3.2. THE BASICS OF LOGIC

In this section we introduce the basic concepts, definitions, and symbolic notation used in logic. We start by explaining the importance of logic and providing some historical context for this old and venerable discipline.

The Importance of Logic

It is difficult to overemphasize the importance of logic. Our species, *Homo Sapiens,* is by far the most intelligent animal to have evolved on Earth thanks, in large part, to the development of our extraordinary brain and our ability to reason. Logic is what allowed humans to develop language, invent the alphabet, communicate effectively, grasp abstract concepts, create systems of numeration, and innovative new and powerful technologies.

While our ability to reason is innate, it is still a skill that requires much practice and honing, just like our writing or motor skills. For this reason, logic is typically presented to us implicitly in a more formal academic setting. Early in our schooling, we learn that mathematics is built within a precise framework in which properties are derived from definitions and theorems are proved from more elementary theorems (or axioms) using the standard rules of logic. These logical principles are, in turn, used in all other disciplines to understand particular concepts and the relationships unifying these particular concepts as coherent wholes. Much of what we see and experience as humans is studied nowadays through structures – whether physical or abstract – that involve various levels of logical sophistication. For example, an organizational flow chart used in business, a chemical table illustrating the properties of molecules under various states, or a definition used in astronomy to identify planets, all use straightforward propositional logic (*e.g.* "When a water molecule is heated to 100 degrees Celsius, it changes from liquid to gas" or "A celestial body is called a planet if it orbits a sun and has enough mass to be nearly spherical.")

Outside the classroom, the principles of logic are applied everywhere in the modern world. They are used, for instance, by jurors to reach verdicts based on trial evidence, by computer scientists to write algorithms and software, by scientists to make claims based on their data and models, by politicians to argue specific policies, by marketers to sell their products, etc. More broadly, logic underpins any cognitive activity that uses language or involves thinking and reasoning. So when you make plans for tomorrow afternoon, go food-shopping, cook a dish, or read this sentence, you are necessarily using logic!

What is Logic?

It is surprisingly difficult to define exactly what constitutes the study of logic. There are technical reasons for this as logic involves various levels of rigor that range from the elementary to the abstract metatheories in use today. More importantly, its *purpose* can be different depending on whether one is a mathematician, a linguist, a pollster, or a scholar of jurisprudence. Russell famously declared logic to be "the subject in which nobody knows what one is talking about, nor whether what one is saying is true." Perhaps, the real difficulty stems from the fact that we – as human beings – have such an innate understanding of the ways in which we reason and reach "correct" conclusions that we simply take the principles of logic for granted. Why study something that we already know? For example, a person does not need to study formal logic to know that if p implies q and q implies r, then p also implies r (this is the famous law of *syllogism* applied to the three statements p, q, r). Trying to explain this principle is, in a sense, just as difficult as trying to describe how we manage to walk: "Well… I just *walk!*" As we will see in this chapter, there are various ways to formalize the rules of logic and check for *valid inferences* (the technical term for "correct reasoning"). To accomplish this, however, we will first need to produce a symbolic writing that can be applied to those particular sentences in our language that contain logic. Logic provides the basis for argumentation and valid reasoning, but it goes much further. As it turns out, the syntax that rules the grammar in our language is also determined by logic.

A (Very) Brief History of Logic

Logic is one of the oldest branches of mathematics and philosophy. While concepts of logic were formally studied in Ancient India and China, it was the Greek philosopher Aristotle (4^{th} century BC) who first systematically studied logic in a collection of treatises titled *The Organon* ("The Instrument"). In this work Aristotle examined the properties of inferential systems, much in the same spirit as the modern study of predicate (or *first-order*) logic. In the late 3rd century BC, Chrysippus, a leading Greek Stoic philosopher, created an original framework for propositional (or *zeroth-order*) logic[1]. Aristotle's sophisticated approach in *The Organon* led to a highly developed logical theory that remained influential for over two thousand years. Much like Euclid's classical geometry, Aristotle's work on logic reigned supreme and uncontested in Western culture until the mid-19th century, when the British logicians George Boole (1815 – 1864) and Augustus De Morgan (1806 – 1871) introduced symbolic logic and linked logic to a binary algebra with 0's and 1's denoting the truth value of propositions. This is the Boolean algebra that provides the current framework for digital computing. Later, the German logician Gottlob Frege (1848 – 1925) presented logic as a more fundamental branch of mathematics and arithmetic. With his work starts the formal axiomatization of logic that would later become the metatheories of Russell, Ludwig Wittgenstein (1889 – 1951), Kurt Gödel (1906 – 1978) and other major figures in 20th century logic.

Statements

All languages feature a variety of sentences, including declarations, commands, opinions, questions, hypotheticals, exclamations, wishes, clichés, paradoxes, etc. Out of all these kinds of sentences, we only consider those that declare, or assert, something that can be determined to be either true or false, but not both. These are called *statements*, or *propositions*. The requirement that a statement be either true or false is called the *Law of Excluded Middle.* The exclusion of paradoxical sentences that can be both true *and* false (or neither) is called the *Law of Contradiction.* These laws are two principles of classic thought that have been known since antiquity.

> DEFINITION: A *statement,* or *proposition*, is a sentence that asserts something that is either true or false, but not both.

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[1] Predicate logic is a more powerful system of logic built upon the principles of propositional logic. We will explain the key differences between these two systems later in this section.

The property of being true or false is called the *truth value*[2] of the statement. What this definition requires then, in accordance with the *Law of Excluded Middle*, is for a statement to have exactly one truth value.

EXAMPLES:

1) "The word *dog* has two vowels."
2) "The number $2017^{2017} + 1$ is prime."
3) "All squares are rectangles."
4) "$x + 1 = 4$."
5) "If I could fly to the moon, then I would spend little time here on Earth."
6) "*Le Bernardin* is the finest seafood restaurant in New York City."
7) "This sentence contains less than seven words."

The first four sentences above are statements. The first sentence is a false statement since the word *dog* has only one vowel. The second sentence is a statement since any natural number greater than 1 is either prime or not. Note that, in this case, the truth value of the statement can only be determined by resorting to sophisticated algorithms and using vast computational resources. (This number has over six thousand digits!) The third sentence is a true statement since all squares are special types of rectangles. Finally, the fourth sentence is called an *open* statement since fixing the value of x makes this algebraic sentence either true or false. Here, the statement is true if $x = 3$ and false if $x \neq 3$.

The last three sentences above are not statements. The fifth sentence is hypothetical in nature and, as a result, there is no straightforward way to determine its truth value[3]. The sixth sentence is an opinion that is subjective in nature and, as a result, does not have a clear truth value. Here the property of being the *finest* restaurant is not a well-defined predicate. Finally, the seventh sentence is a logical paradox (also called an *antinomy*). If assumed true, then it contradicts its original assertion of containing less than seven words (since it has exactly seven words). If assumed false, then it asserts something true, contradicting itself again!

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[2] In Boolean algebra, true and false statements are given, respectively, numerical values of 1 and 0.

[3] Sentences like these can be studied with the tools of *modal logic*, a non-classical branch of logic developed in the 1960's.

Sets and Language

There is a deep link between the theory of sets we examined in the last chapter and statements, the basic units of language. Consider the statement

> "Peewee is a penguin featured in the 2005 movie *March of the Penguins*."

If we define P as the set of all penguins featured in the movie *March of the Penguins* and let p denote Peewee, we can then write this statement using set notation as follows:

> "$p \in P$."

All basic statements consist of a subject (p) followed by a predicate or property that applies to this subject (P). We can therefore reduce all basic statements in language to ones asserting the membership of a subject to a set ($p \in P$). For instance, in statements such as "Peewee likes to eat sardines" and "Peewee looks adorable" P can be defined, respectively, as the set of all creatures that like to eat sardines and the set of all things that look adorable.

If statements become more complex, the situation remains unchanged. Consider the statement

> "Peewee is a bird that doesn't fly and has no claws."

If we now define B as the set of all birds, F as the set of all flying creatures, and C as the set of all clawed things, we can once again write this statement in set notation as follows:

> "$p \in [B - (F \cup C)]$."

It seems then that the logic contained in the statements of language can be handled effectively by the operations of sets. As we shall explore later in this chapter, this idea is far-reaching and can be applied in all areas of propositional logic, including the study of conditional statements and arguments.

Propositional vs. Predicate logic

In chapter seven of Lewis Carroll's *Through the Looking-Glass*, there is a scene in which the White King asks Alice whether she sees any of the two messengers he sent for. Alice tells him "I see nobody on the road" and to this the king fretfully replies "I only wish I had such eyes. To be able to see Nobody!" The funny thing about this exchange is how it plays on the different meanings one can assign to the word "nobody." What exactly are the two characters referring to when they use the word "nobody"? For the King, "Nobody" (note the capital "N") is the name for someone he wished to see, whereas for Alice "nobody" refers to... well... no one! She is using the word as a quantifier with its usual literal meaning "not-a-body," while the king is using it as an actual name for someone walking down the road. This distinction is one that plays a big role in modern logic. Suppose P is the set of all people Alice sees walking down the road. Then, on one hand, the King's perspective is positing that there is an element in P – namely the person called Nobody. On the other hand, Alice is stating that P is empty. These different situations highlight the fact that words like "nobody" have a dual semantic function in language. While propositional logic is limited to statements where a single subject belongs to P ("$p \in P$"), predicate logic considers statements where a whole *class* of subjects (or any subset of it) belongs to P. These statements, which include what are called *quantifiers*, have the form "$All\ p \in P$," "$Some\ p \in P$", or "$No\ p \in P$". The logical study of these statements require the introduction of propositional *functions* and, as a result, lead to a more sophisticated system of logic. In the following discussion on logic, we focus on propositional logic and only touch briefly on the more powerful system of predicate logic.

Logical Connectives and Compound Statements

Basic statements such as "It is raining" and "There are clouds in the sky" can be joined together to form more complex statements such as "It's <u>not</u> raining <u>but</u> there are clouds in the sky," or "<u>If</u> there are <u>no</u> clouds in the sky, <u>then</u> it's <u>not</u> raining." Here, the underlined words are called logical connectives. These words provide the logical relation between the individual components (the basic statements) in the overall statement. More complex statements formed by linking basic statements with such logical connectives are called *compound* statements. Most languages provide myriad ways of expressing the underlying logic of a compound statement. There is, indeed, a long list of possible words you may encounter in a sentence that act as logical connectives. For example, the words "but", "yet", "while", or even a simple comma, could all act as the conjunction "and". Similarly, words like "unless", "provided", and "when", may all act as the conditional connective "if... then..." Luckily, the complexities of language reduce to only five fundamental logical connectives, which are all summarized in the table below. These five connectives are all that is needed to form compound statements out of basic ones.

NEGATION	**"NOT"** (also: "no", "un-", "dis-")
CONJUNCTION	**"AND"** (also: "but", "yet", "while")
DISJUNCTION	**"OR"** (also: "otherwise")
CONDITIONAL (IMPLICATION)	**"IF… THEN…"** (also: "when", "unless", "provided", "implies")
BICONDITIONAL (EQUIVALENCE)	**"IF AND ONLY IF"** (also: "equals", "equivalent")

Table 1: The Five Logical Connectives

Symbolic Notation

In the precise setting of propositional logic we use symbols to represent the five logical connectives listed in Table 1. This is a convenient and elegant way to reduce all compound statements to symbolic sentences. To do so, the basic components of a compound statement are represented by lower-case letters ($p, q, r,$ ....) and the logical connectives are represented by specific symbols ($\sim$, $\wedge$, $\vee$, $\rightarrow, \leftrightarrow$). Below is a table summarizing the symbols used for each of the five logical connectives.

NEGATION	"Not p"	$\sim p$ or $\neg p$
CONJUNCTION	" p and q"	$p \wedge q$
DISJUNCTION	" p or q"	$p \vee q$
CONDITIONAL (IMPLICATION)	"If p, then q" ("p implies q")	$p \rightarrow q$
BICONDITIONAL (EQUIVALENCE)	" p if and only if q" ("p is equivalent to q")	$p \leftrightarrow q$ ($p \equiv q$)

Table 2: Symbols Used for Logical Connectives

Let's revisit the compound statements presented earlier. If we let p represent the basic statement "It is raining" and q represent the basic statement "There are clouds in the sky", then we can write these compound statements symbolically as follows:

Verbal statement: "It's not raining but there are clouds in the sky"
Symbolic statement: $\sim p \wedge q$

Verbal statement: "If there are no clouds in the sky, then it's not raining."
Symbolic statement: $\sim q \rightarrow \sim p$

Note that p and q are basic statements since they are either true or false and both do not include logical connectives.