$

$\ln y = g(x) \ln f(x)$

$\lim_{x \to a} g(x) \ln f(x) = (\infty)(-\infty) = -\infty$

As $x \to a$, $\ln y \implies -\infty$, and hence $y = 0$. Thus,

$\lim_{x \to a} f(x)^{g(x)} = 0$.

107. $\lim_{x \to a} f(x)^{g(x)}$

$y = f(x)^{g(x)}$

$\ln y = g(x) \ln f(x)$

$\lim_{x \to a} g(x) \ln f(x) = (-\infty)(-\infty) = \infty$

As $x \to a$, $\ln y \implies \infty$, and hence $y = \infty$. Thus,

$\lim_{x \to a} f(x)^{g(x)} = \infty$.

108.
$$f'(a)(b - a) - \int_a^b f''(t)(t - b)\,dt = f'(a)(b - a) - \left\{\Big[f'(t)(t - b)\Big]_a^b - \int_a^b f'(t)\,dt\right\}$$
$$= f'(a)(b - a) + f'(a)(a - b) + \Big[f(t)\Big]_a^b = f(b) - f(a)$$

$dv = f''(t)\,dt \implies v = f'(t)$

$u = t - b \implies du = dt$

109. (a) $\lim_{x\to 0^+} x^{(\ln 2)/(1+\ln x)}$ is of form 0^0.

Let $y = x^{(\ln 2)/(1+\ln x)}$

$$\ln y = \frac{\ln 2}{1+\ln x}\ln x$$

$$\lim_{x\to 0^+} \ln y = \frac{\ln 2(1/x)}{1/x} = \ln 2.$$

Thus, $\lim_{x\to 0^+} x^{(\ln 2)/(1+\ln x)} = 2.$

(b) $\lim_{x\to\infty} x^{(\ln 2)/(1+\ln x)}$ is of form ∞^0.

Let $y = x^{(\ln 2)/(1+\ln x)}$

$$\ln y = \frac{\ln 2}{1+\ln x}\ln x$$

$$\lim_{x\to\infty} \ln y = \frac{\ln 2(1/x)}{1/x} = \ln 2.$$

Thus, $\lim_{x\to\infty} x^{(\ln 2)/(1+\ln x)} = 2.$

(c) $\lim_{x\to 0} (x+1)^{(\ln 2)/(x)}$ is of form 1^∞.

Let $y = (x+1)^{(\ln 2)/(x)}$

$$\ln y = \frac{\ln 2}{x}\ln(x+1)$$

$$\lim_{x\to 0} \ln y = \lim_{x\to 0}\frac{(\ln 2)1/(x+1)}{1} = \ln 2.$$

Thus, $\lim_{x\to 0} (x+1)^{(\ln 2)/(x)} = 2.$

110. $\lim_{x\to a}\dfrac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}}$

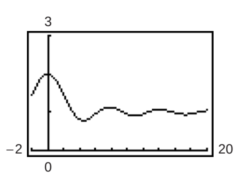
$$= \lim_{x\to a}\frac{\frac{1}{2}(2a^3x - x^4)^{-1/2}(2a^3 - 4x^3) - \frac{a}{3}(a^2x)^{-2/3}a^2}{-\frac{1}{4}(ax^3)^{-3/4}}$$

$$= \frac{\frac{1}{2}(a^4)^{-1/2}(-2a^3) - \frac{a^3}{3}(a^3)^{-2/3}}{-\frac{1}{4}(ax^3)^{-3/4}(3ax^2)}$$

$$= \frac{a + \frac{a}{3}}{\frac{1}{4}(a^{-3})(3a^3)}$$

$$= \frac{\frac{4}{3}a}{\frac{3}{4}} = \frac{16}{9}a$$

111. (a) $h(x) = \dfrac{x + \sin x}{x}$



$$\lim_{x\to\infty} h(x) = 1$$

(b) $h(x) = \dfrac{x+\sin x}{x} = \dfrac{x}{x} + \dfrac{\sin x}{x} = 1 + \dfrac{\sin x}{x}, x > 0$

Hence, $\lim_{x\to\infty} h(x) = \lim_{x\to\infty}\left[1 + \dfrac{\sin x}{x}\right] = 1 + 0 = 1.$

(c) No. $h(x)$ is not an indeterminate form.

112. Let $f(x) = \left[\dfrac{1}{x}\cdot\dfrac{a^x - 1}{a - 1}\right]^{1/x}$.

For $a > 1$ and $x > 0$,

$$\ln f(x) = \frac{1}{x}\left[\ln\frac{1}{x} + \ln(a^x - 1) - \ln(a - 1)\right] = -\frac{\ln x}{x} + \frac{\ln(a^x - 1)}{x} - \frac{\ln(a-1)}{x}.$$

As $x\to\infty$, $\dfrac{\ln x}{x}\to 0$, $\dfrac{\ln(a-1)}{x}\to 0$, and $\dfrac{\ln(a^x - 1)}{x} = \dfrac{\ln[(1 - a^{-x})a^x]}{x} = \dfrac{\ln(1 - a^{-x})}{x} + \ln a \to \ln a$.

Hence, $\ln f(x) \to \ln a$.

For $0 < a < 1$ and $x > 0$,

$$\ln f(x) = \frac{-\ln x}{x} + \frac{\ln(1 - a^x)}{x} - \frac{\ln(1-a)}{x} \to 0 \text{ as } x\to\infty.$$

Combining these results, $\lim_{x\to\infty} f(x) = \begin{cases} a & \text{if } a > 1 \\ 1 & \text{if } 0 < a < 1\end{cases}$.

Section 8.8 Improper Integrals

1. $\int_0^1 \frac{dx}{3x-2}$ is improper because $3x - 2 = 0$ when $x = \frac{2}{3}$.

2. $\int_1^2 \frac{dx}{x^2}$ is not improper because $\frac{1}{x^2}$ is continuous on $[1, 2]$.

3. $\int_0^1 \frac{2x-5}{x^2-5x+6}\,dx = \int_0^1 \frac{2x-5}{(x-2)(x-3)}\,dx$

is not improper because

$\frac{2x-5}{(x-2)(x-3)}$ is continuous on $[0, 1]$.

4. $\int_1^\infty \ln(x^2)dx$

is improper because the upper limit of integration is ∞.

5. Infinite discontinuity at $x = 0$.

$$\int_0^4 \frac{1}{\sqrt{x}}\,dx = \lim_{b\to 0^+}\int_b^4 \frac{1}{\sqrt{x}}\,dx = \lim_{b\to 0^+}\Big[2\sqrt{x}\Big]_b^4 = \lim_{b\to 0^+}\left(4 - 2\sqrt{b}\right) = 4$$

Converges

6. Infinite discontinuity at $x = 3$.

$$\int_3^4 \frac{1}{(x-3)^{3/2}}\,dx = \lim_{b\to 3^+}\int_b^4 (x-3)^{-3/2}\,dx = \lim_{b\to 3^+}\Big[-2(x-3)^{-1/2}\Big]_b^4 = \lim_{b\to 3^+}\left[-2 + \frac{2}{\sqrt{b-3}}\right] = \infty$$

Diverges

7. Infinite discontinuity at $x = 1$.

$$\int_0^2 \frac{1}{(x-1)^2}\,dx = \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^2 \frac{1}{(x-1)^2}\,dx$$
$$= \lim_{b\to 1^-}\int_0^b \frac{1}{(x-1)^2}\,dx + \lim_{c\to 1^+}\int_c^2 \frac{1}{(x-1)^2}\,dx$$
$$= \lim_{b\to 1^-}\left[-\frac{1}{x-1}\right]_0^b + \lim_{c\to 1^+}\left[-\frac{1}{x-1}\right]_c^2 = (\infty - 1) + (-1 + \infty)$$

Diverges

8. Infinite discontinuity at $x = 1$.

$$\int_0^2 \frac{1}{(x-1)^{2/3}}\,dx = \int_0^1 \frac{1}{(x-1)^{2/3}}\,dx + \int_1^2 \frac{1}{(x-1)^{2/3}}\,dx$$
$$= \lim_{b\to 1^-}\int_0^b \frac{1}{(x-1)^{2/3}}\,dx + \lim_{c\to 1^+}\int_c^2 \frac{1}{(x-1)^{2/3}}\,dx$$
$$= \lim_{b\to 1^-}\Big[3\sqrt[3]{x-1}\Big]_0^b + \lim_{c\to 1^+}\Big[3\sqrt[3]{x-1}\Big]_c^2 = (0 + 3) + (3 - 0) = 6$$

Converges

9. Infinite limit of integration.

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty}\Big[-e^{-x}\Big]_0^b = 0 + 1 = 1$$

Converges

10. Infinite limit of integration.

$$\int_{-\infty}^0 e^{2x}\,dx = \lim_{b\to -\infty}\int_b^0 e^{2x}\,dx = \lim_{b\to -\infty}\left[\frac{1}{2}e^{2x}\right]_b^0 = \frac{1}{2} - 0 = \frac{1}{2}$$

Converges

11. $\int_{-1}^{1} \frac{1}{x^2}\,dx \neq -2$

because the integrand is not defined at $x = 0$.
Diverges

12. $\int_{-2}^{2} \frac{-2}{(x-1)^3}\,dx \neq \frac{8}{9}$ because the integral is not defined at $x = 1$. The integral diverges.

13. $\int_0^{\infty} e^{-x}\,dx \neq 0$. You need to evaluate the limit.

$$\lim_{b\to\infty} \int_0^b e^{-x}\,dx = \lim_{b\to\infty} \Big[-e^{-x}\Big]_0^b$$

$$= \lim_{b\to\infty} \Big[-e^{-b} + 1\Big] = 1$$

14. $\int_0^{\pi} \sec x\,dx \neq 0$ because $\sec x$ is not defined at $x = \pi/2$.

The integral diverges.

15. $\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$

$$= \lim_{b\to\infty} \Big[-\frac{1}{x}\Big]_1^b = 1$$

16. $\int_1^{\infty} \frac{5}{x^3}\,dx = \lim_{b\to\infty} \int_1^b \frac{5}{x^3}\,dx$

$$= \lim_{b\to\infty} \Big[-\frac{5}{2}x^{-2}\Big]_1^b = \frac{5}{2}$$

17. $\int_1^{\infty} \frac{3}{\sqrt[3]{x}}\,dx = \lim_{b\to\infty} \int_1^b 3x^{-1/3}\,dx$

$$= \lim_{b\to\infty} \Big[\frac{9}{2}x^{2/3}\Big]_1^b = \infty$$

Diverges

18. $\int_1^{\infty} \frac{4}{\sqrt[4]{x}}\,dx = \lim_{b\to\infty} \int_1^b 4x^{-1/4}\,dx$

$$= \lim_{b\to\infty} \Big[\frac{16}{3}x^{3/4}\Big]_1^b = \infty \quad \text{Diverges}$$

19. $\int_{-\infty}^{0} xe^{-2x}\,dx = \lim_{b\to-\infty} \int_b^0 xe^{-2x}\,dx = \lim_{b\to-\infty} \frac{1}{4}\Big[(-2x-1)e^{-2x}\Big]_b^0 = \lim_{b\to-\infty} \frac{1}{4}\big[-1 + (2b+1)e^{-2b}\big] = -\infty$ (Integration by parts)

Diverges

20. $\int_0^{\infty} xe^{-x/2}\,dx = \lim_{b\to\infty} \int_0^b xe^{-x/2}\,dx = \lim_{b\to\infty} \Big[e^{-x/2}(-2x-4)\Big]_0^b = \lim_{b\to\infty} e^{-b/2}(-2b-4) + 4 = 4$

21. $\int_0^{\infty} x^2e^{-x}\,dx = \lim_{b\to\infty} \int_0^b x^2e^{-x}\,dx = \lim_{b\to\infty} \Big[-e^{-x}(x^2+2x+2)\Big]_0^b = \lim_{b\to\infty} \left(-\frac{b^2+2b+2}{e^b} + 2\right) = 2$

Since $\lim_{b\to\infty} \left(-\frac{b^2+2b+2}{e^b}\right) = 0$ by L'Hôpital's Rule.

22. $\int_0^{\infty} (x-1)e^{-x}\,dx = \lim_{b\to\infty} \int_0^b (x-1)e^{-x}\,dx = \lim_{b\to\infty} \Big[-xe^{-x}\Big]_0^b = \lim_{b\to\infty} \left(\frac{-b}{e^b} + 0\right) = 0$ by L'Hôpital's Rule.

23. $\int_0^{\infty} e^{-x}\cos x\,dx = \lim_{b\to\infty} \frac{1}{2}\Big[e^{-x}(-\cos x + \sin x)\Big]_0^b$

$$= \frac{1}{2}[0 - (-1)] = \frac{1}{2}$$

24. $\int_0^{\infty} e^{-ax}\sin bx\,dx = \lim_{c\to\infty} \Big[\frac{e^{-ax}(-a\sin bx - b\cos bx)}{a^2+b^2}\Big]_0^c$

$$= 0 - \frac{-b}{a^2+b^2} = \frac{b}{a^2+b^2}$$

25. $\displaystyle\int_4^{\infty} \frac{1}{x(\ln x)^3}\,dx = \lim_{b\to\infty}\int_4^b (\ln x)^{-3}\frac{1}{x}\,dx$

$$= \lim_{b\to\infty}\left[-\frac{1}{2}(\ln x)^{-2}\right]_4^b$$

$$= -\frac{1}{2}(\ln b)^{-2} + \frac{1}{2}(\ln 4)^{-2}$$

$$= \frac{1}{2}\frac{1}{(2\ln 2)^2} = \frac{1}{8(\ln 2)^2}$$

26. $\displaystyle\int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x}\,dx$

$$= \lim_{b\to\infty}\left[\frac{(\ln x)^2}{2}\right]_1^b = \infty$$

Diverges

27. $\displaystyle\int_{-\infty}^{\infty} \frac{2}{4+x^2}\,dx = \int_{-\infty}^{0}\frac{2}{4+x^2}\,dx + \int_0^{\infty}\frac{2}{4+x^2}\,dx$

$$= \lim_{b\to-\infty}\int_b^0 \frac{2}{4+x^2}\,dx + \lim_{c\to\infty}\int_0^c \frac{2}{4+x^2}\,dx$$

$$= \lim_{b\to-\infty}\left[\arctan\left(\frac{x}{2}\right)\right]_b^0 + \lim_{c\to\infty}\left[\arctan\left(\frac{x}{2}\right)\right]_0^c$$

$$= \left(0-\left(-\frac{\pi}{2}\right)\right) + \left(\frac{\pi}{2}-0\right) = \pi$$

28. $\displaystyle\int_0^{\infty} \frac{x^3}{(x^2+1)^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{x}{x^2+1}\,dx - \lim_{b\to\infty}\int_0^b \frac{x}{(x^2+1)^2}\,dx$

$$= \lim_{b\to\infty}\left[\frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)}\right]_0^b$$

$$= \infty - \frac{1}{2}$$

Diverges

29. $\displaystyle\int_0^{\infty} \frac{1}{e^x+e^{-x}}\,dx = \lim_{b\to\infty}\int_0^b \frac{e^x}{1+e^{2x}}\,dx$

$$= \lim_{b\to\infty}\left[\arctan(e^x)\right]_0^b$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

30. $\displaystyle\int_0^{\infty} \frac{e^x}{1+e^x}\,dx = \lim_{b\to\infty}\left[\ln(1+e^x)\right]_0^b = \infty - \ln 2$

Diverges

31. $\displaystyle\int_0^{\infty} \cos \pi x\,dx = \lim_{b\to\infty}\left[\frac{1}{\pi}\sin \pi x\right]_0^b$

Diverges since $\sin \pi b$ does not approach a limit as $b\to\infty$.

32. $\displaystyle\int_0^{\infty} \sin\frac{x}{2}\,dx = \lim_{b\to\infty}\left[-2\cos\frac{x}{2}\right]_0^b$

Diverges since $\cos\dfrac{x}{2}$ does not approach a limit as $x\to\infty$.

33. $\displaystyle\int_0^{1} \frac{1}{x^2}\,dx = \lim_{b\to 0^+}\left[\frac{-1}{x}\right]_b^1 = \lim_{b\to 0^+}\left[-1+\frac{1}{b}\right] = -1+\infty$

Diverges

34. $\displaystyle\int_0^{4} \frac{8}{x}\,dx = \lim_{b\to 0^+}\int_b^4 \frac{8}{x}\,dx = \lim_{b\to 0^+}\left[8\ln x\right]_b^4 = \infty$

Diverges

35. $\displaystyle\int_0^{8} \frac{1}{\sqrt[3]{8-x}}\,dx = \lim_{b\to 8^-}\int_0^b \frac{1}{\sqrt[3]{8-x}}\,dx$

$$= \lim_{b\to 8^-}\left[\frac{-3}{2}(8-x)^{2/3}\right]_0^b = 6$$

36. $\displaystyle\int_0^{6} \frac{4}{\sqrt{6-x}}\,dx = \lim_{b\to 6^-}\int_0^b 4(6-x)^{-1/2}\,dx$

$$= \lim_{b\to 6^-}\left[-8(6-x)^{1/2}\right]_0^b$$

$$= -8(0) + 8\sqrt{6}$$

$$= 8\sqrt{6}$$

37. $\displaystyle\int_0^{1} x\ln x\,dx = \lim_{b\to 0^+}\left[\frac{x^2}{2}\ln|x| - \frac{x^2}{4}\right]_b^1$

$$= \lim_{b\to 0^+}\left[\frac{-1}{4} - \frac{b^2\ln b}{2} + \frac{b^2}{4}\right] = \frac{-1}{4}$$

since $\displaystyle\lim_{b\to 0^+}(b^2\ln b) = 0$ by L'Hôpital's Rule.

38. $\displaystyle\int_0^{e} \ln x^2\,dx = \lim_{b\to 0^+}\int_0^e 2\ln x\,dx$

$$= \lim_{b\to 0^+}\left[2x\ln x - 2x\right]_b^e$$

$$= \lim_{b\to 0^+}\left[(2e-2e) - (2b\ln b - 2b)\right]$$

$$= 0$$

39. $\displaystyle\int_0^{\pi/2} \tan\theta\,d\theta = \lim_{b\to(\pi/2)^-}\Big[\ln|\sec\theta|\Big]_0^b = \infty$

Diverges

40. $\displaystyle\int_0^{\pi/2} \sec\theta\,d\theta = \lim_{b\to(\pi/2)}\Big[\ln|\sec\theta + \tan\theta|\Big]_0^b = \infty$

Diverges

41. $$\int_2^4 \frac{2}{x\sqrt{x^2-4}}\,dx = \lim_{b\to2^+}\int_b^4 \frac{2}{x\sqrt{x^2-4}}\,dx$$
$$= \lim_{b\to2^+}\left[\operatorname{arcsec}\left|\frac{x}{2}\right|\right]_b^4$$
$$= \lim_{b\to2^+}\left(\operatorname{arcsec}2 - \operatorname{arcsec}\left(\frac{b}{2}\right)\right)$$
$$= \frac{\pi}{3} - 0 = \frac{\pi}{3}$$

42. $\displaystyle\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx = \lim_{b\to2^-}\left[\arcsin\left(\frac{x}{2}\right)\right]_0^b = \frac{\pi}{2}$

43. $\displaystyle\int_2^4 \frac{1}{\sqrt{x^2-4}} = \lim_{b\to2^+}\Big[\ln\left|x+\sqrt{x^2-4}\right|\Big]_b^4 = \ln\left(4+2\sqrt{3}\right) - \ln 2 = \ln\left(2+\sqrt{3}\right) \approx 1.317$

44. $\displaystyle\int_0^2 \frac{1}{4-x^2}\,dx = \lim_{b\to2^-}\int_0^b \frac{1}{4}\left(\frac{1}{2+x}+\frac{1}{2-x}\right)dx = \lim_{b\to2^-}\left[\frac{1}{4}\ln\left|\frac{2+x}{2-x}\right|\right]_0^b = \infty - 0$

Diverges

45. $$\int_0^2 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx + \int_1^2 \frac{1}{\sqrt[3]{x-1}}\,dx$$
$$= \lim_{b\to1^-}\left[\frac{3}{2}(x-1)^{2/3}\right]_0^b + \lim_{c\to1^+}\left[\frac{3}{2}(x-1)^{2/3}\right]_c^2 = \frac{-3}{2}+\frac{3}{2} = 0$$

46. $$\int_1^3 \frac{2}{(x-2)^{8/3}}\,dx = \int_1^2 2(x-2)^{-8/3}\,dx + \int_2^3 2(x-2)^{-8/3}\,dx$$
$$= \lim_{b\to2^-}\int_1^b 2(x-2)^{-8/3}\,dx + \lim_{c\to2^+}\int_c^3 2(x-2)^{-8/3}\,dx$$
$$= \lim_{b\to2^-}\left[-\frac{6}{5}(x-2)^{-5/3}\right]_1^b + \lim_{c\to2^+}\left[-\frac{6}{5}(x-2)^{-5/3}\right]_c^3 = \infty$$

Diverges

47. $$\int_0^\infty \frac{4}{\sqrt{x}(x+6)}\,dx = \int_0^1 \frac{4}{\sqrt{x}(x+6)}\,dx + \int_1^\infty \frac{4}{\sqrt{x}(x+6)}\,dx$$

Let $u = \sqrt{x}$, $u^2 = x$, $2u\,du = dx$.

$$\int \frac{4}{\sqrt{x}(x+6)}\,dx = \int \frac{4(2u\,du)}{u(u^2+6)} = 8\int \frac{du}{u^2+6} = \frac{8}{\sqrt{6}}\arctan\left(\frac{u}{\sqrt{6}}\right) + C = \frac{8}{\sqrt{6}}\arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right) + C$$

Thus, $$\int_0^\infty \frac{4}{\sqrt{x}(x+6)}\,dx = \lim_{b\to0^+}\left[\frac{8}{\sqrt{6}}\arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right)\right]_b^1 + \lim_{c\to\infty}\left[\frac{8}{\sqrt{6}}\arctan\left(\frac{\sqrt{x}}{\sqrt{6}}\right)\right]_1^c$$
$$= \left(\frac{8}{\sqrt{6}}\arctan\left(\frac{1}{\sqrt{6}}\right) - \frac{8}{\sqrt{6}}0\right) + \left(\frac{8}{\sqrt{6}}\frac{\pi}{2} - \frac{8}{\sqrt{6}}\arctan\left(\frac{1}{\sqrt{6}}\right)\right)$$
$$= \frac{8\pi}{2\sqrt{6}} = \frac{2\pi\sqrt{6}}{3}.$$

48. $\int \frac{1}{x \ln x}\,dx = \ln|\ln|x|| + C$

Thus,

$$\int_1^\infty \frac{1}{x \ln x}\,dx = \int_1^e \frac{1}{x \ln x}\,dx + \int_e^\infty \frac{1}{x \ln x}\,dx$$
$$= \lim_{b\to 1^+}\Big[\ln(\ln x)\Big]_1^e + \lim_{c\to\infty}\Big[\ln(\ln x)\Big]_e^\infty.$$

Diverges

49. If $p = 1$, $\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln x\Big]_1^b$

$$= \lim_{b\to\infty}[\ln b] = \infty.$$

Diverges. For $p \neq 1$,

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^b = \lim_{b\to\infty}\left[\frac{b^{1-p}}{1-p} - \frac{1}{1-p}\right].$$

This converges to $\frac{1}{p-1}$ if $1 - p < 0$ or $p > 1$.

50. If $p = 1$, $\int_0^1 \frac{1}{x}\,dx = \lim_{a\to 0^+} \ln x\Big]_a^1 = \lim_{a\to 0^+} -\ln a = \infty.$

Diverges. If $p \neq 1$,

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{a\to 0^+}\left[\frac{x^{1-p}}{1-p}\right]_a^1 = \lim_{a\to 0^+}\left[\frac{1}{1-p} - \frac{a^{1-p}}{1-p}\right].$$

This converges to $\frac{1}{1-p}$ if $1 - p > 0$ or $p < 1$.

51. For $n = 1$ we have

$$\int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x}\,dx$$
$$= \lim_{b\to\infty}\Big[-e^{-x}x - e^{-x}\Big]_0^b \qquad \text{(Parts: } u = x,\ dv = e^{-x}\,dx)$$
$$= \lim_{b\to\infty}\big[-e^{-b}b - e^{-b} + 1\big]$$
$$= \lim_{b\to\infty}\left[\frac{-b}{e^b} - \frac{1}{e^b} + 1\right] = 1 \quad \text{(L'Hôpital's Rule).}$$

Assume that $\int_0^\infty x^n e^{-x}\,dx$ converges. Then for $n + 1$ we have

$$\int x^{n+1}e^{-x}\,dx = -x^{n+1}e^{-x} + (n+1)\int x^n e^{-x}\,dx$$

by parts ($u = x^{n+1}$, $du = (n+1)x^n\,dx$, $dv = e^{-x}\,dx$, $v = -e^{-x}$).

Thus,

$$\int_0^\infty x^{n+1}e^{-x}\,dx = \lim_{b\to\infty}\Big[-x^{n+1}e^{-x}\Big]_0^b + (n+1)\int_0^\infty x^n e^{-x}\,dx = 0 + (n+1)\int_0^\infty x^n e^{-x}\,dx, \text{ which converges.}$$

52. (a) Assume $\int_a^\infty g(x)\,dx = L$ (converges).

Since $0 \le f(x) \le g(x)$ on $[a, \infty)$, $0 \le \int_a^\infty f(x)\,dx \le \int_a^\infty g(x)\,dx = L$ and $\int_a^\infty f(x)\,dx$ converges.

(b) $\int_a^\infty g(x)\,dx$ diverges, because otherwise, by part (a), if $\int_a^\infty g(x)\,dx$ converges, then so does $\int_a^\infty f(x)\,dx$.

53. $\int_0^1 \frac{1}{x^3}\,dx$ diverges.

(See Exercise 50, $p = 3 \not< 1$.)

54. $\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx = \frac{1}{1 - (1/3)} = \frac{3}{2}$ converges.

(See Exercise 50, $p = \frac{1}{3}$.)

55. $\int_1^{\infty} \frac{1}{x^3}\,dx = \frac{1}{3-1} = \frac{1}{2}$ converges.

(See Exercise 49, $p = 3$.)

56. $\int_0^{\infty} x^4 e^{-x}\,dx$ converges.

(See Exercise 51.)

57. Since $\frac{1}{x^2+5} \le \frac{1}{x^2}$ on $[1, \infty)$ and $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges by Exercise 49, $\int_1^{\infty} \frac{1}{x^2+5}\,dx$ converges.

58. Since $\frac{1}{\sqrt{x-1}} \ge \frac{1}{x}$ on $[2, \infty)$ and $\int_2^{\infty} \frac{1}{x}\,dx$ diverges by Exercise 49, $\int_2^{\infty} \frac{1}{\sqrt{x-1}}\,dx$ diverges.

59. Since $\frac{1}{\sqrt[3]{x(x-1)}} \ge \frac{1}{\sqrt[3]{x^2}}$ on $[2, \infty)$ and $\int_2^{\infty} \frac{1}{\sqrt[3]{x^2}}\,dx$ diverges by Exercise 49, $\int_2^{\infty} \frac{1}{\sqrt[3]{x(x-1)}}\,dx$ diverges.

60. Since $\frac{1}{\sqrt{x}(1+x)} \le \frac{1}{x^{3/2}}$ on $[1, \infty)$ and $\int_1^{\infty} \frac{1}{x^{3/2}}\,dx$ converges by Exercise 49, $\int_1^{\infty} \frac{1}{\sqrt{x}(1+x)}\,dx$ converges.

61. Since $e^{-x^2} \le e^{-x}$ on $[1, \infty)$ and $\int_0^{\infty} e^{-x}\,dx$ converges (see Exercise 9), $\int_0^{\infty} e^{-x^2}\,dx$ converges.

62. $\frac{1}{\sqrt{x}\ln x} \ge \frac{1}{x}$ since $\sqrt{x}\ln x < x$ on $[2, \infty)$. Since $\int_2^{\infty} \frac{1}{x}\,dx$ diverges by Exercise 49, $\int_2^{\infty} \frac{1}{\sqrt{x}\ln x}\,dx$ diverges.

63. Answers will vary.

64. See the definitions, pages 578, 581.

65. $\int_{-1}^{1} \frac{1}{x^3}\,dx = \int_{-1}^{0} \frac{1}{x^3}\,dx + \int_0^1 \frac{1}{x^3}\,dx$

These two integrals diverge by Exercise 50.

66. $\frac{10}{x^2-2x} = \frac{10}{x(x-2)} \implies x = 0, 2.$

You must analyze three improper integrals, and each must converge in order for the original integral to converge.

$$\int_0^3 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx$$

67.
$$\begin{aligned} A &= \int_{-\infty}^{1} e^x\,dx \\ &= \lim_{b\to-\infty} \int_b^1 e^x\,dx \\ &= \lim_{b\to-\infty} \Big[e^x \Big]_b^1 \\ &= \lim_{b\to-\infty} [e - e^b] = e \end{aligned}$$

68.
$$\begin{aligned} A &= \int_0^1 -\ln x\,dx \\ &= -\lim_{b\to0^+} \int_b^1 \ln x\,dx \\ &= -\lim_{b\to0^+} \Big[x\ln x - x \Big]_b^1 \\ &= -\lim_{b\to0^+} [(0-1) - b\ln b + b] \\ &= 1 \end{aligned}$$

Note: $\lim_{b\to0^+} b\ln b = \lim_{b\to0^+} \frac{\ln b}{1/b} = \lim_{b\to0^+} \frac{1/b}{-1/b^2} = 0$

69. $A = \int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$

$= \lim_{b\to-\infty}\int_b^0 \frac{1}{x^2+1}\,dx + \lim_{b\to\infty}\int_0^b \frac{1}{x^2+1}\,dx$

$= \lim_{b\to-\infty}\Big[\arctan(x)\Big]_b^0 + \lim_{b\to\infty}\Big[\arctan(x)\Big]_0^b$

$= \lim_{b\to-\infty}[0 - \arctan(b)] + \lim_{b\to\infty}[\arctan(b) - 0]$

$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi$

70. $A = \int_{-\infty}^{\infty} \frac{8}{x^2+4}\,dx$

$= \lim_{b\to-\infty}\int_b^0 \frac{8}{x^2+4}\,dx + \lim_{b\to\infty}\int_0^b \frac{8}{x^2+4}\,dx$

$= \lim_{b\to-\infty}\left[4\arctan\left(\frac{x}{2}\right)\right]_b^0 + \lim_{b\to\infty}\left[4\arctan\left(\frac{x}{2}\right)\right]_0^b$

$= \lim_{b\to-\infty}\left[0 - 4\arctan\left(\frac{b}{2}\right)\right] + \lim_{b\to\infty}\left[4\arctan\left(\frac{b}{2}\right) - 0\right]$

$= -4\left(\frac{-\pi}{2}\right) + 4\left(\frac{\pi}{2}\right) = 4\pi$

71. (a) $A = \int_0^{\infty} e^{-x}\,dx$

$= \lim_{b\to\infty}\Big[-e^{-x}\Big]_0^b = 0 - (-1) = 1$

(b) **Disk:**

$V = \pi\int_0^{\infty}(e^{-x})^2\,dx$

$= \lim_{b\to\infty}\pi\left[-\frac{1}{2}e^{-2x}\right]_0^b = \frac{\pi}{2}$

(c) **Shell:**

$V = 2\pi\int_0^{\infty} xe^{-x}\,dx$

$= \lim_{b\to\infty}\left\{2\pi\Big[-e^{-x}(x+1)\Big]_0^b\right\} = 2\pi$

72. (a) $A = \int_1^{\infty}\frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^{\infty} = 1$

(b) **Disk:**

$V = \pi\int_1^{\infty}\frac{1}{x^4}\,dx = \lim_{b\to\infty}\left[-\frac{\pi}{3x^3}\right]_1^b = \frac{\pi}{3}$

(c) **Shell:**

$V = 2\pi\int_1^{\infty} x\left(\frac{1}{x^2}\right)dx = \lim_{b\to\infty}\Big[2\pi(\ln x)\Big]_1^b = \infty$

Diverges

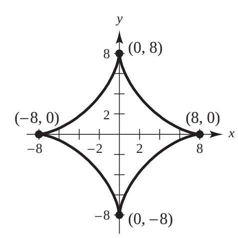
73. $x^{2/3} + y^{2/3} = 4$

$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$

$y' = \frac{-y^{1/3}}{x^{1/3}}$

$\sqrt{1+(y')^2} = \sqrt{1 + \frac{y^{2/3}}{x^{2/3}}} = \sqrt{\frac{x^{2/3}+y^{2/3}}{x^{2/3}}} = \sqrt{\frac{4}{x^{2/3}}} = \frac{2}{x^{1/3}}, \quad (x > 0)$

$s = 4\int_0^8 \frac{2}{x^{1/3}}\,dx = \lim_{b\to0^+}\left[8\cdot\frac{3}{2}x^{2/3}\right]_b^8 = 48$



74. $y = \sqrt{16-x^2},\ 0 \le x \le 4$

$y' = \frac{-x}{\sqrt{16-x^2}}$

$s = \int_0^4\sqrt{1+\frac{x^2}{16-x^2}}\,dx = \int_0^4\frac{4}{\sqrt{16-x^2}}\,dx$

$= \lim_{t\to4^-}\int_0^t\frac{4}{\sqrt{16-x^2}}\,dx$

$= \lim_{t\to4^-}\left[4\arcsin\left(\frac{x}{4}\right)\right]_0^t$

$= \lim_{t\to4^-}4\arcsin\left(\frac{t}{4}\right) = 2\pi$

75. $(x-2)^2 + y^2 = 1$

$2(x-2) + 2yy' = 0$

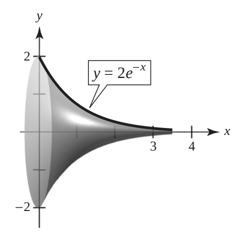
$y' = \dfrac{-(x-2)}{y}$

$\sqrt{1+(y')^2} = \sqrt{1+[(x-2)^2/y^2]} = \dfrac{1}{y}$ (Assume $y > 0$.)

$$S = 4\pi\int_1^3 \frac{x}{y}\,dx = 4\pi\int_1^3 \frac{x}{\sqrt{1-(x-2)^2}}\,dx = 4\pi\int_1^3\left[\frac{x-2}{\sqrt{1-(x-2)^2}} + \frac{2}{\sqrt{1-(x-2)^2}}\right]dx$$

$$= \lim_{\substack{a\to 1^+\\ b\to 3^-}}\left\{4\pi\left[-\sqrt{1-(x-2)^2} + 2\arcsin(x-2)\right]_a^b\right\} = 4\pi[0 + 2\arcsin(1) - 2\arcsin(-1)] = 8\pi^2$$

76. $y = 2e^{-x}$



$y' = -2e^{-x}$

$$S = 2\pi\int_0^\infty (2e^{-x})\sqrt{1+4e^{-2x}}\,dx$$

Let $u = e^{-x}$, $du = -e^{-x}\,dx$.

$$\int e^{-x}\sqrt{1+4e^{-2x}}\,dx = -\int\sqrt{1+4u^2}\,du$$

$$= -\frac{1}{4}\left[2u\sqrt{4u^2+1} + \ln\left|2u + \sqrt{4u^2+1}\right|\right] + C$$

$$= -\frac{1}{4}\left[2e^{-x}\sqrt{4e^{-2x}+1} + \ln\left|2e^{-x} + \sqrt{4e^{-2x}+1}\right|\right] + C$$

$$S = 4\pi\lim_{b\to\infty}\int_0^b (e^{-x})\sqrt{1+4e^{-2x}}\,dx$$

$$= -\pi\lim_{b\to\infty}\left[2e^{-x}\sqrt{4e^{-2x}+1} + \ln\left|2e^{-x} + \sqrt{4e^{-2x}+1}\right|\right]_0^b$$

$$= \pi\left[2\sqrt{5} + \ln\left(2+\sqrt{5}\right)\right] \approx 18.5849$$

77. (a) $F(x) = \dfrac{K}{x^2}$, $5 = \dfrac{K}{(4000)^2}$, $K = 80{,}000{,}000$

$$W = \int_{4000}^\infty \frac{80{,}000{,}000}{x^2}\,dx = \lim_{b\to\infty}\left[\frac{-80{,}000{,}000}{x}\right]_{4000}^b = 20{,}000 \text{ mi-ton}$$

(b) $$\frac{W}{2} = 10{,}000 = \left[\frac{-80{,}000{,}000}{x}\right]_{4000}^b = \frac{-80{,}000{,}000}{b} + 20{,}000$$

$$\frac{80{,}000{,}000}{b} = 10{,}000$$

$$b = 8000$$

Therefore, 4000 miles *above* the earth's surface.

78. (a) $F(x) = \dfrac{k}{x^2}$, $10 = \dfrac{k}{4000^2}$, $k = 10(4000^2)$

$$W = \int_{4000}^\infty \frac{10(4000^2)}{x^2}\,dx = \lim_{b\to\infty}\left[\frac{-10(4000^2)}{x}\right]_{4000}^b$$

$$= \frac{10(4000^2)}{4000} = 40{,}000 \text{ mi-ton}$$

(b) $$\frac{W}{2} = 20{,}000 = \left[\frac{-10(4000^2)}{x}\right]_{4000}^b = \frac{-10(4000^2)}{b} + 40{,}000$$

$$\frac{10(4000^2)}{b} = 20{,}000$$

$$b = 8000$$

Therefore, 4000 miles above the earth's surface.

79. (a) $\int_{-\infty}^{\infty} \frac{1}{7}e^{-t/7}\,dt = \int_{0}^{\infty} \frac{1}{7}e^{-t/7}\,dt = \lim_{b\to\infty}\left[-e^{-t/7}\right]_0^b = 1$

(b) $\int_0^4 \frac{1}{7}e^{-t/7}\,dt = \left[-e^{-t/7}\right]_0^4 = -e^{-4/7} + 1$

$\approx 0.4353 = 43.53\%$

(c) $\int_0^{\infty} t\left[\frac{1}{7}e^{-t/7}\right]dt = \lim_{b\to\infty}\left[-te^{-t/7} - 7e^{-t/7}\right]_0^b$

$= 0 + 7 = 7$

80. (a) $\int_{-\infty}^{\infty} \frac{2}{5}e^{-2t/5}\,dt = \int_{0}^{\infty} \frac{2}{5}e^{-2t/5}\,dt = \lim_{b\to\infty}\left[-e^{-2t5}\right]_0^b = 1$

(b) $\int_0^4 \frac{2}{5}e^{-2t/5}\,dt = \left[-e^{-2t/5}\right]_0^4 = -e^{-8/5} + 1$

$\approx 0.7981 = 79.81\%$

(c) $\int_0^{\infty} t\left[\frac{2}{5}e^{-2t/5}\right]dt = \lim_{b\to\infty}\left[-te^{2t/5} - \frac{5}{2}e^{-2t/5}\right]_0^b = \frac{5}{2}$

81. (a) $C = 650{,}000 + \int_0^5 25{,}000\,e^{-0.06t}\,dt = 650{,}000 - \left[\frac{25{,}000}{0.06}e^{-0.06t}\right]_0^5 \approx \$757{,}992.41$

(b) $C = 650{,}000 + \int_0^{10} 25{,}000e^{-0.06t}\,dt \approx \$837{,}995.15$

(c) $C = 650{,}000 + \int_0^{\infty} 25{,}000e^{-0.06t}\,dt = 650{,}000 - \lim_{b\to\infty}\left[\frac{25{,}000}{0.06}e^{-0.06t}\right]_0^b \approx \$1{,}066{,}666.67$

82. (a) $C = 650{,}000 + \int_0^5 25{,}000(1 + 0.08t)e^{-0.06t}\,dt$

$= 650{,}000 + 25{,}000\left[-\frac{1}{0.06}e^{-0.06t} - 0.08\left(\frac{t}{0.06}e^{-0.06t} + \frac{1}{(0.06)^2}e^{-0.06t}\right)\right]_0^5 \approx \$778{,}512.58$

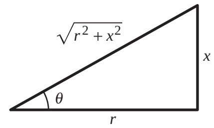
(b) $C = 650{,}000 + \int_0^{10} 25{,}000(1 + 0.08t)e^{-0.06t}\,dt$

$= 650{,}000 + 25{,}000\left[-\frac{1}{0.06}e^{-0.06t} - 0.08\left(\frac{t}{0.06}e^{-0.06t} + \frac{1}{(0.06)^2}e^{-0.06t}\right)\right]_0^{10} \approx \$905{,}718.14$

(c) $C = 650{,}000 + \int_0^{\infty} 25{,}000(1 + 0.08t)e^{-0.06t}\,dt$

$= 650{,}000 + 25{,}000\lim_{b\to\infty}\left[-\frac{t}{0.06}e^{-0.06t} - 0.08\left(\frac{t}{0.06}e^{-0.06t+}\frac{1}{(0.06)^2}e^{-0.06t}\right)\right]_0^b \approx \$1{,}622{,}222.22$

83. Let $K = \frac{2\pi NI r}{k}$. Then



$P = K\int_c^{\infty} \frac{1}{(r^2 + x^2)^{3/2}}\,dx.$

Let $x = r\tan\theta,\ dx = r\sec^2\theta\,d\theta,\ \sqrt{r^2 + x^2} = r\sec\theta.$

$\int \frac{1}{(r^2 + x^2)^{3/2}}\,dx = \int \frac{r\sec^2\theta\,d\theta}{r^3\sec^3\theta} = \frac{1}{r^2}\int \cos\theta\,d\theta$

$= \frac{1}{r^2}\sin\theta + C = \frac{1}{r^2}\frac{x}{\sqrt{r^2 + x^2}} + C$

Hence,

$P = K\frac{1}{r^2}\lim_{b\to\infty}\left[\frac{x}{\sqrt{r^2 + x^2}}\right]_c^b$

$= \frac{K}{r^2}\left[1 - \frac{c}{\sqrt{r^2 + c^2}}\right]$

$= \frac{K\left(\sqrt{r^2 + c^2} - c\right)}{r^2\sqrt{r^2 + c^2}}$

$= \frac{2\pi NI\left(\sqrt{r^2 + c^2} - c\right)}{kr\sqrt{r^2 + c^2}}.$

84. $F = \int_0^{\infty} \frac{GM\delta}{(a+x)^2}\,dx$

$= \lim_{b\to\infty}\left[\frac{-GM\delta}{a+x}\right]_0^b$

$= \frac{GM\delta}{a}$

85. False. $f(x) = 1/(x+1)$ is continuous on $[0, \infty)$, $\lim_{x\to\infty} 1/(x+1) = 0$, but $\int_0^{\infty} \frac{1}{x+1}\,dx = \lim_{b\to\infty}\left[\ln|x+1|\right]_0^b = \infty$.

Diverges

86. False. This is equivalent to Exercise 85.

87. True

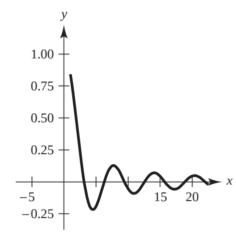
88. True

89. (a) $\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty}\left[\ln|x|\right]_1^b = \infty$

$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = 1$

$\int_1^{\infty} \frac{1}{x^n}\,dx$ will converge if $n > 1$ and will diverge if $n \le 1$.

(b) It would appear to converge.



(c) Let $dv = \sin x\,dx \implies v = -\cos x$

$u = \frac{1}{x} \implies du = -\frac{1}{x^2}\,dx.$

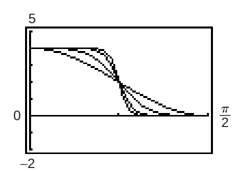
$\int_1^{\infty} \frac{\sin x}{x}\,dx = \lim_{b\to 0}\left[-\frac{\cos x}{x}\right]_1^b - \int_1^{\infty} \frac{\cos x}{x^2}\,dx$

$= \cos 1 - \int_1^{\infty} \frac{\cos x}{x^2}\,dx$

Converges

90. (a) Yes, the integrand is not defined at $x = \pi/2$.

(b)



(c) As $n \to \infty$, the integral approaches $4(\pi/4) = \pi$.

(d) $I_n = \int_0^{\pi/2} \frac{4}{1 + (\tan x)^n}\,dx$

$I_2 \approx 3.14159$

$I_4 \approx 3.14159$

$I_8 \approx 3.14159$

$I_{12} \approx 3.14159$

91. $\Gamma(n) = \int_0^{\infty} x^{n-1}e^{-x}\,dx$

(a) $\Gamma(1) = \int_0^{\infty} e^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-x}\right]_0^b = 1$

$\Gamma(2) = \int_0^{\infty} xe^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-x}(x+1)\right]_0^b = 1$

$\Gamma(3) = \int_0^{\infty} x^2e^{-x}\,dx = \lim_{b\to\infty}\left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^b = 2$

(b) $\Gamma(n+1) = \int_0^{\infty} x^n e^{-x}\,dx = \lim_{b\to\infty}\left[-x^n e^{-x}\right]_0^b + \lim_{b\to\infty} n\int_0^b x^{n-1}e^{-x}\,dx = 0 + n\Gamma(n) \quad (u = x^n,\ dv = e^{-x}\,dx)$

(c) $\Gamma(n) = (n-1)!$

92. For $n = 1$,

$$I_1 = \int_0^\infty \frac{x}{(x^2+1)^4}\,dx = \lim_{b\to\infty} \frac{1}{2}\int_0^b (x^2+1)^{-4}(2x\,dx) = \lim_{b\to\infty}\left[-\frac{1}{6}\frac{1}{(x^2+1)^3}\right]_0^b = \frac{1}{6}.$$

For $n > 1$,

$$I_n = \int_0^\infty \frac{x^{2n-1}}{(x^2+1)^{n+3}}\,dx = \lim_{b\to\infty}\left[\frac{-x^{2n-2}}{2(n+2)(x^2+1)^{n+2}}\right]_0^b + \frac{n-1}{n+2}\int_0^\infty \frac{x^{2n-3}}{(x^2+1)^{n+2}}\,dx = 0 + \frac{n-1}{n+2}(I_{n-1})$$

$$\left(\text{Parts: } u = x^{2n-2},\ du = (2n-2)x^{2n-3}\,dx,\ dv = \frac{x}{(x^2+1)^{n+3}}\,dx,\ v = \frac{-1}{2(n+2)(x^2+1)^{n+2}}\right)$$

(a) $\displaystyle\int_0^\infty \frac{x}{(x^2+1)^4}\,dx = \lim_{b\to\infty}\left[-\frac{1}{6(x^2+1)^3}\right]_0^b = \frac{1}{6}$

(b) $\displaystyle\int_0^\infty \frac{x^3}{(x^2+1)^5}\,dx = \frac{1}{4}\int_0^\infty \frac{x}{(x^2+1)^4}\,dx = \frac{1}{4}\left(\frac{1}{6}\right) = \frac{1}{24}$

(c) $\displaystyle\int_0^\infty \frac{x^5}{(x^2+1)^6} = \frac{2}{5}\int_0^\infty \frac{x^3}{(x^2+1)^5}\,dx = \frac{2}{5}\left(\frac{1}{24}\right) = \frac{1}{60}$

93. $f(t) = 1$

$$F(s) = \int_0^\infty e^{-st}\,dt = \lim_{b\to\infty}\left[-\frac{1}{s}e^{-st}\right]_0^b = \frac{1}{s},\ s > 0$$

94. $f(t) = t$

$$F(s) = \int_0^\infty te^{-st}\,dt = \lim_{b\to\infty}\left[\frac{1}{s^2}(-st-1)e^{-st}\right]_0^b$$

$$= \frac{1}{s^2},\ s > 0$$

95. $f(t) = t^2$

$$F(s) = \int_0^\infty t^2e^{-st}\,dt = \lim_{b\to\infty}\left[\frac{1}{s^3}(-s^2t^2 - 2st - 2)e^{-st}\right]_0^b$$

$$= \frac{2}{s^3},\ s > 0$$

96. $f(t) = e^{at}$

$$F(s) = \int_0^\infty e^{at}e^{-st}\,dt = \int_0^\infty e^{t(a-s)}\,dt$$

$$= \lim_{b\to\infty}\left[\frac{1}{a-s}e^{t(a-s)}\right]_0^b$$

$$= 0 - \frac{1}{a-s} = \frac{1}{s-a},\ s > a$$

97. $f(t) = \cos at$

$$F(s) = \int_0^\infty e^{-st}\cos at\,dt$$

$$= \lim_{b\to\infty}\left[\frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin at)\right]_0^b$$

$$= 0 + \frac{s}{s^2+a^2} = \frac{s}{s^2+a^2},\ s > 0$$

98. $f(t) = \sin at$

$$F(s) = \int_0^\infty e^{-st}\sin at\,dt$$

$$= \lim_{b\to\infty}\left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^b$$

$$= 0 + \frac{a}{s^2+a^2} = \frac{a}{s^2+a^2},\ s > 0$$

99. $f(t) = \cosh at$

$$F(s) = \int_0^\infty e^{-st}\cosh at\,dt = \int_0^\infty e^{-st}\left(\frac{e^{at}+e^{-at}}{2}\right)dt = \frac{1}{2}\int_0^\infty \left[e^{t(-s+a)} + e^{t(-s-a)}\right]dt$$

$$= \lim_{b\to\infty}\frac{1}{2}\left[\frac{1}{(-s+a)}e^{t(-s+a)} + \frac{1}{(-s-a)}e^{t(-s-a)}\right]_0^b = 0 - \frac{1}{2}\left[\frac{1}{(-s+a)} + \frac{1}{(-s-a)}\right]$$

$$= \frac{-1}{2}\left[\frac{1}{(-s+a)} + \frac{1}{(-s-a)}\right] = \frac{s}{s^2-a^2},\ s > |a|$$

100. $f(t) = \sinh at$

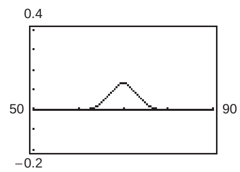
$$F(s) = \int_0^{\infty} e^{-st} \sinh at\, dt = \int_0^{\infty} e^{-st}\left(\frac{e^{at} - e^{-at}}{2}\right) dt = \frac{1}{2}\int_0^{\infty} \left[e^{t(-s+a)} - e^{t(-s-a)}\right] dt$$

$$= \lim_{b\to\infty} \frac{1}{2}\left[\frac{1}{(-s+a)}e^{t(-s+a)} - \frac{1}{(-s-a)}e^{t(-s-a)}\right]_0^b = 0 - \frac{1}{2}\left[\frac{1}{(-s+a)} - \frac{1}{(-s-a)}\right]$$

$$= \frac{-1}{2}\left[\frac{1}{(-s+a)} - \frac{1}{(-s-a)}\right] = \frac{a}{s^2 - a^2}, s > |a|$$

101. (a) $f(x) = \dfrac{1}{3\sqrt{2\pi}} e^{-(x-70)^2/18}$

$$\int_{50}^{90} f(x)\, dx \approx 1.0$$



(b) $P(72 \le x < \infty) \approx 0.2525$

(c) $0.5 - P(70 \le x \le 72) \approx 0.5 - 0.2475 = 0.2525$

These are the same answers because by symmetry,

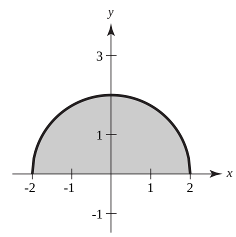
$P(70 \le x < \infty) = 0.5$

and

$0.5 = P(70 \le x < \infty)$

$= P(70 \le x \le 72) + P(72 \le x < \infty).$

102. (a)



(b) Area $= \dfrac{1}{2}\pi(2)^2 = 2\pi$

Arc length is also $\dfrac{1}{2}(2\pi(2)) = 2\pi.$

Hence, the corresponding integrals are equal.

Let $y = \sqrt{4 - x^2}, y' = \dfrac{-x}{\sqrt{4 - x^2}}$

$1 + (y')^2 = \dfrac{4}{4 - x^2} \Rightarrow \sqrt{1 + (y')^2} = \dfrac{2}{\sqrt{4 - x^2}}.$

Thus, $\displaystyle\int_{-2}^{2} \sqrt{4 - x^2}\, dx = \int_{-2}^{2} \frac{2}{\sqrt{4 - x^2}}\, dx.$

(area) (arc length)

103. $$\int_0^{\infty}\left(\frac{1}{\sqrt{x^2+1}} - \frac{c}{x+1}\right) dx = \lim_{b\to\infty}\int_0^b \left(\frac{1}{\sqrt{x^2+1}} - \frac{c}{x+1}\right) dx$$

$$= \lim_{b\to\infty}\left[\ln\left|x + \sqrt{x^2+1}\right| - c\ln|x+1|\right]_0^b$$

$$= \lim_{b\to\infty}\left[\ln\left(b + \sqrt{b^2+1}\right) - \ln(b+1)^c\right] = \lim_{b\to\infty} \ln\left[\frac{b + \sqrt{b^2+1}}{(b+1)^c}\right]$$

This limit exists for $c = 1$, and you have

$$\lim_{b\to\infty} \ln\left[\frac{b + \sqrt{b^2+1}}{(b+1)}\right] = \ln 2.$$

104. $$\int_1^{\infty}\left(\frac{cx}{x^2+2} - \frac{1}{3x}\right) dx = \lim_{b\to\infty}\int_1^b \left(\frac{cx}{x^2+2} - \frac{1}{3x}\right) dx$$

$$= \lim_{b\to\infty}\left[\frac{c}{2}\ln(x^2+2) - \frac{1}{3}\ln|x|\right]_1^b = \lim_{b\to\infty} \ln\left[\frac{(x^2+2)^{c/2}}{x^{1/3}}\right]_1^b = \lim_{b\to\infty}\left[\ln\frac{(b^2+2)^{c/2}}{b^{1/3}} - \ln 3^{c/2}\right]$$

This limit exists if $c = 1/3$, and you have

$$\lim_{b\to\infty}\left[\ln\frac{(b^2+2)^{1/6}}{b^{1/3}} - \ln 3^{1/6}\right] = -\ln 3^{1/6} = \frac{-\ln 3}{6}.$$

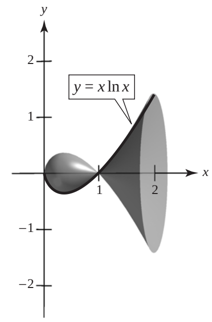
105. $f(x) = \begin{cases} x \ln x, & 0 < x \le 2 \\ 0, & x = 0 \end{cases}$

$$V = \pi \int_0^2 (x \ln x)^2\, dx$$

Let $u = \ln x$, $e^u = x$, $e^u\, du = dx$.

$$V = \pi \int_{-\infty}^{\ln 2} e^{2u} u^2 (e^4\, du) = \pi \int_{-\infty}^{\ln 2} e^{3u} u^2\, du$$

$$= \lim_{b \to -\infty} \left[\pi \left[\frac{u^2}{3} - \frac{2u}{9} + \frac{2}{27} \right] e^{3u} \right]_b^{\ln 2} = \pi \left[\frac{(\ln 2)^2}{3} - \frac{2 \ln 2}{9} + \frac{2}{27} \right] 8 \approx 2.0155$$



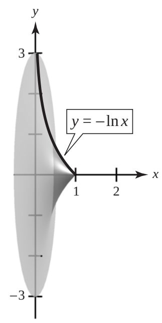
106. $V = \pi \int_0^1 (-\ln x)^2\, dx$

$$= \lim_{b \to 0^+} \pi \int_b^1 (\ln x)^2\, dx$$

$$= \lim_{b \to 0^+} \pi x \left[(\ln x)^2 - 2 \ln x + 2 \right]_b^1$$

$$= \lim_{b \to 0^+} \pi [2 - b(\ln b)^2 - 2b \ln b - 2b]$$

$$= 2\pi$$



107. $u = \sqrt{x}$, $u^2 = x$, $2u\, du = dx$

$$\int_0^1 \frac{\sin x}{\sqrt{x}}\, dx = \int_0^1 \frac{\sin(u^2)}{u} (2u\, du) = \int_0^1 2 \sin(u^2)\, du$$

Trapezoidal Rule $(n = 5)$: 0.6278

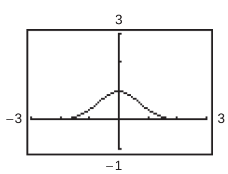
108. $u = \sqrt{1 - x}$, $1 - x = u^2$, $2u\, du = -dx$

$$\int_0^1 \frac{\cos x}{\sqrt{1 - x}}\, dx = \int_1^0 \frac{\cos(1 - u^2)}{u} (-2u\, du)$$

$$= \int_0^1 2 \cos(1 - u^2)\, du$$

Trapezoidal Rule $(n = 5)$: 1.4997

109. (a)



(b) Let $y = e^{-x^2}$, $0 \le x < \infty$.

$\ln y = -x^2$

$x = \sqrt{-\ln y}$ for $0 < y \le 1$

The area bounded by $y = e^{-x^2}$, $x = 0$ and $y = 0$ is

$$\int_0^\infty e^{-x^2}\, dx = \int_0^1 \sqrt{-\ln y}\, dy, \quad \left(= \frac{\sqrt{\pi}}{2} \right).$$

110. Assume $a < b$. The proof is similar if $a > b$.

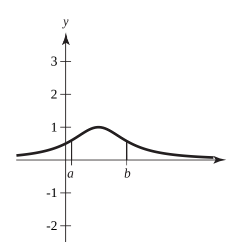
$$\int_{-\infty}^a f(x)\, dx + \int_a^\infty f(x)\, dx = \lim_{c \to -\infty} \int_c^a f(x)\, dx + \lim_{d \to \infty} \int_a^d f(x)\, dx$$

$$= \lim_{c \to -\infty} \int_c^a f(x)\, dx + \lim_{d \to \infty} \left[\int_a^b f(x)\, dx + \int_b^d f(x)\, dx \right]$$

$$= \lim_{c \to -\infty} \int_c^a f(x)\, dx + \int_a^b f(x)\, dx + \lim_{d \to \infty} \int_b^d f(x)\, dx$$

$$= \lim_{c \to -\infty} \left[\int_c^a f(x)\, dx + \int_a^b f(x)\, dx \right] + \lim_{d \to \infty} \int_b^d f(x)\, dx = \lim_{c \to -\infty} \int_c^b f(x)\, dx + \lim_{d \to \infty} \int_b^d f(x)\, dx$$

$$= \int_{-\infty}^b f(x)\, dx + \int_b^\infty f(x)\, dx$$



Review Exercises for Chapter 8

1. $$\int x\sqrt{x^2-1}\,dx = \frac{1}{2}\int (x^2-1)^{1/2}(2x)\,dx = \frac{1}{2}\frac{(x^2-1)^{3/2}}{3/2} + C = \frac{1}{3}(x^2-1)^{3/2} + C$$

2. $$\int xe^{x^2-1}\,dx = \frac{1}{2}\int e^{x^2-1}(2x)\,dx = \frac{1}{2}e^{x^2-1} + C$$

3. $$\int \frac{x}{x^2-1}\,dx = \frac{1}{2}\int \frac{2x}{x^2-1}\,dx = \frac{1}{2}\ln\left|x^2-1\right| + C$$

4. $$\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int (1-x^2)^{-1/2}(-2x)\,dx = -\frac{1}{2}\frac{(1-x^2)^{1/2}}{1/2} + C = -\sqrt{1-x^2} + C$$

5. Let $u = \ln(2x)$, $du = \frac{1}{x}\,dx$.

$$\int_1^e \frac{\ln(2x)}{x}\,dx = \int_{\ln 2}^{1+\ln 2} u\,du = \left.\frac{u^2}{2}\right]_{\ln 2}^{1+\ln 2} = \frac{1}{2}\left[1 + 2\ln 2 + (\ln 2)^2 - (\ln 2)^2\right] = \frac{1}{2} + \ln 2 \approx 1.1931$$

6. Let $u = 2x - 3$, $du = 2\,dx$, $x = \frac{1}{2}(u+3)$.

$$\int_{3/2}^2 2x\sqrt{2x-3}\,dx = \int_0^1 (u+3)u^{1/2}\frac{1}{2}\,du = \frac{1}{2}\int_0^1 (u^{3/2} + 3u^{1/2})\,du = \frac{1}{2}\left[\frac{2}{5}u^{5/2} + 2u^{3/2}\right]_0^1 = \frac{1}{2}\left[\frac{2}{5} + 2\right] = \frac{6}{5}$$

7. $$\int \frac{16}{\sqrt{16-x^2}}\,dx = 16\arcsin\left(\frac{x}{4}\right) + C$$

8. $$\frac{x^4 + 2x^2 + x + 1}{x^4 + 2x^2 + 1} = 1 + \frac{x}{(x^2+1)^2}$$

$$\int \frac{x^4 + 2x^2 + x + 1}{(x^2+1)^2}\,dx = \int dx + \frac{1}{2}\int \frac{2x}{(x^2+1)^2}\,dx = x - \frac{1}{2(x^2+1)} + C$$

9. $$\int e^{2x}\sin 3x\,dx = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\int e^{2x}\cos 3x\,dx$$

$$= -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\left(\frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\int e^{2x}\sin 3x\,dx\right)$$

$$\frac{13}{9}\int e^{2x}\sin 3x\,dx = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x$$

$$\int e^{2x}\sin 3x\,dx = \frac{e^{2x}}{13}(2\sin 3x - 3\cos 3x) + C$$

(1) $dv = \sin 3x\,dx \implies v = -\frac{1}{3}\cos 3x$

$u = e^{2x} \implies du = 2e^{2x}\,dx$

(2) $dv = \cos 3x\,dx \implies v = \frac{1}{3}\sin 3x$

$u = e^{2x} \implies du = 2e^{2x}\,dx$

10. $\int (x^2 - 1)e^x\,dx = (x^2 - 1)e^x - 2\int xe^x\,dx = (x^2 - 1)e^x - 2xe^x + 2\int e^x\,dx = e^x(x^2 - 2x + 1) + 1$

(1) $dv = e^x\,dx \implies v = e^x$ (2) $dv = e^x\,dx \implies v = e^x$

$u = x^2 - 1 \implies du = 2x\,dx$ $u = x \implies du = dx$

11. $u = x,\ du = dx,\ dv = (x - 5)^{1/2}\,dx,\ v = \frac{2}{3}(x - 5)^{3/2}$

$$\int x\sqrt{x - 5}\,dx = \frac{2}{3}x(x - 5)^{3/2} - \int \frac{2}{3}(x - 5)^{3/2}\,dx$$
$$= \frac{2}{3}x(x - 5)^{3/2} - \frac{4}{15}(x - 5)^{5/2} + C$$
$$= (x - 5)^{3/2}\left[\frac{2}{3}x - \frac{4}{15}(x - 5)\right] + C$$
$$= (x - 5)^{3/2}\left[\frac{6}{15}x + \frac{4}{3}\right] + C$$
$$= \frac{2}{15}(x - 5)^{3/2}[3x + 10] + C$$

12. $u = \arctan 2x,\ du = \frac{2}{1 + 4x^2}\,dx,\ dv = dx, v = x$

$$\int \arctan 2x\,dx = x\arctan 2x - \int \frac{2x}{1 + 4x^2}\,dx$$
$$= x\arctan 2x - \frac{1}{4}\ln(1 + 4x^2) + C$$

13. $\int x^2 \sin 2x\,dx = -\frac{1}{2}x^2\cos 2x + \int x\cos 2x\,dx$

$$= -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x\,dx$$
$$= -\frac{1}{2}x^2\cos 2x + \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + C$$

(1) $dv = \sin 2x\,dx \implies v = -\frac{1}{2}\cos 2x$

$u = x^2 \implies du = 2x\,dx$

(2) $dv = \cos 2x\,dx \implies v = \frac{1}{2}\sin 2x$

$u = x \implies du = dx$

14. $\int \ln\sqrt{x^2 - 1}\,dx = \frac{1}{2}\int \ln(x^2 - 1)\,dx$

$$= \frac{1}{2}x\ln|x^2 - 1| - \int \frac{x^2}{x^2 - 1}\,dx$$
$$= \frac{1}{2}x\ln|x^2 - 1| - \int dx - \int \frac{1}{x^2 - 1}\,dx$$
$$= \frac{1}{2}x\ln|x^2 - 1| - x - \frac{1}{2}\ln\left|\frac{x - 1}{x + 1}\right| + C$$

$dv = dx \implies v = x$

$u = \ln(x^2 - 1) \implies du = \frac{2x}{x^2 - 1}\,dx$

15. $\int x\arcsin 2x\,dx = \frac{x^2}{2}\arcsin 2x - \int \frac{x^2}{\sqrt{1 - 4x^2}}\,dx$

$$= \frac{x^2}{2}\arcsin 2x - \frac{1}{8}\int \frac{2(2x)^2}{\sqrt{1 - (2x)^2}}\,dx$$
$$= \frac{x^2}{2}\arcsin 2x - \frac{1}{8}\left(\frac{1}{2}\right)\left[-(2x)\sqrt{1 - 4x^2} + \arcsin 2x\right] + C$$ (by Formula 43 of Integration Tables)
$$= \frac{1}{16}\left[(8x^2 - 1)\arcsin 2x + 2x\sqrt{1 - 4x^2}\right] + C$$

$dv = x\,dx \implies v = \frac{x^2}{2}$

$u = \arcsin 2x \implies du = \frac{2}{\sqrt{1 - 4x^2}}\,dx$

16. $\displaystyle\int e^x \arctan(e^x)\,dx = e^x \arctan(e^x) - \int \frac{e^{2x}}{1+e^{2x}}\,dx$

$\displaystyle= e^x \arctan(e^x) - \frac{1}{2}\ln(1 + e^{2x}) + C$

$dv = e^x\,dx \quad \Longrightarrow \quad v = e^x$

$\displaystyle u = \arctan e^x \Longrightarrow du = \frac{e^x}{1+e^{2x}}\,dx$

17. $\displaystyle\int \cos^3(\pi x - 1)\,dx = \int [1 - \sin^2(\pi x - 1)]\cos(\pi x - 1)\,dx$

$\displaystyle= \frac{1}{\pi}\left[\sin(\pi x - 1) - \frac{1}{3}\sin^3(\pi x - 1)\right] + C$

$\displaystyle= \frac{1}{3\pi}\sin(\pi x - 1)[3 - \sin^2(\pi x - 1)] + C$

$\displaystyle= \frac{1}{3\pi}\sin(\pi x - 1)[3 - (1 - \cos^2(\pi x - 1))] + C$

$\displaystyle= \frac{1}{3\pi}\sin(\pi x - 1)[2 + \cos^2(\pi x - 1)] + C$

18. $\displaystyle\int \sin^2\frac{\pi x}{2}\,dx = \int \frac{1}{2}(1 - \cos \pi x)\,dx = \frac{1}{2}\left[x - \frac{1}{\pi}\sin \pi x\right] + C = \frac{1}{2\pi}[\pi x - \sin \pi x] + C$

19. $\displaystyle\int \sec^4\left(\frac{x}{2}\right)dx = \int\left[\tan^2\left(\frac{x}{2}\right) + 1\right]\sec^2\left(\frac{x}{2}\right)dx$

$\displaystyle= \int \tan^2\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)dx + \int \sec^2\left(\frac{x}{2}\right)dx$

$\displaystyle= \frac{2}{3}\tan^3\left(\frac{x}{2}\right) + 2\tan\left(\frac{x}{2}\right) + C = \frac{2}{3}\left[\tan^3\left(\frac{x}{2}\right) + 3\tan\left(\frac{x}{2}\right)\right] + C$

20. $\displaystyle\int \tan\theta \sec^4\theta\,d\theta = \int (\tan^3\theta + \tan\theta)\sec^2\theta\,d\theta = \frac{1}{4}\tan^4\theta + \frac{1}{2}\tan^2\theta + C_1$

or

$\displaystyle\int \tan\theta\sec^4\theta\,d\theta = \int \sec^3\theta(\sec\theta\tan\theta)\,d\theta = \frac{1}{4}\sec^4\theta + C_2$

21. $\displaystyle\int \frac{1}{1-\sin\theta}\,d\theta = \int \frac{1}{1-\sin\theta}\cdot\frac{1+\sin\theta}{1+\sin\theta}\,d\theta = \int \frac{1+\sin\theta}{\cos^2\theta}\,d\theta = \int (\sec^2\theta + \sec\theta\tan\theta)\,d\theta = \tan\theta + \sec\theta + C$

22. $\displaystyle\int \cos 2\theta(\sin\theta + \cos\theta)^2\,d\theta = \int (\cos^2\theta - \sin^2\theta)(\sin\theta + \cos\theta)^2\,d\theta$

$\displaystyle= \int (\sin\theta + \cos\theta)^3(\cos\theta - \sin\theta)\,d\theta = \frac{1}{4}(\sin\theta + \cos\theta)^4 + C$

23. $A = \int_{\pi/4}^{3\pi/4} \sin^4 x\,dx$. Using the Table of Integrals,

$$\int \sin^4 x\,dx = -\frac{\sin^3 x\cos x}{4} + \frac{3}{4}\int \sin^2 x\,dx$$

$$= \frac{-\sin^3 x\cos x}{4} + \frac{3}{4}\left[\frac{1}{2}(x - \sin x\cos x)\right] + C$$

$$\int_{\pi/4}^{3\pi/4} \sin^4 x\,dx = \left[\frac{-\sin^3 x\cos x}{4} + \frac{3}{8}x - \frac{3}{8}\sin x\cos x\right]_{\pi/4}^{3\pi/4}$$

$$= \left(\frac{1}{16} + \frac{9\pi}{32} + \frac{3}{16}\right) - \left(\frac{-1}{16} + \frac{3\pi}{32} - \frac{3}{16}\right)$$

$$= \frac{3\pi}{16} + \frac{1}{2} \approx 1.0890$$

24. $A = \int_0^{\pi/6} \cos(3x)\cos x\,dx$

$$= \int_0^{\pi/6} \frac{1}{2}[\cos 2x + \cos 4x]\,dx$$

$$= \left[\frac{\sin 2x}{4} + \frac{\sin 4x}{8}\right]_0^{\pi/6}$$

$$= \left(\frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{16}\right) - 0$$

$$= \frac{3\sqrt{3}}{16}$$

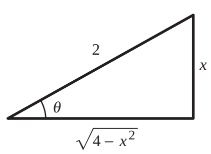
25. $\displaystyle\int \frac{-12}{x^2\sqrt{4-x^2}}\,dx = \int \frac{-24\cos\theta\,d\theta}{(4\sin^2\theta)(2\cos\theta)}$

$$= -3\int \csc^2\theta\,d\theta$$

$$= 3\cot\theta + C$$

$$= \frac{3\sqrt{4-x^2}}{x} + C$$

$x = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$, $\sqrt{4-x^2} = 2\cos\theta$



26. $\displaystyle\int \frac{\sqrt{x^2-9}}{x}\,dx = \int \frac{3\tan\theta}{3\sec\theta}(3\sec\theta\tan\theta\,d\theta)$

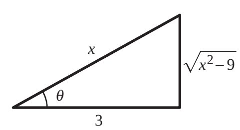
$$= 3\int \tan^2\theta\,d\theta$$

$$= 3\int (\sec^2\theta - 1)\,d\theta$$

$$= 3(\tan\theta - \theta) + C$$

$$= \sqrt{x^2-9} - 3\,\text{arcsec}\left(\frac{x}{3}\right) + C$$

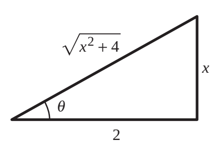
$x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-9} = 3\tan\theta$



27. $x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$4 + x^2 = 4\sec^2\theta$



$$\int \frac{x^3}{\sqrt{4+x^2}}\,dx = \int \frac{8\tan^3\theta}{2\sec\theta}\,2\sec^2\theta\,d\theta$$

$$= 8\int \tan^3\theta\sec\theta\,d\theta$$

$$= 8\int (\sec^2\theta - 1)\tan\theta\sec\theta\,d\theta$$

$$= 8\left[\frac{\sec^3\theta}{3} - \sec\theta\right] + C$$

$$= 8\left[\frac{(x^2+4)^{3/2}}{24} - \frac{\sqrt{x^2+4}}{2}\right] + C$$

$$= \sqrt{x^2+4}\left[\frac{1}{3}(x^2+4) - 4\right] + C$$

$$= \frac{1}{3}x^2\sqrt{x^2+4} - \frac{8}{3}\sqrt{x^2+4} + C$$

$$= \frac{1}{3}(x^2+4)^{1/2}(x^2-8) + C$$

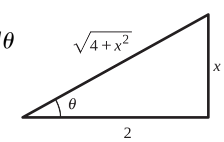
28. $$\int \sqrt{9-4x^2}\,dx = \frac{1}{2}\int \sqrt{9-(2x)^2}\,(2)\,dx$$
$$= \frac{1}{2}\cdot\frac{1}{2}\left[9\arcsin\frac{2x}{3} + 2x\sqrt{9-4x^2}\right] + C$$
$$= \frac{9}{4}\arcsin\frac{2x}{3} + \frac{x}{2}\sqrt{9-4x^2} + C$$

29. $$\int_{-2}^{0}\sqrt{4-x^2}\,dx = \frac{1}{2}\left[4\arcsin\left(\frac{x}{2}\right) + x\sqrt{4-x^2}\right]_{-2}^{0}$$
$$= \frac{1}{2}[0 - 4\arcsin(-1)]$$
$$= \frac{1}{2}\left[-4\left(\frac{-\pi}{2}\right)\right]$$
$$= \pi$$

Note: The integral represents the area of a quarter circle of radius 2: $A = \frac{1}{4}(\pi 2^2) = \pi$.

30. Let $u = \cos\theta$, $du = -\sin\theta\,d\theta$.
$$\int_0^{\pi/2}\frac{\sin\theta}{1+2\cos^2\theta}\,d\theta = \int_1^0 \frac{1}{1+2u^2}(-du)$$
$$= \int_0^1 \frac{1}{1+2u^2}\,du$$
$$= \frac{1}{2}\int_0^1 \frac{1}{(1/2)+u^2}\,du, \qquad a = \frac{1}{\sqrt{2}}$$
$$= \frac{1}{2}\sqrt{2}\arctan\left(\sqrt{2}u\right)\Big]_0^1$$
$$= \frac{\sqrt{2}}{2}\arctan\sqrt{2}$$

31. (a) Let $x = 2\tan\theta$, $dx = 2\sec^2\theta\,d\theta$.
$$\int\frac{x^3}{\sqrt{4+x^2}}\,dx = \int\frac{8\tan^3\theta}{2\sec\theta}2\sec^2\theta\,d\theta$$
$$= 8\int\tan^3\theta\sec\theta\,d\theta$$



$$= 8\int\frac{\sin^3\theta}{\cos^4\theta}\,d\theta$$
$$= 8\int(1-\cos^2\theta)\cos^{-4}\theta\sin\theta\,d\theta$$
$$= 8\int(\cos^{-4}\theta - \cos^{-2}\theta)\sin\theta\,d\theta$$
$$= 8\left[\frac{\cos^{-3}\theta}{3} - \frac{\cos^{-1}\theta}{-1}\right] + C$$
$$= \frac{8}{3}\sec\theta(\sec^2\theta - 3) + C$$
$$= \frac{8}{3}\frac{\sqrt{4+x^2}}{2}\left(\frac{4+x^2}{4} - 3\right) + C$$
$$= \frac{1}{3}\sqrt{4+x^2}\,(x^2-8) + C$$

(b) $$\int\frac{x^3}{\sqrt{4+x^2}}\,dx = \int\frac{x^2}{\sqrt{4+x^2}}\,x\,dx$$
$$= \int\frac{(u^2-4)u\,du}{u}$$
$$= \int(u^2-4)\,du$$
$$= \frac{1}{3}u^3 - 4u + C$$
$$= \frac{u}{3}(u^2-12) + C$$
$$= \frac{\sqrt{4+x^2}}{3}(x^2-8) + C$$

$u^2 = 4 + x^2$, $2u\,du = 2x\,dx$

(c) $$\int\frac{x^3}{\sqrt{4+x^2}}\,dx = x^2\sqrt{4+x^2} - \int 2x\sqrt{4+x^2}\,dx$$
$$= x^2\sqrt{4+x^2} - \frac{2}{3}(4+x^2)^{3/2} + C = \frac{\sqrt{4+x^2}}{3}(x^2-8) + C$$

$$dv = \frac{x}{\sqrt{4+x^2}}\,dx \implies v = \sqrt{4+x^2}$$
$$u = x^2 \implies du = 2x\,dx$$

32. (a) $$\int x\sqrt{4+x}\,dx = 64\int \tan^3\theta\sec^3\theta\,d\theta$$
$$= 64\int (\sec^4\theta - \sec^2\theta)\sec\theta\tan\theta\,d\theta$$
$$= \frac{64\sec^3\theta}{15}(3\sec^3\theta - 5) + C$$
$$= \frac{2(4+x)^{3/2}}{15}(3x-8) + C$$

$x = 4\tan^2\theta,\ dx = 8\tan\theta\sec^2\theta\,d\theta,$

$\sqrt{4+x} = 2\sec\theta$

(b) $$\int x\sqrt{4+x}\,dx = 2\int (u^4 - 4u^2)\,du$$
$$= \frac{2u^3}{15}(3u^2 - 20) + C$$
$$= \frac{2(4+x)^{3/2}}{15}(3x-8) + C$$

$u^2 = 4 + x,\ dx = 2u\,du$

(c) $$\int x\sqrt{4+x}\,dx = \int (u^{3/2} - 4u^{1/2})\,du$$
$$= \frac{2u^{3/2}}{15}(3u - 20) + C$$
$$= \frac{2(4+x)^{3/2}}{15}(3x-8) + C$$

$u = 4 + x,\ du = dx$

(d) $$\int x\sqrt{4+x}\,dx = \frac{2x}{3}(4+x)^{3/2} - \frac{2}{3}\int (4+x)^{3/2}\,dx$$
$$= \frac{2x}{3}(4+x)^{3/2} - \frac{4}{15}(4+x)^{5/2} + C$$
$$= \frac{2(4+x)^{3/2}}{15}(3x-8) + C$$

$dv = \sqrt{4+x}\,dx \implies v = \frac{2}{3}(4+x)^{3/2}$

$u = x \implies du = dx$

33. $$\frac{x-28}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$$

$x - 28 = A(x+2) + B(x-3)$

$x = -2 \implies -30 = B(-5) \implies B = 6$

$x = 3 \implies -25 = A(5) \implies A = -5$

$$\int \frac{x-28}{x^2-x-6}\,dx = \int\left(\frac{-5}{x-3} + \frac{6}{x+2}\right)dx = -5\ln|x-3| + 6\ln|x+2| + C$$

34. $$\frac{2x^3 - 5x^2 + 4x - 4}{x^2 - x} = 2x - 3 + \frac{4}{x} - \frac{3}{x-1}$$

$$\int \frac{2x^3 - 5x^2 + 4x - 4}{x^2 - x}\,dx = \int\left(2x - 3 + \frac{4}{x} - \frac{3}{x-1}\right)dx = x^2 - 3x + 4\ln|x| - 3\ln|x-1| + C$$

35. $$\frac{x^2+2x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$x^2 + 2x = A(x^2+1) + (Bx+C)(x-1)$

Let $x = 1$: $3 = 2A \implies A = \frac{3}{2}$ Let $x = 0$: $0 = A - C \implies C = \frac{3}{2}$ Let $x = 2$: $8 = 5A + 2B + C \implies B = -\frac{1}{2}$

$$\int \frac{x^2+2x}{x^3 - x^2 + x - 1}\,dx = \frac{3}{2}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{x-3}{x^2+1}\,dx$$
$$= \frac{3}{2}\int \frac{1}{x-1}\,dx - \frac{1}{4}\int \frac{2x}{x^2+1}\,dx + \frac{3}{2}\int \frac{1}{x^2+1}\,dx$$
$$= \frac{3}{2}\ln|x-1| - \frac{1}{4}\ln|x^2+1| + \frac{3}{2}\arctan x + C$$
$$= \frac{1}{4}\left[6\ln|x-1| - \ln(x^2+1) + 6\arctan x\right] + C$$

36. $\dfrac{4x-2}{3(x-1)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2}$

$4x - 2 = 3A(x-1) + 3B$

Let $x = 1$: $2 = 3B \implies B = \dfrac{2}{3}$

Let $x = 2$: $6 = 3A + 3B \implies A = \dfrac{4}{3}$

$$\int \frac{4x-2}{3(x-1)^2}\,dx = \frac{4}{3}\int \frac{1}{x-1}\,dx + \frac{2}{3}\int \frac{1}{(x-1)^2}\,dx = \frac{4}{3}\ln|x-1| - \frac{2}{3(x-1)} + C = \frac{2}{3}\left(2\ln|x-1| - \frac{1}{x-1}\right) + C$$

37. $\dfrac{x^2}{x^2+2x-15} = 1 + \dfrac{15-2x}{x^2+2x-15}$

$\dfrac{15-2x}{(x-3)(x+5)} = \dfrac{A}{x-3} + \dfrac{B}{x+5}$

$15 - 2x = A(x+5) + B(x-3)$

Let $x = 3$: $9 = 8A \implies A = \dfrac{9}{8}$

Let $x = -5$: $25 = -8B \implies B = -\dfrac{25}{8}$

$$\int \frac{x^2}{x^2+2x-15}\,dx = \int dx + \frac{9}{8}\int \frac{1}{x-3}\,dx - \frac{25}{8}\int \frac{1}{x+5}\,dx = x + \frac{9}{8}\ln|x-3| - \frac{25}{8}\ln|x+5| + C$$

38. $$\int \frac{\sec^2\theta}{\tan\theta(\tan\theta-1)}\,d\theta = \int \frac{1}{u(u-1)}\,du = \int \frac{1}{u-1}\,du - \int \frac{1}{u}\,du$$

$$= \ln|u-1| - \ln|u| + C = \ln\left|\frac{\tan\theta-1}{\tan\theta}\right| + C = \ln|1-\cot\theta| + C$$

$u = \tan\theta$, $du = \sec^2\theta\,d\theta$

$\dfrac{1}{u(u-1)} = \dfrac{A}{u} + \dfrac{B}{u-1}$

$1 = A(u-1) + Bu$

Let $u = 0$: $1 = -A \implies A = -1$

Let $u = 1$: $1 = B$

39. $$\int \frac{x}{(2+3x)^2}\,dx = \frac{1}{9}\left[\frac{2}{2+3x} + \ln|2+3x|\right] + C$$

(Formula 4)

40. $$\int \frac{x}{\sqrt{2+3x}}\,dx = \frac{-2(4-3x)}{27}\sqrt{2+3x} + C \quad \text{(Formula 21)}$$

$$= \frac{6x-8}{27}\sqrt{2+3x} + C$$

41. Let $u = x^2$, $du = 2x\,dx$.

$$\int_0^{\sqrt{\pi/2}} \frac{x}{1+\sin x^2}\,dx = \frac{1}{2}\int_0^{\pi/4} \frac{1}{1+\sin u}\,du$$

$$= \frac{1}{2}\Big[\tan u - \sec u\Big]_0^{\pi/4}$$

$$= \frac{1}{2}\left[\left(1-\sqrt{2}\right) - (0-1)\right]$$

$$= 1 - \frac{\sqrt{2}}{2}$$

42. Let $u = x^2$, $du = 2x\,dx$.

$$\int_0^1 \frac{x}{1+e^{x^2}}\,dx = \frac{1}{2}\int_0^1 \frac{1}{1+e^u}\,du$$

$$= \frac{1}{2}\Big[u - \ln(1+e^u)\Big]_0^1$$

$$= \frac{1}{2}[(1 - \ln(1+e)) + \ln 2]$$

$$= \frac{1}{2}\left[1 + \ln\left(\frac{2}{1+e}\right)\right]$$

43. $\int \frac{x}{x^2+4x+8}\,dx = \frac{1}{2}\left[\ln|x^2+4x+8| - 4\int \frac{1}{x^2+4x+8}\,dx\right]$ (Formula 15)

$= \frac{1}{2}[\ln|x^2+4x+8|] - 2\left[\frac{2}{\sqrt{32-16}}\arctan\left(\frac{2x+4}{\sqrt{32-16}}\right)\right] + C$ (Formula 14)

$= \frac{1}{2}\ln|x^2+4x+8| - \arctan\left(1+\frac{x}{2}\right) + C$

44. $\int \frac{3}{2x\sqrt{9x^2-1}}\,dx = \frac{3}{2}\int \frac{1}{3x\sqrt{(3x)^2-1}}\,3\,dx$ $(u = 3x)$

$= \frac{3}{2}\operatorname{arcsec}|3x| + C$ (Formula 33)

45. $\int \frac{1}{\sin \pi x \cos \pi x}\,dx = \frac{1}{\pi}\int \frac{1}{\sin \pi x \cos \pi x}(\pi)\,dx$ $(u = \pi x)$

$= \frac{1}{\pi}\ln|\tan \pi x| + C$ (Formula 58)

46. $\int \frac{1}{1+\tan \pi x}\,dx = \frac{1}{\pi}\int \frac{1}{1+\tan \pi x}(\pi)dx$ $(u = \pi x)$

$= \frac{1}{\pi}\frac{1}{2}\left[\pi x + \ln|\cos \pi x + \sin \pi x|\right] + C$ (Formula 71)

47. $dv = dx \implies v = x$

$u = (\ln x)^n \implies du = n(\ln x)^{n-1}\frac{1}{x}dx$

$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$

48. $\int \tan^n x\,dx = \int \tan^{n-2}x(\sec^2 x - 1)\,dx$

$= \int \tan^{n-2}x \sec^2 x\,dx - \int \tan^{n-2}x\,dx$

$= \frac{1}{n-1}\tan^{n-1}x - \int \tan^{n-2}x\,dx$

49. $\int \theta \sin\theta \cos\theta\,d\theta = \frac{1}{2}\int \theta \sin 2\theta\,d\theta$

$= -\frac{1}{4}\theta\cos 2\theta + \frac{1}{4}\int \cos 2\theta\,d\theta = -\frac{1}{4}\theta\cos 2\theta + \frac{1}{8}\sin 2\theta + C = \frac{1}{8}(\sin 2\theta - 2\theta\cos 2\theta) + C$

$dv = \sin 2\theta\,d\theta \implies v = -\frac{1}{2}\cos 2\theta$

$u = \theta \implies du = d\theta$

50. $\int \frac{\csc\sqrt{2x}}{\sqrt{x}}\,dx = \sqrt{2}\int \csc\sqrt{2x}\left(\frac{1}{\sqrt{2x}}\right)dx$

$= -\sqrt{2}\ln|\csc\sqrt{2x} + \cot\sqrt{2x}| + C$

$u = \sqrt{2x},\ du = \frac{1}{\sqrt{2x}}\,dx$

51. $\int \frac{x^{1/4}}{1+x^{1/2}}\,dx = 4\int \frac{u(u^3)}{1+u^2}\,du$

$= 4\int\left(u^2 - 1 + \frac{1}{u^2+1}\right)du$

$= 4\left(\frac{1}{3}u^3 - u + \arctan u\right) + C$

$= \frac{4}{3}[x^{3/4} - 3x^{1/4} + 3\arctan(x^{1/4})] + C$

$u = \sqrt[4]{x},\ x = u^4,\ dx = 4u^3\,du$

52. $\int \sqrt{1+\sqrt{x}}\,dx = \int u(4u^3 - 4u)\,du = \int (4u^4 - 4u^2)\,du = \frac{4u^5}{5} - \frac{4u^3}{3} + C = \frac{4}{15}(1+\sqrt{x})^{3/2}(3\sqrt{x} - 2) + C$

$u = \sqrt{1+\sqrt{x}},\ x = u^4 - 2u^2 + 1,\ dx = (4u^3 - 4u)\,du$

53. $\displaystyle\int \sqrt{1 + \cos x}\, dx = \int \frac{\sqrt{1 + \cos x}}{1} \cdot \frac{\sqrt{1 - \cos x}}{\sqrt{1 - \cos x}}\, dx$

$\displaystyle= \int \frac{\sin x}{\sqrt{1 - \cos x}}\, dx$

$\displaystyle= \int (1 - \cos x)^{-1/2}(\sin x)\, dx$

$= 2\sqrt{1 - \cos x} + C$

$u = 1 - \cos x,\ du = \sin x\, dx$

54. $\dfrac{3x^3 + 4x}{(x^2 + 1)^2} = \dfrac{Ax + B}{x^2 + 1} + \dfrac{Cx + D}{(x^2 + 1)^2}$

$3x^3 + 4x = (Ax + B)(x^2 + 1) + Cx + D$

$= Ax^3 + Bx^2 + (A + C)x + (B + D)$

$A = 3, B = 0, A + C = 4 \implies C = 1,$

$B + D = 0 \implies D = 0$

$\displaystyle\int \frac{3x^3 + 4x}{(x^2 + 1)^2}\, dx = 3\int \frac{x}{x^2 + 1}\, dx + \int \frac{x}{(x^2 + 1)^2}\, dx$

$\displaystyle= \frac{3}{2}\ln(x^2 + 1) - \frac{1}{2(x^2 + 1)} + C$

55. $\displaystyle\int \cos x \ln(\sin x)\, dx = \sin x \ln(\sin x) - \int \cos x\, dx = \sin x \ln(\sin x) - \sin x + C$

$dv = \cos x\, dx \implies v = \sin x$

$u = \ln(\sin x) \implies du = \dfrac{\cos x}{\sin x}\, dx$

56. $\displaystyle\int (\sin\theta + \cos\theta)^2\, d\theta = \int (\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta)\, d\theta$

$\displaystyle= \int (1 + \sin 2\theta)\, d\theta = \theta - \frac{1}{2}\cos 2\theta + C = \frac{1}{2}(2\theta - \cos 2\theta) + C$

57. $\displaystyle y = \int \frac{9}{x^2 - 9}\, dx = \frac{3}{2}\ln\left|\frac{x - 3}{x + 3}\right| + C$ (by Formula 24 of Integration Tables)

58. $\displaystyle y = \int \frac{\sqrt{4 - x^2}}{2x}\, dx = \int \frac{2\cos\theta(2\cos\theta)\, d\theta}{4\sin\theta}$

$\displaystyle= \int (\csc\theta - \sin\theta)\, d\theta$

$= \left[-\ln|\csc\theta + \cos\theta| + \cos\theta\right] + C$

$\displaystyle= -\ln\left|\frac{2 + \sqrt{4 - x^2}}{x}\right| + \frac{\sqrt{4 - x^2}}{2} + C$

$x = 2\sin\theta,\ dx = 2\cos\theta\, d\theta,\ \sqrt{4 - x^2} = 2\cos\theta$

59. $\displaystyle y = \int \ln(x^2 + x)dx = x\ln|x^2 + x| - \int \frac{2x^2 + x}{x^2 + x}\, dx$

$\displaystyle= x\ln|x^2 + x| - \int \frac{2x + 1}{x + 1}\, dx$

$\displaystyle= x\ln|x^2 + x| - \int 2dx + \int \frac{1}{x + 1}\, dx$

$= x\ln|x^2 + x| - 2x + \ln|x + 1| + C$

$dv = dx \implies v = x$

$u = \ln(x^2 + x) \implies du = \dfrac{2x + 1}{x^2 + x}\, dx$

60. $\displaystyle y = \int \sqrt{1 - \cos\theta}\, d\theta = \int \frac{\sin\theta}{\sqrt{1 + \cos\theta}}\, d\theta = -\int (1 + \cos\theta)^{-1/2}(-\sin\theta)d\theta = -2\sqrt{1 + \cos\theta} + C$

$u = 1 + \cos\theta,\ du = -\sin\theta\, d\theta$

61. $\displaystyle\int_2^{\sqrt{5}} x(x^2 - 4)^{3/2}\, dx = \left[\frac{1}{5}(x^2 - 4)^{5/2}\right]_2^{\sqrt{5}} = \frac{1}{5}$

62. $\displaystyle\int_0^1 \frac{x}{(x - 2)(x - 4)}\, dx = \Big[2\ln|x - 4| - \ln|x - 2|\Big]_0^1$

$= 2\ln 3 - 2\ln 4 + \ln 2$

$= \ln\dfrac{9}{8} \approx 0.118$

63. $\int_1^4 \frac{\ln x}{x}\,dx = \left[\frac{1}{2}(\ln x)^2\right]_1^4 = \frac{1}{2}(\ln 4)^2 = 2(\ln 2)^2 \approx 0.961$

64. $\int_0^2 xe^{3x}\,dx = \left[\frac{e^{3x}}{9}(3x-1)\right]_0^2 = \frac{1}{9}(5e^6+1) \approx 224.238$

65. $\int_0^{\pi} x\sin x\,dx = \left[-x\cos x + \sin x\right]_0^{\pi} = \pi$

66. $\int_0^3 \frac{x}{\sqrt{1+x}}\,dx = \left[\frac{-2(2-x)}{3}\sqrt{1+x}\right]_0^3 = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$

67. $A = \int_0^4 x\sqrt{4-x}\,dx = \int_2^0 (4-u^2)u(-2u)\,du$

$= \int_2^0 2(u^4 - 4u^2)\,du$

$= \left[2\left(\frac{u^5}{5} - \frac{4u^3}{3}\right)\right]_2^0 = \frac{128}{15}$

$u = \sqrt{4-x},\ x = 4 - u^2,\ dx = -2u\,du$

68. $A = \int_0^4 \frac{1}{25 - x^2}\,dx$

$= \left[-\frac{1}{10}\ln\left|\frac{x-5}{x+5}\right|\right]_0^4 = -\frac{1}{10}\ln\frac{1}{9} = \frac{1}{10}\ln 9 \approx 0.220$

69. By symmetry, $\bar{x} = 0$, $A = \frac{1}{2}\pi$.

$\bar{y} = \frac{2}{\pi}\left(\frac{1}{2}\right)\int_{-1}^{1}\left(\sqrt{1-x^2}\right)^2 dx = \frac{1}{\pi}\left[x - \frac{1}{3}x^3\right]_{-1}^{1} = \frac{4}{3\pi}$

$(\bar{x}, \bar{y}) = \left(0, \frac{4}{3\pi}\right)$

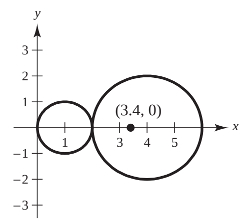
70. By symmetry, $\bar{y} = 0$.

$A = \pi + 4\pi = 5\pi$

$\bar{x} = \frac{1(\pi) + 4(4\pi)}{\pi + 4\pi}$

$= \frac{17\pi}{5\pi} = 3.4$

$(\bar{x}, \bar{y}) = (3.4, 0)$



71. $s = \int_0^{\pi}\sqrt{1 + \cos^2 x}\,dx \approx 3.82$

72. $s = \int_0^{\pi}\sqrt{1 + \sin^2 2x}\,dx \approx 3.82$

73. $\lim_{x\to 1}\left[\frac{(\ln x)^2}{x-1}\right] = \lim_{x\to 1}\left[\frac{2(1/x)\ln x}{1}\right] = 0$

74. $\lim_{x\to 0}\frac{\sin \pi x}{\sin 2\pi x} = \lim_{x\to 0}\frac{\pi\cos\pi x}{2\pi\cos 2\pi x} = \frac{\pi}{2\pi} = \frac{1}{2}$

75. $\lim_{x\to\infty}\frac{e^{2x}}{x^2} = \lim_{x\to\infty}\frac{2e^{2x}}{2x} = \lim_{x\to\infty}\frac{4e^{2x}}{2} = \infty$

76. $\lim_{x\to\infty} xe^{-x^2} = \lim_{x\to\infty}\frac{x}{e^{x^2}} = \lim_{x\to\infty}\frac{1}{2xe^{x^2}} = 0$

77. $y = \lim_{x\to\infty}(\ln x)^{2/x}$

$\ln y = \lim_{x\to\infty}\frac{2\ln(\ln x)}{x} = \lim_{x\to\infty}\left[\frac{2/(x\ln x)}{1}\right] = 0$

Since $\ln y = 0$, $y = 1$.

78. $y = \lim_{x\to 1^+}(x-1)^{\ln x}$

$\ln y = \lim_{x\to 1^+}\left[(\ln x)\ln(x-1)\right]$

$= \lim_{x\to 1^+}\left[\frac{\ln(x-1)}{\frac{1}{\ln x}}\right] = \lim_{x\to 1^+}\left[\frac{\frac{1}{x-1}}{\left(\frac{1}{x}\right)\frac{-1}{\ln^2 x}}\right] = \lim_{x\to 1^+}\left[\frac{-\ln^2 x}{\frac{x-1}{x}}\right] = \lim_{x\to 1^+}\left[\frac{-2\left(\frac{1}{x}\right)(\ln x)}{\frac{1}{x^2}}\right]$

$= \lim_{x\to 1^+} 2x(\ln x) = 0$

Since $\ln y = 0$, $y = 1$.

79. $\lim_{n\to\infty} 1000\left(1 + \frac{0.09}{n}\right)^n = 1000 \lim_{n\to\infty}\left(1 + \frac{0.09}{n}\right)^n$

Let $y = \lim_{n\to\infty}\left(1 + \frac{0.09}{n}\right)^n$.

$$\ln y = \lim_{n\to\infty} n \ln\left(1 + \frac{0.09}{n}\right) = \lim_{n\to\infty}\frac{\ln\left(1 + \frac{0.09}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{\frac{-0.09/n^2}{1 + (0.09/n)}}{-\frac{1}{n^2}}\right) = \lim_{n\to\infty}\frac{0.09}{1 + \left(\frac{0.09}{n}\right)} = 0.09$$

Thus, $\ln y = 0.09 \implies y = e^{0.09}$ and $\lim_{n\to\infty} 1000\left(1 + \frac{0.09}{n}\right)^n = 1000e^{0.09} \approx 1094.17$.

80. $\lim_{x\to 1^+}\left(\frac{2}{\ln x} - \frac{2}{x-1}\right) = \lim_{x\to 1^+}\left[\frac{2x - 2 - 2\ln x}{(\ln x)(x-1)}\right]$

$$= \lim_{x\to 1^+}\left[\frac{2 - (2/x)}{(x-1)(1/x) + \ln x}\right]$$

$$= \lim_{x\to 1^+}\frac{2x - 2}{(x-1) + x\ln x} = \lim_{x\to 1^+}\frac{2}{1 + 1 + \ln x} = 1$$

81. $\int_0^{16}\frac{1}{\sqrt[4]{x}}\,dx = \lim_{b\to 0^+}\left[\frac{4}{3}x^{3/4}\right]_b^{16} = \frac{32}{3}$

Converges

82. $\int_0^1 \frac{6}{x-1}\,dx = \lim_{b\to 1^-}\Big[6\ln|x-1|\Big]_0^b = -\infty$

Diverges

83. $\int_1^\infty x^2 \ln x\,dx = \lim_{b\to\infty}\left[\frac{x^3}{9}(-1 + 3\ln x)\right]_1^b = \infty$

Diverges

84. $\int_0^\infty \frac{e^{-1/x}}{x^2}\,dx = \lim_{\substack{a\to 0^+\\ b\to\infty}}\left[e^{-1/x}\right]_a^b = 1 - 0 = 1$

85. Let $u = \ln x$, $du = \frac{1}{x}\,dx$, $dv = x^{-2}\,dx$, $v = -x^{-1}$.

$$\int \frac{\ln x}{x^2}\,dx = \frac{-\ln x}{x} + \int \frac{1}{x^2}\,dx = \frac{-\ln x}{x} - \frac{1}{x} + C$$

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\left[\frac{-\ln x}{x} - \frac{1}{x}\right]_1^b$$

$$= \lim_{b\to\infty}\left(\frac{-\ln b}{b} - \frac{1}{b}\right) - (-1)$$

$$= 0 + 1 = 1$$

86. $\int_1^\infty \frac{1}{\sqrt[4]{x}}\,dx = \lim_{b\to\infty}\int_1^b x^{-1/4}\,dx$

$$= \lim_{b\to\infty}\left[\frac{4}{3}x^{3/4}\right]_1^b$$

$$= \lim_{b\to\infty}\left[\frac{4}{3}b^{3/4} - \frac{4}{3}\right]$$

Diverges

87. $\int_0^{t_0} 500{,}000e^{-0.05t}\,dt = \left[\frac{500{,}000}{-0.05}e^{-0.05t}\right]_0^{t_0}$

$$= \frac{-500{,}000}{0.05}\left(e^{-0.05t_0} - 1\right)$$

$$= 10{,}000{,}000\left(1 - e^{-0.05t_0}\right)$$

(a) $t_0 = 20$: \$6,321,205.59

(b) $t_0 \to \infty$: \$10,000,000

88. $V = \pi\int_0^\infty (xe^{-x})^2\,dx$

$$= \pi\int_0^\infty x^2 e^{-2x}\,dx$$

$$= \lim_{b\to\infty}\left[-\frac{\pi e^{-2x}}{4}(2x^2 + 2x + 1)\right]_0^b = \frac{\pi}{4}$$

89. (a) $P(13 \le x < \infty) = \frac{1}{0.95\sqrt{2\pi}}\int_{13}^\infty e^{-(x-12.9)^2/2(0.95)^2}\,dx \approx 0.4581$

(b) $P(15 \le x < \infty) = \frac{1}{0.95\sqrt{2\pi}}\int_{15}^\infty e^{-(x-12.9)^2/2(0.95)^2}\,dx \approx 0.0135$

Problem Solving for Chapter 8

1. (a) $\displaystyle\int_{-1}^{1}(1-x^2)\,dx = \left[x-\frac{x^3}{3}\right]_{-1}^{1} = 2\left(1-\frac{1}{3}\right) = \frac{4}{3}$

$$\int_{-1}^{1}(1-x^2)^2\,dx = \int_{-1}^{1}(1-2x^2+x^4)\,dx = \left[x-\frac{2x^3}{3}+\frac{x^5}{5}\right]_{-1}^{1} = 2\left(1-\frac{2}{3}+\frac{1}{5}\right) = \frac{16}{15}$$

(b) Let $x = \sin u$, $dx = \cos u\,du$, $1-x^2 = 1-\sin^2 u = \cos^2 u$.

$$\begin{aligned}\int_{-1}^{1}(1-x^2)^n\,dx &= \int_{-\pi/2}^{\pi/2}(\cos^2 u)^n\cos u\,du\\ &= \int_{-\pi/2}^{\pi/2}\cos^{2n+1}u\,du\\ &= 2\left[\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{(2n)}{(2n+1)}\right] \qquad \text{(Wallis's Formula)}\\ &= 2\left[\frac{2^2\cdot 4^2\cdot 6^2\cdots(2n)^2}{2\cdot 3\cdot 4\cdot 5\cdots(2n)(2n+1)}\right]\\ &= \frac{2(2^{2n})(n!)^2}{(2n+1)!} = \frac{2^{2n+1}(n!)^2}{(2n+1)!}\end{aligned}$$

2. (a) $\displaystyle\int_{0}^{1}\ln x\,dx = \lim_{b\to 0^+}\Big[x\ln - x\Big]_{b}^{1}$

$$= (-1) - \lim_{b\to 0^+}(b\ln b - b) = -1$$

Note: $\displaystyle\lim_{b\to 0^+} b\ln b = \lim_{b\to 0^+}\frac{\ln b}{b^{-1}} = \lim_{b\to 0^+}\frac{1/b}{-1/b^2} = 0$

$$\int_{0}^{1}(\ln x)^2\,dx = \lim_{b\to 0^+}\Big[x(\ln x)^2 - 2x\ln x + 2x\Big]_{b}^{1}$$

$$= 2 - \lim_{b\to 0^+}\left(b(\ln b)^2 - 2b\ln b + 2b\right) = 2$$

(b) Note first that $\displaystyle\lim_{b\to 0^+} b(\ln b)^n = 0$ (Mathematical induction).

Also, $\displaystyle\int(\ln x)^{n+1}\,dx = x(\ln x)^{n+1} - (n+1)\int(\ln x)^n\,dx.$

Assume $\displaystyle\int_{0}^{1}(\ln x)^n\,dx = (-1)^n n!.$

Then, $\displaystyle\int_{0}^{1}(\ln x)^{n+1}\,dx = \lim_{b\to 0^+}\Big[x(\ln x)^{n+1}\Big]_{b}^{1} - (n+1)\int_{0}^{1}(\ln x)^n\,dx$

$$= 0 - (n+1)(-1)^n n! = (-1)^{n+1}(n+1)!.$$

3.

$$\lim_{x\to\infty}\left(\frac{x+c}{x-c}\right)^x = 9$$

$$\lim_{x\to\infty} x\ln\left(\frac{x+c}{x-c}\right) = \ln 9$$

$$\lim_{x\to\infty}\frac{\ln(x+c)-\ln(x-c)}{1/x} = \ln 9$$

$$\lim_{x\to\infty}\frac{\dfrac{1}{x+c}-\dfrac{1}{x-c}}{-\dfrac{1}{x^2}} = \ln 9$$

$$\lim_{x\to\infty}\frac{-2c}{(x+c)(x-c)}(-x^2) = \ln 9$$

$$\lim_{x\to\infty}\left(\frac{2cx^2}{x^2-c^2}\right) = \ln 9$$

$$2c = \ln 9$$

$$2c = 2\ln 3$$

$$c = \ln 3$$

4.

$$\lim_{x\to\infty}\left(\frac{x-c}{x+c}\right)^x = \frac{1}{4}$$

$$\lim_{x\to\infty} x\ln\left(\frac{x-c}{x+c}\right) = \ln\frac{1}{4}$$

$$\lim_{x\to\infty}\frac{\ln(x-c)-\ln(x+c)}{1/x} = -\ln 4$$

$$\lim_{x\to\infty}\frac{\dfrac{1}{x-c}-\dfrac{1}{x+c}}{-\dfrac{1}{x^2}} = -\ln 4$$

$$\lim_{x\to\infty}\frac{2c}{(x-c)(x+c)}(-x^2) = -\ln 4$$

$$\lim_{x\to\infty}\frac{2cx^2}{x^2-c^2} = \ln 4$$

$$2c = \ln 4$$

$$2x = 2\ln 2$$

$$c = \ln 2$$

5. $\sin\theta = \dfrac{PB}{OP} = PB$, $\cos\theta = OB$

$AQ = \overparen{AP} = \theta$

$BR = OR + OB = OR + \cos\theta$

The triangles $\triangle AQR$ and $\triangle BPR$ are similar:

$$\frac{AR}{AQ} = \frac{BR}{BP} \Longrightarrow \frac{OR+1}{\theta} = \frac{OR+\cos\theta}{\sin\theta}$$

$$\sin\theta(OR) + \sin\theta = (OR)\theta + \theta\cos\theta$$

$$OR = \frac{\theta\cos\theta - \sin\theta}{\sin\theta - \theta}$$

$$\lim_{\theta\to 0^+} OR = \lim_{\theta\to 0^+}\frac{\theta\cos\theta - \sin\theta}{\sin\theta - \theta}$$

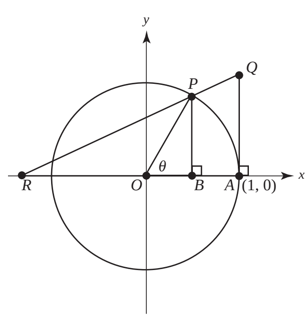
$$= \lim_{\theta\to 0^+}\frac{-\theta\sin\theta + \cos\theta - \cos\theta}{\cos\theta - 1}$$

$$= \lim_{\theta\to 0^+}\frac{-\theta\sin\theta}{\cos\theta - 1}$$

$$= \lim_{\theta\to 0^+}\frac{-\sin\theta - \theta\cos\theta}{-\sin\theta}$$

$$= \lim_{\theta\to 0^+}\frac{\cos\theta + \cos\theta - \theta\sin\theta}{\cos\theta}$$

$$= 2$$



6. $\sin\theta = BD$, $\cos\theta = OD$

$$\text{Area } \triangle DAB = \frac{1}{2}(DA)(BD) = \frac{1}{2}(1-\cos\theta)\sin\theta$$

$$\text{Shaded area} = \frac{\theta}{2} - \frac{1}{2}(1)(BD) = \frac{\theta}{2} - \frac{1}{2}\sin\theta$$

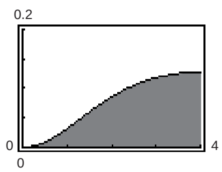
$$R = \frac{\triangle DAB}{\text{Shaded area}} = \frac{1/2(1-\cos\theta)\sin\theta}{1/2(\theta - \sin\theta)}$$

$$\lim_{\theta\to 0^+} R = \lim_{\theta\to 0^+}\frac{(1-\cos\theta)\sin\theta}{\theta-\sin\theta} = \lim_{\theta\to 0^+}\frac{(1-\cos\theta)\cos\theta + \sin^2\theta}{1-\cos\theta}$$

$$= \lim_{\theta\to 0^+}\frac{(1-\cos\theta)(-\sin\theta) + \cos\theta\sin\theta + 2\sin\theta\cos\theta}{\sin\theta}$$

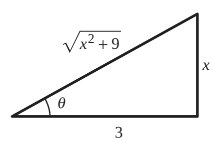
$$= \lim_{\theta\to 0^+}\frac{-\sin\theta - 4\cos\theta\sin\theta}{\sin\theta} = \lim_{\theta\to 0}\frac{4\cos\theta - 1}{1} = 3$$

7. (a)



Area ≈ 0.2986

(b) Let $x = 3\tan\theta$, $dx = 3\sec^2\theta\,d\theta$, $x^2+9 = 9\sec^2\theta$.



$$\int \frac{x^2}{(x^2+9)^{3/2}}\,dx = \int \frac{9\tan^2\theta}{(9\sec^2\theta)^{3/2}}(3\sec^2\theta\,d\theta)$$

$$= \int \frac{\tan^2\theta}{\sec\theta}\,d\theta$$

$$= \int \frac{\sin^2\theta}{\cos\theta}\,d\theta$$

$$= \int \frac{1-\cos^2\theta}{\cos\theta}\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| - \sin\theta + C$$

$$\text{Area} = \int_0^4 \frac{x^2}{(x^2+9)^{3/2}}\,dx = \Big[\ln|\sec\theta+\tan\theta| - \sin\theta\Big]_0^{\tan^{-1}(4/3)}$$

$$= \left[\ln\left(\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right) - \frac{x}{\sqrt{x^2+9}}\right]_0^4$$

$$= \ln\left(\frac{5}{3}+\frac{4}{3}\right) - \frac{4}{5} = \ln 3 - \frac{4}{5}$$

(c) $x = 3\sinh u$, $dx = 3\cosh u\,du$, $x^2+9 = 9\sinh^2 u + 9 = 9\cosh^2 u$

$$A = \int_0^4 \frac{x^2}{(x^2+9)^{3/2}}\,dx = \int_0^{\sinh^{-1}(4/3)} \frac{9\sinh^2 u}{(9\cosh^2 u)^{3/2}}(3\cosh u\,du) = \int_0^{\sinh^{-1}(4/3)} \tanh^2 u\,du$$

$$= \int_0^{\sinh^{-1}(4/3)} (1-\operatorname{sech}^2 u)\,du = \Big[u - \tanh u\Big]_0^{\sinh^{-1}(4/3)}$$

$$= \sinh^{-1}\left(\frac{4}{3}\right) - \tanh\left(\sinh^{-1}\left(\frac{4}{3}\right)\right) = \ln\left(\frac{4}{3} + \sqrt{\frac{16}{9}+1}\right) - \tanh\left[\ln\left(\frac{4}{3}+\sqrt{\frac{16}{9}+1}\right)\right]$$

$$= \ln\left(\frac{4}{3}+\frac{5}{3}\right) - \tanh\left(\ln\left(\frac{4}{3}+\frac{5}{3}\right)\right) = \ln 3 - \tanh(\ln 3)$$

$$= \ln 3 - \frac{3-(1/3)}{3+(1/3)} = \ln 3 - \frac{4}{5}$$

8. $u = \tan \frac{x}{2}, \cos x = \frac{1 - u^2}{1 + u^2}, 2 + \cos x = 2 + \frac{1 - u^2}{1 + u^2} = \frac{3 + u^2}{1 + u^2}$

$$dx = \frac{2\,du}{1 + u^2}$$

$$\begin{aligned}\int_0^{\pi/2} \frac{1}{2 + \cos x}\,dx &= \int_0^1 \left(\frac{1 + u^2}{3 + u^2}\right)\left(\frac{2}{1 + u^2}\right) du\\ &= \int_0^1 \frac{2}{3 + u^2}\,du\\ &= \left[2\frac{1}{\sqrt{3}} \arctan\left(\frac{u}{\sqrt{3}}\right)\right]_0^1\\ &= \frac{2}{\sqrt{3}} \arctan\left(\frac{1}{\sqrt{3}}\right)\\ &= \frac{2}{\sqrt{3}}\frac{\pi}{6} = \frac{\pi\sqrt{3}}{9} \approx 0.6046\end{aligned}$$

9. $y = \ln(1 - x^2), y' = \frac{-2x}{1 - x^2}$

$$1 + (y')^2 = 1 + \frac{4x^2}{(1 - x^2)^2} = \frac{1 - 2x^2 + x^4 + 4x^2}{(1 - x^2)^2} = \left(\frac{1 + x^2}{1 - x^2}\right)^2$$

$$\begin{aligned}\text{Arc length} &= \int_0^{1/2} \sqrt{1 + (y')^2}\,dx\\ &= \int_0^{1/2} \left(\frac{1 + x^2}{1 - x^2}\right) dx\\ &= \int_0^{1/2} \left(-1 + \frac{2}{1 - x^2}\right) dx\\ &= \int_0^{1/2} \left(-1 + \frac{1}{x + 1} + \frac{1}{1 - x}\right) dx\\ &= \Big[-x + \ln(1 + x) - \ln(1 - x)\Big]_0^{1/2}\\ &= \left(-\frac{1}{2} + \ln\frac{3}{2} - \ln\frac{1}{2}\right)\\ &= -\frac{1}{2} + \ln 3 - \ln 2 + \ln 2\\ &= \ln 3 - \frac{1}{2} \approx 0.5986\end{aligned}$$

10. Let $u = cx$, $du = c\,dx$.

$$\int_0^b e^{-c^2x^2}\,dx = \int_0^{cb} e^{-u^2}\,\frac{du}{c} = \frac{1}{c}\int_0^{cb} e^{-u^2}\,du$$

As $b \to \infty$, $cb \to \infty$. Hence, $\displaystyle\int_0^{\infty} e^{-c^2x^2}\,dx = \frac{1}{c}\int_0^{\infty} e^{-x^2}\,dx$.

$\bar{x} = 0$ by symmetry.

$$\bar{y} = \frac{M_x}{m} = \frac{2\displaystyle\int_0^{\infty}\frac{(e^{-c^2x^2})}{2}\,dx}{2\displaystyle\int_0^{\infty} e^{-c^2x^2}\,dx}$$

$$= \frac{1}{2}\,\frac{\displaystyle\int_0^{\infty} e^{-2c^2x^2}\,dx}{\displaystyle\int_0^{\infty} e^{-c^2x^2}\,dx}$$

$$= \frac{1}{2}\,\frac{\dfrac{1}{\sqrt{2}c}\displaystyle\int_0^{\infty} e^{-x^2}\,dx}{\dfrac{1}{c}\displaystyle\int_0^{\infty} e^{-x^2}\,dx}$$

$$= \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

Thus, $(\bar{x}, \bar{y}) = \left(0, \dfrac{\sqrt{2}}{4}\right)$.

11. Consider $\displaystyle\int \frac{1}{\ln x}\,dx$.

Let $u = \ln x$, $du = \dfrac{1}{x}\,dx$, $x = e^u$. Then $\displaystyle\int \frac{1}{\ln x}\,dx = \int \frac{1}{u}e^u\,du = \int \frac{e^u}{u}\,du$.

If $\displaystyle\int \frac{1}{\ln x}\,dx$ were elementary, then $\displaystyle\int \frac{e^u}{u}\,du$ would be too, which is false.

Hence, $\displaystyle\int \frac{1}{\ln x}\,dx$ is not elementary.

12. (a) Let $y = f^{-1}(x)$, $f(y) = x$, $dx = f'(y)\,dy$.

$$\int f^{-1}(x)\,dx = \int y f'(y)\,dy$$

$$= y f(y) - \int f(y)\,dy \qquad \left[\begin{array}{l} u = y,\ du = dy \\ dv = f'(y)\,dy,\ v = f(y) \end{array}\right]$$

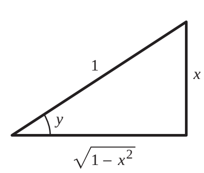
$$= x f^{-1}(x) - \int f(y)\,dy$$

(b) $f^{-1}(x) = \arcsin x = y$, $f(x) = \sin x$

$$\int \arcsin x\,dx = x\arcsin x - \int \sin y\,dy$$

$$= x\arcsin x + \cos y + C$$

$$= x\arcsin x + \sqrt{1 - x^2} + C$$



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12. —CONTINUED—

(c) $f(x) = e^x, f^{-1}(x) = \ln x = y \qquad x = 1 \Leftrightarrow y = 0; x = e \Leftrightarrow y = 1$

$$\int_1^e \ln x\,dx = \Big[x \ln x\Big]_1^e - \int_0^1 e^y\,dy$$

$$= e - \Big[e^y\Big]_0^1$$

$$= e - (e - 1) = 1$$

13. $x^4 + 1 = (x^2 + ax + b)(x^2 + cx + d)$

$$= x^4 + (a + c)x^3 + (ac + b + d)x^2 + (ad + bc)x + bd$$

$a = -c, b = d = 1, a = \sqrt{2}$

$$x^4 + 1 = \left(x^2 + \sqrt{2}x + 1\right)\left(x^2 - \sqrt{2}x + 1\right)$$

$$\int_0^1 \frac{1}{x^4 + 1}\,dx = \int_0^1 \frac{Ax + B}{x^2 + \sqrt{2}x + 1}\,dx + \int_0^1 \frac{Cx + D}{x^2 - \sqrt{2}x + 1}\,dx$$

$$= \int_0^1 \frac{\frac{1}{2} + \frac{\sqrt{2}}{4}x}{x^2 + \sqrt{2}x + 1}\,dx - \int_0^1 \frac{-\frac{1}{2} + \frac{\sqrt{2}}{4}x}{x^2 + \sqrt{2}x + 1}\,dx$$

$$= \frac{\sqrt{2}}{4}\Big[\arctan\left(\sqrt{2}x + 1\right) + \arctan\left(\sqrt{2}x - 1\right)\Big]_0^1 + \frac{\sqrt{2}}{8}\Big[\ln\left(x^2 + \sqrt{2}x + 1\right) - \ln\left(x^2 - \sqrt{2}x + 1\right)\Big]_0^1$$

$$= \frac{\sqrt{2}}{4}\left[\arctan\left(\sqrt{2} + 1\right) + \arctan\left(\sqrt{2} - 1\right)\right] + \frac{\sqrt{2}}{8}\left[\ln\left(2 + \sqrt{2}\right) - \ln\left(2 - \sqrt{2}\right)\right] - \frac{\sqrt{2}}{4}\left[\frac{\pi}{4} - \frac{\pi}{4}\right] - \frac{\sqrt{2}}{8}[0]$$

$$\approx 0.5554 + 0.3116$$

$$\approx 0.8670$$

14. (a) Let $x = \frac{\pi}{2} - u, dx = du.$

$$I = \int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x}\,dx = \int_{\pi/2}^0 \frac{\sin\left(\frac{\pi}{2} - u\right)}{\cos\left(\frac{\pi}{2} - u\right) + \sin\left(\frac{\pi}{2} - u\right)}(-du)$$

$$= \int_0^{\pi/2} \frac{\cos u}{\sin u + \cos u}\,du$$

Hence,

$$2I = \int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x}\,dx + \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\,dx$$

$$= \int_0^{\pi/2} 1\,dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}.$$

(b) $$I = \int_{\pi/2}^0 \frac{\sin^n\left(\frac{\pi}{2} - u\right)}{\cos^n\left(\frac{\pi}{2} - u\right) + \sin^n\left(\frac{\pi}{2} - u\right)}(-du)$$

$$= \int_0^{\pi/2} \frac{\cos^n u}{\sin^n u + \cos^n u}\,du$$

Thus, $2I = \int_0^{\pi/2} 1\,dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}.$

15. Using a graphing utility,

(a) $\lim_{x\to 0^+}\left(\cot x + \frac{1}{x}\right) = \infty$

(b) $\lim_{x\to 0^+}\left(\cot x - \frac{1}{x}\right) = 0$

(c) $\lim_{x\to 0^+}\left(\cot x + \frac{1}{x}\right)\left(\cot x - \frac{1}{x}\right) \approx -\frac{2}{3}.$

Analytically,

(a) $\lim_{x\to 0^+}\left(\cot x + \frac{1}{x}\right) = \infty + \infty = \infty$

(b) $$\lim_{x\to 0^+}\left(\cot x - \frac{1}{x}\right) = \lim_{x\to 0^+}\frac{x\cot x - 1}{x} = \lim_{x\to 0^+}\frac{x\cos x - \sin x}{x\sin x}$$

$$= \lim_{x\to 0^+}\frac{\cos x - x\sin x - \cos x}{\sin x + x\cos x} = \lim_{x\to 0^+}\frac{-x\sin x}{\sin x + x\cos x}$$

$$= \lim_{x\to 0^+}\frac{-\sin x - x\cos x}{\cos x + \cos x - x\sin x} = 0.$$

(c) $$\left(\cot x + \frac{1}{x}\right)\left(\cot x - \frac{1}{x}\right) = \cot^2 x - \frac{1}{x^2}$$

$$= \frac{x^2\cot^2 x - 1}{x^2}$$

$$\lim_{x\to 0^+}\frac{x^2\cot^2 x - 1}{x^2} = \lim_{x\to 0^+}\frac{2x\cot^2 x - 2x^2\cot x\csc^2 x}{2x}$$

$$= \lim_{x\to 0^+}\frac{\cot^2 x - x\cot x\csc^2 x}{1}$$

$$= \lim_{x\to 0^+}\frac{\cos^2 x\sin x - x\cos x}{\sin^3 x}$$

$$= \lim_{x\to 0^+}\frac{(1 - \sin^2 x)\sin x - x\cos x}{\sin^3 x}$$

$$= \lim_{x\to 0^+}\frac{\sin x - x\cos x}{\sin^3 x} - 1$$

Now, $$\lim_{x\to 0^+}\frac{\sin x - x\cos x}{\sin^3 x} = \lim_{x\to 0^+}\frac{\cos x - \cos x + x\sin x}{3\sin^2 x\cos x}$$

$$= \lim_{x\to 0^+}\frac{x}{3\sin x\cdot\cos x}$$

$$= \lim_{x\to 0^+}\left(\frac{x}{\sin x}\right)\frac{1}{3\cos x} = \frac{1}{3}.$$

Thus, $\lim_{x\to 0^+}\left(\cot x + \frac{1}{x}\right)\left(\cot x - \frac{1}{x}\right) = \frac{1}{3} - 1 = -\frac{2}{3}.$

The form $0\cdot\infty$ is indeterminant.

16. $\dfrac{N(x)}{D(x)} = \dfrac{P_1}{x - c_1} + \dfrac{P_2}{x - c_2} + \cdots + \dfrac{P_n}{x - c_n}$

$N(x) = P_1(x - c_2)(x - c_3)\ldots(x - c_n) + P_2(x - c_1)(x - c_3)\ldots(x - c_n) + \cdots + P_n(x - c_1)(x - c_2)\ldots(x - c_{n-1})$

Let $x = c_1$: $N(c_1) = P_1(c_1 - c_2)(c_1 - c_3)\ldots(c_1 - c_n)$

$$P_1 = \frac{N(c_1)}{(c_1 - c_2)(c_1 - c_3)\ldots(c_1 - c_n)}$$

Let $x = c_2$: $N(c_2) = P_2(c_2 - c_1)(c_2 - c_3)\ldots(c_2 - c_n)$

$$P_2 = \frac{N(c_2)}{(c_2 - c_1)(c_2 - c_3)\ldots(c_2 - c_n)}$$

$\vdots \qquad \vdots$

Let $x = c_n$: $N(c_n) = P_n(c_n - c_1)(c_n - c_2)\ldots(c_n - c_{n-1})$

$$P_n = \frac{N(c_n)}{(c_n - c_1)(c_n - c_2)\ldots(c_n - c_{n-1})}$$

If $D(x) = (x - c_1)(x - c_2)(x - c_3)\ldots(x - c_n)$, then by the Product Rule

$$D'(x) = (x - c_2)(x - c_3)\ldots(x - c_n) + (x - c_1)(x - c_3)\ldots(x - c_n) + \cdots + (x - c_1)(x - c_2)(x - c_3)\ldots(x - c_{n-1})$$

and

$$D'(c_1) = (c_1 - c_2)(c_1 - c_3)\ldots(c_1 - c_n)$$
$$D'(c_2) = (c_2 - c_1)(c_2 - c_3)\ldots(c_2 - c_n)$$
$$\vdots$$
$$D'(c_n) = (c_n - c_1)(c_n - c_2)\ldots(c_n - c_{n-1}).$$

Thus, $P_k = N(c_k)/D'(c_k)$ for $k = 1, 2, \ldots, n$.

17. $\dfrac{x^3 - 3x^2 + 1}{x^4 - 13x^2 + 12x} = \dfrac{P_1}{x} + \dfrac{P_2}{x - 1} + \dfrac{P_3}{x + 4} + \dfrac{P_4}{x - 3} \Longrightarrow c_1 = 0,\ c_2 = 1,\ c_3 = -4,\ c_4 = 3$

$N(x) = x^3 - 3x^2 + 1$

$D'(x) = 4x^3 - 26x + 12$

$$P_1 = \frac{N(0)}{D'(0)} = \frac{1}{12}$$

$$P_2 = \frac{N(1)}{D'(1)} = \frac{-1}{-10} = \frac{1}{10}$$

$$P_3 = \frac{N(-4)}{D'(-4)} = \frac{-111}{-140} = \frac{111}{140}$$

$$P_4 = \frac{N(3)}{D'(3)} = \frac{1}{42}$$

Thus, $\dfrac{x^3 - 3x^2 + 1}{x^4 - 13x^2 + 12x} = \dfrac{1/12}{x} + \dfrac{1/10}{x - 1} + \dfrac{111/140}{x + 4} + \dfrac{1/42}{x - 3}$.

18. $$s(t) = \int\left[-32t + 12{,}000 \ln \frac{50{,}000}{50{,}000 - 400t}\right] dt$$

$$= -16t^2 + 12{,}000\int[\ln 50{,}000 - \ln(50{,}000 - 400t)]\, dt$$

$$= 16t^2 + 12{,}000t \ln 50{,}000 - 12{,}000\left[t \ln(50{,}000 - 400t) - \int \frac{-400t}{50{,}000 - 400t}\, dt\right]$$

$$= -16t^2 + 12{,}000t \ln \frac{50{,}000}{50{,}000 - 400t} + 12{,}000t\int\left[1 - \frac{50{,}000}{50{,}000 - 400t}\right] dt$$

$$= -16t^2 + 12{,}000t \ln \frac{50{,}000}{50{,}000 - 400t} + 12{,}000t + 1{,}500{,}000 \ln(50{,}000 - 400t) + C$$

$$s(0) = 1{,}500{,}000 \ln 50{,}000 + C = 0$$

$$C = -1{,}500{,}000 \ln 50{,}000$$

$$s(t) = -16t^2 + 12{,}000t\left[1 + \ln \frac{50{,}000}{50{,}000 - 400t}\right] + 1{,}500{,}000 \ln \frac{50{,}000 - 400t}{50{,}000}$$

When $t = 100$, $s(100) \approx 557{,}168.626$ feet.

19. By parts,

$$\int_a^b f(x)g''(x)\, dx = \Big[f(x)g'(x)\Big]_a^b - \int_a^b f'(x)g'(x)\, dx [u = f(x),\, dv = g''(x)\, dx]$$

$$= -\int_a^b f'(x)g'(x)\, dx$$

$$= \Big[-f'(x)g(x)\Big]_a^b + \int_a^b g(x)f''(x)\, dx [u = f'(x),\, dv = g'(x)\, dx]$$

$$= \int_a^b f''(x)g(x)\, dx.$$

20. Let $u = (x - a)(x - b)$, $du = [(x - a) + (x - b)]\, dx$, $dv = f''(x)\, dx$, $v = f'(x)$.

$$\int_a^b (x - a)(x - b)\, dx = \Big[(x - a)(x - b)f'(x)\Big]_a^b - \int_a^b [(x - a) + (x - b)]f'(x)\, dx$$

$$= -\int_a^b (2x - a - b)f'(x)\, dx \qquad \begin{pmatrix} u = 2x - a - b \\ dv = f'(x)\, dx \end{pmatrix}$$

$$= \Big[-(2x - a - b)f(x)\Big]_a^b + \int_a^b 2f(x)\, dx$$

$$= 2\int_a^b f(x)\, dx$$

21. $$\int_2^\infty \left[\frac{1}{x^5} + \frac{1}{x^{10}} + \frac{1}{x^{15}}\right] dx < \int_2^\infty \frac{1}{x^5 - 1}\, dx < \int_2^\infty \left[\frac{1}{x^5} + \frac{1}{x^{10}} + \frac{2}{x^{15}}\right] dx$$

$$\lim_{b\to\infty}\left[-\frac{1}{4x^4} - \frac{1}{9x^9} - \frac{1}{14x^{14}}\right]_2^b < \int_2^\infty \frac{1}{x^5 - 1}\, dx < \lim_{b\to\infty}\left[-\frac{1}{4x^4} - \frac{1}{9x^9} - \frac{1}{7x^{14}}\right]_2^b$$

$$0.015846 < \int_2^\infty \frac{1}{x^5 - 1}\, dx < 0.015851$$