Analysis II: homework # 5
Due day: Lecture, Monday April 25, 2016

NAME (print):

Circle the problems that you have solved:

47 48 49 50 51 52 53 54 55

The solutions must be written in a **legible** form. The front page **must** be returned. All the papers **must** be stapled. **If any of the conditions is not satisfied, the homework will burned and flushed away.** The homework **will not** be returned so you better have a copy.

Problem 47. Prove that:

(a) If $K = \mathbb{R}$, then $\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$,

(b) If $K = \mathbb{C}$, then $\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) - \frac{i}{4}(\|ix+y\|^2 - \|ix-y\|^2)$,

Problem 48. Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{R}$ or $\mathbb{C}$. Prove that there is an inner product $\langle\cdot,\cdot\rangle$ such that $\|x\| = \sqrt{\langle x, x\rangle}$ if and only if $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ for all $x, y \in X$. **Hint:** *Use previous problem.*

Problem 49. Prove that $L^1(\mathbb{R}^n)$ is not a Hilbert space. **Hint:** *Use previous problem.*

Problem 50. Show an example of a linear subspace of ℓ^2 which is not closed.

Problem 51. Prove that ℓ^∞ is a Banach space. Then prove that $c_0 \subset \ell^\infty$ is a closed subspace.

Problem 52. Prove that if f is absolutely continuous on S^1 and $f' \in L^2(S^1)$, then $s_n(f)$ converges uniformly to f. **Hint:** *Modify the proof of Theorem 6.3 on p.50 in my* Functional Analysis *notes.*

Problem 53. Find an example of $f \in L^1(S^1) \setminus L^2(S^1)$.

Problem 54. Prove that for any $f \in L^2(S^1)$ the sequence of functions

$$g_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right)$$

converges in $L^2(S^1)$ to the constant function $g(x) \equiv \int_0^1 f$. **Hint:** *Compute the Fourier coefficients of the sum and use the Plancherel identity.*

Problem 55. Use Fourier series to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$