Introduction to Complex Analysis
by Hilary Priestley
Unofficial Solutions Manual

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Dedicated to my Parents

Preface

This is an ongoing Solutions Manual for *Introduction to Complex Analysis* by Hilary Priestley [1]. The main reason for taking up such a project is to have an electronic backup of my own handwritten solutions.

Mathematics cannot be done without actually *doing* it. However at the undergraduate level many students are put off attempting problems unless they have access to written solutions. Thus I am making my work publicly available in the hope that it will encourage undergraduates (or even dedicated high school students) to attempt the exercises and gain confidence in their own problem-solving ability.

I am aware that questions from textbooks are often set as assessed homework for students. Thus in making available these solutions there arises the danger of plagiarism. In order to address this issue I have attempted to write the solutions in a manner which conveys the general idea, but leaves it to the reader to fill in the details.

At the time of writing this work is far from complete. While I will do my best to add additional solutions whenever possible, I can not guarantee that any one solution will be available at a given time. Updates will be made whenever I am free to do so.

I should point out that my solutions are not the only ways to tackle the questions. It is possible that many 'better' solutions exist for any given problem. Additionally my work has not been peer reviewed, so it is not guaranteed to be free of errors. Anyone using these solutions does so at their own risk.

I also wish to emphasize that this is an *unofficial* work, in that it has nothing to do with the original author or publisher. However, in respect of their copyright, I have chosen to omit statements of all the questions. Indeed it should be quite impossible for one to read this work without having a copy of the book [1] present.

I hope that the reader will find this work useful and wish him the best of luck in his Mathematical studies.

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Project started on 29 June 2008

The end of a solution is indicated by ■. Any reference such as 'Theorem 13.1', 'Question 23.10' refers to the relevant numbered item in Priestly's book [1]. This work has been prepared using LaTeX.

The latest version of this file can be found at: `http://akhtarmath.wordpress.com/`

Cite this file as follows:

Akhtar, M.E. *Unofficial Solutions Manual for Introduction to Complex Analysis by H.A.Priestley*. Online book available at: `http://akhtarmath.wordpress.com/`

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Chapter 1

The Complex Plane

1.12) Given $z, w \in \mathbb{C}$ we have that :

$$\begin{aligned}|1-\bar{z}w|^2 - |z-w|^2 &= (1-\bar{z}w)(1-z\bar{w}) - (z-w)(\bar{z}-\bar{w}) \\ &= 1 + |z|^2|w|^2 - |z|^2 - |w|^2 = \left(1-|z|^2\right)\left(1-|w|^2\right)\end{aligned}$$

as required. Now suppose $|z| < 1$ and $|w| < 1$. We observe that :

$$\left|\frac{z-w}{1-\bar{z}w}\right| < 1 \iff |z-w| < |1-\bar{z}w| \iff |z-w|^2 < |1-\bar{z}w|^2$$

where the last equivalence follows because both $|z-w| \geq 0$ and $|1-\bar{z}w| \geq 0$. Furthermore $|z-w|^2 < |1-\bar{z}w|^2$ if and only if $\left(1-|z|^2\right)\left(1-|w|^2\right) > 0$ by the previous part. This last inequality is true because both $|z| < 1$ and $|w| < 1$ by assumption. The result follows. ■

1.13) (i) Recall that for a complex number a, we have $2\text{Re}(a) = a + \bar{a}$, and if $a \neq 0$ then $\overline{(a^{-1})} = (\bar{a})^{-1}$. With these ideas in mind, select $z, w \in \mathbb{C}$ with $z \neq w$. Then :

$$\begin{aligned}\text{Re}\left(\frac{w+z}{w-z}\right) &= \frac{1}{2}\left[\left(\frac{w+z}{w-z}\right) + \overline{\left(\frac{w+z}{w-z}\right)}\right] = \frac{1}{2}\left[\left(\frac{w+z}{w-z}\right) + \frac{(\bar{w}+\bar{z})}{\overline{(w-z)}}\right] \\ &= \frac{1}{2}\left[\left(\frac{w+z}{w-z}\right) + \frac{(\bar{w}+\bar{z})(w-z)}{\overline{(w-z)}(w-z)}\right] \\ &= \frac{1}{2}\left[\frac{(w+z)(\bar{w}-\bar{z}) + (\bar{w}+\bar{z})(w-z)}{|w-z|^2}\right] \\ &= \frac{1}{2}\left[\frac{2\left(|w|^2-|z|^2\right)}{|w-z|^2}\right] = \frac{|w|^2-|z|^2}{|w-z|^2}\end{aligned}$$

as required. (ii) We observe that $|w-z|^2 = (w-z)(\bar{w}-\bar{z}) = |w|^2 - 2\text{Re}(z\bar{w}) + |z|^2$. Writing $z = re^{i\theta}$ and $w = Re^{i\varphi}$ we find that $|w-z|^2 = R^2 - 2\text{Re}\left(Rre^{i(\theta-\varphi)}\right) + r^2$ and since $e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$ for $\alpha \in \mathbb{R}$ we conclude that $|w-z|^2 = R^2 - 2Rr\cos(\theta-\varphi) + r^2$. By substituting this result, along with $|z| = r$ and $|w| = R$ into the expression obtained Part (i) we deduce that

$$\text{Re}\left(\frac{w+z}{w-z}\right) = \frac{R^2-r^2}{R^2-2Rr\cos(\theta-\varphi)+r^2}$$

which was what we wanted to show. ■

1.14) Suppose there exists a relation $>$ on $\mathbb{C}$ satisfying both (a) and (b). Since $i \neq 0$, condition (a) implies that exactly one of $i > 0$ or $-i > 0$ holds. If $i > 0$ then condition (b) implies that

$i^2 = -1 > 0$. Now[1] since both $i > 0$ and $-1 > 0$, it follows once again from condition (b)that $(-1)(i) = -i > 0$. But now both $i > 0$ and $-i > 0$, which contradicts condition (a). On the other hand, if $-i > 0$ then we once again find that $(-i)^2 = -1 > 0$ which, using condition (b) implies $(-1)(-i) = i > 0$. This once again contradicts condition (a) as before. ■

[1]Note that the fact that $-1 > 0$ does not in itself yield a contradiction. This is because $>$ may not be the usual order relation on $\mathbb{R}$.

Chapter 2

Geometry in the Complex Plane

2.4) We shall use the same notation as in [1] : (i) Here $\alpha = -i$, $\beta = 3i$ and $\lambda = 1$. The given equation describes the straight line $\mathrm{Im}(z) = 1$ in $\mathbb{C}$. (ii) Here $\alpha = -1$, $\beta = 1$ and $\lambda = 4$. Since $\lambda > 0$ and $\lambda \neq 1$, the given equation describes a circle in $\mathbb{C}$. The diameter of this circle through α and β intersects the circle at z_1 and z_2 where $z_1 - \alpha = \lambda(z_1 - \beta) \iff z_1 + 1 = 4(z_1 - 1)$ and $z_2 - \alpha = -\lambda(z_2 - \beta) \iff z_2 + 1 = -4(z_2 - 1)$. We conclude that $z_1 = (5/3)$ and $z_2 = (3/5)$. Therefore, the centre of the circle is : $(1/2)(z_1 + z_2) = (17/15)$ and the radius of the circle is : $(1/2)|z_1 - z_2| = (8/15)$. (iii) Here $\alpha = i$, $\beta = 0$ and $\lambda = 2$. Proceeding as in (ii), the equation describes a circle in $\mathbb{C}$ centered at $(-i/3)$ with radius $(2/3)$. (iv) Here $\alpha = 0$, $\beta = i$ and $\lambda = 2$. Proceeding as in (ii), the equation describes a circle in $\mathbb{C}$ centered at $(4i/3)$ with radius $(2/3)$. The sketches are left to the reader. ■

2.14) Remark. There seem to be two *misprints* in this question as stated in [1]. The first is in part (ii)(b) : The equation of the arc is probably meant to be $\arg((z-\alpha)/(z-\beta)) = \mu \,(\mathrm{mod}\, 2\pi)$. The second is in part (iii) : K should be $c/(a - c\alpha)$, not $1/(a - c\alpha)$.
(i) A complex number $z \in \mathbb{C}$ is a fixed point of the Möbius transformation f if and only if $z = f(z) = (az + b)/(cz + d)$, if and only if z is a root of the polynomial : $cz^2 + (d - a)z - b$ $(a, b, c, d \in \mathbb{C})$ The stated polynomial has either one or two roots in $\mathbb{C}$; Therefore, f has either one or two fixed points.
(ii) Suppose that $\alpha, \beta \in \mathbb{C}$ are two distinct fixed points of f. The inverse map of f is $g : z \mapsto (dz - b)/(a - cz)$. Since $f(\alpha) = \alpha$ and $f(\beta) = \beta$, it must be the case that $\alpha = g(\alpha) = (d\alpha - b)/(a - c\alpha)$ and $\beta = g(\beta) = (d\beta - b)/(a - c\beta)$. Using these facts we observe :

$$\begin{aligned}\frac{w-\alpha}{w-\beta} &= \frac{(a-c\alpha)z - (d\alpha - b)}{(a-c\beta)z - (d\beta - b)} \\ &= \frac{(a-c\alpha)}{a-c\beta}\left[\frac{z-g(\alpha)}{z-g(\beta)}\right] = k\,\frac{z-\alpha}{z-\beta}\end{aligned}$$

where $k = (a - c\alpha)/(a - c\beta)$ as required. From this we see that $\frac{w-\alpha}{w-\beta} = \frac{1}{k}\left(\frac{z-\alpha}{z-\beta}\right)$. Therefore,
(a) The image of the given circline under f is the circline : $|(w - \alpha)/(w - \beta)| = |k|\lambda$. For (b) we assume that the arc is described as in the remark above. Then the image of the arc under f is the arc : $\arg((w - \alpha)/(w - \beta)) - \arg(k) = \mu \,(\mathrm{mod}\, 2\pi)$.
(iii) Suppose that f has a single fixed point α. Then α is a repeated root of the polynomial stated in part (i). That is : $\alpha = (a - d)/2c$. Equivalently, $d = a - 2c\alpha$. Also, as in part(ii),

$g(\alpha) = \alpha$. Using these facts we observe :

$$\begin{aligned}\frac{1}{w-\alpha} &= \frac{cz+d}{(a-c\alpha)z-(d\alpha-b)}\\ &= \frac{cz+a-2c\alpha}{(a-c\alpha)(z-g(\alpha))} = \frac{1}{z-\alpha}+K\end{aligned}$$

where $K = c/(a-c\alpha)$ as required (see the remark at the beginning of this solution). ■

Chapter 3

Topology and analysis in the complex plane

3.9) (a)(i) Since $|z_n| = (1/n) \to 0$, the sequence $\{(1/n)i^n\}$ converges to 0. (a)(ii) Note that $|1+i| = \left|\sqrt{2}e^{i\frac{\pi}{4}}\right| = \sqrt{2}$. Therefore $|z_n| = \left(1/\sqrt{2}\right)^n \to 0$ and so the sequence $\{(1+i)^{-n}\}$ converges to 0. (a)(iii) Since $(n^2+in)/(n^2+i) = \left(1+in^{-1}\right)/\left(1+in^{-2}\right)$, the sequence $\left\{(n^2+in)/(n^2+i)\right\}$ converges to 1. (b)(i) $\{i^n\}$ does not converge because it contains the divergent sequence $\{(-1)^n\}$ as a subsequence. (b)(ii) Since $|z_n| = |(1+i)^n| = \left(\sqrt{2}\right)^n \to \infty$, it follows that $\{(1+i)^n\}$ does not converge. (b)(iii) The real part of $\{n(-1)^n/(n+i)\}$ is the divergent sequence $\left\{n^2(-1)^n/\left(n^2+1\right)\right\}$. Thus the original sequence does not converge. ■

Chapter 5

Holomorphic functions

5.1) (i) For any $z = x + iy \in \mathbb{C}$, $\operatorname{Im} z = u(x,y) + iv(x,y)$, where $u(x,y) = 0$ and $v(x,y) = y$ for all $(x,y) \in \mathbb{R}^2$. In particular $u_x = 0 \neq 1 = v_y$ at any $(x,y) \in \mathbb{R}^2$. So the Cauchy-Riemann equations fail at every $z = x + iy \in \mathbb{C}$. We conclude $\operatorname{Im} z$ is not differentiable anywhere.
(ii) For any $z = x + iy \in \mathbb{C}$, $\bar{z} = u(x,y) + iv(x,y)$, where $u(x,y) = x$ and $v(x,y) = -y$ for all $(x,y) \in \mathbb{R}^2$. In particular, $u_x = 1 \neq -1 = v_y$. So the Cauchy-Riemann equations fail at every $z = x + iy \in \mathbb{C}$. We conclude $\bar{z}$ is not differentiable anywhere. ■

5.3) If $f = u + iv$ is differentiable at $z = a + ib \in \mathbb{C}$, then all the partial derivatives u_x, u_y, v_x, v_y exist at z, and in particular, $v_x = -u_y$ there. Now $f'(z) = u_x + iv_x$ and also $f'(z) = v_y - iu_y$. Substituting $v_x = -u_y$ into both of these yields $f'(z) = u_x - iu_y$ and $f'(z) = v_y + iv_x$. ■

5.4) In each part of this question, we shall denote the given function by f for convenience. (i) If $h \neq 0$ then $\frac{1}{h}(f(0+h) - f(0)) = |h|^2/h = h \to 0$ as $h \to 0$. Therefore, $f(z) = |z|^2$ is differentiable at $z = 0$ and $f'(0) = 0$. (ii) If $z = x + iy$ then $f(z) = x + y$. Therefore, $f = u + iv$ where $u(x,y) = x + y$ and $v(x,y) = 0$. If f is differentiable at $z = 0$, then all the first partial derivatives of u and v must exist at $z = 0$ and must satisfy the Cauchy-Riemann equations at $z = 0$. However, $\frac{\partial u}{\partial x}(0,0) = 1 \neq 0 = \frac{\partial v}{\partial y}(0,0)$, and thus we conclude that the function f is not differentiable at $z = 0$. (iii) If $z = x + iy$ then $f(z) = xy$. Therefore, $f = u + iv$ where $u(x,y) = xy$ and $v(x,y) = 0$. One can check that all the first partial derivatives of u and v are continuous in $\mathbb{C}$ and that they satisfy the Cauchy-Riemann equations at $z = 0 \in \mathbb{C}$. Therefore, the function $f(z) = (\operatorname{Re} z)(\operatorname{Im} z)$ is differentiable at $z = 0$. ■

5.6) Suppose that there exists $c \in [1, i]$ such that the given equation holds. This is equivalent to saying that there exists $t \in [0,1]$ such that $c = (1-t) + it$ and $(1+i)/(1-i) = 3c^2$. Taking modulus on both sides, the second equation implies that:

$$\left| \frac{1+i}{1-i} \right| = 3|c|^2 \Rightarrow |c| = \frac{1}{\sqrt{3}}$$

Now also $c = (1-t) + it$ for some $t \in [0,1]$. So $\sqrt{1 - 2t + 2t^2} = |c| = 1/\sqrt{3}$, which implies that t must be a root of the quadratic: $6t^2 - 6t + 2$. But since the discriminant $(-6)^2 - 4(6)(2)$ is negative, t must have nonzero imaginary part. This contradicts the assumption that $t \in [0,1]$. Therefore, there exists no $c \in [1, i]$ satisfying the given equation. ■

5.7) (a)(i) This is a polynomial and so is holomorphic at all $z \in \mathbb{C}$. (a)(ii) Holomorphic at all $z \in \mathbb{C}$ except at $z = 0, 1$ and 2. (a)(iii) Holomorphic at all $z \in \mathbb{C}$ except at $z = 1, \omega, \omega^2, \omega^3$ and ω^4, where $\omega = \exp(2\pi i/5)$. (b)(i) First select any $a \in \mathbb{C}\backslash\{0\}$. If $1/|z|$ is holomorphic at a then in particular $1/|z|$ is differentiable at a. Also $1/|a| \neq 0$ so $(1/|z|)^{-1} = |z|$ is also

differentiable at a. This is a contradiction since $|z|$ does not satisfy the Cauchy-Riemann equations at a $(\neq 0)$. So $1/|z|$ is not holomorphic at any $a \in \mathbb{C}\backslash\{0\}$. It follows that $1/|z|$ can not be holomorphic in any disc centered at 0. Thus $1/|z|$ is not holomorphic at any $a \in \mathbb{C}$.
(b)(ii) Suppose that $z|z|$ is holomorphic at $a \in \mathbb{C}\backslash\{0\}$. The function $1/z$ is also holomorphic at a $(\neq 0)$. Therefore $(1/z)(z|z|) = |z|$ is also holomorphic at a. This is a contradiction since $|z|$ is not even differentiable at a. The argument now follows as in (b)(i). ■

Chapter 6

Complex series and power series

6.3) Viewing $\sum z^n$ as a power series expansion of $(1-z)^{-1}$ for $|z|<1$, we have:

(i)(a) $\frac{1}{1-z} = \frac{1}{2\left(1-\frac{(z+1)}{2}\right)} = \sum_{n=0}^{\infty} \frac{(z+1)^n}{2^{n+1}}$; this converges for $|z+1|<2$.

(i)(b) $\frac{1}{1-z} = \frac{1}{(1-i)\left(1-\left(\frac{z-i}{1-i}\right)\right)} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{1-i}^{n+1}$; this converges for $|z-i|<|1-i|=\sqrt{2}$.

(ii)(a) $\frac{1}{z(z+2)} = \frac{1}{z^2+2z} = \frac{1}{1+(z+1)^2} = \sum_{n=0}^{\infty}(-1)^n(z+1)^{2n}$; this converges for $|z+1|<1$.

(ii)(b) For this part we first express $1/z$ and $1/(z+2)$ individually as power series in powers of $z-i$, and then use the multiplication rule for power series. The resulting series will converge for $\min\{r_1, r_2\}$, where r_1 and r_2 are the radii of convergence of each of the series we have just obtained. Hence:

$$\frac{1}{z(z+2)} = \left(-i\sum_{n=0}^{\infty} i^n(z-i)^n\right) \cdot \left(\frac{1}{2+i}\sum_{m=0}^{\infty}\frac{(-1)^m(z-i)^m}{(2+i)^m}\right) = \frac{-i}{2+i}\sum_{n=0}^{\infty} c_n(z-i)^n$$

where $c_n = \sum_{k=0}^{n} \frac{i^{n-k}(-1)^k}{(2+i)^k}$. This series will converge for $|z-i| < \min\{1, \sqrt{5}\} = 1$. ■

6.4) (i) By the ratio test, the series has infinite radius of convergence, and thus[1] it defines a holomorphic function at every point of $\mathbb{C}$. (ii) By the ratio test, the series has radius of convergence equal to 1, and thus it defines a holomorphic function on the open unit disc. (iii) By Cauchy's root test, the series has infinite radius of convergence, and thus it defines a holomorphic function on $\mathbb{C}$. (iv) By the ratio test, the series has radius of convergence equal to 0. Therefore, it does not define a holomorphic function at any point[2] of $\mathbb{C}$. ■

6.7) By the differentiation theorem for power series:

(i) $(1+z)^{-2} = -\frac{d}{dz}(1+z)^{-1} = -\sum_{n=1}^{\infty}(-1)^n n z^{n-1} = \sum_{n=1}^{\infty}(-1)^{n+1} n z^{n-1}$; and

(ii) $(1+z)^{-3} = -(1/2)\frac{d}{dz}(1+z)^{-2} = (1/2)\sum_{n=2}^{\infty}(-1)^n n(n-1)z^{n-2}$

where both of the obtained expansions are valid for $|z|<1$. ■

[1] Using the differentiation theorem for power series

[2] Since such a function will not even be defined, let alone differentiable, in an open disc about any point of $\mathbb{C}$.

Bibliography

[1] Priestley, H.A. *Introduction to Complex Analysis, 2nd Ed.*, 2003. Oxford University Press.