UNIVERSITY OF WATERLOO

Department of Mechanical Engineering
ME 303 Advanced Engineering Mathematics

Spring Term 1999 June 7, 1999
M.M. Yovanovich 4:30–6:30 P.M.

• Open book examination. Spiegel's Text and Spiegel's Handbook of Mathematical Formulas.

• All aids are allowed. Lecture notes, material obtained from the ME 303 Web site, and a calculator.

• All questions must be answered. The marks for each question are clearly stated.

• Show all steps and state clearly all assumptions made.

• Material which is not legible will not be considered.

• The mathematical problems considered in this exam are related to problems in fluid mechanics, heat conduction, dynamics, etc. Solution to one of the problems of the exam can be applied to wire drawing, for example.

• Good luck.

Problem 1.
[30 Marks]

The velocity potential of an infinitely long *line source* located on the $z-$axis is defined by

$$\phi = -C \ln r = -C \ln \sqrt{x^2 + y^2}$$

where C is a constant which is related to the strength of the line source. The corresponding stream function is defined as

$$\psi = -C\theta = -C \arctan \frac{y}{x}$$

Both functions are solutions of the two-dimensional Laplace equations $\nabla^2\phi = 0$ and $\nabla^2\psi = 0$, respectively.

(a) Verify that the velocity potential is the solution of the two-dimensional Laplace equation:

$$\phi_{xx} + \phi_{yy} = 0$$

You may require the relation:

$$\frac{d}{dx} \ln u = \frac{1}{u}\frac{du}{dx}$$

It can be verified that the stream function is the solution of the two-dimensional Laplace equation:

$$\psi_{xx} + \psi_{yy} = 0$$

A Computer Algebra System such as Maple can be used to show that this is true. Do not attempt to verify this fact.

(b) The velocity components in cylindrical coordinates (r, θ) are related to the stream function:

$$u_r = -\frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad \text{and} \qquad u_\theta = \frac{\partial\psi}{\partial r}$$

Determine u_r and u_θ.

(c) The volumetric flow rate $Q\,[m^3/s]$ per unit length of the line source is defined by the relation

$$Q = 2\pi r u_r$$

Obtain the relation for Q.

Problem 2.
[40 Marks]

The following linear, homogeneous, second-order, partial differential equation PDE can be derived by means of an energy balance over a differential control volume when there is transient heat conduction, motion of the material with constant velocity along the positive $x-$axis, with convection heat transfer (the third term), and energy storage (the time dependent term):

$$kA_c \frac{\partial^2 u}{\partial x^2} - \rho c_p V A_c \frac{\partial u}{\partial x} - hPu = \rho c_p A_c \frac{\partial u}{\partial t}, \quad 0 < x < L, \quad t > 0$$

where k is the thermal conductivity, A_c is the cross-sectional area and P is the perimeter of the system. The thermophysical properties: ρ, the mass density, and c_p, the specific heat capacity are assumed to be constants. The velocity V in the positive $x-$direction, and h is the convection heat transfer coefficient are also assumed to be constants. The units of the dependent variable $u(x,t)$ and the independent variables x, t are K, m, s respectively.

(a) All terms of the given PDE must have identical units. Determine the units by examination of the time dependent term

$$\rho c_p A_c \frac{\partial u}{\partial t}$$

given the units of ρ $[kg/m^3]$ and c_p $[(W \cdot s)/(kg \cdot K)]$.

(b) (i) Divide through by the product kA_c, (ii) introduce the thermophysical parameter $\alpha = k/(\rho c_p)$ and set $\bar{m}^2 = (hP)/(kA_c)$, and (iii) rewrite the PDE. (iv) Classify the PDE as elliptic, hyperbolic or parabolic type?

(c) From the *rewritten* PDE determine the units of α, the groups: (V/α), and $\bar{m}^2 = (hP)/(kA_c)$, and (h/k). Finally, what are the units of the thermophysical parameters h and k?

(d) If $V = 0$, $h > 0$, and the temperature is steady-state, specify the corresponding ODE for $u(x)$, and obtain its *general* solution. Do not apply the boundary conditions.

(e) If $V > 0$, $h = 0$, and the temperature is steady-state, specify the corresponding ODE for $u(x)$, and obtain its *general* solution. Do not apply the boundary conditions.

(f) The Separation of Variables Method (SVM) based on $u(x,t) = X(x)T(t)$ can be used to separate the homogeneous PDE into two independent ODEs. Select the sign of the separation constant λ^2 such that the separated time ODE is $T' + \lambda^2 \alpha T = 0$. Obtain the separated spatial ODE for the general case where $V > 0$ and $m > 0$.

Do not attempt to solve this ODE.

Problem 3.
[30 Marks]

When the Separation of Variables Method (SVM) is applied to the two-dimensional Laplace equation $u_{xx} + u_{yy} = 0$ for the potential $u(x, y)$, and the one-dimensional wave equation $u_{yy} = \frac{1}{c^2} u_{tt}$, where $u(y, t)$ is the displacement of the vertical string from its equilibrium position, and the one-dimensional Heat equation, $u_{yy} = \frac{1}{\alpha} u_t$ where $u(y, t)$ represents the temperature, the Sturm-Liouville Problem (SLP) arises:

$$Y'' + \gamma^2 Y = 0, \qquad 0 < y < H$$

where $Y(y)$, and γ^2 is the separation constant.

The homogeneous Dirichlet and Robin boundary conditions at $y = 0$ and $y = H$, respectively, are

$$Y(0) = 0 \qquad \text{and} \qquad -kY'(H) = hY(H)$$

where $h > 0$ and $k > 0$ are physical parameters of the system.

(a) Obtain the *eigenfunctions* and the corresponding *eigenvalues* for the case where $h/k = \infty$.

(b) Obtain the *eigenfunctions* and the corresponding *eigenvalues* for the case where $h/k = 0$.

(c) From the results obtained in parts (a) and (b) above where does the first eigenvalue (root of characteristic equation) lie? Verify that the value of the first eigenvalue $\delta_1 = \gamma H$ is approximately 2.0002 when $hH/k = 0.916$
