Section 3.6--A Summary of Curve Sketching

Analyzing the Graph of a Function

1) Determine the domain and range of the function
2) Determine the intercepts, asymptotes, and symmetry of the graph
3) Locate the x-values for which $f'(x)$ and $f''(x)$ either are zero or do not exist. Use the results to determine relative extrema and points of inflection

Example 1

Analyze and sketch the graph of $f(x)=\dfrac{2(x^2-9)}{x^2-4}$

$f(x)=\dfrac{2(x^2-9)}{x^2-4}=\dfrac{2(x+3)(x-3)}{(x+2)(x-2)}$ $\quad x\neq\pm2$

$f'(x)=\dfrac{20x}{(x^2-4)^2}$ $\quad$ c.n. $x=0$

$f''(x)=\dfrac{-20(3x^2+4)}{(x^2-4)^3}$ $\quad$ possible poi: none

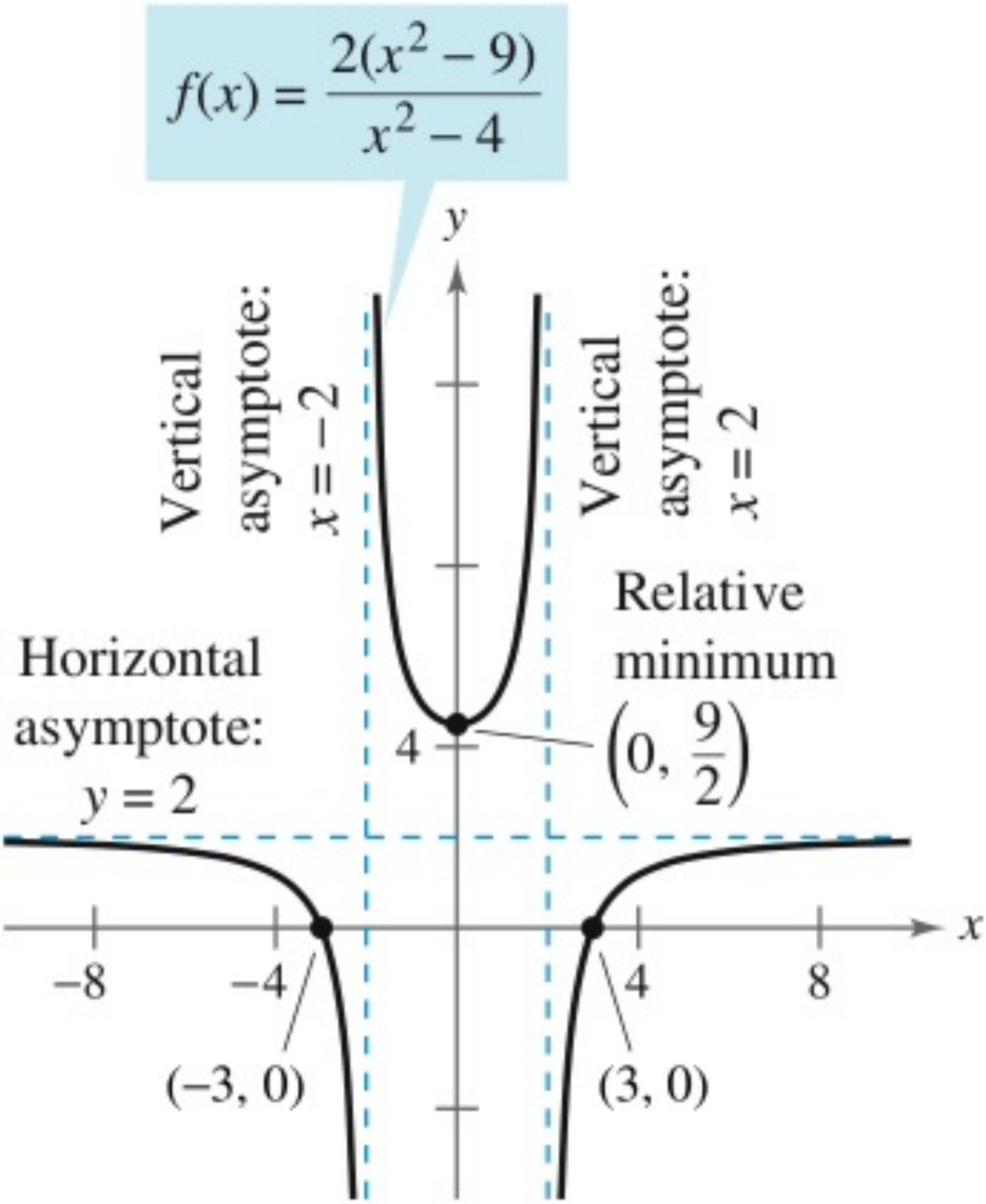
$f''(0)=\frac{5}{4}>0$

graph this info 1st

1) Domain: all $\mathbb{R}$ except $x=-2,2$ — check denominator of f(x)
2) x-int: $(-3,0)$ $(3,0)$ — set $f(x)=0$ & solve
   y-int: $(0,\frac{9}{2})$ — find $f(0)$
   v.a. $x=-2$, $x=2$ — set denominator $=0$
   h.a. $y=2$ — $\lim_{x\to\infty} f(x)$
3) relative minimum $(0,\frac{9}{2})$ — Derivative tests

Ti	x	f(x)	f'(x)	conc	f''(x)	conc
$(-\infty,-3)$	-5	1.524	$-$	dec	$-$	conc ↓
$(-3,-2)$	-2.5	-2.444	$-$	dec	$-$	conc ↓
$(-2,0)$	-1	5.333	$-$	dec	$+$	conc ↑
$(0,2)$	1	5.333	$+$	inc	$+$	conc ↑
$(2,3)$	2.5	-2.444	$+$	inc	$-$	conc ↓
$(3,\infty)$	5	1.524	$+$	inc	$-$	conc ↓

Use asymptotes, intercepts, and critical numbers to help set up test intervals. Big table helps to find extra points to plot in addition to learning the behavior of the graph (inc/dec, concave up/down).


$f(x) = \dfrac{2(x^2 - 9)}{x^2 - 4}$
Vertical asymptote: $x = -2$
Vertical asymptote: $x = 2$
Relative minimum $\left(0, \dfrac{9}{2}\right)$
Horizontal asymptote: $y = 2$
$(-3, 0)$
$(3, 0)$
y
x
4
−8
−4
4
8

Example 2

Analyze and sketch the graph of $\frac{x^2-2x+4}{x-2}$

$f(x)=\frac{x^2-2x+4}{x-2}$ $\quad f'(x)=\frac{x(x-4)}{(x-2)^2}$ $\quad f''(x)=\frac{8}{(x-2)^3}$

$x\neq 2$ $\quad$ c.n. $x=0,4$ $\quad$ c.n. none

1) Domain: all $\mathbb{R}$ except $x=2$

2) x-int: none $\qquad x=\frac{2\pm\sqrt{4-16}}{2}$

y-int: $(0,-2)$

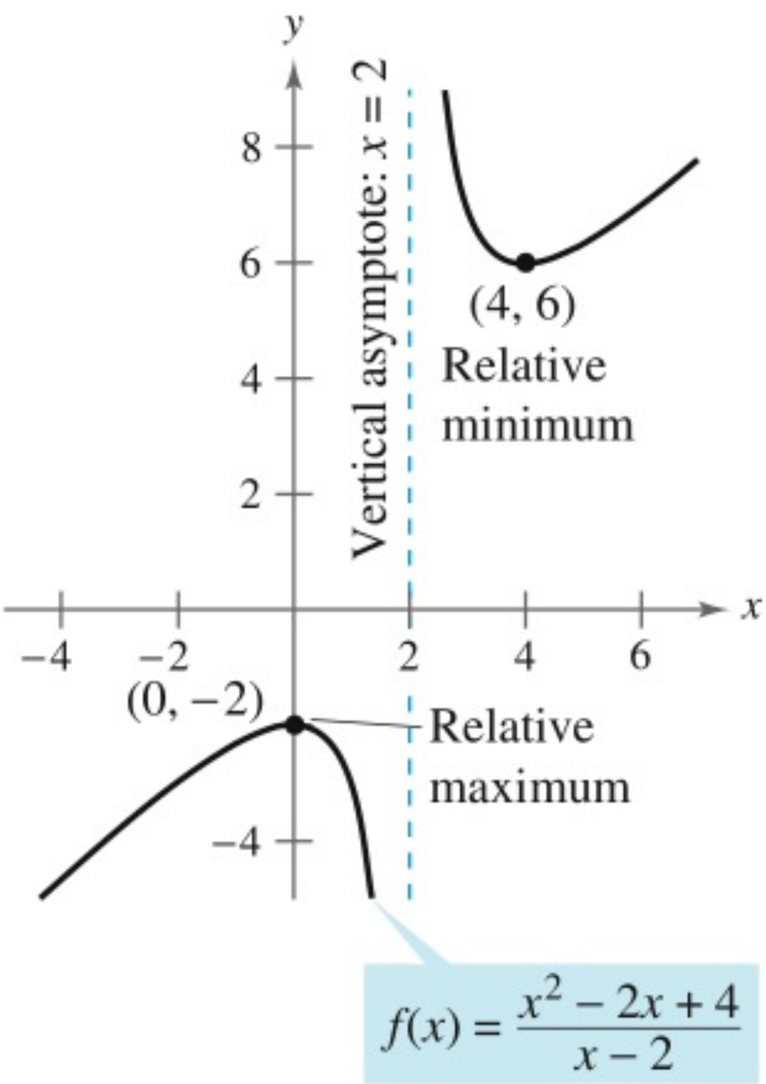
v.a. $x=2$

h.a. none

3) relative maximum $(0,-2)$

relative minimum $(4,6)$

TI	x	f(x)	f'(x)	conc	f''(x)	conc
$(-\infty,0)$	-1	$-\frac{5}{3}$	+	inc	−	concave ↓
$(0,2)$	1	-3	−	dec	−	concave ↓
$(2,4)$	3	7	−	dec	+	concave ↑
$(4,\infty)$	5	$\frac{19}{3}$	+	inc	+	concave ↑



Example 3

Analyze and sketch the graph of $f(x)=\frac{x}{\sqrt{x^2+2}}$

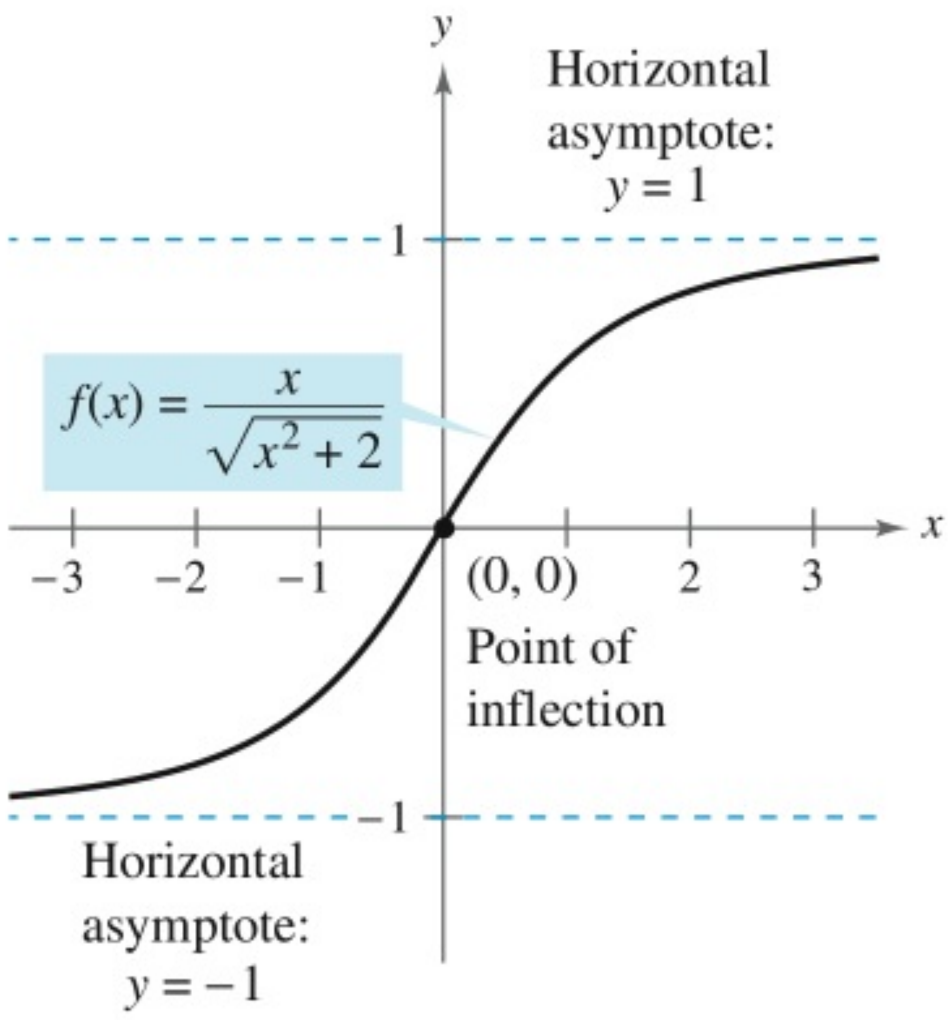
$f(x)=\frac{x}{\sqrt{x^2+2}}$ no rest.

$f'(x)=\frac{2}{(x^2+2)^{3/2}}$ c.n. none

$f''(x)=\frac{6x}{(x^2+2)^{5/2}}$ c.n. $x=0$

1) Domain: all $\mathbb{R}$
2) x-int: $(0,0)$
   y-int: $(0,0)$
   no v.a.
   h.a. $y=-1$, $y=1$ $\quad \lim_{x\to\infty} f(x)=1 \quad \lim_{x\to-\infty} f(x)=-1$
3) pt of inflection $(0,0)$

Ti	x	f(x)	f'(x)	conc	f''(x)	conc
$(-\infty,0)$	-1	$-\frac{1}{\sqrt{3}}$	+	inc	−	concave ↓
$(0,\infty)$	1	$\frac{1}{\sqrt{3}}$	+	inc	+	concave ↑



Example 4

Analyze and sketch the graph of $f(x) = 2x^{5/3} - 5x^{4/3}$

$f(x) = x^{4/3}(2x^{1/3} - 5)$ no restr.

$f'(x) = \frac{10}{3}x^{1/3}(x^{1/3} - 2)$ c.n. $x = 0, 8$

$f''(x) = \frac{20(x^{1/3} - 1)}{9x^{2/3}}$ c.n. $x = 0, 1$

1) Domain: all $\mathbb{R}$

2) x-int: $(0,0)$, $(0, \frac{125}{8})$

y-int: $(0,0)$

v.a. none

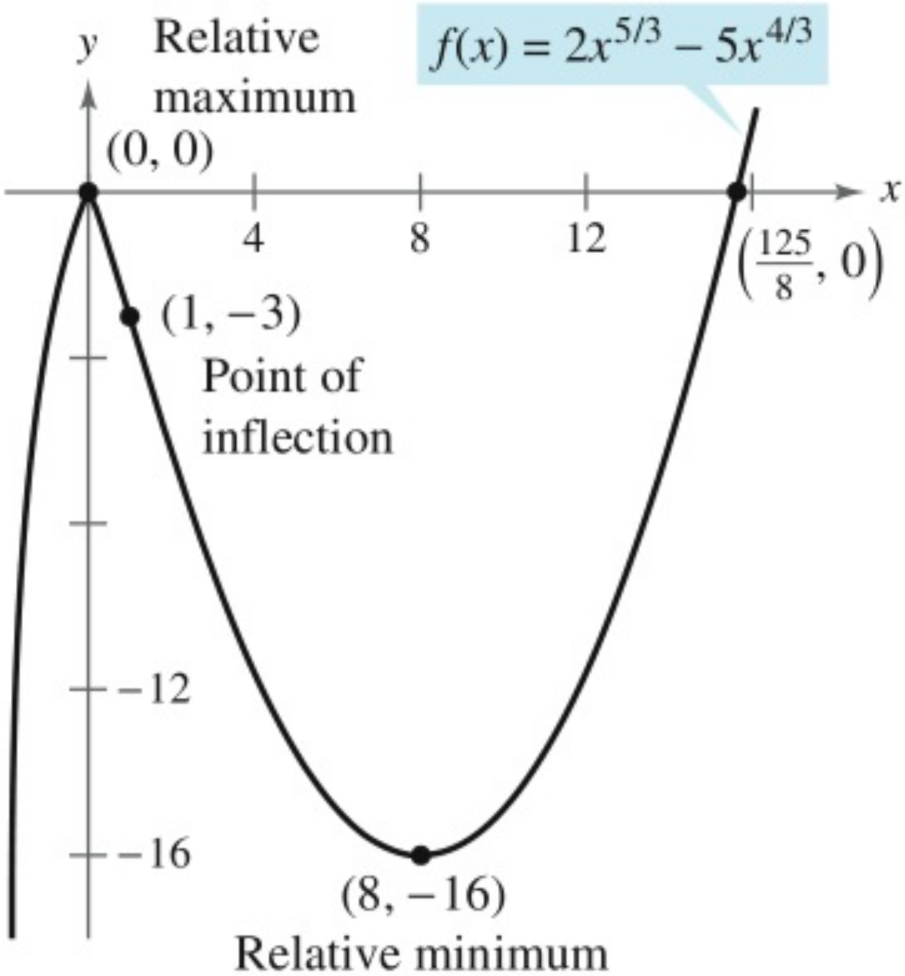
h.a. none

3) relative min: $(8, -16)$

relative max: $(0,0)$

pt. of inflection: $(1, -3)$

Ti	x	f(x)	f'(x)	conc	f''(x)	conc
$(-\infty, 0)$	-1	-7	+	inc	−	concave ↓
$(0, 1)$	$\frac{1}{2}$		−	dec	−	concave ↓
$(1, 8)$	2		−	dec	+	concave ↑
$(8, \infty)$	9		+	inc	+	concave ↑



Example 5

Analyze and sketch the graph of $f(x) = x^4 - 12x^3 + 48x^2 - 64x$

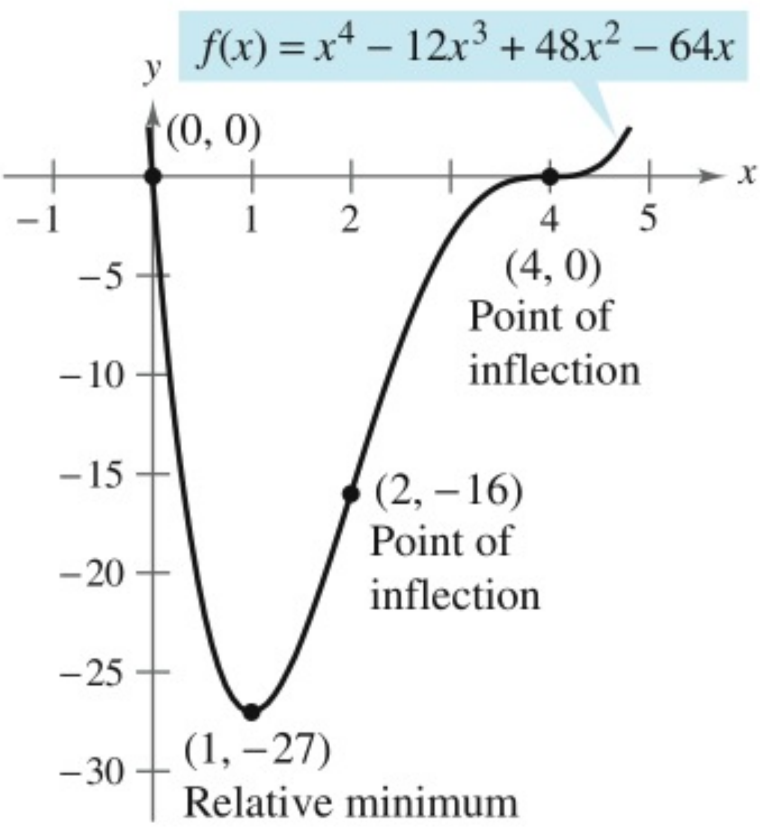
$f(x) = x(x-4)^3$ — no rest.

$f'(x) = 4(x-1)(x-4)^2$ — c.n. $x = 1, 4$

$f''(x) = 12(x-4)(x-2)$ — c.n. $x = 2, 4$

1) Domain: all $\mathbb{R}$
2) x-int: $(0,0)$, $(4,0)$
   y-int: $(0,0)$
   v.a. none
   h.a. none
3) relative min: $(1,-27)$
   pts. of inflection: $(2,-16)$, $(4,0)$

Ti	x	f(x)	f'(x)	conc	f''(x)	conc
$(-\infty,0)$	-1	-125	$-$	dec	$+$	concave ↑
$(0,1)$	$\frac{1}{2}$		$-$	dec	$+$	concave ↑
$(1,2)$	$\frac{3}{2}$		$+$	inc	$+$	concave ↑
$(2,4)$	3	-3	$+$	inc	$-$	concave ↓
$(4,\infty)$	5	5	$+$	inc	$+$	concave ↑



Factoring work for Example 5:

$f(x) = x(x^3 - 12x^2 + 48x - 64) = x(x-4)^3$

$f'(x) = 3x(x-4)^2 + (x-4)^3(1) = (x-4)^2(4x-4) = 4(x-4)^2(x-1)$ cn $x = 1, 4$

$f''(x) = 8(x-4)(x-1) + 4(x-4)^2 = 4(x-4)(2x-2+x-4) = 4(x-4)(3x-6)$

$= 12(x-4)(x-2)$ possible pt of inf $x = 2, 4$

Example 7

Analyze and sketch the graph of $f(x) = \frac{\cos x}{1+\sin x}$ Period: 2π Restrict domain $(-\frac{\pi}{2}, \frac{3\pi}{2})$

$f(x) = \frac{\cos x}{1+\sin x}$ $f'(x) = -\frac{1}{1+\sin x}$ $f''(x) = \frac{\cos x}{(1+\sin x)^2}$

$x \neq \frac{3\pi}{2},$ c.n. $x = \frac{3\pi}{2},$ c.n. $x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$

x-int: $(\frac{\pi}{2}, 0)$

y-int: $(0, 1)$

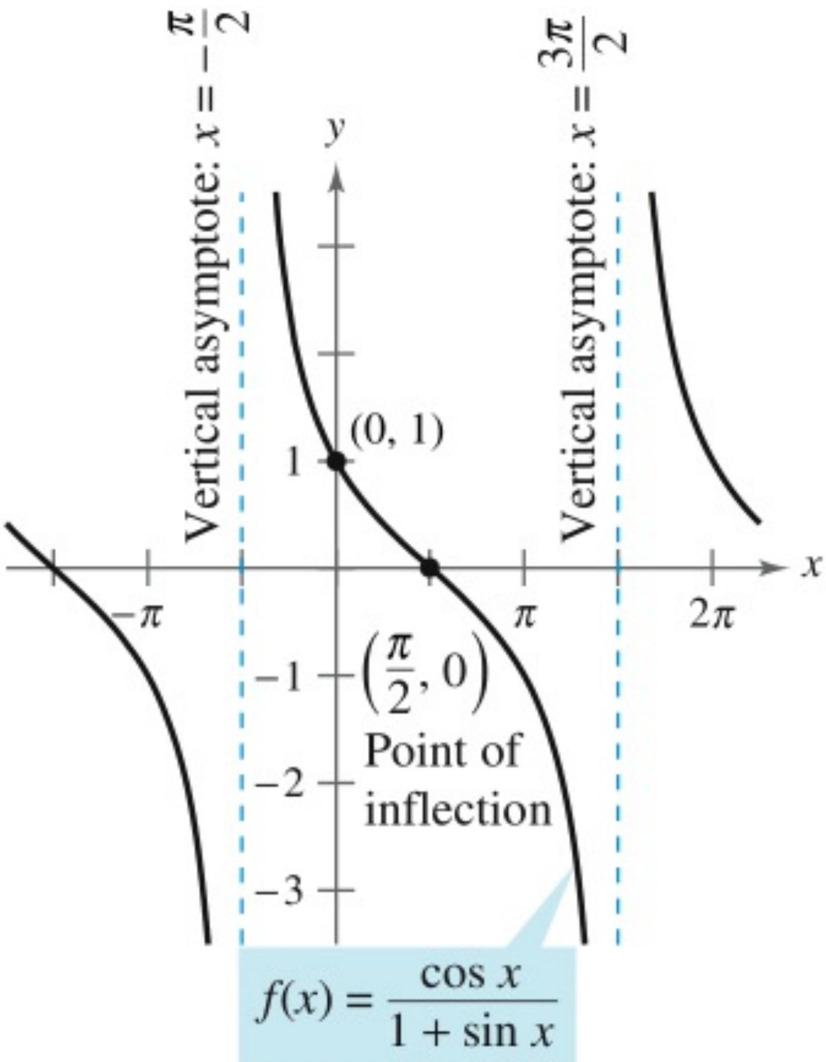
v.a. $x = -\frac{\pi}{2}, \frac{3\pi}{2}$

h.a. none

no extrema

pt. of inflection $(\frac{\pi}{2}, 0)$

TI	x	f(x)	f'(x)	conc	f''(x)	conc
$(-\frac{\pi}{2}, \frac{\pi}{2})$	0	1	−	dec	+	concave ↑
$(\frac{\pi}{2}, \pi)$	$\frac{3\pi}{4}$		−	dec	−	concave ↓
$(\pi, \frac{3\pi}{2})$	$\frac{5\pi}{4}$		−	dec	−	concave ↓



Assignment: