INTRODUCTION TO LINEAR ALGEBRA

Sixth Edition

SOLUTIONS TO PROBLEM SETS

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Problem Set 3.1, page 79

Note An interesting "max-plus" vector space comes from the real numbers $\mathbf{R}$ combined with $-\infty$. Change addition to give $x + y = \mathbf{max}(\boldsymbol{x}, \boldsymbol{y})$ and change multiplication to $xy = \textbf{usual } \boldsymbol{x}+\boldsymbol{y}$. Which y is the zero vector that gives $x + \mathbf{0} = \max(x, \mathbf{0}) = x$ for every x?

1 $\boldsymbol{x} + \boldsymbol{y} \neq \boldsymbol{y} + \boldsymbol{x}$ and $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) \neq (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$ and $(c_1 + c_2)\boldsymbol{x} \neq c_1\boldsymbol{x} + c_2\boldsymbol{x}$.

2 When $c(x_1, x_2) = (cx_1, 0)$, the only broken rule is 1 times $\boldsymbol{x}$ equals $\boldsymbol{x}$. Rules (1)-(4) for addition $\boldsymbol{x} + \boldsymbol{y}$ still hold since addition is not changed.

3 (a) $c\boldsymbol{x}$ may not be in our set: not closed under multiplication. Also no $\mathbf{0}$ and no $-\boldsymbol{x}$
(b) $c(\boldsymbol{x} + \boldsymbol{y})$ is the usual $(xy)^c$, while $c\boldsymbol{x} + c\boldsymbol{y}$ is the usual $(x^c)(y^c)$. Those are equal. With $c = 3$, $x = 2$, $y = 1$ this is $3(\mathbf{2} + \mathbf{1}) = 8$. The zero vector is the number 1.

4 The zero vector in matrix space $\mathbf{M}$ is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; $\frac{1}{2}A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ and $-A = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}$. The smallest subspace of $\mathbf{M}$ containing the matrix A consists of all matrices cA.

5 (a) One possibility: The matrices cA form a subspace not containing B (b) Yes: the subspace must contain $A - B = I$ (c) Matrices whose main diagonal is all zero.

6 When $\boldsymbol{f}(x) = x^2$ and $\boldsymbol{g}(x) = 5x$, the combination $3\boldsymbol{f} - 4\boldsymbol{g}$ in function space is $\boldsymbol{h}(x) = 3\boldsymbol{f}(x) - 4\boldsymbol{g}(x) = 3x^2 - 20x$.

7 Rule 8 is broken: If $c\boldsymbol{f}(x)$ is defined to be the usual $\boldsymbol{f}(cx)$ then $(c_1 + c_2)\boldsymbol{f} = \boldsymbol{f}((c_1 + c_2)x)$ is not generally the same as $c_1\boldsymbol{f} + c_2\boldsymbol{f} = \boldsymbol{f}(c_1x) + \boldsymbol{f}(c_2x)$.

8 (a) The vectors with integer components allow addition, but not multiplication by $\frac{1}{2}$
(b) Remove the x axis from the xy plane (but leave the origin). Multiplication by any c is allowed but not all vector additions: $(1, 1) + (-1, 1) = (0, 2)$ is removed.

9 The only subspaces are (a) the plane with $b_1 = b_2$ (d) the linear combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ (e) the plane with $b_1 + b_2 + b_3 = 0$.

10 (a) All matrices $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ (b) All matrices $\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}$ (c) All diagonal matrices.

11 For the plane $x + y - 2z = 4$, the sum of $(4, 0, 0)$ and $(0, 4, 0)$ is not on the plane. (The key is that this plane does not go through $(0, 0, 0)$.)

12 The parallel plane $\mathbf{P}_0$ has the equation $x + y - 2z = 0$. Pick two points, for example $(2, 0, 1)$ and $(0, 2, 1)$, and their sum $(2, 2, 2)$ is in $\mathbf{P}_0$.

13 The smallest subspace containing a plane $\mathbf{P}$ and a line $\mathbf{L}$ is *either* $\mathbf{P}$ (when the line $\mathbf{L}$ is in the plane $\mathbf{P}$) *or* $\mathbf{R}^3$ (when $\mathbf{L}$ is not in $\mathbf{P}$).

14 (a) The invertible matrices do not include the zero matrix, so they are not a subspace (b) The sum of singular matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is not singular: not a subspace.

15 (a) *True*: The symmetric matrices do form a subspace (b) *True*: The matrices with $A^{\mathrm{T}} = -A$ do form a subspace (c) *True*: Any set of vectors from a vector space will span a subspace of that space.

16 The column space of A is the x-axis = all vectors $(x, 0, 0)$: a *line*. The column space of B is the xy plane = all vectors $(x, y, 0)$. The column space of C is the line of vectors $(x, 2x, 0)$.

17 (a) Elimination leads to $0 = b_2 - 2b_1$ and $0 = b_1 + b_3$ in equations 2 and 3: Solution only if $b_2 = 2b_1$ and $b_3 = -b_1$ (b) Elimination leads to $0 = b_1 + b_3$ in equation 3: Solution only if $b_3 = -b_1$.

18 A combination of the columns of C is also a combination of the columns of A. Then $C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ have the same column space. $B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ has a different column space. The key word is "space".

19 (a) Solution for every $\boldsymbol{b}$ (b) Solvable only if $b_3 = 0$ (c) Solvable only if $b_3 = b_2$.

20 The extra column $\boldsymbol{b}$ enlarges the column space unless $\boldsymbol{b}$ is *already in* the column space.

$$[A \;\; \boldsymbol{b}] = \begin{bmatrix} 1 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} \end{bmatrix} \begin{matrix} \text{(larger column space)} \\ \text{(no solution to } A\boldsymbol{x} = \boldsymbol{b}) \end{matrix} \begin{bmatrix} 1 & 0 & \mathbf{1} \\ 0 & 1 & \mathbf{1} \end{bmatrix} \begin{matrix} (\boldsymbol{b} \text{ is in column space}) \\ (A\boldsymbol{x} = \boldsymbol{b} \text{ has a solution}) \end{matrix}$$

21 The column space of AB is *contained in* (possibly equal to) the column space of A. The example B = zero matrix and $A \neq 0$ is a case when AB = zero matrix has a smaller column space (it is just the zero space **Z**) than A.

22 The solution to $A\boldsymbol{z} = \boldsymbol{b} + \boldsymbol{b}^*$ is $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$. If $\boldsymbol{b}$ and $\boldsymbol{b}^*$ are in $\mathbf{C}(A)$ so is $\boldsymbol{b} + \boldsymbol{b}^*$.

23 The column space of any invertible 5 by 5 matrix is $\mathbf{R}^5$. The equation $A\boldsymbol{x} = \boldsymbol{b}$ is always solvable (by $\boldsymbol{x} = A^{-1}\boldsymbol{b}$) so every $\boldsymbol{b}$ is in the column space of that invertible matrix.

24 (a) *False*: Vectors that are *not* in a column space don't form a subspace. (b) *True*: Only the zero matrix has $\mathbf{C}(A) = \{\mathbf{0}\}$. (c) *True*: $\mathbf{C}(A) = \mathbf{C}(2A)$. (d) *False*: $\mathbf{C}(A - I) \neq \mathbf{C}(A)$ when $A = I$ or $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (or other examples).

25 $A = \begin{bmatrix} 1 & 1 & \mathbf{0} \\ 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & \mathbf{2} \\ 1 & 0 & \mathbf{1} \\ 0 & 1 & \mathbf{1} \end{bmatrix}$ do not have $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in $\mathbf{C}(A)$. $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$ has $\mathbf{C}(A) =$ line in $\mathbf{R}^3$.

26 When $A\boldsymbol{x} = \boldsymbol{b}$ is solvable for all $\boldsymbol{b}$, every $\boldsymbol{b}$ is in the column space of A. So that space is $\mathbf{C}(A) = \mathbf{R}^9$.

27 (a) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are both in $\mathbf{S} + \mathbf{T}$, then $\boldsymbol{u} = \boldsymbol{s}_1 + \boldsymbol{t}_1$ and $\boldsymbol{v} = \boldsymbol{s}_2 + \boldsymbol{t}_2$. So $\boldsymbol{u} + \boldsymbol{v} = (\boldsymbol{s}_1 + \boldsymbol{s}_2) + (\boldsymbol{t}_1 + \boldsymbol{t}_2)$ is also in $\mathbf{S} + \mathbf{T}$. And so is $c\boldsymbol{u} = c\boldsymbol{s}_1 + c\boldsymbol{t}_1$: $\mathbf{S} + \mathbf{T} =$ *subspace*.

(b) If $\mathbf{S}$ and $\mathbf{T}$ are different lines, then $\mathbf{S} \cup \mathbf{T}$ is just the two lines (*not a subspace*) but $\mathbf{S} + \mathbf{T}$ is the whole plane that they span.

28 If $\mathbf{S} = \mathbf{C}(A)$ and $\mathbf{T} = \mathbf{C}(B)$ then $\mathbf{S} + \mathbf{T}$ is the column space of $M = [\, A \quad B \,]$.

29 The columns of AB are combinations of the columns of A. So all columns of $[\, A \quad AB \,]$ are already in $\mathbf{C}(A)$. But $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has a larger column space than $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. For square matrices, the column space is $\mathbf{R}^n$ exactly when A is *invertible*.

30 $y - e^{-x}$ and $y = e^x$ are independent solutions to $d^2y/dx^2 = y$. Also $y = \cos x$ and $y = \sin x$ are independent solutions to $d^2y/dx^2 = -y$. The solution space contains all combinations $A\cos x + B\sin x$.

31 If $\boldsymbol{x}$ and $\boldsymbol{y}$ are in the vector space $\mathbf{V} \cap \mathbf{W}$, then they are in both $\mathbf{V}$ and $\mathbf{W}$. So all combinations $c\boldsymbol{x} + d\boldsymbol{y}$ are in both $\mathbf{V}$ and $\mathbf{W}$. So all combinations are in $\mathbf{V} \cap \mathbf{W}$.

Problem Set 3.2, page 100

1 If $A\boldsymbol{x} = \mathbf{0}$ then $EA\boldsymbol{x} = \mathbf{0}$. If $EA\boldsymbol{x} = \mathbf{0}$, multiply by E^{-1} to find $A\boldsymbol{x} = \mathbf{0}$.

2 (a) If $c = 4$ then A has rank 1 and column 1 is its pivot column and $(-2, 1, 0)$ and $(-1, 0, 1)$ are special solutions to $A\boldsymbol{x} = \mathbf{0}$. If $c \neq 4$ then A has rank 2 and columns 1 and 3 are pivot columns and $(-2, 1, 0)$ is a special solution. If $c = 0$ then $B =$ zero matrix with rank 0 and $(1, 0)$ and $(0, 1)$ are special solutions to $B\boldsymbol{x} = \mathbf{0}$. If $c \neq 0$ then B has rank 1 and column 1 is its pivot column and $(-1, 1)$ is the special solution to $B\boldsymbol{x} = \mathbf{0}$.

3 $R = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 6 \end{bmatrix}$. All matrices $A = CR$ with $C = 2$ by 2 invertible matrix have the same nullspace as R.

4 (a) $R = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$ Free variables x_2, x_4, x_5 / Pivot variables x_1, x_3 (b) $R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ Free x_3 / Pivot x_1, x_2

5 Free variables x_2, x_4, x_5 and solutions $(-2, 1, 0, 0, 0)$, $(0, 0, -2, 1, 0)$, $(0, 0, -3, 0, 1)$.

6 (a) *False*: Any singular square matrix would have free variables (b) *True*: An invertible square matrix has *no* free variables. (c) *True* (only n columns to hold pivots) (d) *True* (only m rows to hold pivots)

7 $A = \begin{bmatrix} C \end{bmatrix} \begin{bmatrix} I & I \end{bmatrix}$ (notice that $F = I$). The r special solutions to $A\boldsymbol{x} = \mathbf{0}$ are the r columns of $\begin{bmatrix} -I \\ I \end{bmatrix}$.

8 $$R = \begin{bmatrix} \mathbf{1} & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{bmatrix}, \begin{bmatrix} 0 & \mathbf{1} & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Notice the identity matrix in the pivot columns of these *reduced* row echelon forms R.

9 If column 4 of a 3 by 5 matrix is all zero then x_4 is a *free* variable. Its special solution is $\boldsymbol{x} = (0, 0, 0, 1, 0)$, because 1 will multiply that zero column to give $A\boldsymbol{x} = \mathbf{0}$.

10 If column $1 =$ column 5 then x_5 is a free variable. Its special solution is $(-1, 0, 0, 0, 1)$.

11 The nullspace contains only $\boldsymbol{x} = \mathbf{0}$ when A has 5 pivots. Also the column space is $\mathbf{R}^5$, because we can always solve $A\boldsymbol{x} = \boldsymbol{b}$ and every $\boldsymbol{b}$ is in the column space.

12 If a matrix has n columns and r pivots, there are $\boldsymbol{n - r}$ special solutions. The nullspace contains only $\boldsymbol{x} = \mathbf{0}$ when $\boldsymbol{r = n}$. The column space is all of $\mathbf{R}^m$ when $r = m$. All those statements are important!

13 Fill in $\mathbf{12}$ then $\mathbf{3}$ then $\mathbf{1}$ to get the complete solution in $\mathbf{R}^3$ to $x - 3y - z = 12$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{12} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \mathbf{3} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \mathbf{1} \\ 0 \\ 1 \end{bmatrix} = \text{one particular solution} + \text{all nullspace solutions.}$$

14 Column 5 is sure to have no pivot since it is a combination of earlier columns. With 4 pivots in the other columns, the special solution is $\boldsymbol{s} = (1, 0, 1, 0, 1)$. The nullspace contains all multiples of this vector $\boldsymbol{s}$ (this nullspace is a line in $\mathbf{R}^5$).

15 To produce special solutions $(2, 2, 1, 0)$ and $(3, 1, 0, 1)$ with free variables x_3, x_4:

$$R = \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -2 & -1 \end{bmatrix} \text{ and } A \text{ can be any invertible 2 by 2 matrix times this } R.$$

16 The nullspace of $A = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ The rank is 3 is the line through the special solution $\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$.

17 $A = \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$ has $\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$ in $\mathbf{C}(A)$ and $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ in $\mathbf{N}(A)$. *Which other* A's?

18 $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

19 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $\mathbf{N}(A) = \mathbf{C}(A)$. Notice that $\mathsf{rref}(A^{\mathrm{T}}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not A^{T}.

20 If nullspace = column space (with r pivots) then $n - r = r$. If $n = 3$ then $3 = 2r$ is impossible. Only possible when n is even.

21 If A times every column of B is zero, the column space of B is contained in the *nullspace* of A. An example is $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. Here $\mathbf{C}(B)$ *equals* $\mathbf{N}(A)$. For $B = 0$, $\mathbf{C}(B)$ is smaller than $\mathbf{N}(A)$.

22 For A = random 3 by 3 matrix, R is almost sure to be I. For 4 by 3, R is most likely to be I with a fourth row of zeros. What is R for a random 3 by 4 matrix?

23 If $\mathbf{N}(A)$ = line through $\boldsymbol{x} = (2, 1, 0, 1)$, A has *three pivots* (4 columns and 1 special solution). Its reduced echelon form can be $R = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (add any zero rows).

24 $R = [\,1 \quad -2 \quad -3\,]$, $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $R = I$. Any zero rows come after those rows.

25 (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (b) All 8 matrices are R's !

26 The nullspace of $B = [\,A \quad A\,]$ contains all vectors $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{y} \\ -\boldsymbol{y} \end{bmatrix}$ for $\boldsymbol{y}$ in $\mathbf{R}^4$.

One reason that R is the same for A and $-A$: They have the same nullspace. (They also have the same row space. They also have the same column space, but that is not required for two matrices to share the same R. R tells us the nullspace and row space.)

27 If $C\boldsymbol{x} = \mathbf{0}$ then $A\boldsymbol{x} = \mathbf{0}$ and $B\boldsymbol{x} = \mathbf{0}$. So $\mathbf{N}(C) = \mathbf{N}(A) \cap \mathbf{N}(B)$ = *intersection*.

28 A has $R_0 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ and $R = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$. B and C have $R_0 = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

And $R = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{bmatrix}$.

29 $R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $N = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

30 A and A^{T} have the same rank $r =$ number of pivots. But the pivot column is column 2 for this matrix A and column 1 for A^{T}: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

31 The new entries keep rank 1 : $A = \begin{bmatrix} a & b & c \\ d & \frac{bd}{a} & \frac{cd}{a} \\ g & \frac{bg}{a} & \frac{cg}{a} \end{bmatrix}$ if $a \neq 0$, $\quad B = \begin{bmatrix} \mathbf{3} & 9 & \mathbf{-4.5} \\ 1 & \mathbf{3} & \mathbf{-1.5} \\ 2 & 6 & -3 \end{bmatrix}$,

$M = \begin{bmatrix} a & b \\ c & \boldsymbol{bc/a} \end{bmatrix}$ if $a \neq 0$.

32 With rank 1, the second row of R does not exist !

33 Invertible r by r submatrices
Use pivot rows and columns $\quad S = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $S = [\,1\,]$ and $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

34 (a) A and B will both have the same nullspace and row space as the R they share.

(b) A equals an *invertible* matrix times B, when they share the same R. A key fact!

35 CORRECTED : $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$: $y_1 - y_3 + y_4 = -y_1 + y_2 + y_5 = -y_2 + y_3 + y_6 = -y_4 - y_5 - y_6 = 0$.

These equations add to $0 = 0$. Free variables y_3, y_5, y_6: watch for flows around loops. The solutions to $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ are combinations of $(-1, 0, 0, 1, -1, 0)$ and $(0, 0, -1, -1, 0, 1)$ and $(0, -1, 0, 0, 1, -1)$. Those are flows around the 3 small loops.

36 $C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 2 & 7 \end{bmatrix}$ C^{T} has pivot columns $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$. The invertible S inside C is $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$

37 The column space of AB contains all vectors $(AB)\boldsymbol{x}$. Those vectors are the same as $A(B\boldsymbol{x})$ so they are also in the column space of A.

38 By matrix multiplication, each column of AB is A times the corresponding column of B. So if column j of B is a combination of earlier columns of B, then column j of AB is the same combination of earlier columns of AB. Then rank $(AB) \leq$ rank (B). No new pivot columns !

39 We are given $AB = I$ which has rank n. Then rank$(AB) \leq$ rank(A) forces rank$(A) = n$. This means that A is invertible. The right-inverse B is also a left-inverse: $BA = I$ and $B = A^{-1}$.

40 Certainly A and B have at most $\text{rank}\, 2$. Then their product AB has at most $\text{rank}\, 2$. Since BA is 3 by 3, it cannot be I even if $AB = I$. *Example* $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$,

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

41 $A = \begin{bmatrix} I & I \end{bmatrix}$ has $N = \begin{bmatrix} I \\ -I \end{bmatrix}$; $B = \begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix}$ has the same N; $C = \begin{bmatrix} I & I & I \end{bmatrix}$ has

$$N = \begin{bmatrix} -I & -I \\ I & 0 \\ 0 & I \end{bmatrix}.$$

42 The m by n matrix Z has r ones to start its main diagonal. Otherwise Z is all zeros.

43 $R_0 = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} r \text{ by } r & r \text{ by } n-r \\ m-r \text{ by } r & m-r \text{ by } n-r \end{bmatrix}$; (b) $B = \begin{bmatrix} I \\ 0 \end{bmatrix}$ (c) $C = \begin{bmatrix} I & 0 \end{bmatrix}$

$\textbf{rref}(R_0^{\mathrm{T}}) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$; $\textbf{rref}(R_0^{\mathrm{T}} R_0) = \text{same } R_0$

44 $R_0 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has $R_0^{\mathrm{T}} R_0 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and this matrix row reduces to $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} =$ $\begin{bmatrix} R_0 \\ \text{zero row} \end{bmatrix}$. Always $R_0^{\mathrm{T}} R_0$ has the same nullspace as R_0, so its row reduced form must be R_0 with $n - m$ extra zero rows. R_0 is determined by its nullspace and shape !

45 $$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix}$$

Notice 2 rows of A are in the matrix B.

46 Multiply block row 1 by JW^{-1} to produce row 2.

Problem Set 3.3, page 111

1 $$\begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 2 & 5 & 7 & 6 & \mathbf{b}_2 \\ 2 & 3 & 5 & 2 & \mathbf{b}_3 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 0 & 1 & 1 & 2 & \mathbf{b}_2 - \mathbf{b}_1 \\ 0 & -1 & -1 & -2 & \mathbf{b}_3 - \mathbf{b}_1 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 0 & 1 & 1 & 2 & \mathbf{b}_2 - \mathbf{b}_1 \\ 0 & 0 & 0 & 0 & \mathbf{b}_3 + \mathbf{b}_2 - \mathbf{2b}_1 \end{bmatrix} \begin{matrix} 4 \\ -1 \\ 0 \end{matrix}$$

$A\boldsymbol{x} = \boldsymbol{b}$ has a solution when $b_3 + b_2 - 2b_1 = 0$; the column space contains all combinations of $(2, 2, 2)$ and $(4, 5, 3)$. **This is the plane** $b_3 + b_2 - 2b_1 = 0$ (!). The nullspace contains all combinations of $\boldsymbol{s}_1 = (-1, -1, 1, 0)$ and $\boldsymbol{s}_2 = (2, -2, 0, 1)$; $\boldsymbol{x}_{complete} = \boldsymbol{x}_p + c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2$;

$$\begin{bmatrix} R_0 & \boldsymbol{d} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ gives the particular solution $x_p = (4, -1, 0, 0)$.

2 $$\begin{bmatrix} 2 & 1 & 3 & \mathbf{b}_1 \\ 6 & 3 & 9 & \mathbf{b}_2 \\ 4 & 2 & 6 & \mathbf{b}_3 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 3 & \mathbf{b}_1 \\ 0 & 0 & 0 & \mathbf{b}_2 - \mathbf{3b}_1 \\ 0 & 0 & 0 & \mathbf{b}_3 - \mathbf{2b}_1 \end{bmatrix} \quad \text{Then } [\, R_0 \;\; \boldsymbol{d} \,] = \begin{bmatrix} 1 & 1/2 & 3/2 & \mathbf{5} \\ 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$$

$A\boldsymbol{x} = \boldsymbol{b}$ has a solution when $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; $\mathbf{C}(A) =$ line through $(2, 6, 4)$ which is the intersection of the planes $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; the nullspace contains all combinations of $\boldsymbol{s}_1 = (-1/2, 1, 0)$ and $\boldsymbol{s}_2 = (-3/2, 0, 1)$; particular solution $\boldsymbol{x}_p = \boldsymbol{d} = (5, 0, 0)$ and complete solution $\boldsymbol{x}_p + c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2$.

3 (a) $$\begin{aligned} x + 3y &= 7 \\ 2x + 6y &= 14 \end{aligned} \qquad \begin{aligned} x + 3y &= 7 \\ 0 &= 0 \end{aligned} \qquad \boldsymbol{x}_p = \begin{bmatrix} 7 \\ 0 \end{bmatrix} \qquad \boldsymbol{x}_n = cs = c\begin{bmatrix} -3 \\ 1 \end{bmatrix} \text{ for any } c\,.$$

(b) $$\boldsymbol{x}_{\text{complete}} = \begin{bmatrix} 7 \\ 0 \end{bmatrix} + c\begin{bmatrix} -3 \\ 1 \end{bmatrix}; \qquad \boldsymbol{x}_{\text{complete}} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + c\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}.$$

4 $\boldsymbol{x}_{\text{complete}} = \boldsymbol{x}_p + \boldsymbol{x}_n = (\frac{1}{2}, 0, \frac{1}{2}, 0) + x_2(-3, 1, 0, 0) + x_4(0, 0, -2, 1)$.

5 $\begin{bmatrix} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 - b_2 \end{bmatrix}$ solvable if $b_3 - 2b_1 - b_2 = 0$.

Back-substitution gives the particular solution to $A\boldsymbol{x} = \boldsymbol{b}$ and the special solution to $A\boldsymbol{x} = \boldsymbol{0}$: $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. $\begin{bmatrix} 2 & 0 & 2 & b_1 \\ 4 & 4 & 0 & b_2 \\ 8 & 8 & 0 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & b_1/2 \\ 0 & 1 & -1 & b_2/4 - b_1/2 \\ 0 & 0 & 0 & b_3 - 2b_2 \end{bmatrix}$

is solvable if $b_3 = 2b_2$. Then $\boldsymbol{x} = \begin{bmatrix} b_1/2 \\ b_2/4 - b_1/2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

6 (a) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. Then $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \end{bmatrix} = \boldsymbol{x}_p$

(b) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

7 $\begin{bmatrix} 1 & 3 & 1 & b_1 \\ 3 & 8 & 2 & b_2 \\ 2 & 4 & 0 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & b_2 \\ 0 & -1 & -1 & b_2 - 3b_1 \\ 0 & -2 & -2 & b_3 - 2b_1 \end{bmatrix}$ One more step gives $[\,0\ 0\ 0\ 0\,] =$ row 3 − 2 (row 2) + 4(row 1) **provided $\boldsymbol{b_3 - 2b_2 + 4b_1 = 0}$**.

8 (a) Every $\boldsymbol{b}$ is in $\mathbf{C}(A)$: *independent rows*, only the zero combination gives $\mathbf{0}$.

(b) We need $b_3 = 2b_2$, because $(\text{row}\,3) - 2(\text{row}\,2) = \mathbf{0}$.

9 (a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. The second equation in part (b) removed one special solution from the nullspace.

10 $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ has $\boldsymbol{x}_p = (2, 4, 0)$ and $\boldsymbol{x}_{\text{null}} = (c, c, c)$. Many possible A !

11 A 1 by 3 system has at least **two** free variables. But $\boldsymbol{x}_{\text{null}}$ in Problem 10 only has **one**.

12 (a) If $A\boldsymbol{x}_1 = \boldsymbol{b}$ and $A\boldsymbol{x}_2 = \boldsymbol{b}$ then $\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$ and also $\boldsymbol{x} = \mathbf{0}$ solve $A\boldsymbol{x} = \mathbf{0}$

(b) $A(2\boldsymbol{x}_1 - 2\boldsymbol{x}_2) = \mathbf{0}, A(2\boldsymbol{x}_1 - \boldsymbol{x}_2) = \boldsymbol{b}$

13 (a) The particular solution $\boldsymbol{x}_p$ is always multiplied by 1. $2\boldsymbol{x}_p$ would solve $A\boldsymbol{x} = 2\boldsymbol{b}$

(b) Any solution can be $\boldsymbol{x}_p$. If A has rank $= m$, the only $\boldsymbol{x}_p$ is $\mathbf{0}$.

(c) $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$. Then $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is shorter (length $\sqrt{2}$) than $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ (length 2)

(d) The only "homogeneous" solution in the nullspace is $\boldsymbol{x}_n = \mathbf{0}$ when A is invertible.

14 If column 5 has no pivot, x_5 is a *free* variable. The zero vector *is not* the only solution to $A\boldsymbol{x} = \mathbf{0}$. If this system $A\boldsymbol{x} = \boldsymbol{b}$ has a solution, it has *infinitely many* solutions.

15 If row 3 of U has no pivot, that is a *zero row*. $U\boldsymbol{x} = \boldsymbol{c}$ is only solvable provided $\boldsymbol{c}_3 = 0$. $A\boldsymbol{x} = \boldsymbol{b}$ *might not be solvable*, because U may have other zero rows needing more $c_i = 0$.

16 The largest rank is 3. Then there is a pivot in every *row*. The solution *always exists*. The column space is $\mathbf{R}^3$. An example is $A = [\, I \quad F \,]$ for any 3 by 2 matrix F.

17 The largest rank of a 6 by 4 matrix is 4. Then there is a pivot in every *column*. The columns are independent. The solution is *unique* (if there is a solution). The nullspace contains only the *zero vector*. Then $\mathbf{R_0} = \mathbf{rref}(A) = \begin{bmatrix} I \ \ (4 \text{ by } 4) \\ 0 \ \ (2 \text{ by } 4) \end{bmatrix}$.

18 Rank $= 2$; rank $= 3$ unless $q = 2$ (then rank $= 2$). Transpose has the same rank!

19 If $A\boldsymbol{x}_1 = \boldsymbol{b}$ and also $A\boldsymbol{x}_2 = \boldsymbol{b}$ then $A(\boldsymbol{x}_1 - \boldsymbol{x}_2) = \mathbf{0}$ and we can add $\boldsymbol{x}_1 - \boldsymbol{x}_2$ to any solution of $A\boldsymbol{x} = \boldsymbol{B}$: the solution $\boldsymbol{x}$ is not unique. But there will be **no solution** to $A\boldsymbol{x} = \boldsymbol{B}$ if $\boldsymbol{B}$ is not in the column space.

20 For $A, q = 3$ gives rank 1, every other q gives rank 2. For $B, q = 6$ gives rank 1, every other q gives rank 2. These matrices cannot have rank 3.

21 (a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix} [\, x \,] = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ has 0 or 1 solutions, depending on $\boldsymbol{b}$ (b) $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\, b \,]$ has infinitely many solutions for every b (c) There are 0 or ∞ solutions when A has rank $r < m$ and $r < n$: the simplest example is a zero matrix. (d) *one* solution for all $\boldsymbol{b}$ when A is square and invertible (like $A = I$).

22 (a) $r < m$, always $r \le n$ (b) $r = m, r < n$ (c) $r < m, r = n$ (d) $r = m = n$.

23 $\begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \to R_0 = \begin{bmatrix} \mathbf{1} & 0 & -2 \\ 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix} \to R_0 = I = R$ and

$\begin{bmatrix} 0 & 0 & 4 \\ 0 & 1 & 0 \end{bmatrix} \to R_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R.$

24 $R_0 = I$ when A is square and invertible—so for a triangular matrix, all diagonal entries must be nonzero.

25 $\begin{bmatrix} 1 & 2 & 3 & \mathbf{0} \\ 0 & 0 & 4 & \mathbf{0} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \end{bmatrix}$; $\boldsymbol{x}_n = \begin{bmatrix} -2 \\ \mathbf{1} \\ 0 \end{bmatrix}$; $\begin{bmatrix} 1 & 2 & 3 & \mathbf{5} \\ 0 & 0 & 4 & \mathbf{8} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -\mathbf{1} \\ 0 & 0 & 1 & \mathbf{2} \end{bmatrix}$.

Free $x_2 = 0$ gives $\boldsymbol{x}_p = (-1, 0, 2)$ because the pivot columns contain I. Note : $R_0 = R$.

26 $[\,R_0 \;\; \boldsymbol{d}\,] = \begin{bmatrix} 1 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$ leads to $\boldsymbol{x}_n = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$; $[\,R_0 \;\; \boldsymbol{d}\,] = \begin{bmatrix} 1 & 0 & 0 & -\mathbf{1} \\ 0 & 0 & 1 & \mathbf{2} \\ 0 & 0 & 0 & \mathbf{5} \end{bmatrix}$

leads to no solution because of the 3rd equation $0 = 5$.

27 $\begin{bmatrix} 1 & 0 & 2 & 3 & \mathbf{2} \\ 1 & 3 & 2 & 0 & \mathbf{5} \\ 2 & 0 & 4 & 9 & \mathbf{10} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 3 & \mathbf{2} \\ 0 & 3 & 0 & -3 & \mathbf{3} \\ 0 & 0 & 0 & 3 & \mathbf{6} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 & -\mathbf{4} \\ 0 & 1 & 0 & 0 & \mathbf{3} \\ 0 & 0 & 0 & 1 & \mathbf{2} \end{bmatrix}$; $\begin{bmatrix} -4 \\ 3 \\ 0 \\ 2 \end{bmatrix}$; $\boldsymbol{x}_n = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

28 For $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 3 \end{bmatrix}$, the only solution to $A\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is $\boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

B cannot exist since 2 equations in 3 unknowns cannot have a unique solution.

29 $A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 5 \end{bmatrix}$ factors into $LU = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 2 & 2 & 1 & \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and the rank is $r = 2$. The special solution to $A\boldsymbol{x} = \boldsymbol{0}$ and $U\boldsymbol{x} = \boldsymbol{0}$ is $\boldsymbol{s} = (-7, 2, 1)$. Since $\boldsymbol{b} = (1, 3, 6, 5)$ is also the last column of A, a particular solution to $A\boldsymbol{x} = \boldsymbol{b}$ is $(0, 0, 1)$ and the complete solution is $\boldsymbol{x} = (0, 0, 1) + c\boldsymbol{s}$. (Another particular solution is $\boldsymbol{x}_p = (7, -2, 0)$ with free variable $x_3 = 0$.)

For $\boldsymbol{b} = (1, 0, 0, 0)$ elimination leads to $U\boldsymbol{x} = (1, -1, 0, 1)$ and the fourth equation is $0 = 1$. No solution for this $\boldsymbol{b}$.

30 If the complete solution to $A\boldsymbol{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c \end{bmatrix}$ then $A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$.

31 (a) If $\boldsymbol{s} = (2, 3, 1, 0)$ is the only special solution to $A\boldsymbol{x} = \boldsymbol{0}$, the complete solution is $\boldsymbol{x} = c\boldsymbol{s}$ (a line of solutions). The rank of A must be $4 - 1 = 3$.

(b) The fourth variable x_4 is *not free* in $\boldsymbol{s}$, and R_0 must be $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(c) $A\boldsymbol{x} = \boldsymbol{b}$ can be solved for all $\boldsymbol{b}$, because A and R_0 have *full row rank* $r = 3$.

32 If $A\boldsymbol{x} = \boldsymbol{b}$ and $C\boldsymbol{x} = \boldsymbol{b}$ have the same solutions, A and C have the same shape and the same nullspace (take $\boldsymbol{b} = \boldsymbol{0}$). If $\boldsymbol{b} =$ column 1 of A, $\boldsymbol{x} = (1, 0, \ldots, 0)$ solves $A\boldsymbol{x} = \boldsymbol{b}$ so it solves $C\boldsymbol{x} = \boldsymbol{b}$. Then A and C share column 1. Other columns too: $\boldsymbol{A = C}$!

33 The column space of R_0 (m by n with rank r) is spanned by its r pivot columns (the first r columns of an m by m identity matrix). The column space of R (after $m - r$ zero rows are removed from R_0) is r-dimensional space $\mathbf{R}^r$.

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1 $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$ gives $c_3 = c_2 = c_1 = 0$. So those 3 column vectors are independent: no other combination gives $\mathbf{0}$

$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is solved by $\boldsymbol{c} = \begin{bmatrix} 1 \\ 1 \\ -4 \\ 1 \end{bmatrix}$. Then $\boldsymbol{v}_1 + \boldsymbol{v}_2 - 4\boldsymbol{v}_3 + \boldsymbol{v}_4 = \mathbf{0}$ (dependent).

2 $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are independent (the -1's are in different positions). All six vectors in $\mathbf{R}^4$ are on the plane $(1,1,1,1) \cdot \boldsymbol{v} = 0$ so no four of these six vectors can be independent.

3 If $a = 0$ then column $1 = \mathbf{0}$; if $d = 0$ then $b(\text{column } 1) - a(\text{column } 2) = \mathbf{0}$; if $f = 0$ then all columns end in zero (they are all in the xy plane, they must be dependent).

4 $U\boldsymbol{x} = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ gives $z = 0$ then $y = 0$ then $x = 0$ (by back substitution). A square triangular matrix has independent columns (invertible matrix) *when its diagonal has no zeros*.

5 (a) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & -1 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & -18/5 \end{bmatrix}$: invertible $\Rightarrow$ independent columns.

(b) $\begin{bmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & -7 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix}$; $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ columns add to $\mathbf{0}$.

6 Columns 1, 2, 4 are independent. Also 1, 3, 4 and 2, 3, 4 and others (but not 1, 2, 3). Same column numbers (not same columns!) for A. This is because $EA = U$ for the matrix E that subtracts 2 times row 1 from row 4. So A and U have the same nullspace (same dependencies of columns).

7 The sum $\boldsymbol{v}_1 - \boldsymbol{v}_2 + \boldsymbol{v}_3 = \mathbf{0}$ because $(\boldsymbol{w}_2 - \boldsymbol{w}_3) - (\boldsymbol{w}_1 - \boldsymbol{w}_3) + (\boldsymbol{w}_1 - \boldsymbol{w}_2) = \mathbf{0}$. So the differences are *dependent* and the difference matrix is singular: $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$.

8 If $c_1(\boldsymbol{w}_2 + \boldsymbol{w}_3) + c_2(\boldsymbol{w}_1 + \boldsymbol{w}_3) + c_3(\boldsymbol{w}_1 + \boldsymbol{w}_2) = \mathbf{0}$ then $(c_2 + c_3)\boldsymbol{w}_1 + (c_1 + c_3)\boldsymbol{w}_2 + (c_1 + c_2)\boldsymbol{w}_3 = \mathbf{0}$. Since the $\boldsymbol{w}$'s are independent, $c_2 + c_3 = c_1 + c_3 = c_1 + c_2 = 0$. The only solution is $c_1 = c_2 = c_3 = 0$. Only this combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ gives $\mathbf{0}$.

(changing -1's to 1's for the matrix A in solution **7** above makes A invertible.)

9 (a) The four vectors in $\mathbf{R}^3$ are the columns of a 3 by 4 matrix A. There is a nonzero solution to $A\boldsymbol{x} = \mathbf{0}$ because there is at least one free variable (b) Two vectors are dependent if $[\, \boldsymbol{v}_1 \;\; \boldsymbol{v}_2 \,]$ has rank 0 or 1. (OK to say "they are on the same line" or "one is a multiple of the other" but *not* "$\boldsymbol{v}_2$ is a multiple of $\boldsymbol{v}_1$" —since $\boldsymbol{v}_1$ might be $\mathbf{0}$.) (c) A nontrivial combination of $\boldsymbol{v}_1$ and $\mathbf{0}$ gives $\mathbf{0}$: $0\boldsymbol{v}_1 + 3(0,0,0) = (0,0,0)$.

10 The plane is the nullspace of $A = [1 \;\; 2 \;\; -3 \;\; -1]$. Three free variables give three independent solutions $(x, y, z, t) = (-2, 1, 0, 0)$ and $(3, 0, 1, 0)$ and $(1, 0, 0, 1)$. Combinations of those special solutions give more solutions (all solutions).

11 (a) Line in $\mathbf{R}^3$ (b) Plane in $\mathbf{R}^3$ (c) All of $\mathbf{R}^3$ (d) All of $\mathbf{R}^3$.

12 $\boldsymbol{b}$ is in the column space when $A\boldsymbol{x} = \boldsymbol{b}$ has a solution; $\boldsymbol{c}$ is in the row space when $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{c}$ has a solution. *False* because the zero vector is always in the row space.

13 The column space and row space of A and U all have the same dimension = 2. *The row spaces of A and U are the same*, because the rows of U are combinations of the rows of A (and vice versa !).

14 $\boldsymbol{v} = \frac{1}{2}(\boldsymbol{v} + \boldsymbol{w}) + \frac{1}{2}(\boldsymbol{v} - \boldsymbol{w})$ and $\boldsymbol{w} = \frac{1}{2}(\boldsymbol{v} + \boldsymbol{w}) - \frac{1}{2}(\boldsymbol{v} - \boldsymbol{w})$. The two pairs *span* the same space. They are a basis for the same space when $\boldsymbol{v}$ and $\boldsymbol{w}$ are *independent*.

15 The n independent vectors span a space of dimension n. They are a *basis* for that space. If they are the columns of A then m is *not less* than n $(m \geq n)$. *Invertible* if $m = n$.

16 These bases are not unique! (a) $(1,1,1,1)$ for the space of all constant vectors (c,c,c,c) (b) $(1,-1,0,0),(1,0,-1,0),(1,0,0,-1)$ for the space of vectors with sum of components = 0 (c) $(1,-1,-1,0),(1,-1,0,-1)$ for the space perpendicular to $(1,1,0,0)$ and $(1,0,1,1)$ (d) The columns of I are a basis for its column space, the empty set is a basis (by convention) for $\mathbf{N}(I) = \mathbf{Z} = \{\text{zero vector}\}$.

17 The column space of $U = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$ is $\mathbf{R}^2$ so take any bases for $\mathbf{R}^2$; (row 1 and row 2) or (row 1 and row $1+$ row 2) or (row 1 and $-$ row 2) are bases for the row space of U.

18 (a) The 6 vectors *might not* span $\mathbf{R}^4$ (b) The 6 vectors *are not* independent (c) Any four *might be* a basis.

19 n independent columns $\Rightarrow$ rank n. Columns span $\mathbf{R}^m \Rightarrow$ rank m. Columns are basis for $\mathbf{R}^m \Rightarrow$ *rank* $= m = n$. The rank counts the number of *independent* columns.

20 One basis is $(2,1,0)$, $(-3,0,1)$. A basis for the intersection with the xy plane is $(2,1,0)$. The normal vector $(1,-2,3)$ is a basis for the line perpendicular to the plane.

21 (a) The only solution to $A\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = \mathbf{0}$ because *the columns are independent* (b) $A\boldsymbol{x} = \boldsymbol{b}$ is solvable because *the columns span* $\mathbf{R}^5$. Their combinations give every $\boldsymbol{b}$. Key point: A basis gives exactly one solution for every $\boldsymbol{b}$.

22 (a) True (b) False because the basis vectors for $\mathbf{R}^6$ might not be in $\mathbf{S}$.

23 Columns 1 and 2 are bases for the (**different**) column spaces of A and U; rows 1 and 2 are bases for the (**equal**) row spaces of A and U; $(1,-1,1)$ is a basis for the (**equal**) nullspaces. **Row spaces and nullspaces** stay fixed in elimination.

24 (a) *False* $A = [\,1 \;\; 1\,]$ has dependent columns, independent row (b) *False* Column space $\neq$ row space for $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (c) *True*: Both dimensions $= 2$ if A is invertible, dimensions $= 0$ if $A = 0$, otherwise dimensions $= 1$ (d) *False*, columns may be dependent, in that case not a basis for $\mathbf{C}(A)$.

25 (a) Make $\boldsymbol{v}_1, \dots \boldsymbol{v}_k$ the columns of A. Then find the first n independent columns (we are told they span $\mathbf{R}^n$).

(b) Make $\boldsymbol{v}_1, \dots \boldsymbol{v}_j$ the rows of A and then include the n rows of the identity matrix. Row elimination will keep the first j independent rows and find $n - j$ more rows to form a basis for $\mathbf{R}^n$.

26 A has rank 2 if $c = 0$ and $d = 2$; $B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$ has rank 2 except when $c = d$ or $c = -d$.

27 (a) Basis for all diagonal matrices: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) Add $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ = basis for symmetric matrices.

(c) $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$.

These are simple bases (among many others) for (a) diagonal matrices (b) symmetric matrices (c) skew-symmetric matrices. The dimensions are $3, 6, 3$.

28 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$;

Echelon matrices do *not* form a subspace; they ***span*** the upper triangular matrices (not every U is an echelon matrix).

29 $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}; \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}$.

30 (a) The invertible matrices span the space of all 3 by 3 matrices (b) The rank one matrices also span the space of all 3 by 3 matrices (c) I by itself spans the space of all multiples cI.

31 $\begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix}$. **Dimension** $= 4$.

32 (a) $y(x) =$ constant C (b) $y(x) = 3x$. (c) $y(x) = 3x + C = y_p + y_n$ solves $y' = 3$.

33 $y(0) = 0$ requires $A + B + C = 0$. One basis is $\cos x - \cos 2x$ and $\cos x - \cos 3x$.

34 (a) $y(x) = e^{2x}$ is a basis for all solutions to $y' = 2y$ (b) $y = x$ is a basis for all solutions to $dy/dx = y/x$ (First-order linear equation $\Rightarrow$ 1 basis function in solution space).

35 $y_1(x), y_2(x), y_3(x)$ can be $x, 2x, 3x$ (dim 1) or $x, 2x, x^2$ (dim 2) or x, x^2, x^3 (dim 3).

36 Basis $1, x, x^2, x^3$, for cubic polynomials; basis $x - 1, x^2 - 1, x^3 - 1$ for the subspace with $p(1) = 0$. (4-dimensional space and 3-dimensional subspace).

37 Basis for $\mathbf{S}$: $(1, 0, -1, 0), (0, 1, 0, 0), (1, 0, 0, -1)$; basis for $\mathbf{T}$: $(1, -1, 0, 0)$ and $(0, 0, 2, 1)$; $\mathbf{S} \cap \mathbf{T} =$ multiples of $(3, -3, 2, 1) =$ nullspace for 3 equations in $\mathbf{R}^4$ has dimension 1.

38 If the 5 by 5 matrix $[\,A \;\; \boldsymbol{b}\,]$ is invertible, $\boldsymbol{b}$ is not a combination of the columns of A: no solution to $A\boldsymbol{x} = \boldsymbol{b}$. If $[\,A \;\; \boldsymbol{b}\,]$ is singular, and the 4 columns of A are independent (rank 4), $\boldsymbol{b}$ *is* a combination of those columns. In this case $A\boldsymbol{x} = \boldsymbol{b}$ has a solution.

39 One basis for $y'' = y$ is $y = e^x$ and $y = e^{-x}$. One basis for $y'' = -y$ is $y = \cos x$ and $y = \sin x$.

40 $\boldsymbol{I} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} - \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} + \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} + \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} - \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}$. The six P's are dependent.

Those five are independent: The 4th has $P_{11} = 1$ and cannot be a combination of the others. Then the 3rd cannot be (from $P_{22} = 1$) and also 1st ($P_{33} = 1$). Continuing, a nonzero combination of all five could not be zero. Further challenge: How many independent 4 by 4 permutation matrices?

41 The dimension of $\boldsymbol{S}$ spanned by all rearrangements of $\boldsymbol{x}$ is (a) zero when $\boldsymbol{x} = \boldsymbol{0}$ (b) one when $\boldsymbol{x} = (1, 1, 1, 1)$ (c) three when $\boldsymbol{x} = (1, 1, -1, -1)$ because all rearrangements of this $\boldsymbol{x}$ are perpendicular to $(1, 1, 1, 1)$ (d) four when the $\boldsymbol{x}$'s are not equal and don't add to zero. **No $\boldsymbol{x}$ gives** dim $\boldsymbol{S} = 2$. I owe this nice problem to Mike Artin—the answers are the same in higher dimensions: 0 or 1 or $n-1$ or n.

42 The problem is to show that the $\boldsymbol{u}$'s, $\boldsymbol{v}$'s, $\boldsymbol{w}$'s together are independent. We know the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s together are a basis for $\boldsymbol{V}$, and the $\boldsymbol{u}$'s and $\boldsymbol{w}$'s together are a basis for $\boldsymbol{W}$. Suppose a combination of $\boldsymbol{u}$'s, $\boldsymbol{v}$'s, $\boldsymbol{w}$'s gives $\mathbf{0}$. **To be proved**: All coefficients $=$ zero.

Key idea: In that combination giving $\mathbf{0}$, the part $\boldsymbol{x}$ from the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s is in $\boldsymbol{V}$. So the part from the $\boldsymbol{w}$'s is $-\boldsymbol{x}$. This part is now in $\boldsymbol{V}$ and also in $\boldsymbol{W}$. But if $-\boldsymbol{x}$ is in $\boldsymbol{V} \cap \boldsymbol{W}$ it is a combination of $\boldsymbol{u}$'s only. Now the combination giving $\mathbf{0}$ uses only $\boldsymbol{u}$'s and $\boldsymbol{v}$'s (independent in $\boldsymbol{V}$!) so all coefficients of $\boldsymbol{u}$'s and $\boldsymbol{v}$'s must be zero. Then $\boldsymbol{x} = \mathbf{0}$ and the coefficients of the $\boldsymbol{w}$'s are also zero.

43 If the left side of $\dim(\mathbf{V}) + \dim(\mathbf{W}) = \dim(\mathbf{V} \cap \mathbf{W}) + \dim(\mathbf{V} + \mathbf{W})$ is greater than n, then $\dim(\mathbf{V} \cap \mathbf{W})$ must be greater than zero. So $\mathbf{V} \cap \mathbf{W}$ contains nonzero vectors.

Here is a more basic approach : Put a basis for **V** and then a basis for **W** in the columns of a matrix A. Then A has more columns than rows and there is a nonzero solution to $A\boldsymbol{x} = \mathbf{0}$. That $\boldsymbol{x}$ gives a combination of the **V** columns $=$ a combination of the **W** columns.

44 If $A^2 =$ zero matrix, this says that each column of A is in the nullspace of A. If the column space has dimension r, the nullspace has dimension $10 - r$ by the Counting Theorem. So we must have $r \le 10 - r$ and this leads to $r \le 5$.

Problem Set 3.5, page 137

1 (a) Row and column space dimensions $9-5=5$, nullspace dimension $=4$, $\dim(\mathbf{N}(A^{\mathrm{T}}))$ $=9-7=2$ sum $5+5+4+2=16=m+n$

(b) Column space is $\mathbf{R}^3$; left nullspace contains only $\mathbf{0}$ (dimension zero).

2 A: Row space basis $=$ row $1=(1,2,4)$; nullspace $(-2,1,0)$ and $(-4,0,1)$; column space basis $=$ column $1=(1,2)$; left nullspace $(-2,1)$. B: Row space basis $=$ both rows $=(1,2,4)$ and $(2,5,8)$; column space basis $=$ two columns $=(1,2)$ and $(2,5)$; nullspace $(-4,0,1)$; left nullspace basis is empty because the space contains only $\boldsymbol{y}=\mathbf{0}$: the rows of B are independent.

3 Row space basis $=$ first two rows of R; column space basis $=$ pivot columns (of A not R) $=(1,1,0)$ and $(3,4,1)$; nullspace basis $(1,0,0,0,0)$, $(0,2,-1,0,0)$, $(0,2,0,-2,1)$; left nullspace $(1,-1,1)=$ last row of the elimination matrix $E^{-1}=L$.

4 (a) $\begin{bmatrix}1&0\\1&0\\0&1\end{bmatrix}$ (b) Impossible: $r+(n-r)$ must be 3 (c) $[1\ \ 1]$ (d) $\begin{bmatrix}9&-3\\3&-1\end{bmatrix}$

(e) *Impossible* Row space $=$ column space requires $m=n$. Then $m-r=n-r$; nullspaces have the same dimension. Section 4.1 will prove $\mathbf{N}(A)$ and $\mathbf{N}(A^{\mathrm{T}})$ orthogonal to the row and column spaces respectively—here those are the same space.

5 $A=\begin{bmatrix}1&1&1\\2&1&0\end{bmatrix}$ has those rows spanning its row space. $B=\begin{bmatrix}1&-2&1\end{bmatrix}$ has the same vectors spanning its nullspace and $AB^{\mathrm{T}}=$ zero matrix (*not* AB).

6 A: dim $\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1}$: Rows $(0,3,3,3)$ and $(0,1,0,1)$; columns $(3,0,1)$ and $(3,0,0)$; nullspace $(1,0,0,0)$ and $(0,-1,0,1)$; $\mathbf{N}(A^{\mathrm{T}})$ $(0,1,0)$. B: dim $\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{2}$ Row space (1), column space $(1,4,5)$, nullspace: empty basis, $\mathbf{N}(A^{\mathrm{T}})$ $(-4,1,0)$ and $(-5,0,1)$.

7 Invertible 3 by 3 matrix A: row space basis = column space basis = $(1,0,0)$, $(0,1,0)$, $(0,0,1)$; nullspace basis and left nullspace basis are *empty*. Matrix $B = \begin{bmatrix} A & A \end{bmatrix}$: row space basis $(1,0,0,1,0,0)$, $(0,1,0,0,1,0)$ and $(0,0,1,0,0,1)$; column space basis $(1,0,0)$, $(0,1,0)$, $(0,0,1)$; nullspace basis $(-1,0,0,1,0,0)$ and $(0,-1,0,0,1,0)$ and $(0,0,-1,0,0,1)$; left nullspace basis is empty.

8 $\begin{bmatrix} I & 0 \end{bmatrix}$ and $\begin{bmatrix} I & I; & 0^{\mathrm{T}} & 0^{\mathrm{T}} \end{bmatrix}$ and $\begin{bmatrix} 0 \end{bmatrix}$ $= 3$ by 2 have *row space dimensions* $= 3, 3, 0 =$ column space dimensions; *nullspace dimensions* $2, 3, 2$; *left nullspace dimensions* $0, 2, 3$.

9 (a) Same row space and nullspace. So rank (dimension of row space) is the same

(b) Same column space and left nullspace. Same rank (dimension of column space).

10 For **rand**(3), almost surely rank$= 3$, nullspace and left nullspace contain only $(0,0,0)$. For **rand**$(3,5)$ the rank is almost surely 3 and the dimension of the nullspace is 2.

11 (a) No solution means that $r < m$. Always $r \leq n$. Can't compare m and n here.

(b) Since $m - r > 0$, the left nullspace must contain a nonzero vector.

12 A neat choice is $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{bmatrix}$; $r + (n - r) = n = 3$ does not match $2 + 2 = 4$. Only $\boldsymbol{v} = \mathbf{0}$ is in both $\mathbf{N}(A)$ and $\mathbf{C}(A^{\mathrm{T}})$.

13 (a) *False*: Usually row space $\neq$ column space.

(b) *True*: A and $-A$ have the same four subspaces

(c) *False* (choose A and B same size and invertible: then they have the same four subspaces)

14 Row space basis can be the nonzero rows of U: $(1,2,3,4)$, $(0,1,2,3)$, $(0,0,1,2)$; nullspace basis $(0,1,-2,1)$ as for U; column space basis $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ (happen to have $\mathbf{C}(A) = \mathbf{C}(U) = \mathbf{R}^3$); left nullspace has empty basis.

15 After a row exchange, the row space and nullspace stay the same; $(2,1,3,4)$ is in the new left nullspace after the row exchange.

16 If $A\boldsymbol{v} = \mathbf{0}$ and $\boldsymbol{v}$ is a row of A then $\boldsymbol{v} \cdot \boldsymbol{v} = 0$. So $\boldsymbol{v}$ is perpendicualr to $\boldsymbol{v} : \boldsymbol{v} = \mathbf{0}$.

17 Row space of $A = yz$ plane; column space of $A = xy$ plane; nullspace $= x$ axis; left nullspace $= z$ axis. For $I + A$: Row space $=$ column space $= \mathbf{R}^3$, both nullspaces contain only the zero vector.

18 $a_{11} = 1, a_{12} = 0, a_{13} = 1, a_{22} = 0, a_{32} = 1, a_{31} = 0, a_{23} = 1, a_{33} = 0, a_{21} = 1.$ (Need to specify the five moves).

19 Row $3-2$ row $2+$ row $1 =$ zero row so the vectors $c(1, -2, 1)$ are in the left nullspace. The same vectors happen to be in the nullspace (an accident for this matrix).

20 The steps from A to R_0 are described on page 96 (Section 3.2). I don't think I can do better—but you could put those ideas into different words. By all means give an example that needs row exchanges.

21 (a) $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) $\boldsymbol{v}$ and $\boldsymbol{z}$ (c) rank < 2 if $\boldsymbol{u}$ and $\boldsymbol{w}$ are dependent or if $\boldsymbol{v}$ and $\boldsymbol{z}$ are dependent (d) The rank of $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} + \boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$ is 2.

22 $A = \begin{bmatrix} \\ \boldsymbol{u} & \boldsymbol{w} \\ \\ \end{bmatrix} \begin{bmatrix} \boldsymbol{v}^{\mathrm{T}} \\ \boldsymbol{z}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 2 \\ 5 & 1 \end{bmatrix}$ $\boldsymbol{u}, \boldsymbol{w}$ span column space; $\boldsymbol{v}, \boldsymbol{z}$ span row space

23 As in Problem 22: Row space basis $(3, 0, 3), (1, 1, 2)$; column space basis $(1, 4, 2)$, $(2, 5, 7)$; the rank of (3 by 2) times (2 by 3) cannot be larger than the rank of either factor, so rank ≤ 2 and the 3 by 3 product is not invertible.

24 $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{d}$ puts $\boldsymbol{d}$ in the *row space* of A; unique solution if the *left nullspace* (nullspace of A^{T}) contains only $\boldsymbol{y} = \boldsymbol{0}$.

25 (a) *True* (A and A^{T} have the same rank) (b) *False* $A = [\,1 \quad 0\,]$ and A^{T} have very different left nullspaces (c) *False* (A can be invertible and unsymmetric even if $\boldsymbol{C}(A) = \boldsymbol{C}(A^{\mathrm{T}})$) (d) *True* (The subspaces for A and $-A$ are always the same. If $A^{\mathrm{T}} = A$ or $A^{\mathrm{T}} = -A$ they are also the same for A^{T})

26 Choose $d = bc/a$ to make $\left[\begin{smallmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{smallmatrix}\right]$ a rank-1 matrix. Then the row space has basis (a, b) and the nullspace has basis $(-b, a)$. Those two vectors are perpendicular !

27 B and C (checkers and chess) both have rank 2 if $p \neq 0$. Row 1 and 2 are a basis for the row space of C, $B^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ has 6 special solutions with -1 and 1 separated by a zero; $\mathbf{N}(C^{\mathrm{T}})$ has $(-1,0,0,0,0,0,0,1)$ and $(0,-1,0,0,0,0,1,0)$ and columns $3,4,5,6$ of I; $\mathbf{N}(C)$ is a challenge : one vector in $\mathbf{N}(C)$ is $(1,0,\ldots,0,-1)$.

28 The subspaces for $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ are pairs of orthogonal lines ($\boldsymbol{v}$ and $\boldsymbol{v}^{\perp}$, $\boldsymbol{u}$ and $\boldsymbol{u}^{\perp}$). If B has those same four subspaces then $B = cA$ with $c \neq 0$.

29 (a) $AX = 0$ if each column of X is a multiple of $(1,1,1)$; $\dim(\text{nullspace}) = 3$. (b) If $AX = B$ then all columns of B add to zero; dimension of the B's $= 6$. (c) $3 + 6 = \dim(M^{3\times 3}) = 9$ entries in a 3 by 3 matrix.

30 The key is equal row spaces. First row of A = combination of the rows of B : the only possible combination (notice I) is 1 (row 1 of B). Same for each row so $F = G$.

31 $A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$ $\mathbf{N}(A) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ Row space $\mathbf{C}(A^{\mathrm{T}}) \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$\mathbf{C}(A) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\mathbf{N}(A^{\mathrm{T}}) \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

32 (a) $\mathbf{N}(BA)$ contains $\mathbf{N}(A)$.

(b) $\mathbf{C}(AB)$ is contained in $\mathbf{C}(A)$.

33 (a) $\mathbf{N}(A)$ and $\mathbf{N}(B)$ contain $\mathbf{N}(T)$.

(b) Row spaces of A and B are contained in the row space of T.

34 Fundamental subspaces for $A \quad (m \times n)$.

Row space $\mathbf{C}(A^{\mathrm{T}})$ perpendicular to Nullspace $\mathbf{N}(A)$: Dimensions r and $n - r$.

Column space $\mathbf{C}(A)$ perpendicular to $\mathbf{N}(A^{\mathrm{T}})$: Dimensions r and $m - r$.

Subspaces for $W = \begin{bmatrix} A & A \end{bmatrix}$ – same rank r.

Row space of W contains all $\begin{bmatrix} \boldsymbol{v} & \boldsymbol{v} \end{bmatrix}$ $\quad \boldsymbol{v}$ in $\mathbf{C}(A^{\mathrm{T}})$ (Dimension r).

Nullspace of W contains all $\begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{z} \end{bmatrix}$ with $\boldsymbol{y} + \boldsymbol{z}$ in $\mathbf{N}(W)$ (Dimension $2n - r$).

Column space of W = Column space of A (Dimension r).

Nullspace of W^{T} = Nullspace of A (Dimension $m - r$).

35 Please send a proof or counterexample. Thank you.