Notes on Synchronisation
I. Control Theory Perspective

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1 Introduction

Synchronisation, as its etymology suggests, may be defined as the adjustment of rhythms of repetitive events (phenomena, processes, ...) through weak interaction.

Let us underline the following concepts related to synchronisation:

Repetition of a process or an event. We consider the synchronisation of phenomena that occur independently of each other *i.e.*, by their own "force" and which repeat themselves through time. Let us stress that *repetition* does not imply periodicity (events repeat themselves every exact constant amount of time measures) but it indicates that the processes happens over and over again, regularly. The

rhythm marks the pace of the repetition for a process. Therefore, two processes will be synchronised if they have the same rhythm. To fix the ideas, one may think of two pendulum clocks oscillating with different frequencies and out of phase; each at its own rhythm. If the pendula are decoupled physically, that is, if the movement of one does not exert any influence on the movement of the second whatsoever, the clocks will continue oscillating at their own pace. Instead, the clocks may adjust their rhythms one another if there exists a

weak interaction *i.e.*, a physical coupling. For instance, two pendulum clocks hanging from a beam will transmit each other vibrations of very small intensity through the beam. As a result, and this depending on other factors, the two clocks may start to oscillate at the same frequency, after some time.

Such was the observation that C. Huygens, the Dutch Natural Philosopher made accidentally when attempting to construct a clock capable to measure time at sea, back in the 17th century.

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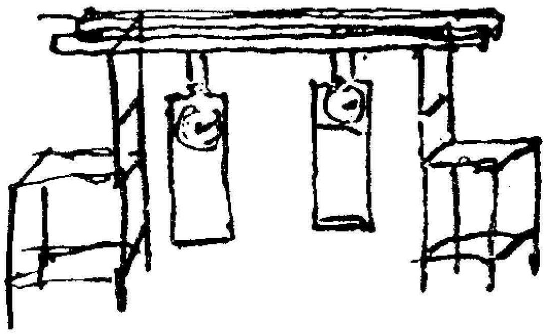

Figure 1. Huijgens's synchronised pendula

One can identify various types of synchronisation. Firstly, one can make the distinction between self and controlled synchronisation.

Self-synchronisation pertains to the case when two or more otherwise free oscillators interact through a weak coupling without external stimuli and influence each other's rhythm until all attain a synchronised motion; an adjustment of rhythms.

Controlled synchronisation pertains to the case when via external stimuli, two or more systems are forced to enter in synchrony. Two forms of controlled synchronisation may be distinguished:

mutual synchronisation, in which case two or more processes adjust their rhythms with no particular priority on either one's rhythm;

master-slave synchronisation, in which case on system imposes its own rhythm of motion to the second. The dominant process is commonly called the *master* and the entrained system is called the *slave*.

Self-synchronisation may be observed in a number of natural phenomena. As a matter of fact, as soon as the universal (nonetheless abstract) concept of $time$ enters into the equation one immediately can make a link to the concept of time-keeper, *clock*. From here to synchrony is but a small step. Endless examples of synchronised clocks may be found in nature: the movements of planets, the circadian rhythm synchronised with the day/night cycle, the heart pace maker cells, neuron firing, *etc*. Synchronised processes which may cause bigger astonishment and curiosity are the famous examples of fireflies' synchronised lightening. It has been observed that a group of fireflies initially lightening "randomly" will eventually light on and off in a perfect synchrony as if led by a director yet, it has been proved that this is a case of mutual self synchronisation.

Other situations that relate to the concept of synchronisation are group formation of animals such as banks of fish, flocks of migrating birds, *etc.*

Such natural phenomena have inspired engineers and scientists to study both, self and controlled synchronisation, for a number of decades and within a variety of domains including (nonlinear) physics, biology, medicine, mechanical engineering, mechatronics, robotics, computer science, *etc.*

Through these Lecture Notes we intend to give a broad yet introductory perspective on Synchronisation. We explore synchronisation as understood by physicists by studying the particular but extensive topic of synchronisation of chaotic oscillators; we study the consensus paradigm, a subject of research highly popular in computer science and network communications engineering; we briefly discuss consequences of synchronisation phenomena of biological cells, in particular, as cause of certain brain diseases; we study controlled synchronisation of mechanical systems. The leading thread is control and stability theory. We often approach the analysis of synchronisation using stability theory tools while we design mechanisms of forced synchronisation based on modern nonlinear control techniques such as but not exclusively, observer design.

2 Synchronisation, mathematically speaking

We have spoken so far, of synchronisation (of oscillators) as being the *adjustment* of rhythms due to weak interaction. At this point, we emphasise the word adjustment and give it a mathematical meaning. Consider a set of N general nonlinear systems

$$\begin{aligned}
\dot{x}_1 &= F_1(t,x_1) + G_1(t,x_1,\dots,x_N) \qquad (t,x_i) \in \mathbb{R}\times\mathbb{R}^n, \quad i \in 1,\dots,N \\
&\vdots \\
\dot{x}_i &= F_i(t,x_i) + G_i(t,x_1,\dots,x_N) \\
&\vdots \\
\dot{x}_N &= F_N(t,x_i) + G_N(t,x_1,\dots,x_N)
\end{aligned}$$

with drifts F_i and interconnections G_i. Let $Q : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ be a continuous function. We say that the systems are synchronised with respect to Q if

$$\lim Q(t, x_1(t), x_2(t)) = 0$$

In the case, of autonomous systems that is, the synchronisation of autonomous oscillators one may restate the problem for systems of the form

$$\begin{aligned}
\dot{x}_1 &= F_1(x_1) + G_1(x_1,\dots,x_N) \qquad x_i \in \mathbb{R}^n, \quad i \in 1,\dots,N \\
&\vdots \\
\dot{x}_i &= F_i(x_i) + G_i(x_1,\dots,x_N) \\
&\vdots \\
\dot{x}_N &= F_N(x_i) + G_N(x_1,\dots,x_N)
\end{aligned}$$

and a function $Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that the trajectories satisfy

$$\lim Q(x_1(t), x_2(t)) = 0\,.$$

Indeed, the latter is a fairly general way of stating the synchronisation paradigm. It concerns couplings of dynamic systems which may be forced or unforced. In a number of situations the significant synchronisation problem may concern the whole state for instance, Q may be defined as $[x_1 - x_2 \ \cdots \ x_{N-1} - x_N \ x_N - x_1]^\top$ or possibly, if only two states are significant to be sinchronised, $Q = [x_{1i} - x_{2i} \ \cdots \ x_{N-1,i} - x_{N,i} \ x_{N,i} - x_{1,i}]^\top$.

Master-slave synchronisation

This situation pertains to the case in which a slave system, say described by the dynamical equation

$$\dot{x}_s(t) = f(t, x_s(t)) \quad x_s(t_\circ) \in \mathbb{R}^n \, t \geq t'_\circ \geq 0 \tag{1}$$

is synchronised with a master system

$$\dot{x}_m(t) = f(t, x_m(t)) \quad x_m(t_\circ) \in \mathbb{R}^n \, t \geq t_\circ \geq 0 \tag{2}$$

in the sense that the master system performs a free motion whereas the slave system must follow the movement of the master's. Hence, we say that the systems are synchronised if for any initial conditions $t_{m\circ}$, $t_{s\circ}$, $x_{m\circ}$, and $x_{s\circ}$ the respective motions of systems (2) and (1) satisfy:

$$\lim_{t\to\infty} |x_s(t) - x_m(t)| = 0. \tag{3}$$

Case-study: synchronisation of two forced pendula

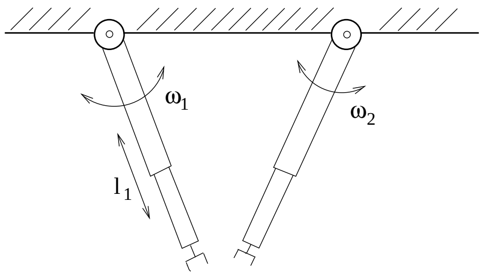

Figure 2. Synchronisation of two mechanical pendula

Let us consider the problem of synchronising the two pendula showed in Fig. 2. That is, we consider the problem of making the "slave" pendulum oscillate at the frequency of the "master", while assuming that *no control action* is available *at the joints*. Instead, we desire to achieve our task by modifying on line the length of the slave pendulum thereby, its oscillating frequency which is given by

$$\omega_1 = \sqrt{\frac{l_1}{9.81}}.$$

The dynamic equations are

$$\begin{aligned} \text{slave:} \qquad & \ddot{y} + 2\zeta_1\omega_1\dot{y} + \omega_1^2 y &=& \; a_1 \cos\omega_1 t \\ \text{master:} \qquad & \ddot{y}_d + 2\zeta_2\omega_2\dot{y}_d + \omega_2^2 y_d &=& \; a_2 \cos\omega_2 t \end{aligned}$$

where the control input corresponds to the change in ω_1, i.e.,

$$\dot{\omega}_1 = u\,.$$

If $\omega_2 > 0$, $\zeta_1 = \zeta_2$, $a_1 = a_2$, using a cascades approach it is easy to prove that the linear control law $u = -k\tilde{\omega}$, with $k > 0$ makes that

$$\lim_{t\to\infty} \tilde{\omega}(t) = 0 \qquad \lim_{t\to\infty} \tilde{y}(t) = 0\,.$$

where $\tilde{\omega}(t) := \omega_1(t) - \omega_2$ and $\tilde{y}(t) := y(t) - y_d(t)$. One only needs to observe that, defining

$$v := \ddot{y}_d + 2\zeta\omega_2\dot{y}_d + \omega_2^2 y_d - a\cos\omega_2 t = 0$$

and $z := [\tilde{y},\, \dot{\tilde{y}}]^\top$, the two pendula dynamic equations

$$\begin{array}{l} \ddot{y} + 2\zeta\omega_1\dot{y} + \omega_1^2 y = a\cos\omega_1 t + v \\ \ddot{y}_d + 2\zeta\omega_2\dot{y}_d + \omega_2^2 y_d = a\cos\omega_2 t \end{array}$$

and the control law, are equivalent to

$$\begin{array}{rcl} \left\{ \begin{array}{l} \dot{z}_1 \\ \dot{z}_2 \end{array} \right. & \begin{array}{c} = \\ = \end{array} & \begin{array}{l} z_2 \\ -2\zeta\omega_2 z_2 - \omega_2^2 z_1 + g_2(t, z, \tilde{\omega}) \end{array} \\ \dot{\tilde{\omega}} & = & -k\tilde{\omega} \end{array}$$

where $g_2(t, z, \tilde{\omega}) = 2\zeta\tilde{\omega}z_2 + 2\zeta\tilde{\omega}\dot{y}_d(t) - \tilde{\omega}^2(z_1 + y_d(t)) - 2\tilde{\omega}\omega_2 y_d + a(cos\omega_1 t - \cos\omega_2 t)$. Notice that this system is of the form

$$\dot{z} = f_1(z) + g(t,\, z,\, \tilde{\omega}) \qquad (4)$$

$$\dot{\tilde{\omega}} = -k\tilde{\omega} \qquad (5)$$

where $g(t, z, \tilde{\omega}) := [0,\, g_2(t, z, \tilde{\omega})]^\top$ and clearly, $\dot{z} = f_1(z)$ is exponentially stable. It occurs that, since the "growth" of the function $g(t, \cdot, \tilde{\omega})$ is linear for each fixed $\tilde{\omega}$ uniformly in t that is, for each fixed $\tilde{\omega}$ there exists $c > 0$ such that $|g(t, z, \tilde{\omega})| \leq c\,|z|$ for all $t \geq 0$, the *cascaded system* (4) is globally asymptotically stable. Roughly speaking, the exponentially stable dynamics $\dot{z} = f_1(z)$ dominates over the interconnection term, no matter how large the input $\tilde{\omega}(t)$ gets.

Mutual synchronisation

Let us consider a simple but intuitive example. The synchronisation of two harmonic oscillators

$$\begin{array}{rclcrcl} \dot{x}_1 & = & -\omega x_2 & \qquad & \dot{y}_1 & = & -\omega y_2 \\ \dot{x}_2 & = & \omega x_1 & & \dot{y}_2 & = & \omega y_1 \end{array}$$

The solutions are given by $x_1(t) = A(x_o)\cos(\omega t)$ and $x_1(t) = A(x_o)\sin(\omega t)$ where the amplitude of the oscillations A, depends on the initial conditions; similarly for the "y-oscillator.

By definition, synchronisation takes place if there exists a weak interaction between the systems. Say that the interaction occurs as the value of the state x_1 is somehow "transmitted" to the second oscillator. Then, the dynamics of the latter becomes

$$\begin{aligned}\dot{y}_1 &= -\omega y_2 - \ell(y_1 - x_1), \quad \ell > 0\\ \dot{y}_2 &= \omega y_1 + \ell(y_1 - x_1)\end{aligned}$$

We shall say that the systems are synchronised if

$$\lim_{t\to\infty} |x_i - y_i| = 0, \quad i \in 1, 2.$$

One way of establishing whether synchronisation takes place is by analysing the behaviour of the *error* trajectories. That is, by establishing the convergence property[2] for the error system. Let

$$V(x-y) = \frac{1}{2}|x_1 - y_1|^2 + \frac{1}{2}|x_2 - y_2|^2$$

First, it is clear that

$$\rho_1 |z|^2 \leq V(z) \leq \rho_2 |z|^2$$

$$\frac{dV}{dz} V\Big|_{z=x-y}^{\top} \Big[F(x) - F(x,y)\Big] \leq -\rho_3 |x-y|$$

$$\left|\frac{dV}{dz}(z)\right| \leq \rho_4 |z|$$

where

$$F(x) = \begin{bmatrix} -\omega x_2 \\ \omega x_1 \end{bmatrix}, \qquad F(x,y) = \begin{bmatrix} -\omega y_2 - \ell(y_1 - x_1) \\ \omega y_1 + \ell(y_1 - x_1) \end{bmatrix},$$

Then,

$$F(x) - F(x,y) = \begin{bmatrix} -\ell & -\omega \\ \omega + \ell & 0 \end{bmatrix} \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \end{bmatrix}$$

Now, define

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$$

with $p_1 > 0$ and $p_1 p_3 > p_2^2$ hence P is positive definite and symmetric. We have

$$PA + A^T P = \begin{bmatrix} -2\ell p_1 + 2(\omega + \ell) p_2 & -2\ell p_2 + p_3(\ell + \omega) - \omega p_1 \\ -2\ell p_2 + p_3(\ell + \omega) - \omega p_1 & -2\omega p_2 \end{bmatrix}$$

Let

$$p_3 = \frac{\omega p_1 + \ell p_2}{\ell + \omega}$$

then, the terms in the off-diagonal of $PA + A^T P$ equal zero and the latter is negative definite provided that

$$\frac{\omega p_1^2 + \ell p_1 p_2}{\ell + \omega} > p_2^2$$

[2] See the Notes on Oscillations

which holds for sufficiently large p_1 and sufficiently small p_2. Thus,

$$\frac{dV}{dz}V\Big|_{z=x-y}^{\top}\Big[F(x) - F(x,y)\Big] \leq -\rho_3 \, |x-y|$$

with $\rho_3 > 0$ being a lower-bound on the $\min\{2\ell p_1 - 2(\omega+\ell)p_2,\ 2\omega p_2\}$.

We conclude the exponential convergence of $x(t) \to y(t)$ for any values of ℓ and ω in particular, under weak interaction (small ℓ).

3 Observer-based synchronisation

The paradigm of master-slave synchronisation may be broached, from a control-theory viewpoint, via observer and more generally, estimation theory. After all, note that the problem for the slave system consists in reproducing the motion of the master. Mathematically, it is required that $x_s(t)$ converges to $x_m(t)$ asymptotically[3]. Such problem is well-studied among control theorists as an observer-design paradigm. Moreover, while an abundant and solid theory has been completed for linear systems, there is a considerable bulk of literature on nonlinear observers which nevertheless, is far from being completed.

On Luenberger observers

Let us start our exposition with linear systems particularly, by briefly recalling what a Luenberger observer is. Consider the *linear* system of differential equations[4]

$$\dot{x} = Ax \quad y = Cx \tag{6}$$

where $x \in \mathbb{R}^n$ is the state column vector and C is a *row* vector in $\mathbb{R}^n$. Assume that the *observability* matrix

$$\mathcal{O} := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{7}$$

is of full rank n, which corresponds to the dimension of the state. Indeed, if this *structural* condition holds we are guaranteed to be able to construct and observer for system (6). The Luenberger observer is given by

$$\dot{\hat{x}} = A\hat{x} - L(\hat{y} - y), \quad \hat{y} = C\hat{x} \tag{8}$$

where y is a measured output and L is the observer gain, a design parameter to be chosen appropriately. We are concerned with the problem of ensuring that (3) holds, to that end,

[3]Here, we concentrate on this type of synchronisation otherwise, recall that the synchronisation paradigm even in the master-slave setting is more general since one typically requires a function of (the difference between) x_m and x_s to converge to zero.

[4]By dropping the argument t.

we are interested in studying the (Lyapunov) stability of the error system dynamics:

$$\dot{\bar{x}} = A\bar{x} - L(\hat{y} - y) \tag{9}$$

where we have defined $\bar{x} := \hat{x} - x$. Defining, accordingly, $\bar{A} := A - LC$ we see that Equation (8) also has the form

$$\dot{\bar{x}} = \bar{A}\bar{x}\,. \tag{10}$$

According to classical Lyapunov stability theory the trivial solution $x \equiv 0$ of system (10) is exponentially stable if the following holds: there exist positive definite symmetric matrices P and Q such that

$$\bar{A}^\top P + P\bar{A} = -Q\,. \tag{11}$$

For time-invariant systems (A constant) this is equivalent to having the eigen-values of $\bar{A}$ to be negative. For time-varying systems, the condition to satisfy becomes:

$$\bar{A}(t)^\top P + P\bar{A}(t) + \dot{P} = -Q(t) \tag{12}$$

and, in general, that the eigen-values of $\bar{A}(t)$ are (strictly) negative for all t is not sufficient for stability.

To be on the safe side, we recall that the trivial solution $x \equiv 0$ of a dynamical system (2) is said to be exponentially stable if there exist positive real numbers k and γ such that

$$|x(t)| \le k\ |x_\circ|\,\mathrm{e}^{-\gamma(t-t_\circ)} \quad \forall\, t \ge t_\circ \ge 0\,. \tag{13}$$

Obviously, if x denotes the synchronisation errors between two signals, exponential stability of $x = 0$ is sufficient but not necessary for synchronisation.

The **Example** of the two harmonic oscillators synchronised through weak interaction gives a clear illustration of what observer theory may bring to the study of synchronisation. Consider again the oscillator

$$\begin{aligned} \dot{x}_{m1} &= -\omega x_{m2} \\ \dot{x}_{m2} &= \omega x_{m1} \end{aligned}$$

which may be represented in the compact form

$$\dot{x}_m = Ax_m, \quad A = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}.$$

Suppose that it is required to design a slave system that follows its motion based on the measurement of the trajectory $x_{m1}(t)$. Such a slave system is

$$\dot{x}_s = Ax_s - L(x_{s1} - x_{m1}), \quad \begin{bmatrix} \ell \\ \ell \end{bmatrix} \quad \ell > 0.$$

By defining $C = [1\ 0]$ we immediately recognise the similarity of the slave system above and a Luenberger observer. Only the notation is different. However, the synchronisation error dynamics with state $\bar{x} := x_s - x_m$ is given by

$$\dot{\bar{x}} = [A - LC]\bar{x}, \quad [A - LC] = \begin{bmatrix} -\ell & -\omega \\ \omega + \ell & 0 \end{bmatrix}$$

and it is clear that the eigen-values of $[A - LC]$ are negative for any positive values of ℓ and ω hence, the synchronisation errors converge to zero exponentially.

This is evidently, the same conclusion that we had drawn previously under the same arguments but from a different perspective.

The simplicity with which we solved the previous synchronisation problem is motivating to push the observer technique further. That is, we are left wondering whether it is applicable to nonlinear systems or at least, to nonlinear oscillators such as chaotic systems. Although a similar reasoning may guide us in the synchronisation of nonlinear systems, nonlinear observer design is extremely challenging. In the next chapter we broach some techniques of observer design for systems which are linear in the unmeasured variable. Such a class is in general narrow however, it encompasses a large number of chaotic oscillators.

In anticipation of what we shall study in the chapters on Oscillations and Synchronisation of Chaotic systems, let us have a look at the observer-design problem the famous Lorenz system[5] whose dynamics is given by the bilinear system

$$\begin{aligned} \dot{x}_{m1} &= \sigma(x_{m2} - x_{m1}) \\ \dot{x}_{m2} &= r x_{m1} - x_{m2} - x_{m1}x_{m3} \\ \dot{x}_{m3} &= x_{m1}x_{m2} - b x_{m3} \end{aligned}$$

where σ, b and r are constant parameters. The model is nonlinear but may be rewritten in the affine form

$$\dot{x}_m = A(y)x_m \tag{14a}$$

$$y := x_{m1} \tag{14b}$$

$$A(y) := \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & -y \\ 0 & y & -b \end{bmatrix} \tag{14c}$$

that is, assuming that x_{m1} is measurable. What is remarkable and cannot be overestimated is that the system may be written in a form where the unknown states appear linearly.

The measured output x_{m1} entrains the possible interaction between this and a similar oscillator. Then, the observer (slave) system is given by

$$\dot{x}_s = A(y)x_s - L(y_s - y) \tag{15a}$$

$$y_s := x_{s1}\,. \tag{15b}$$

[5]See Notes on Oscillations.

To ensure synchronisation it is sufficient to guarantee that there exists P positive definite and symmetric such that

$$[A(y) - L]^\top P + P[A(y) - L]$$

is negative definite uniformly in y. This is of course, a difficult-to-verify condition however, it is only sufficient. In the chapter on synchronisation of chaotic systems we shall study methods which require more relaxed assumptions.

4 Synchronisation as a Stabilisation Problem

Now, let us study the problem of synchronisation as a tracking control problem. It should be clear to readers familiarized with the latter that synchronisation in the sense that one motion is *imposed* to another, has all the flavour of a standard tracking control problem.

To fix the ideas we place the discussion in the context of master-slave synchronisation that is, we assume to be given a master system which evolves freely according to a nonlinear dynamics

$$\dot{x}_m = f_m(t, x_m) \quad x_m(t_\circ) \in \mathbb{R}^n\, t \geq t_\circ \geq 0 \tag{16}$$

and a *forced* slave system

$$\dot{x_s} = f_s(t, x_s) + u \quad x_s(t_\circ) \in \mathbb{R}^n\, t \geq t'_\circ \geq 0 \tag{17}$$

which is affected by an external control input.

The problem may be posed in the context of a standard tracking control problem for the system (17) in which it is required that it follows the desired reference trajectory $x_m(t)$. That is, synchronisation is reached if the trajectory tracking control problem is solved. The importance of this simple observation cannot be overestimated. Common knowledge regarding tracking control is vast, especially concerning mechanical and electro-mechanical systems.

Another way to present the problem is the following. Define $e = x_s - x_m$ then,

$$\dot{e} = f_s(t, e + x_m(t)) - f_m(t, x_m(t)) + u$$

that is, the error dynamics depends on the state e and time. Note that the master system states have been replaced by the trajectories generated for a particular pair of initial conditions t_0 and $x_m(t_0)$. The control problem comes to designing u so that the origin $\{e = 0\}$ be stabilised asymptotically.

Depending on the variables that may be measured and the dimension of the control inputs, one may be able to apply different *nonlinear* control techniques such as feedback-linearisation, Lyapunov-based, passivity-based, *etc.* We shall not develop any of these since it is out of scope in this Notes however, we recall next an interesting structure-based technique called cascades-based control. This technique is particularly well-suited for mechanical systems and oscillators such as chaotic systems, in view of their intrinsic stability[6].

[6] See the Notes on Oscillations

Cascades-based synchronisation

The cascaded controlled synchronisation design consists in writing an error-system by subtracting the dynamics of one system from the other and then, to perform a cascades-based control of the error system, to zero. In such manner we bring the synchronisation problem to that of stabilisation of a time-varying system. To accomplish the stability analysis hence, to investigate whether synchronisation is achieved, we use modern tools of qualitative analysis of time-varying systems known as *cascaded* systems.

On cascaded systems

Cascaded systems have the form

$$\dot{\xi}_1 = f_1(t, \xi_1) + g(t, \xi_1, \xi_2) \tag{18a}$$

$$\dot{\xi}_2 = f_2(t, \xi_2) \tag{18b}$$

with state $\xi = [\xi_1^\top,\ \xi_2^\top] \in \mathbb{R}^{n_1+n_2}$ and the interconnection is such that $g(t, \xi_1, 0) \equiv 0$.

Analysis

The origin of the system (18) is uniformly globally asymptotically stable if:

1. the origin of $\dot{\xi}_1 = f_1(t, \xi_1)$ is uniformly globally asymptotically stable;
2. the origin of $\dot{\xi}_2 = f_2(t, \xi_2)$ is uniformly globally asymptotically stable;
3. the solutions of (18) are uniformly globally bounded.

We recall that uniform global asymptotic stability means that:

- the origin is stable in the sense of Lyapunov and uniformly in the initial conditions;
- the solutions are bounded uniformly in the initial conditions and for any size of the initial states;
- the norm of the solutions converge to zero at a rate independent of the initial conditions.

Hence, in particular, if one ensures uniform asymptotic stability of the origin of (18) it follows that

$$\lim_{t\to\infty} |\xi(t)| = 0.$$

From the three numbered items above, boundedness of solutions is necessary and sufficient and is, in general, hard to verify. Fortunately, for certain classes of systems, simple conditions are well established. For cascaded systems (18) it suffices to observe the following extra conditions to guarantee boundedness of the solutions:

C1.– for the system $\dot{\xi}_1 = f_1(t, \xi_1)$ we dispose of a Lyapunov function V_1 such that

$$\left|\frac{\partial V_1(t,\xi_1)}{\partial \xi_1}\right| |\xi_1| \leq c_1 V_1(t,\xi_1) \quad \forall\ |\xi_1| \geq \eta_1$$
$$\left|\frac{\partial V_1(t,\xi_1)}{\partial \xi_1}\right| \leq c_2 \quad \forall\, |\xi_1| \leq \eta_2$$
$$\frac{\partial V_1(t,\xi_1)}{\partial \xi_1} f_1(t,\xi_1) \leq -c_4 \left|\xi_1\right|^2 ;$$

C2.– the interconnection term $g(t,\xi)$ is once continuously differentiable and satisfies

$$|g(t,\xi_1,\xi_2)| \leq \theta_1(|\xi_2|) + \theta_2(|\xi_2|)\, |\xi_1| ;$$

C3.– the trajectories $\xi_2(t)$ are integrable:

$$\int_{t_0}^{\infty} |\xi_2(t)|\, dt \leq c_5 .$$

In the previous conditions all constants are independent of the initial conditions, C1 needs to be verified for a particular choice of η_1 and η_2 and the functions θ_1 and θ_2 are continuous non-decreasing. Condition C1 holds for instance by Lyapunov functions satisfying polynomial bounds such as common quadratic functions. Condition C2 simply imposes that the interconnection term g has linear growth in the variable ξ_1. Condition C3 imposes a speed of convergence on $\xi_2(t)$, it holds in particular if the trajectories $\xi_2(t)$ converge exponentially fast.

Controlled synchronisation

Consider the equations of two identical hyper chaotic systems; the master system with general equations

$$\dot{w}_m = f_w^m(w_m) + g_w^m(w_m, x_m, y_m, z_m) \tag{19a}$$
$$\dot{x}_m = f_x^m(w_x) + g_x^m(w_m, x_m, y_m, z_m) \tag{19b}$$
$$\dot{y}_m = f_y^m(w_y) + g_y^m(w_m, x_m, y_m, z_m) \tag{19c}$$
$$\dot{z}_m = f_z^m(w_y) + g_z^m(w_m, x_m, y_m, z_m) \tag{19d}$$

and a slave system be described by the equations

$$\dot{w}_s = f_w^s(w_s) + g_w^s(w_s, x_s, y_s, z_s) + u_w \tag{20a}$$
$$\dot{x}_s = f_x^s(w_x) + g_x^s(w_s, x_s, y_s, z_s) + u_x \tag{20b}$$
$$\dot{y}_s = f_y^s(w_y) + g_y^s(w_s, x_s, y_s, z_s) + u_y \tag{20c}$$
$$\dot{z}_s = f_z^s(w_y) + g_z^s(w_s, x_s, y_s, z_s) + u_z . \tag{20d}$$

Define $e_{(\cdot)} := (\cdot)_s - (\cdot)_m$. Then, the synchronisation error dynamics takes the form

$$\dot{e}_w = f_w(t, e_w) + g_w(t, e_w, e_x, e_y, e_z) + u_w \quad (21a)$$
$$\dot{e}_x = f_x(t, e_x) + g_x(t, e_w, e_x, e_y, e_z) + u_x \quad (21b)$$
$$\dot{e}_y = f_y(t, e_y) + g_y(t, e_w, e_x, e_y, e_z) + u_y \quad (21c)$$
$$\dot{e}_z = f_z(t, e_z) + g_z(t, e_w, e_x, e_y, e_z) + u_z \quad (21d)$$

where $f_w(t, e_w) = f_w^s(e_w + w_m(t)) - f_w^m(w_m(t))$ and similarly for all the other functions. Note that $f_w(t, 0) \equiv 0$ provided that $f_w^m = f_w^s$ hence, we assume that the two systems are equal (if they are not, they may be rendered equal via control).

Cascaded-based controlled synchronisation reduces to design u_w, u_x, u_y and u_z in a way that the four equations above are constituted of two subsystems of second order, interconnected in cascade. Moreover, the conditions C1–C3 in the Appendix must be verified. For instance, one may choose without much loss of generality, $\xi_1 = [e_w,\ e_x]$ and $\xi_2 = [e_y,\ e_z]$ then, we design $u_w = u_w^a + u_w^b$ and $u_x = u_x^a + u_x^b$ so that:

- the systems $\dot{e}_w = f_w(t, e_w) + u_w^a$ and $\dot{e}_x = f_x(t, e_x) + u_x^a$ be asymptotically stable and locally exponentially stable;
- the systems (21c) and (21d) be asymptotically stable;
- the function $g_w(t, e_w, e_x, e_y, e_z) + u_w^b$ has terms of at most linear growth in e_w and e_x;
- the function $g_x(t, e_w, e_x, e_y, e_z) + u_x^b$ has terms of at most linear growth in e_w and e_x.

Roughly speaking, the previous guidelines lead to the satisfaction of conditions C1–C3 to establish asymptotic stability of the error synchronisation system.

Cascades-based design is structure-oriented and is not systematic. Instead, it relies on the designer's ability to construct a controller for each subsystem. The method's advantage lies in its essential feature: "dividing to conquer"; designing controllers for much simpler systems than originally. As it is generic, it applies to a number of chaotic systems. A key feature is that typically, the nonlinearities $g_{(\cdot)}^m$ are of the second order that is, they consist of bilinear and quadratic terms so the growth restrictions on the nonlinearities is a mild assumption.

5 Bibliographical remarks

A good approach to start the study of synchronisation is to read general science education texts. For instance, we may cite the enjoyable book

- S. H. Strogatz, *Sync: How Order Emerges From Chaos In the Universe, Nature, and Daily Life* Hyperion, 2003

which describes how synchrony is present in nature. The author is a recognized mathematics and physics professor, author of several other texts for instance, on chaos, nonlinear physics and several popular science monographs.

A somewhat more technical yet extremely accessible to any student of science or engineering is

- A. Pikovsky, M. Rosenblum and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences*. Cambridge Nonlinear Science Series, 2003.

The first chapters of this book have greatly inspired this lecturer to organise the introductory material.

An early inspirational text for the control theorist as it is written by a Russian mathematician and researcher on theoretical mechanics, is

- I. I. Blekhman, *Synchronisation in Science and Technology*, ASME Press translations, New York, 1988

The method of observers for synchronisation was originally "exported" to disciplines within physics and engineering other than control theory, by the milestone paper

- H. Nijmeijer and I. Mareels. An observer looks at synchronization. *IEEE Trans. on Circ. Syst. I: Fundamental Theory and Applications*, 44(10):882–890, 1997.

which, for the automatic-control researcher, remains a good survey and tutorial.

Interested readers on the stabilisation approach to synchronisation are invited to see the rapidly growing literature on the topic, in physics and circuits fora. To mention a few:

- *Physical Review Letters*;
- *Physical Review A*;
- *Physics Letters A and E*;
- *Chaos, Solitons and Fractals*;
- *Nonlinearity*;
- *IEEE Trans. Circ. Syst. I and II*;
- *J. Bifurcation and Chaos* . . .

Finally, interest readers may consult the introductory tutorial on cascaded systems in

- A. Loria and E. Panteley, "Cascaded nonlinear time-varying systems: analysis and design, ch. in Advanced topics in control systems theory", *Lecture Notes in Control and Information Sciences*, F. Lamnabhi-Lagarrigue, A. Loria, E. Panteley, eds., London: Springer Verlag, 2005.

The latter is the compendium of lecture notes from the 3rd edition of the Paris Graduate School, precursor of the the EECI Graduate School.