SCAN 2002

Affine Arithmetic:

Concepts and Applications

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Outline

- What is affine arithmetic?
- The dependency problem in interval arithmetic
- Main concepts in affine arithmetic
- Comparing affine arithmetic with interval arithmetic
- Exploiting the correlations given by affine arithmetic
- Other approaches to the dependency problem

What is affine arithmetic?

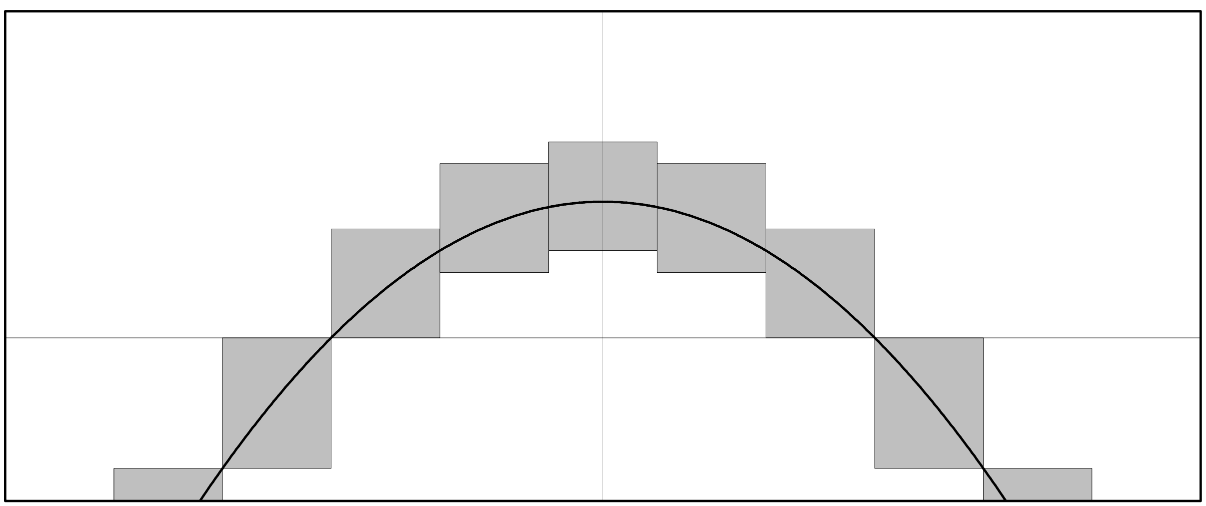
- AA is a tool for validated numerics
  - ◇ introduced by Comba and Stolfi in 1993
- AA is designed to handle the dependency problem in IA
  - ◇ AA keeps track of *first-order correlations*
- AA has been used successfully as a replacement for interval arithmetic
  - ◇ AA provides tighter interval estimates in many cases
  - ◇ AA provides additional information that can be exploited

The dependency problem in interval arithmetic

IA can't see correlations between operands

$$g(x) = (10 + x)(10 - x) \text{ for } x \in [-2, 2]$$

$$\begin{aligned} 10 + x &= [8, 12] \\ 10 - x &= [8, 12] \\ (10 + x)(10 - x) &= [64, 144] \qquad \text{diam} = 80 \\ \text{Exact range} &= [96, 100] \qquad \text{diam} = 4 \end{aligned}$$

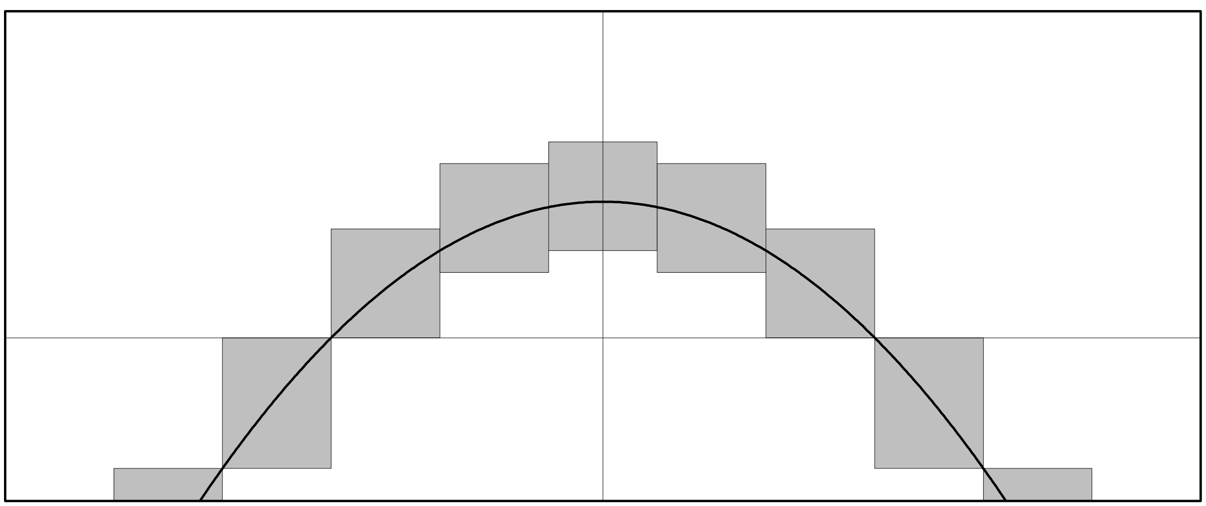


The dependency problem in interval arithmetic

IA can't see correlations between operands

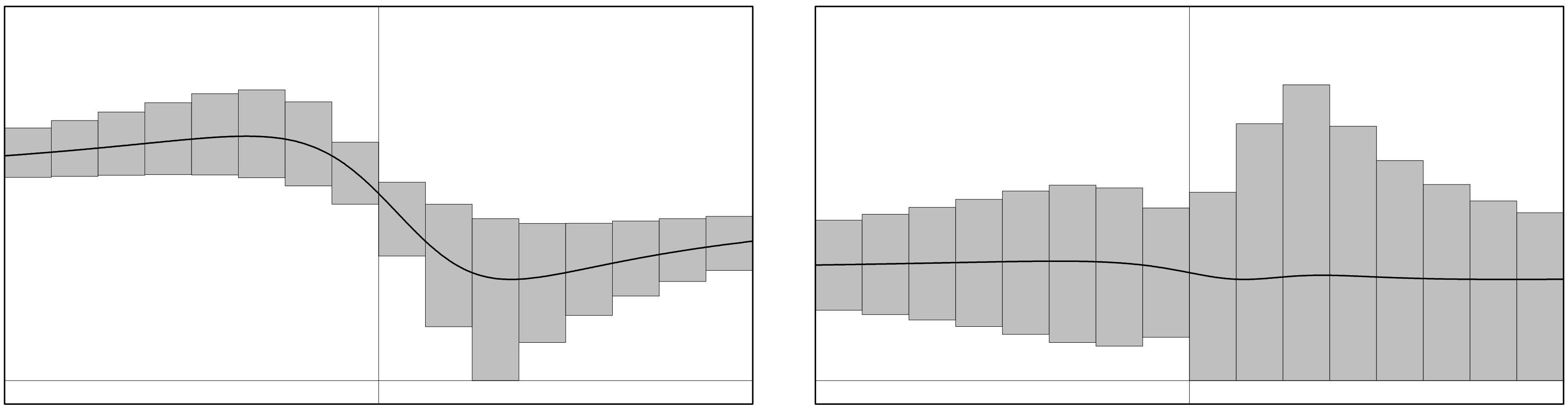
$$g(x) = (10 + x)(10 - x) \text{ for } x \in [-u, u]$$

$$\begin{aligned}
10 + x &= [10 - u, 10 + u] \\
10 - x &= [10 - u, 10 + u] \\
(10 + x)(10 - x) &= [(10 - u)^2, (10 + u)^2] \qquad \text{diam} = 40u \\
\text{Exact range} &= [100 - u^2, 100] \qquad \text{diam} = u^2
\end{aligned}$$



The dependency problem in interval arithmetic

$$g(x) = \sqrt{x^2 - x + 1/2}/\sqrt{x^2 + 1/2}$$



g $\qquad$ $g \circ g$

$g^n \to c =$ fixed point of $g \approx 0.5586$, but intervals diverge

Interval estimates may get too large in long computations

Affine arithmetic: concepts

Affine arithmetic: representation

AA represents a quantity x with an *affine form*

$$\hat{x} = x_0 + x_1\varepsilon_1 + \cdots + x_n\varepsilon_n$$

- *noise symbols* $\varepsilon_i \in \mathbf{U} = [-1, +1]$

  independent, but otherwise unknown

- *central value* $x_0 \in \mathbf{R}$

- *partial deviations* $x_i \in \mathbf{R}$

- n is *not* fixed

  new noise symbols created during computation

Intervals in affine arithmetic

Affine forms imply interval bounds:

$$x \sim \hat{x} = x_0 + x_1\varepsilon_1 + \cdots + x_n\varepsilon_n \;\Rightarrow\; x \in [x_0 - r,\, x_0 + r]$$

$r = |x_1| + \cdots + |x_n|$ is the *total deviation* of $\hat{x}$

Conversely,

$$x \in [a, b]$$

$$x \sim \hat{x} = x_0 + x_1\varepsilon_1$$

$$x_0 = (b + a)/2$$

$$x_1 = (b - a)/2$$

AA algorithms can input and output intervals, but affine forms give more information.

Geometry of affine arithmetic

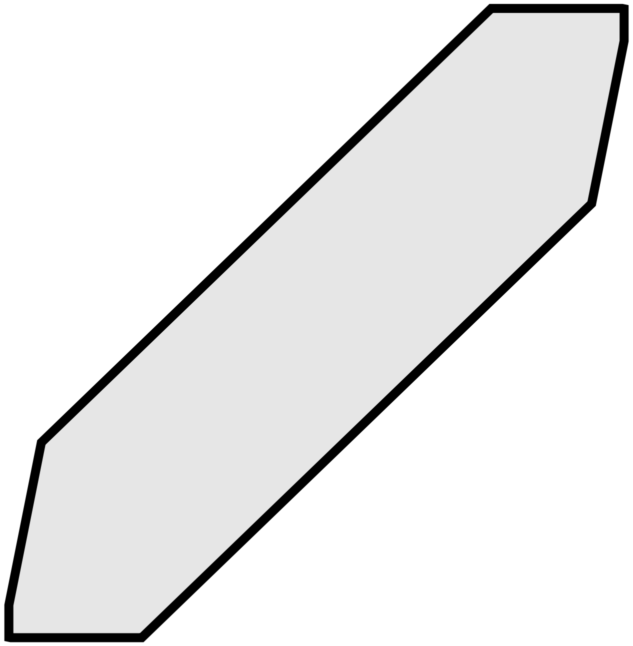
Affine forms that share noise symbols are not independent:

$$\hat{x} = x_0 + x_1\varepsilon_1 + \cdots + x_n\varepsilon_n$$
$$\hat{y} = y_0 + y_1\varepsilon_1 + \cdots + y_n\varepsilon_n$$

The region containing (x, y) is

$$Z = \{(x, y) : \varepsilon_i \in \mathbf{U}\}$$

This region is the image of $\mathbf{U}^n$ under an affine map $\mathbf{R}^n \to \mathbf{R}^2$. It's a centrally symmetric convex polygon, a *zonotope*.



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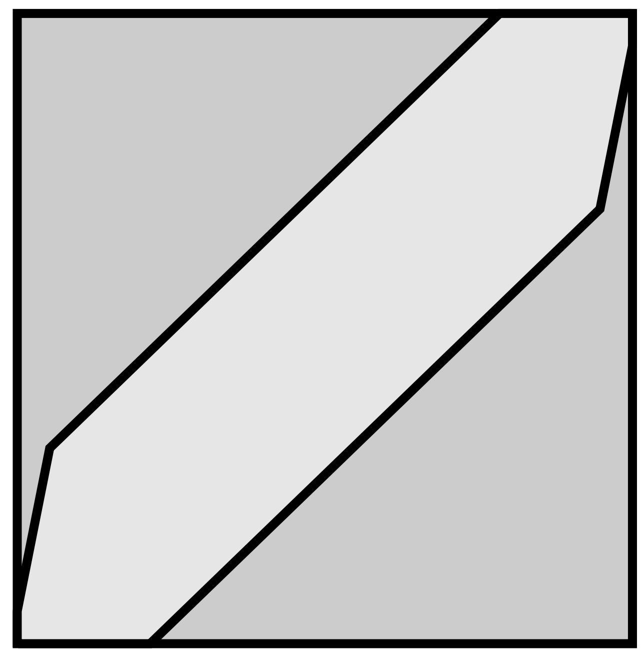
The region would be a rectangle if x and y were independent.



Computing with affine arithmetic

- Affine operations are straightforward by design:

$$\begin{aligned} \hat{x} \pm \hat{y} &= (x_0 \pm y_0) + \cdots + (x_n \pm y_n)\varepsilon_n \\ \alpha\hat{x} &= (\alpha x_0) + (\alpha x_1)\varepsilon_1 + \cdots + (\alpha x_n)\varepsilon_n \\ \hat{x} \pm \alpha &= (x_0 \pm \alpha) + x_1\varepsilon_1 + \cdots + x_n\varepsilon_n \end{aligned}$$

- For non-affine operations, use good affine approximation and append extra term to represent error:

$$\hat{f} = f_0 + f_1\varepsilon_1 + \cdots + f_n\varepsilon_n + f_k\varepsilon_k$$

  (new noise symbol created)

- Add new noise noise symbol on each operation (affine or not) to account for rounding errors.

Non-affine operations in AA

To compute $z = f(x, y)$, when f is not an affine operation, write:

$$\begin{aligned} f(x,y) &= f(\hat{x},\hat{y}) \\ &= f(x_0 + x_1\varepsilon_1 + \cdots + x_n\varepsilon_n,\ y_0 + y_1\varepsilon_1 + \cdots + y_n\varepsilon_n) \\ &= f^*(\varepsilon_1, \ldots, \varepsilon_n) \end{aligned}$$

where f^*: $\mathbf{U}^n \to \mathbf{R}$. Now approximate

$$f^*(\varepsilon_1, \ldots, \varepsilon_n) = f^{\mathsf{a}}(\varepsilon_1, \ldots, \varepsilon_n) + z_k\varepsilon_k$$

where f^{a} is some affine approximation of f^* with error bounded by z_k :

$$\left| f^*(\varepsilon) - f^{\mathsf{a}}(\varepsilon) \right| \leq |z_k| \quad \text{for all } \varepsilon \in \mathbf{U}^n$$

Easiest to take $f^{\mathsf{a}} = \alpha\hat{x} + \beta\hat{y} + \gamma$. (Exact for univariate operations.)

$$\hat{z} = \hat{f}(\hat{x},\hat{y}) = z_0 + z_1\varepsilon_1 + \cdots + z_n\varepsilon_n + z_k\varepsilon_k$$

Multiplication in AA

$$\begin{aligned}\hat{x}\cdot\hat{y} &= (x_0+\sum_{i=1}^{n} x_i\varepsilon_i)\cdot(y_0+\sum_{i=1}^{n} y_i\varepsilon_i)\\ &= x_0y_0+\sum_{i=1}^{n}(x_0y_i+y_0x_i)\,\varepsilon_i+\sum_{i=1}^{n} x_i\varepsilon_i\cdot\sum_{i=1}^{n} y_i\varepsilon_i\end{aligned}$$

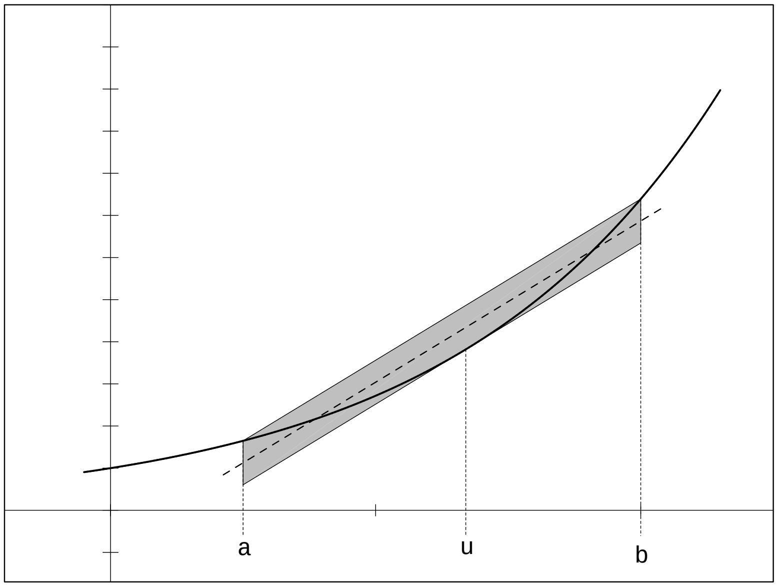
So $\hat{x}\cdot\hat{y} = x_0y_0+\sum_{i=1}^{n}(x_0y_i+y_0x_i)\,\varepsilon_i+z_k\varepsilon_k,$ where

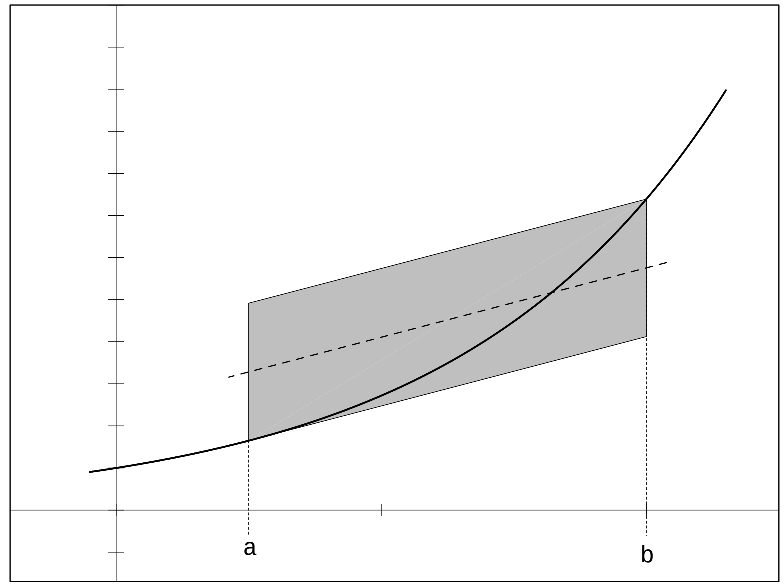
$$|z_k| \geq \left|\sum_{i=1}^{n} x_i\varepsilon_i\cdot\sum_{i=1}^{n} y_i\varepsilon_i\right|, \qquad \varepsilon_i\in\mathbf{U}$$

Easiest to take

$$z_k = \sum_{i=1}^{n}|x_i|\cdot\sum_{i=1}^{n}|y_i|$$

Choice of affine approximations



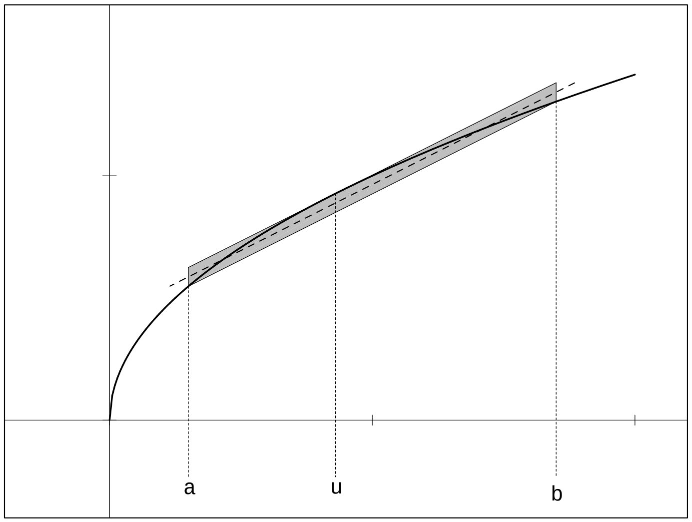


Chebyshev

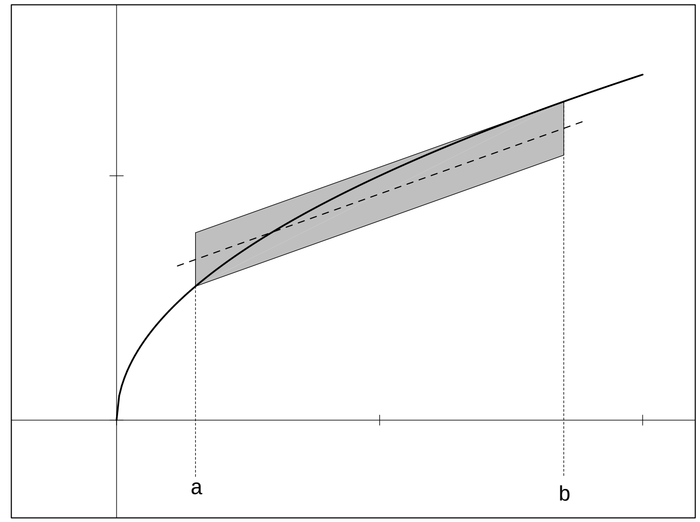
Minimum range

- Chebyshev minimizes error — best approximation
- Minimum range minimizes range `:-)` — preserves signs
- Both have quadratic approximation errors

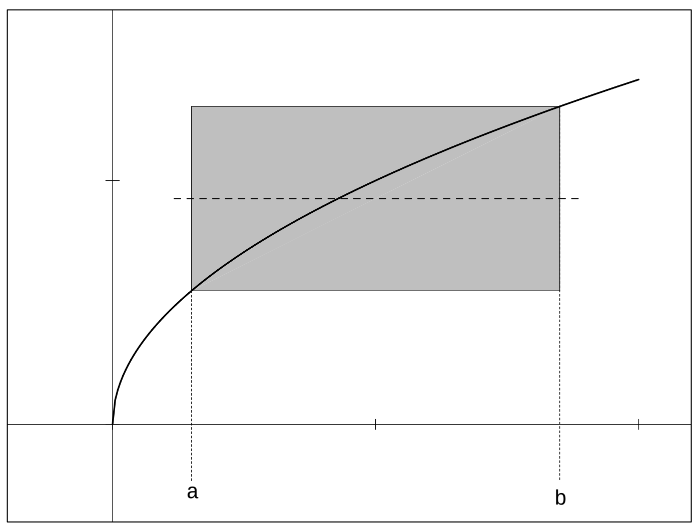
Choice of affine approximations: square root



C



MR



IA

Comparing AA with IA:

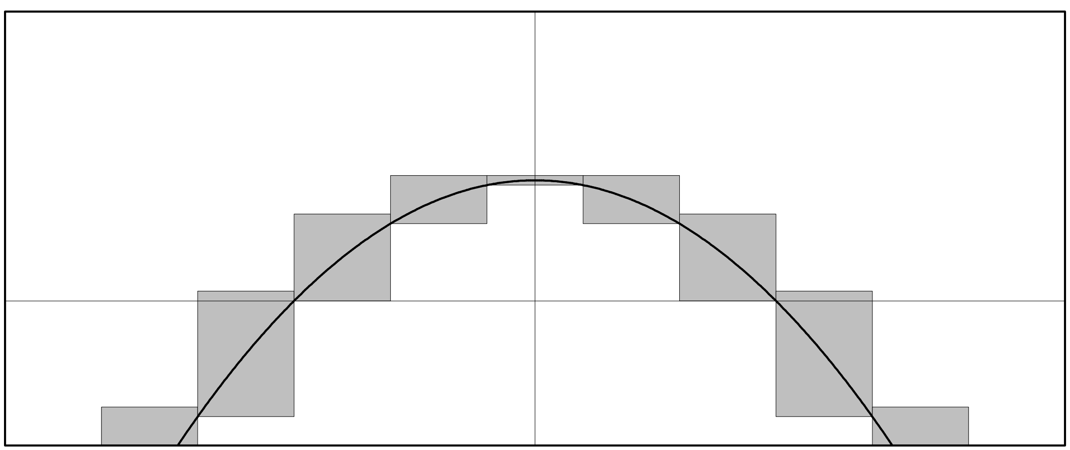
The dependency problem

The dependency problem in interval arithmetic – AA version

AA can see correlations between operands

$$g(x) = (10 + x)(10 - x) \text{ for } x \in [-u, u], \qquad x = 0 + u\,\varepsilon$$

$$\begin{aligned} 10 + x &= 10 - u\,\varepsilon \\ 10 - x &= 10 + u\,\varepsilon \\ (10 + x)(10 - x) &= 100 - u^2\,\varepsilon \\ \text{range} &= [100 - u^2, 100 + u^2] \qquad \text{diam} = 2u^2 \\ \text{Exact range} &= [100 - u^2, 100] \qquad \text{diam} = u^2 \end{aligned}$$



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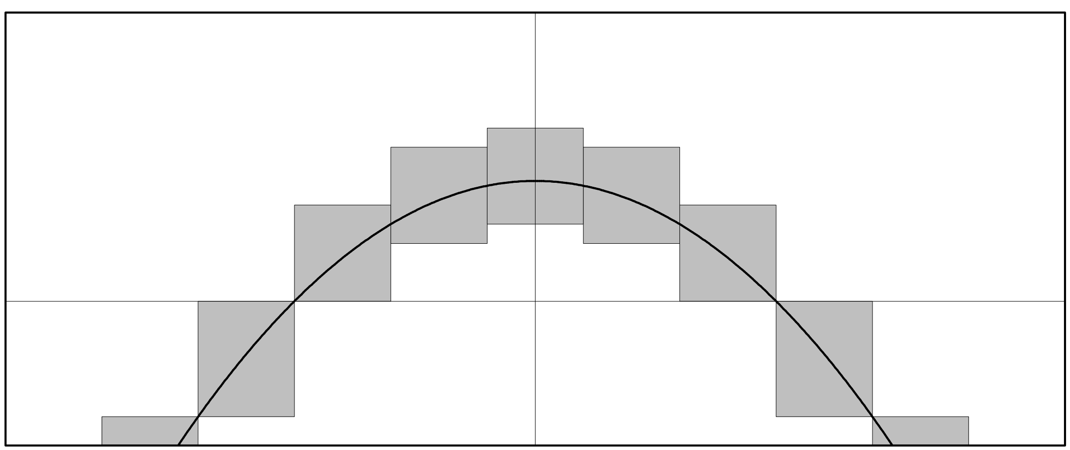
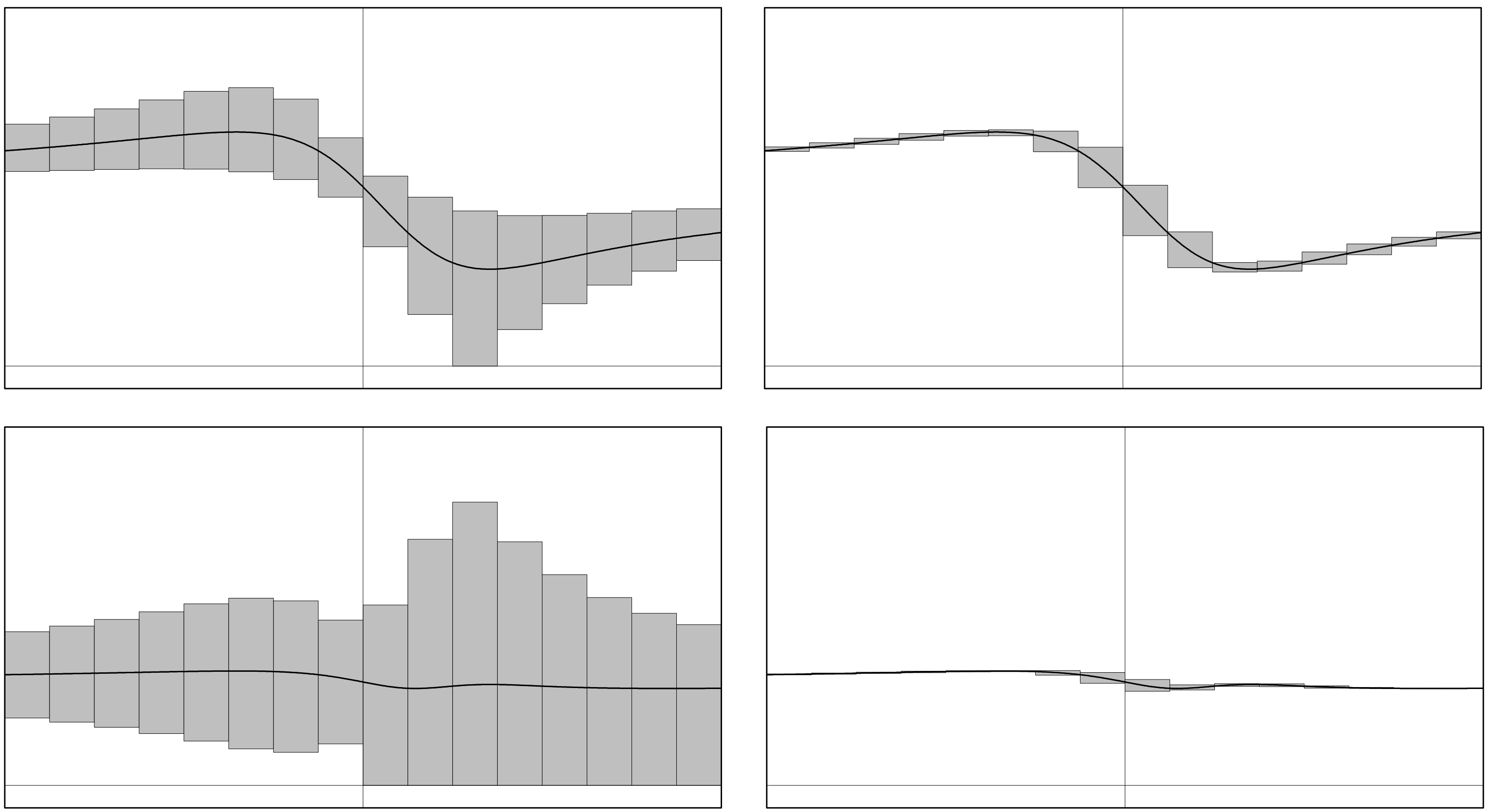


IA

Comparing AA with IA

$$g(x) = \sqrt{x^2 - x + 1/2} / \sqrt{x^2 + 1/2}$$



g

$g \circ g$

IA

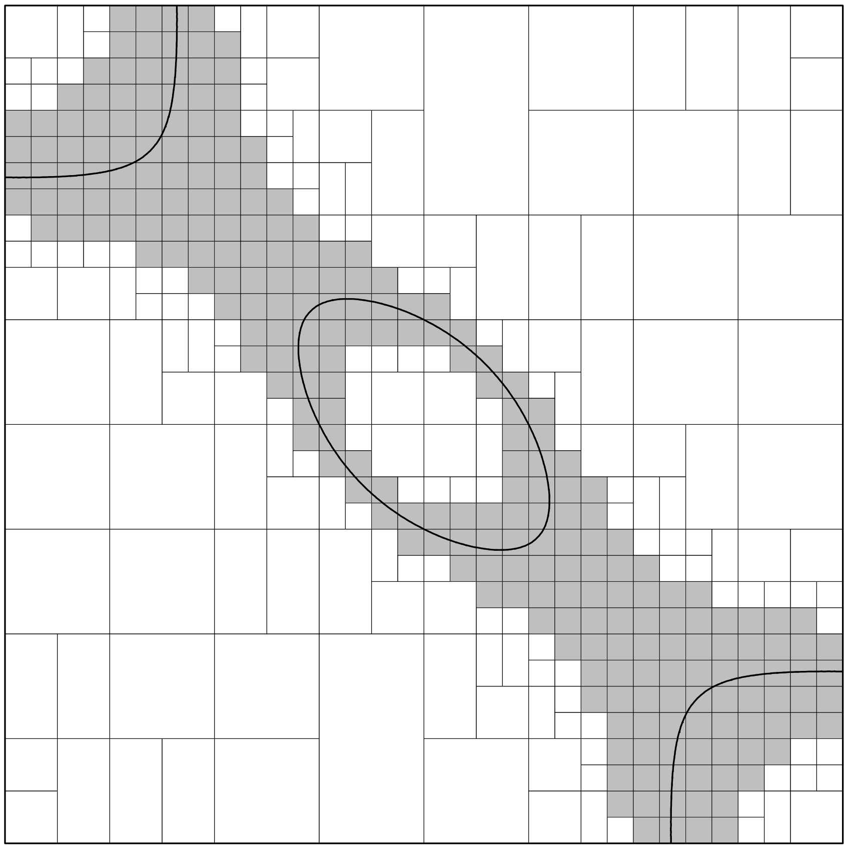
AA

Comparing AA with IA:

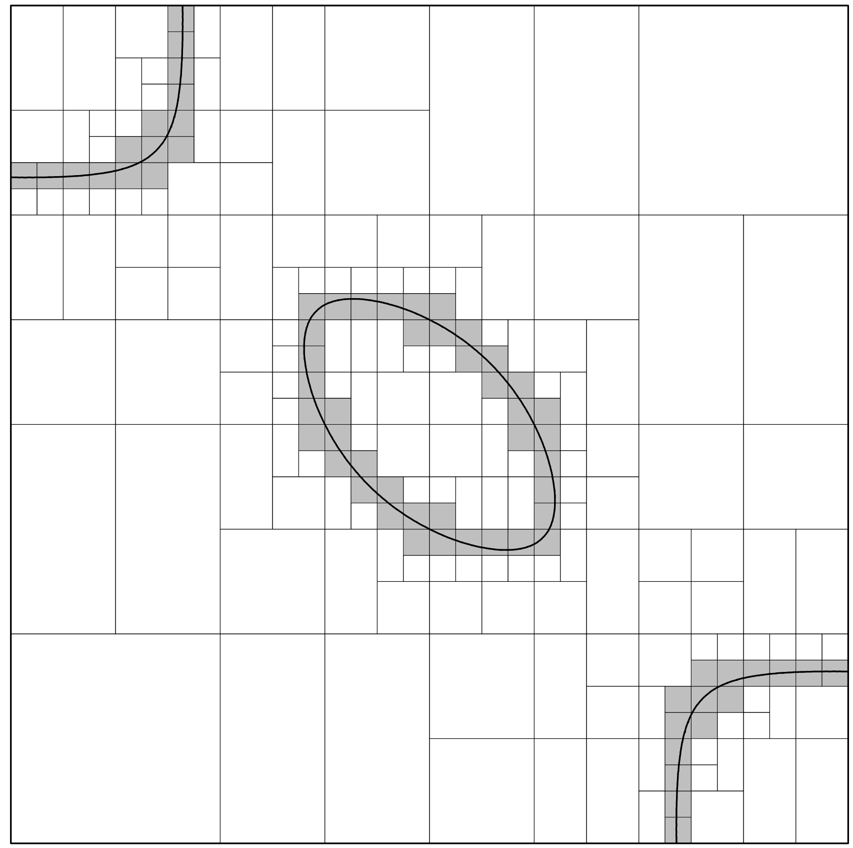
Examples in computer graphics

Comparing AA with IA: plotting implicit curves

$$x^2 + y^2 + xy - (xy)^2/2 - 1/4 = 0$$



IA (246 cells, 66 exact)

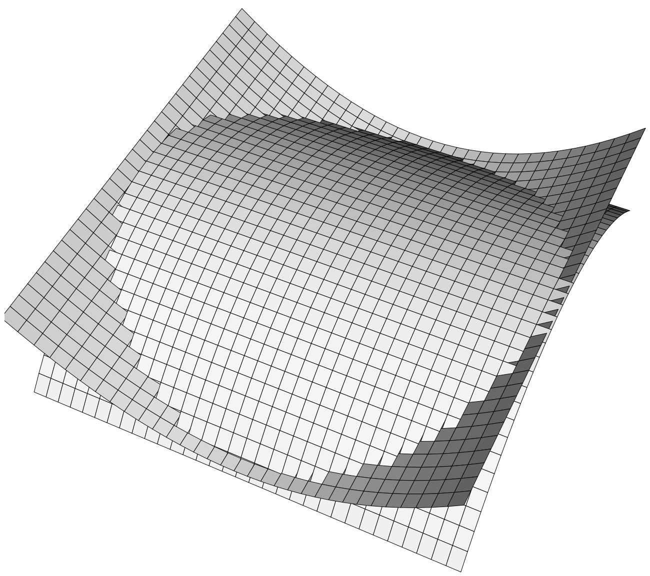


(70 cells) AA

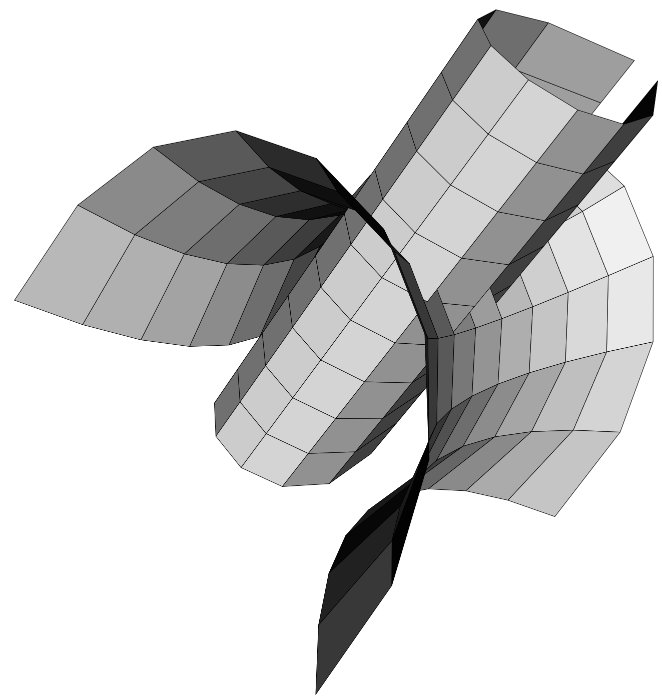
Comparing AA with IA: surface intersection

Tensor product Bézier surfaces of degree (p, q):

$$f(u,v) = \sum_{i=0}^{p} \sum_{j=0}^{q} a_{ij} B_i^p(u) B_j^q(v), \quad B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$



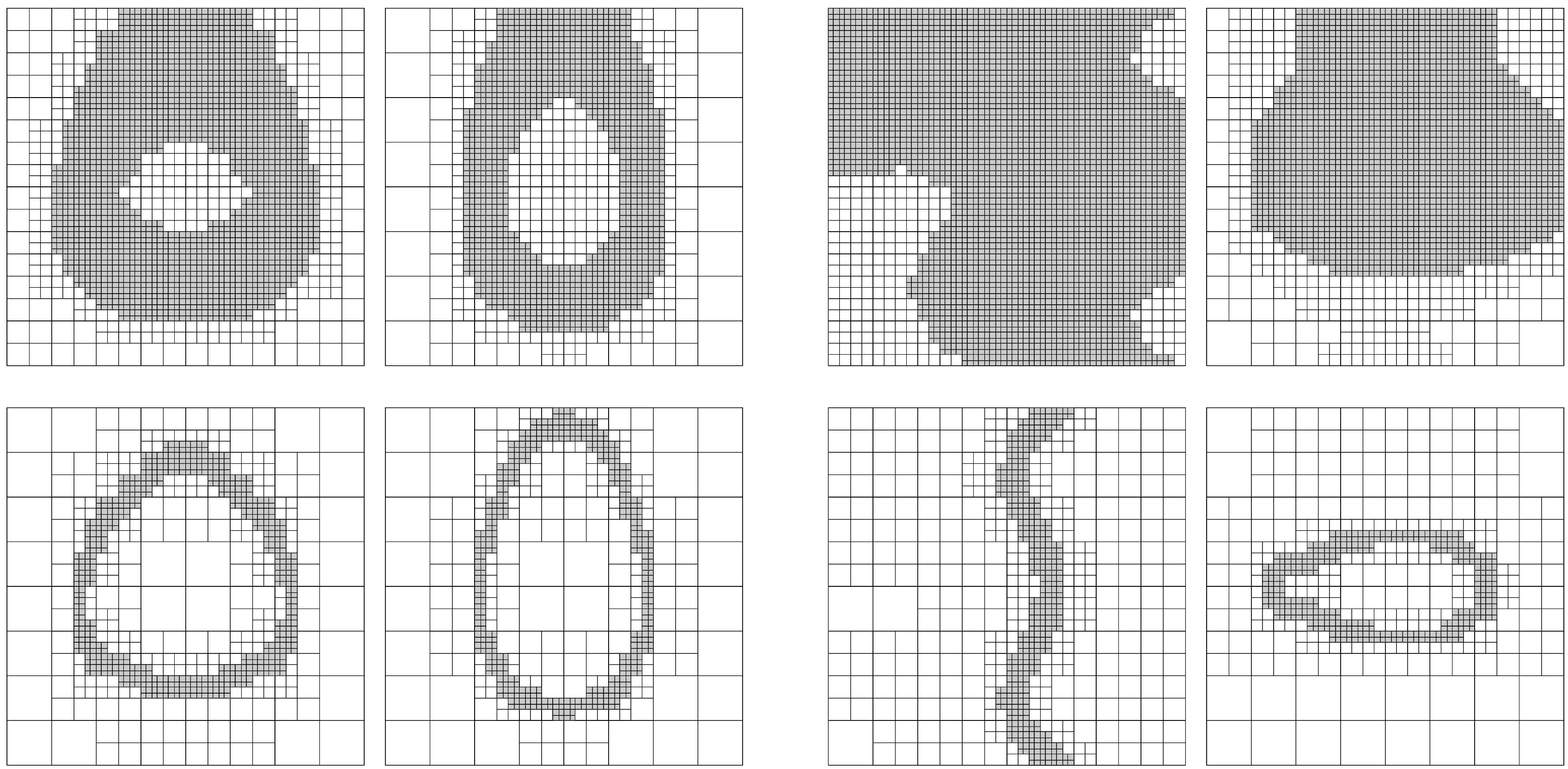
$(2,1)$



$(3,3)$

Surface intersection – domain decompositions

$(2,1)$ IA $(3,3)$

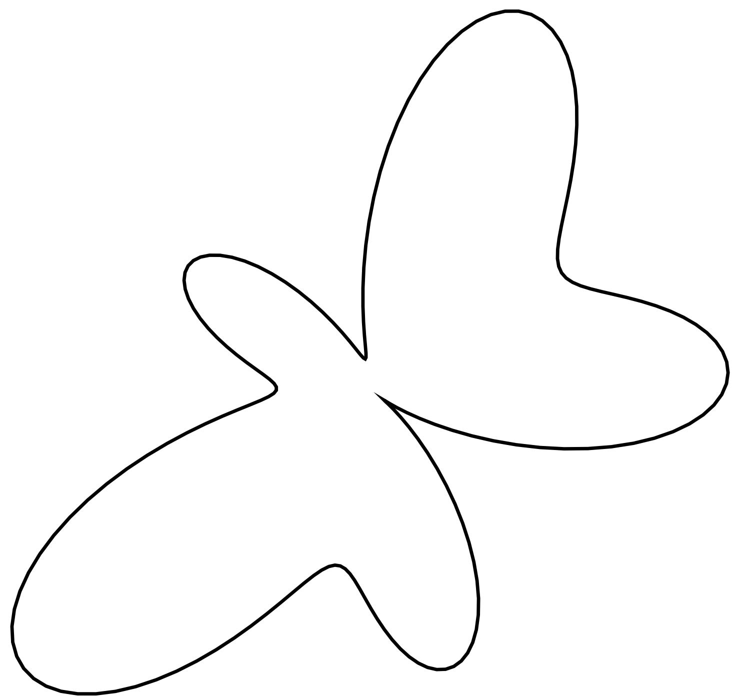


AA

Exploiting the correlations given by AA

Approximating parametric curves

Given a parametric curve $\mathcal{C} = \gamma(I)$, where $\gamma\colon I \to \mathbf{R}^2$ and $T \subseteq I$, compute a bounding rectangle for $\mathcal{P} = \gamma(T)$.



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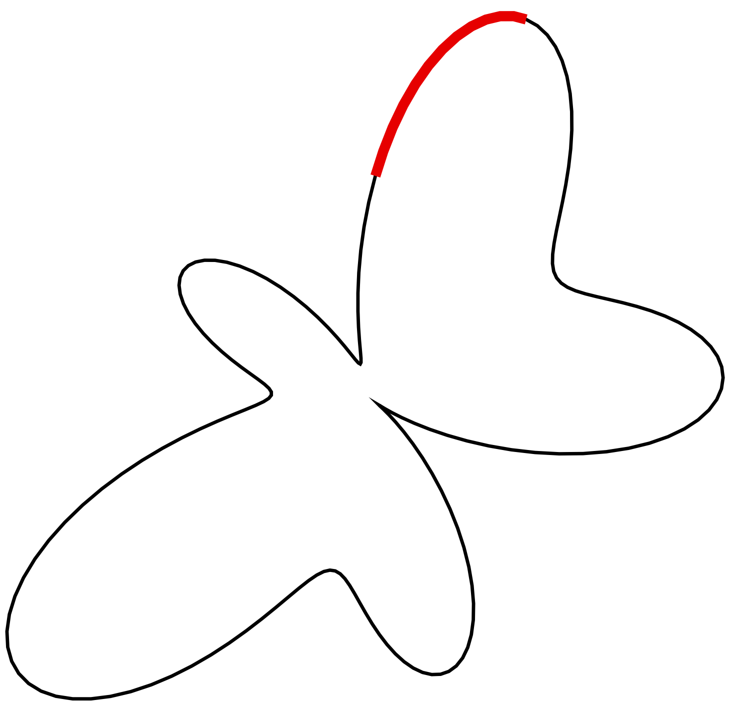

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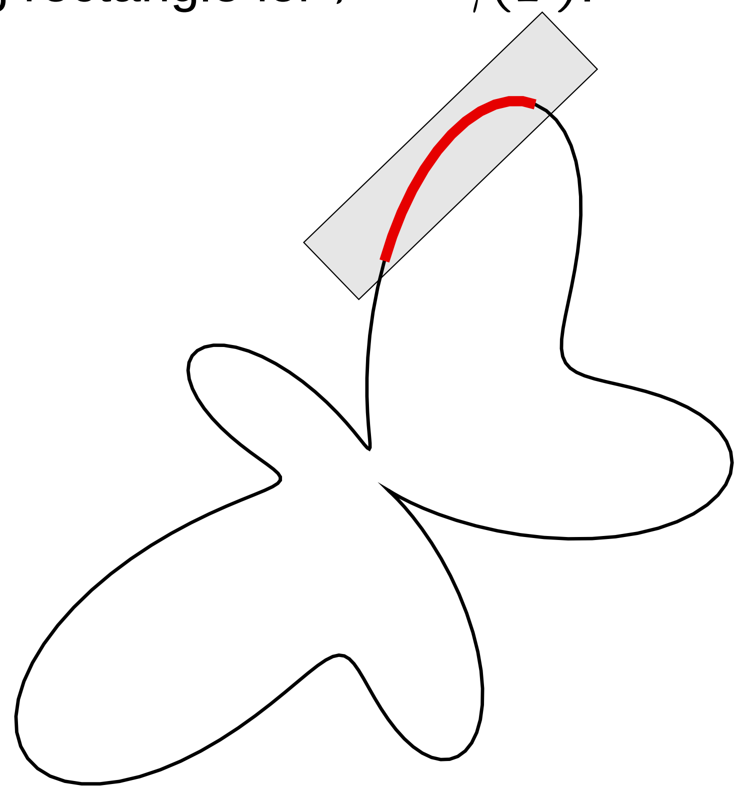


Approximating parametric curves

Given a parametric curve $\mathcal{C} = \gamma(I)$, where $\gamma\colon I \to \mathbf{R}^2$ and $T \subseteq I$, compute a bounding rectangle for $\mathcal{P} = \gamma(T)$.

Solution:

- Write $\gamma(t) = (x(t), y(t))$.
- Represent $t \in T$ with an affine form:

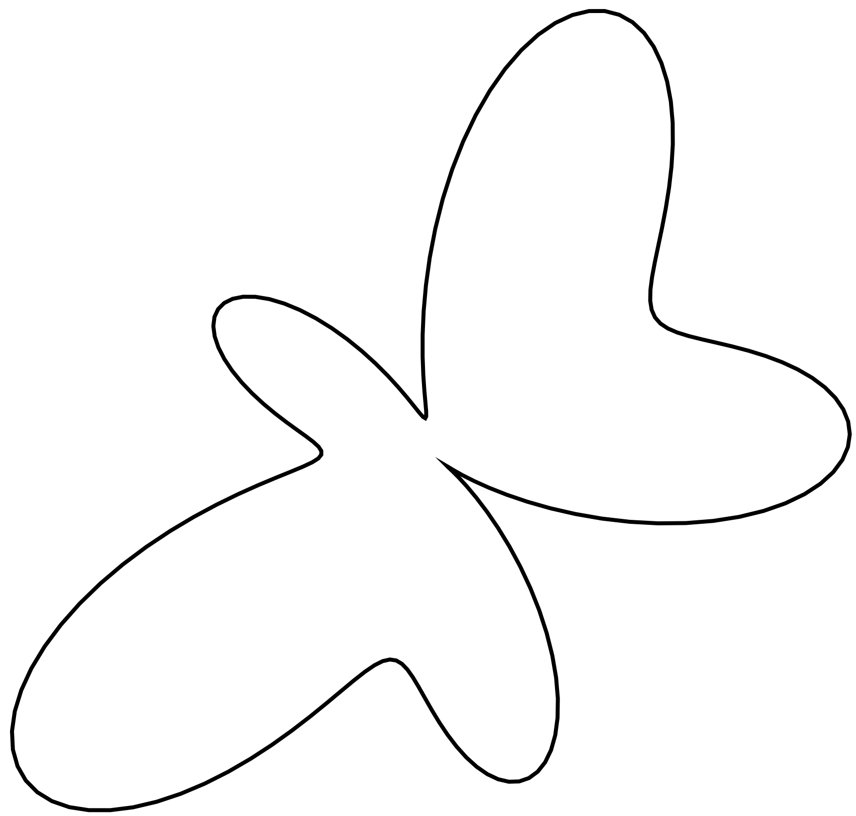
$$\hat{t} = t_0 + t_1\,\varepsilon_1, \qquad t_0 = (b+a)/2, \quad t_1 = (b-a)/2$$

- Compute coordinate functions x and y at $\hat{t}$ using AA:

$$\hat{x} = x_0 + x_1\varepsilon_1 + \cdots + x_n\varepsilon_n$$
$$\hat{y} = y_0 + y_1\varepsilon_1 + \cdots + y_n\varepsilon_n$$

- Use bounding rectangle of the xy zonotope.

Approximating parametric curves



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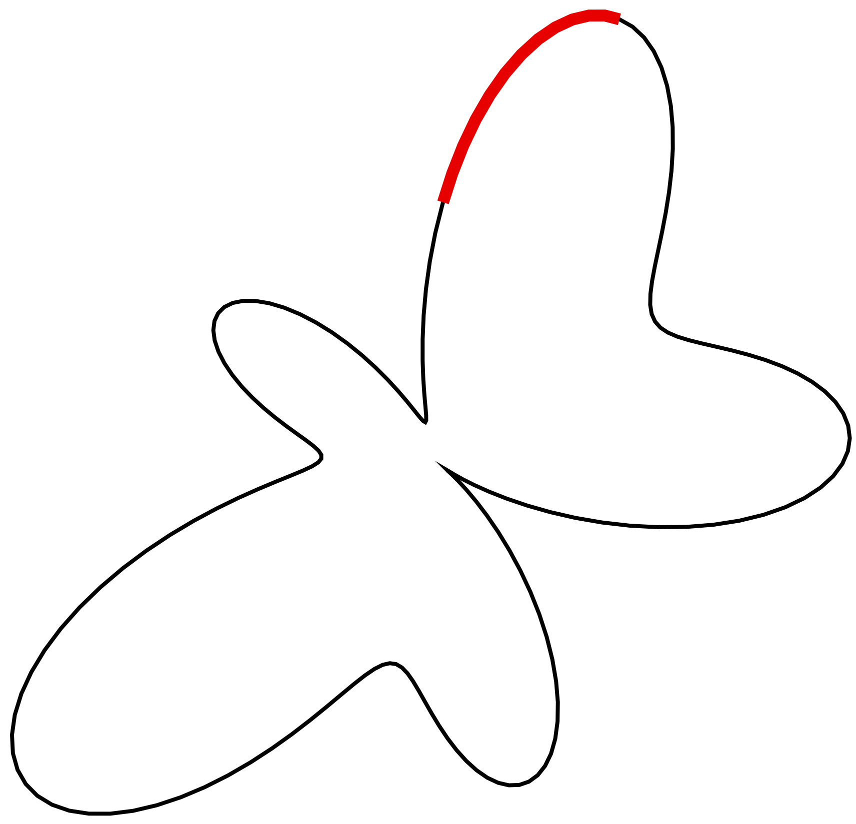
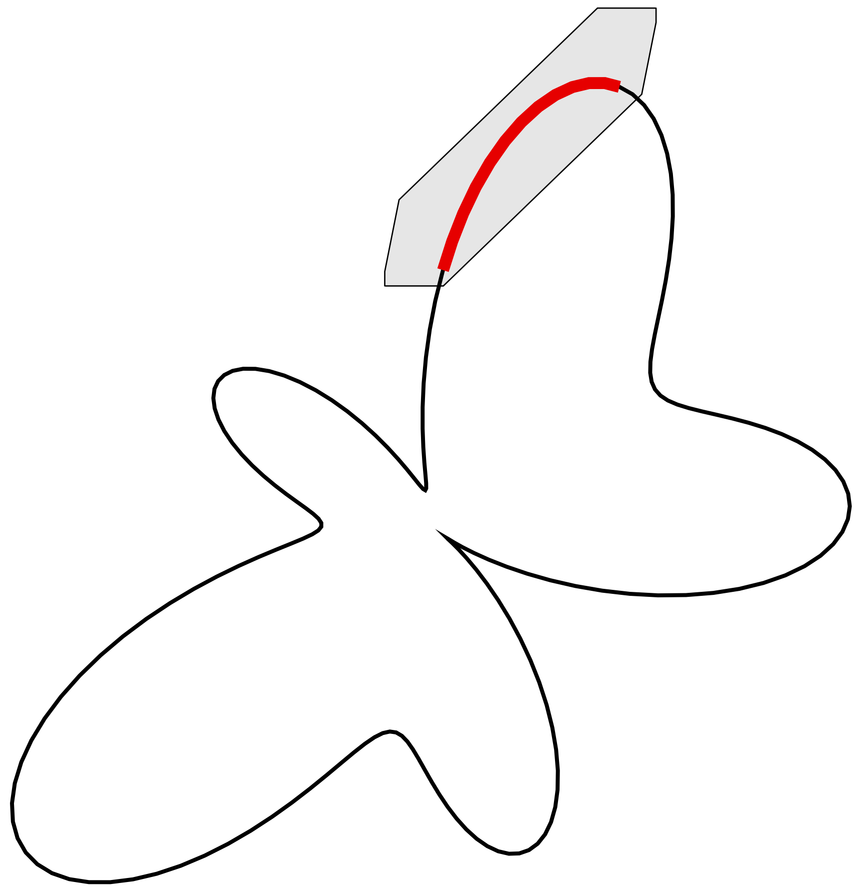


Approximating parametric curves



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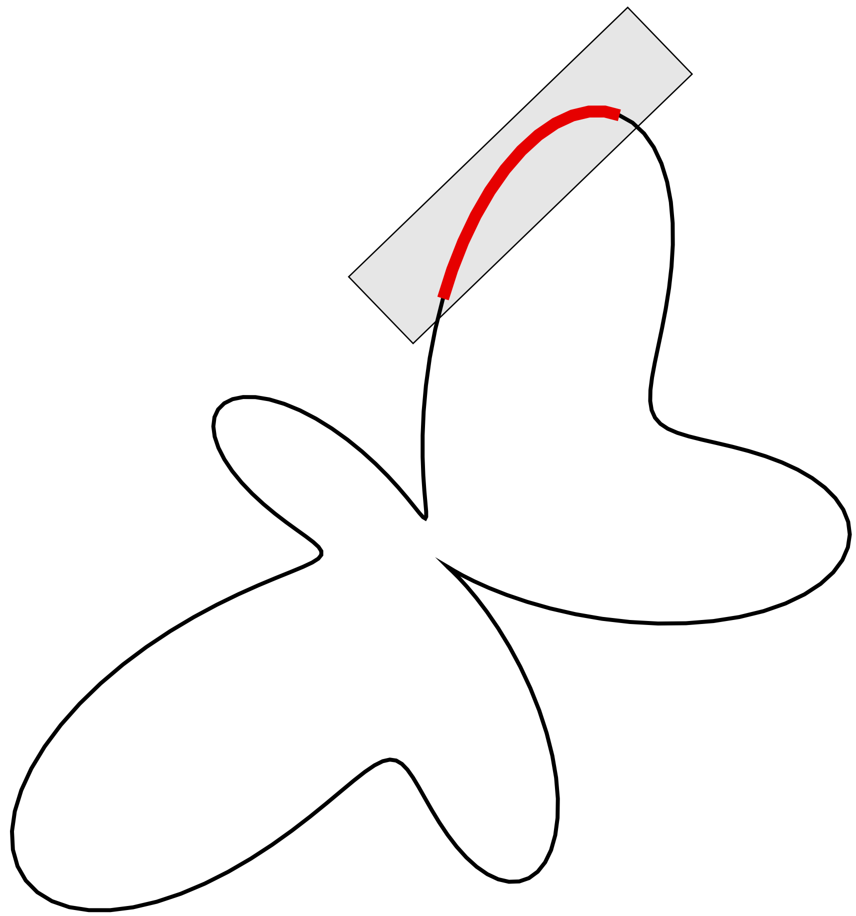

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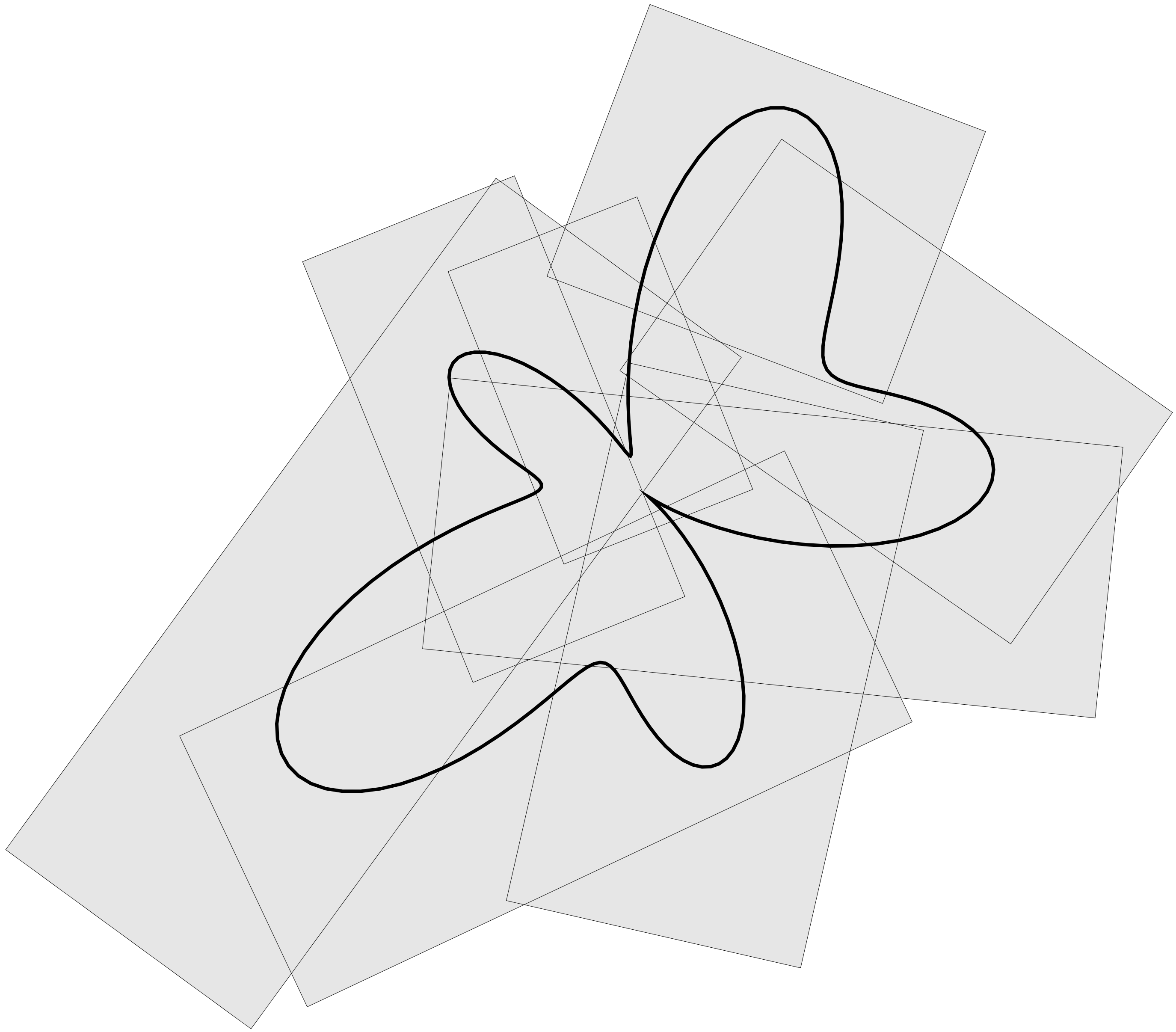

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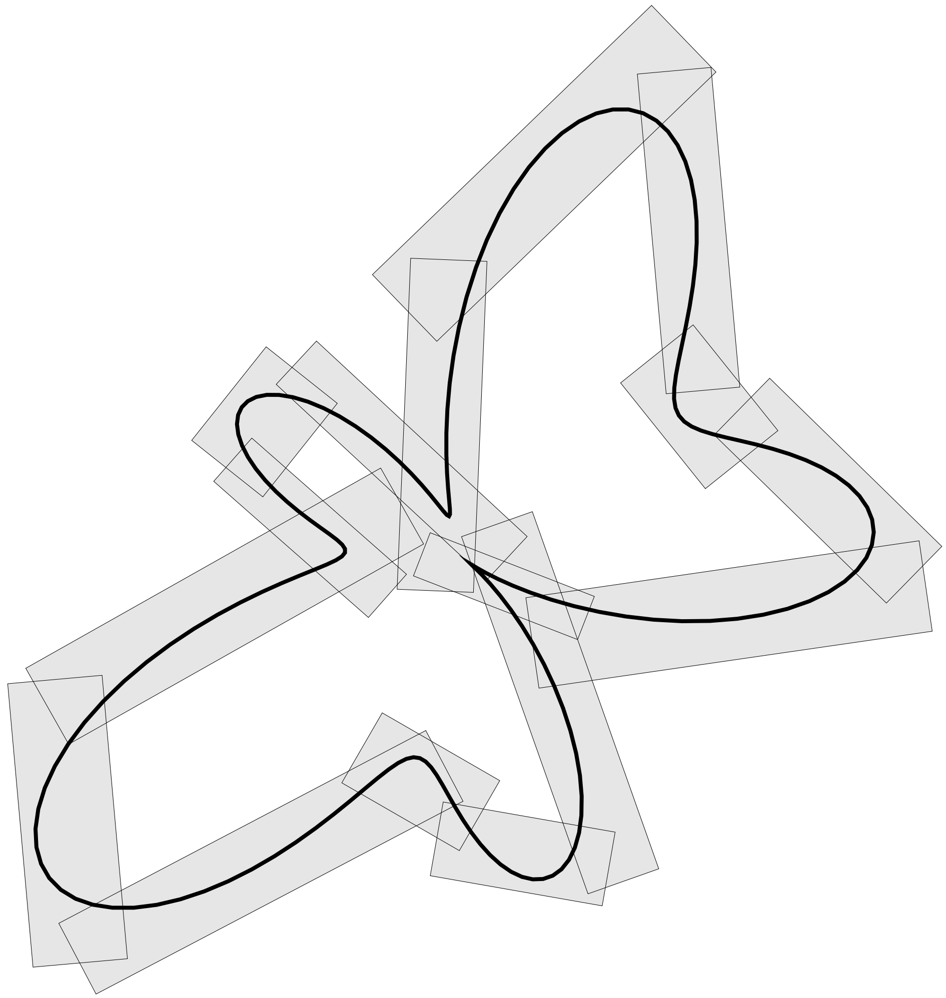

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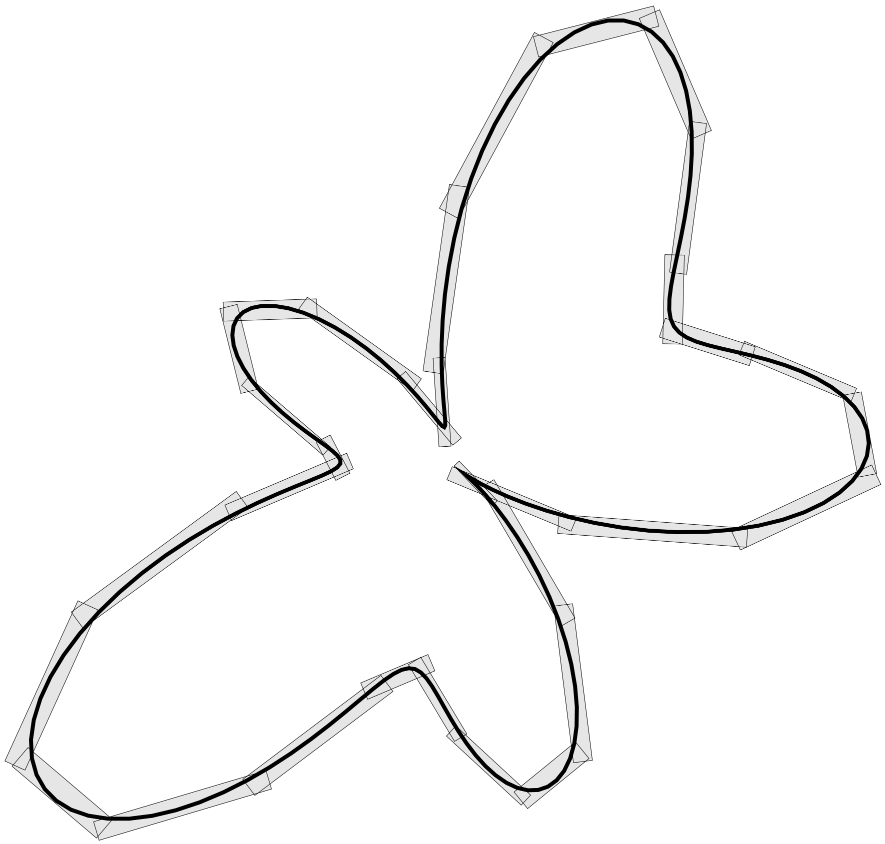

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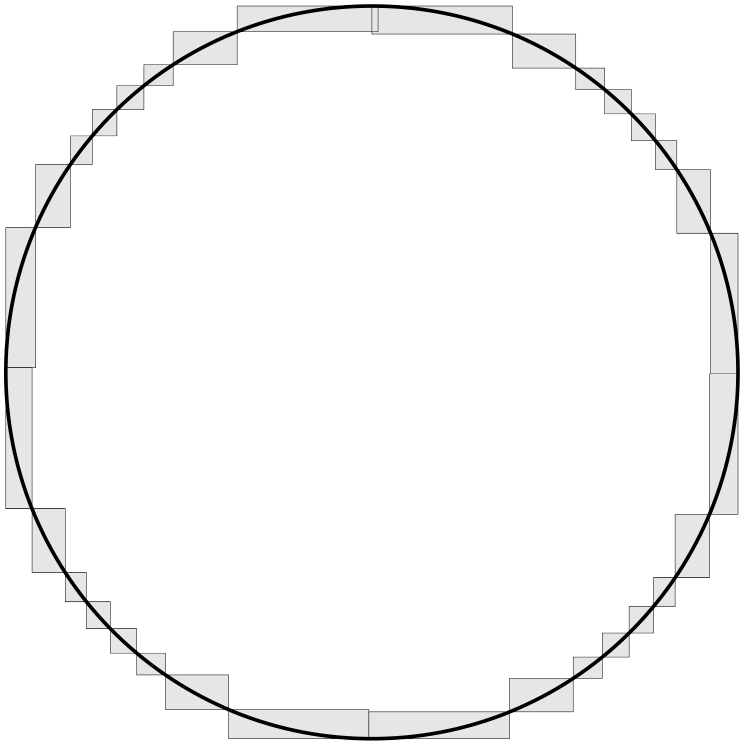
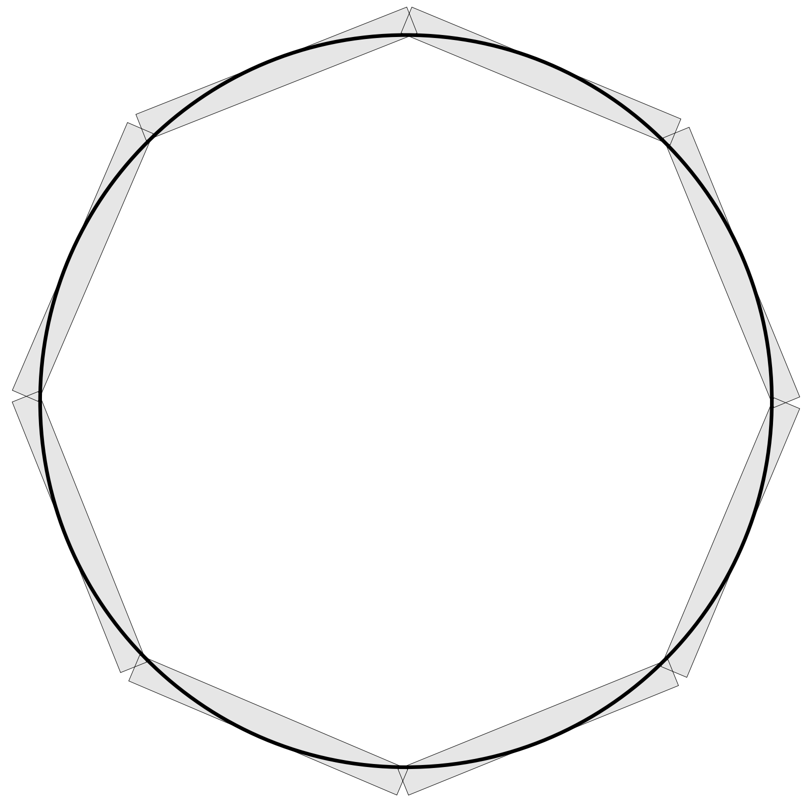


IA (32 boxes)



(8 boxes) AA

Rotated rectangles computed from AA zonotopes

Ray casting implicit surfaces

- Implicit surface

  $h: \mathbf{R}^3 \to \mathbf{R}$

  $S = \{p \in \mathbf{R}^3 : h(p) = 0\}$

- Ray

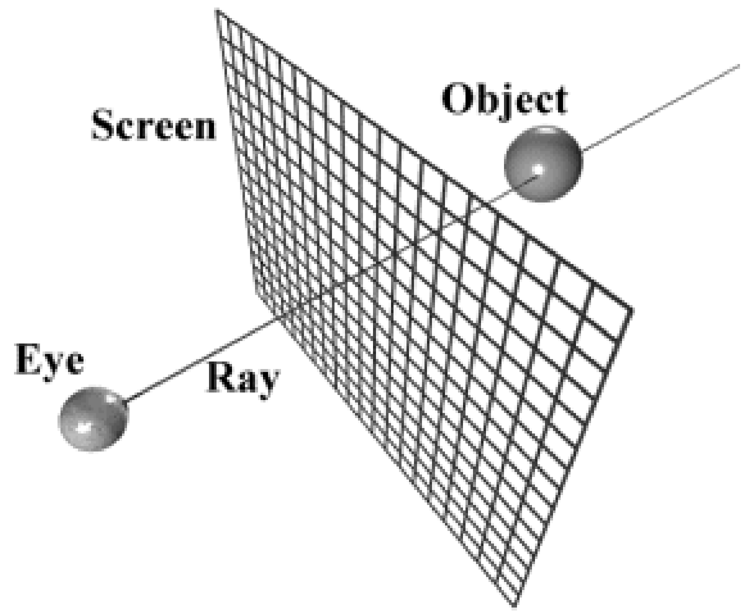
  $r(t) = E + t \cdot v, \quad t \in [0, \infty)$

- Ray intersects S when

  $f(t) = h(r(t)) = 0$

- First intersection occurs at *smallest* zero of f in $[0, \infty)$.

- Paint pixel with color based on normal at first intersection point



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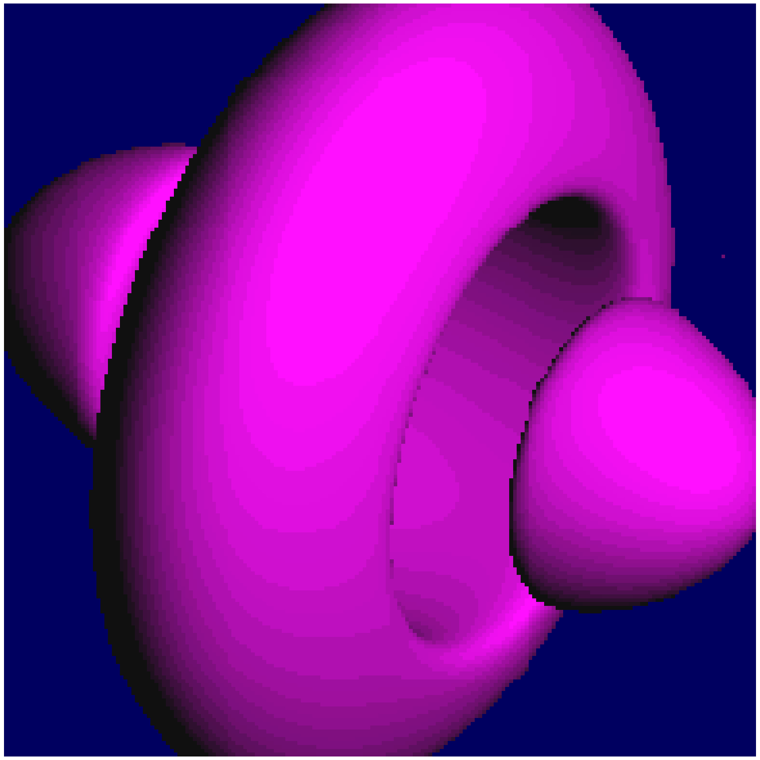


$$4(x^4 + (y^2 + z^2)^2) + 17x^2(y^2 + z^2) - 20(x^2 + y^2 + z^2) + 17 = 0$$

Interval bisection

- Solve $f(t) = 0$ using inclusion function F for f:

$$F(T) \supseteq f(T) = \{f(t) : t \in T\}, \quad T \subseteq I$$

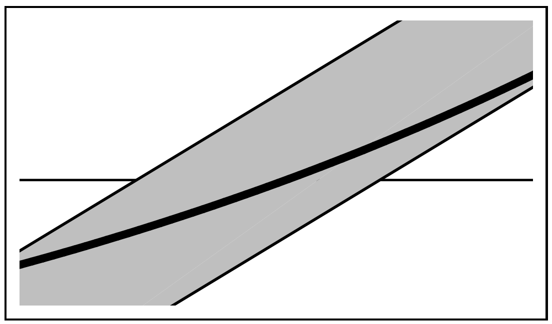
- $0 \notin F(T) \Rightarrow$ *no* solutions of $f(t) = 0$ in T
- $0 \in F(T) \Rightarrow$ there *may* be solutions in T

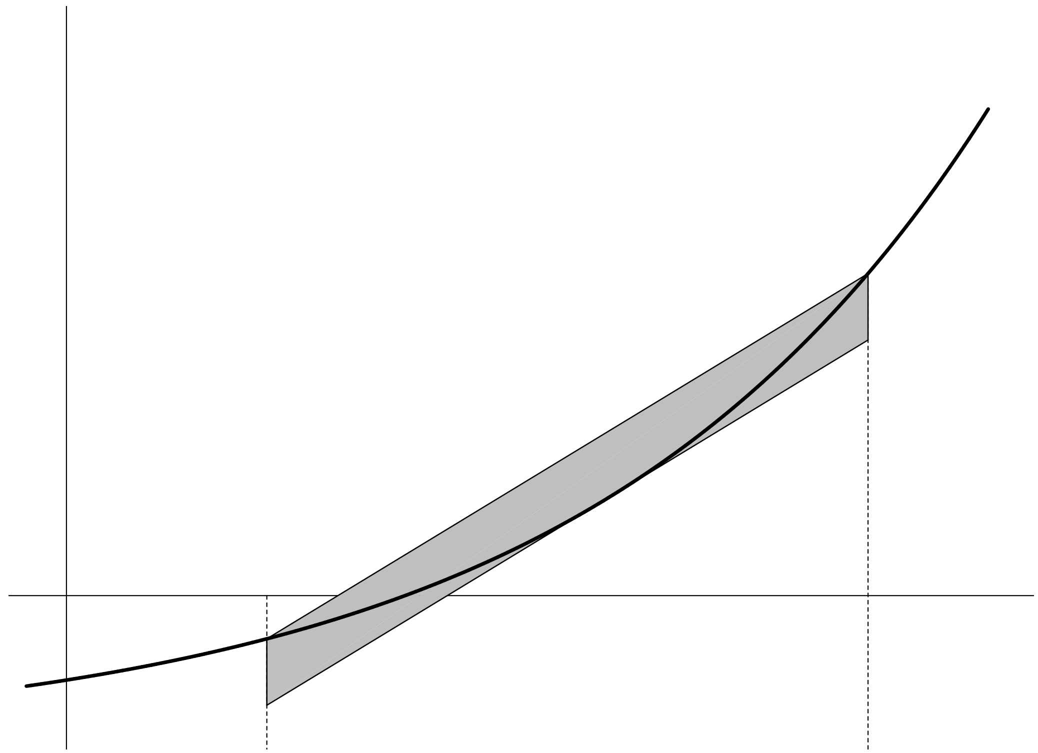
```
interval-bisection([a, b]):
    if 0 ∈ F([a, b]) then
        c ← (a + b)/2
        if (b − a) < ε then
            return c
        else
            interval-bisection([a, c])        ← try left half first!
            interval-bisection([c, b])
```

Start with interval-bisection($[0, t_\infty]$) to find the *first* zero.

Ray casting implicit surfaces with affine arithmetic

- AA exploits linear correlations of x, y, z in $f(t) = h(r(t))$
- AA provides additional information
  - ◇ root must lie in smaller interval
  - ◇ quadratic convergence near simple zeros





Conclusion

Summary

- AA useful replacement for IA
  - ◇ AA more accurate than IA
  - ◇ AA provides additional information that can be exploited
  - ◇ AA locally more expensive than IA but globally more eficient

- AA algorithms not always faster
  - ◇ AA overestimates squares if implemented naively
  - ◇ AA range estimates not always better

- AA has geometric flavor
  - ◇ good for computer graphics!

Other approaches to the dependency problem

- Generalized interval arithmetic (Hansen, 1975)
  - ◇ affine expressions on a fixed set of “noise symbols” with *interval* coefficients
  - ◇ affine operations *not* exact
  - ◇ does not exploit direct correlations of intermediate values
- Linear interval arithmetic (Tupper, 1996) — basis of GrafEq
- Centered forms
- Slopes
- Taylor forms (Berz)
- Zonotope enclosures (Kühn, 1998)
  - ◇ AA is zonotope arithmetic!

Other talks about AA at SCAN 2002

- "A Numerical Method of Proving the Existence of Solutions for Nonlinear ODEs Using Affine Arithmetic", by Yuchi Kanzawa and Shin'ichi Oishi (yesterday at 12:00)

- "Interval Arithmetic, Affine Arithmetic, Taylor Series Methods: Why? What Next?", by Nedialko Nedialkov, Vladik Kreinovich, and Scott Starks (today at 11:00)