Transfinity

A Source Book

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Dear Reader:

Transfinity is the realm of numbers larger than every natural number: For every natural number k there are infinitely many natural numbers $n > k$. For a transfinite number t there is no natural number $n \geq t$.

We will first present the theory of actual infinity, mainly sustained by quotes, in chapter I and then transfinite set theory as far as necessary to understand the following chapters. In addition the attitude of the founder of transfinite set theory, Georg Cantor, with respect to sciences and religion (his point of departure) will be illuminated by various quotes of his as well as of his followers in chapter IV. Also the set of applications of set theory will be summarized there. All this is a prerequisite to judge the social and scientific environment and the importance of set theory. Quotes expressing a sceptical attitude against transfinity or addressing questionable points of current mathematics based on it are collected in chapter V. For a brief overview see also Kritik der transfiniten Mengenlehre or Critics of transfinity. The critique is scrutinized in chapter VI, the main part of this source book. It contains over 100 arguments against actual infinity – from doubtful aspects to clear contradictions – among others applying the newly devised powerful method of ArithmoGeometry. Finally we will present in chapter VII a theory, MatheRealism, that shows that in real mathematics, consisting of monologue, dialogue, and discourse between real thinking-devices, via necessarily physical means, infinite sets cannot exist other than as names. This recognition removes transfinity together with all its problems from mathematics – although the application of mathematics based on MatheRealism would raise a lot of technical problems.

Page numbers of subsections in chapters V and VI have not been listed because these chapters are continuously extended. All subsections can easily be found by their headings.

Bibliographical references have been given, within square brackets, in the plain text. The only exception, because of its frequent appearance throughout this source book, is Cantor's collected works edited by E. Zermelo: "Georg Cantor – Gesammelte Abhandlungen mathematischen und philosophischen Inhalts", Springer, Berlin (1932), quoted as [Cantor, p. *n*]. Internet links which may expire sooner or later, as experience shows, do not always produce the original document.

Cantor's correspondence mainly has been adopted from the following sources:
• E. Noether, J. Cavaillès: "Briefwechsel Cantor-Dedekind", Hermann, Paris (1937).
• I. Grattan-Guinness: "The correspondence between Georg Cantor and Philip Jourdain", Jahresbericht der Deutschen Mathematiker-Vereinigung 73 (1971) pp. 111-130.
• I. Grattan-Guinness: "The rediscovery of the Cantor-Dedekind correspondence", Jahresbericht der Deutschen Mathematiker-Vereinigung 76 (1974) pp. 104-139.
• H. Meschkowski: "Georg Cantor – Leben, Werk und Wirkung", 2nd ed., Bibliographisches Institut, Mannheim (1983).
• W. Scharlau: "Rudolf Lipschitz: Briefwechsel mit Cantor, Dedekind, Helmholtz, Kronecker, Weierstrass und anderen", Vieweg, Braunschweig (1986).
• W. Purkert, H.J. Ilgauds: "Georg Cantor 1845-1918", Birkhäuser, Basel (1987).
• J.W. Dauben: "Georg Cantor: His mathematics and philosophy of the infinite", Princeton University Press, Princeton (1990).
• H. Meschkowski, W. Nilson: "Georg Cantor – Briefe", Springer, Berlin (1991).
• C. Tapp: "Kardinalität und Kardinäle", Franz Steiner Verlag, Stuttgart (2005).

Quotes that were not or not in sufficient quality available in English have been translated by myself. When different sources deviated, the most comprehensive version has been quoted. My own remarks, notes, and comments within quotes are included in double curly brackets {{comment}}.

I am indebted to David Petry, Norman Wildberger, and Doron Zeilberger who have kindly agreed to include complete essays of theirs in this source book.

If you think that something important is missing and should appear in this source book, please do not hesitate to inform me: wolfgang[dot]mueckenheim[at]hs[minus]augsburg[dot]de.

Regards, WM

I Distinguishing actual and potential infinity

During the history of mankind the infinite nearly unanimously has been accepted as potential. [W. Mückenheim: "Die Mathematik des Unendlichen", Shaker-Verlag, Aachen (2006) chapter 8] This situation has been reversed with the rise of transfinite set theory. But many mathematicians and many set theorists are not sure what the difference is and whether it is important for pursuing mathematics. Many set theorists even proudly boast not to know the difference. Others claim that the term "actual infinity" is not a term of set theory. In fact, set theorists try to avoid it in order to keep set theory attractive for newcomers who could be repulsed by words like "finished infinity". "You use terms like completed versus potential infinity, which are not part of the modern vernacular." [P.L. Clark in "Physicists can be wrong", tea.MathOverflow (2 Jul 2010)] This is the typical reproach to be expected when the different kinds of infinity are analyzed and taught.

But set theory clearly needs actual, i.e., completed infinity and uses it heavily. "Cantor's work was well received by some of the prominent mathematicians of his day, such as Richard Dedekind. But his willingness to regard infinite sets as objects to be treated in much the same way as finite sets was bitterly attacked by others, particularly Kronecker. There was no objection to a 'potential infinity' in the form of an unending process, but an 'actual infinity' in the form of a completed infinite set was harder to accept." [H.B. Enderton: "Elements of set theory", Academic Press, New York (1977) p. 14f] Therefore we will meticulously distinguish the two types of infinity in the following. This cannot be done better than by quoting those scholars who have been concerned with this topic.

The first known scholar who in great detail dealt with the infinite in written text was Aristotle (384-322). He denied the actual infinite (αφωρισμενον) in philosophy and mathematics ascribing it only to the Gods. Addition or division repeated without end could only happen in potential infinity (απειρον). With respect to mathematics he concluded: "Our account does not rob the mathematicians of their science, by disproving the actual existence of the infinite in the direction of increase, in the sense of the untraversable. In point of fact they do not need the infinite and do not use it. They postulate only that the finite straight line may be produced as far as they wish. It is possible to have divided in the same ratio as the largest quantity another magnitude of any size you like. Hence, for the purposes of proof, it will make no difference to them to have such an infinite instead, while its existence will be in the sphere of real magnitudes." [Aristotle: "Physics", Book III, Part 7 (≈350 BC)]

Before Cantor only a small minority of scholars believed in actual infinity. Robert Grosseteste, Bishop of Lincoln in the 13th century, claimed: "The number of points in a segment one ell long is its true measure." Also John Baconthorpe in the 14th century, called Doctor resolutus, brought honour on his epithet and courageously opposed the contemporary scholastic opinion "infinitum actu non datur" by stating: "There is the actual infinite in number, time, quantity."

Carl Friedrich Gauß like Augustin-Louis Cauchy opposed actual infinity. Concerning a proof of Schumacher's for the angular sum of 180° in triangles with two infinitely long sides Gauß wrote: "I protest firstly against the use of an infinite magnitude as a *completed* one, which never has been allowed in mathematics. The infinite is only a mode of speaking, when we in principle talk about limits which are approached by certain ratios as closely as desired whereas others are allowed to grow without reservation." [C.F. Gauß, letter to H.C. Schumacher (12 Jul 1831)]

Gauß' last doctoral student Richard Dedekind, although contributing a lot to set theory and accepting the actual infinite, tended to potential infinity nevertheless. "Every time when there is a cut (A_1, A_2) which is not produced by a rational number, we *create* a new, an *irrational* number α which we consider to be completely defined by this cut (A_1, A_2);" [R. Dedekind: "Stetigkeit und irrationale Zahlen", 6th ed., Vieweg, Braunschweig (1960) p. 13] This is clearly potential infinity, because never more than a finite number of cuts can have been created.

"There are infinite systems. Proof (a similar reflection can be found in § 13 of the Paradoxien des Unendlichen by Bolzano (Leipzig 1851))[1]. The world of my thoughts, i.e., the collection *S* of all things which can be objects of my thinking, is infinite. For, if *s* is an element of *S*, then the thought *s'* that *s* can be an object of my thinking is itself an object of my thinking." [R. Dedekind: "Was sind und was sollen die Zahlen?", 8th ed., Vieweg, Braunschweig (1960) p. 14] This is potential infinity too, because never more than a finite number of thoughts can have been thought.

"A system *S* is called *infinite*, if it is similar to a proper part of itself; otherwise *S* is called a *finite* system. [...] *S* is called infinite if there is a proper part of *S* into which *S* can distinctly (similarly) be mapped." [R. Dedekind: "Was sind und was sollen die Zahlen?", 8th ed., Vieweg, Braunschweig (1960) p. 13] A complete infinite system *S* means actual infinity. But a mapping at Dedekind's times could be incomplete. Not *all* elements of a set need exist. It is sufficient when *every* element of one set has a partner in the other set, and vice versa. (Note that *every* element of an infinite set is followed by infinitely many other elements whereas no element follows upon *all* elements.)

Bernard Bolzano, referred to by Dedekind, explained the different types of the infinite: "A multitude which is larger than every finite one, i.e., a multitude which has the property that every finite set is only part of it, I shall call an *infinite multitude*. [...] If they, like *Hegel*, *Erdmann*, and others, imagine the mathematical infinite only as a magnitude which is *variable* and only has no *limit* in its growth (like some mathematicians, as we will see soon, have assumed to explain their notion) so I agree in their reproach of this notion of a magnitude only *growing* into the infinite but never *reaching* it. A *really infinite* magnitude, for instance the length of the line not ending on both sides (i.e., the magnitude of that spatial object containing all points which are determined by the purely mentally imagined relation with respect to two points) need not be variable, as indeed it is not in this example. And a magnitude that only can be considered to be larger than imagined before and being capable of becoming larger than every given (finite) magnitude, may as well permanently remain a merely finite magnitude, like each of the numbers 1, 2, 3, 4 ....." [Bernard Bolzano: "Paradoxien des Unendlichen", Reclam, Leipzig (1851) p. 6ff]

[1] *Die Menge der Sätze und Wahrheiten an sich* ist, wie sich sehr leicht einsehen läßt, unendlich; denn wenn wir irgend eine Wahrheit, etwa den Satz, daß es Wahrheiten überhaupt gebe, oder sonst jeden beliebigen, den ich durch *A* bezeichnen will, betrachten: finden wir, daß der Satz, welchen die Worte "*A ist wahr*" ausdrücken, ein von *A* selbst verschiedener sei; denn dieser hat offenbar ein ganz anderes Subject als jener. Sein Subject nämlich ist der ganze Satz *A* selbst. Allein nach eben dem Gesetze, wie wir hier aus dem Satz *A* diesen von ihm verschiedenen, den ich *B* nennen will, ableiten, läßt sich aus *B* wieder ein dritter Satz *C* ableiten, und so ohne Ende fort. Der Inbegriff all dieser Sätze, deren jeder folgende zu dem nächst vorhergehenden in dem nur eben angegebenen Verhältnisse steht, daß er denselben zu seinem Subjecte erhebt und von demselben aussagt, daß er ein wahrer Satz sei, dieser Inbegriff – sage ich – umfaßt eine Menge von Theilen (Sätzen), die größer als jede endliche Menge ist. [Bernard Bolzano: "Paradoxien des Unendlichen", Reclam, Leipzig (1851) p. 13f]

Most statements on the two types of infinity have been given by Georg Cantor who did not get tired to explain the difference and the importance of the actual infinite over and over again.

Cantor gained his first ideas from the teachings of St Augustin and from the Holy Bible. "*Every* single *finite* cardinal number (1 or 2 or 3 etc.) is contained *in the divine intellect* (from St Augustin: De civitate Dei, lib. XII, ch. 19)." [G. Cantor, letter to I. Jeiler (20 May 1888, Whitsun)] "Dominus regnabit in infinitum (aeternum) *et ultra*." {{The Lord rules in eternity *and beyond*, from Exodus 15,18.}} [G. Cantor, letter to R. Lipschitz (19 Nov 1883)]

"To the idea to consider the infinite large not only in form of the unlimited growing and the closely connected form of the convergent infinite series introduced first in the seventeenth century but also to fix it by numbers in the definite form of the completed-infinite I have been forced logically almost against my own will, because in opposition to highly esteemed tradition, by the development of many years of scientific efforts and attempts, and therefore I do not believe that reasons could be raised which I would not be able to answer." [Cantor, p. 175]

"In spite of significant difference between the notions of the *potential* and *actual* infinite, where the former is a *variable* finite magnitude, *growing* above all limits, the latter a *constant* quantity *fixed in itself* but beyond all finite magnitudes, it happens deplorably often that the one is confused with the other." [Cantor, p. 374]

"By the actual infinite we have to understand a quantity that on the one hand is *not variable* but fixed and definite in all its parts, a real *constant*, but at the same time. on the other hand, exceeds *every finite size* of the same kind by size. As an example I mention the totality, the embodiment of *all* finite positive integers; this set is a *self-contained thing* and forms, apart from the natural sequence of its numbers, a fixed, definite quantity, an αφωρισμενον, which we obviously have to call larger than every finite number." [G. Cantor, letter to A. Eulenburg (28 Feb 1886)]

"It is even allowed to comprehend the newly created number ω as a *limit* which the numbers ν converge to, if thereby nothing else is seen but that ω is the *first* whole number following upon all numbers ν, i.e., which has to be called greater than each of the numbers ν." [Cantor, p. 195]

"Here α is the number *following* by magnitude *next* upon all numbers α_ν." [Cantor, p. 331]

Therefore Cantor condemned Gauß' rejection of the actual infinite in several places: "The *erroneous* in Gauss' letter consists in his sentence that the finished infinite could not become an object of mathematical consideration. [...] The finished infinite can be found, in a sense, in the numbers ω, $\omega + 1$, ... , ω^ω, ..." [G. Cantor, letter to R. Lipschitz (19 Nov 1883)]

"My opposition to *Gauss* consists in the fact that Gauss rejects as *inconsistent* (I mean he does so *unconsciously*, i.e., without knowing this notion) *all multitudes with exception of* the *finite* and therefore *categorically* and *basically* discards the actual infinite which I call *transfinitum*, and together with this he declares the transfinite numbers as *impossible*, the existence of which I have established." [G. Cantor, letter to D. Hilbert (27 Jan 1900)]

"it seems that the ancients haven't had any clue of the transfinite, the possibility of which is even strongly rejected by Aristotle and his school like in newer times by d'Alembert, Lagrange, Gauss, Cauchy, and their adherents." [G. Cantor, letter to G. Peano (21 Sep 1895)]

While normally numbers are created by the first creation principle, namely addition of 1, "the logical function which has supplied the two numbers ω and 2ω does obviously differ from the *first* creation principle. Therefore I call it the *second creation principle* of whole real numbers and define it more closely as follows: If there is a definite succession of defined whole numbers, none of which is the greatest, then, based on the second creation principle, a new number is created which can be understood as the *limit* of those numbers, i.e., defined as the next greater number to all of them." [Cantor, p. 196]

"The notion of ω, for example, contains *nothing shaky*, *nothing indefinite*, *nothing varying*, *nothing potential*, it is not an απειρον but an αφωρισμενον, and the same is true for all other transfinite numbers. [...] *Wundt's* treatise shows that he has no clear grasp of the fundamental difference of 'improper infinite = variable finite = syncategorematice infinitum (απειρον)', and 'proper infinite = transfinitum = completed infinite = *being infinite* = categorematice infinitum (αφωρισμενον)'. Otherwise he would not call the *former* as well as the *latter* a *limit*; limit is always something *fixed*, *invariable*. Therefore of the two notions of infinity only the *transfinitum* can be thought of as *being* and in certain circumstances and in some sense as a *fixed limit*. [...]

There is no further justification necessary when I in the 'Grundlagen', just at the beginning, distinguish two notions toto genere different from each other, which I call the improper-infinite and the proper-infinite; they need *not* be considered as in *any* way compatible with or related to each other. The often, at all times, admitted union or confusion of these two completely disparate notions causes, to my firm conviction, innumerable errors; in particular I see herein the reason why the *transfinite* numbers have not been discovered before.

To exclude this confusion from the outset, I denote the smallest transfinite number by the symbol ω which *differs* from the ordinary symbol corresponding to the improper-infinite ∞.

Indeed ω can be considered, so to speak, as the limit which the variable integer ν converges to beyond all limits, albeit only in the sense that ω is the *smallest* transfinite ordinal number, i.e., the smallest *firmly determined* number which is greater than *all* finite numbers ν; just as √2 is the limit of certain variable, growing rational numbers. But here we have in addition that the difference of √2 and the approximating fractions gets arbitrarily small, whereas ω - ν is always equal to ω; but this difference does *not* change the fact that ω is just as determined and completed as √2, and it does not change the fact that ω carries as little traces of the converging numbers ν as √2 carries anything of the rational approximating fractions." [G. Cantor, letter to K. Laßwitz (15 Feb 1884). Cantor, pp. 390f & 395. (The sources differ in many places.)]

"On account of the matter I would like to add that in conventional mathematics, in particular in differential- and integral calculus, you can gain little or no information about the transfinite because here the potential infinite plays the important role, I don't say the only role but the role emerging to surface (which most mathematicians are readily satisfied with). Even Leibniz with whom I don't harmonize in many other respects too, has [...] fallen into most spectacular contradictions with respect to the actual infinite." [G. Cantor, letter to A. Schmid (26 Mar 1887)]

"Allow me to remark that the reality and the absolute principles of the integers appear to be *much stronger* than those of the world of sensations. And this fact has precisely one very simple reason, namely that the integers separately as well as in their actually infinite totality exist as eternal ideas in intellectu Divino in the highest degree of reality." [G. Cantor, letter to C. Hermite (30 Nov 1895)]

"Nevertheless the transfinite cannot be considered a subsection of what is usually called 'potentially infinite'. Because the latter is *not* (like every individual transfinite and in general everything due to an 'idea divina') *determined in itself*, *fixed*, and *unchangeable*, but a *finite in the process of change*, having in each of its current states a *finite size*; like, for instance, the *temporal duration* since the beginning of the world, which, when measured in some time-unit, for instance a year, is *finite* in every moment, *but always growing beyond all finite limits*, *without ever becoming really infinitely large*." [G. Cantor, letter to I. Jeiler (13 Oct 1895)]

It remains true for transfinite ordinal numbers too that every subset (α') – Cantor did not add the phrase "non-empty" – has a smallest element: "Among the numbers of the set (α') there is always a *smallest one*. In particular, if we have a sequence of numbers {{of the second number class}} $\alpha_1, \alpha_2, ..., \alpha_\beta, ...$, that continuously decrease in magnitude (such that $\alpha_\beta > \alpha_{\beta'}$ if $\beta' > \beta$), then the sequence will necessarily terminate after a finite number of terms and finish with the smallest of the numbers. The sequence cannot be infinite." [Cantor, p. 200]

"That in every set (α') of transfinite numbers there is a smallest one, can be shown as follows." [Cantor, p. 208, footnote by E. Zermelo] Also Zermelo did not add the phrase "non-empty".

"*Every embodiment of different numbers of the first and the second number class has a smallest number*, *a minimum*." [Cantor, p. 332]

"Here we use again and again the theorem [...] that *every* embodiment of numbers, i.e., *every* partial multitude of Ω has a *minimum*, a *smallest number*." [Cantor, p. 445]

"Cantor's belief in the actual existence of the infinite of Set Theory still predominates in the mathematical world today." [A. Robinson: "The metaphysics of the calculus", in I. Lakatos (ed.): "Problems in the philosophy of mathematics", North Holland, Amsterdam (1967) p. 39]

"Should we briefly characterize the new view of the infinite introduced by Cantor, we could certainly say: In analysis we have to deal only with the infinitely small and the infinitely large as a limit-notion, as something becoming, emerging, produced, i.e., as we put it, with the *potential infinite*. But this is not the proper infinite. That we have for instance when we consider the entirety of the numbers 1, 2, 3, 4, ... itself as a completed unit, or the points of a line as an entirety of things which is completely available. That sort of infinity is named *actual infinite*." [D. Hilbert: "Über das Unendliche", Mathematische Annalen 95 (1925) p. 167]

A recent report by the author about the infinite in mathematics was concerned with excluding actually infinite magnitudes from the limit methods, in particular from infinitesimal calculus. The present continuation "should show that this exclusion does in no way mean to refrain from considering actual infinity in mathematics. On the contrary, the example of nondenumerability of the continuum should show the *possibility* to distinguish different cardinalities, and Cantor's resulting proof of the existence of transcendental numbers should show the *practical importance* of this distinction. [...] it has been shown that neither in the elementary chapters of mathematics nor in those denoted by 'infinitesimal calculus' a really infinite '*magnitude*' occurs, but that rather the word 'infinite' is merely used as an abbreviating description of important facts of the finite." [G. Hessenberg: "Grundbegriffe der Mengenlehre", offprint from "Abhandlungen der Fries'schen Schule", Vol. I. no. 4, Vandenhoeck & Ruprecht, Göttingen (1906) Preface and § 1]

"We'll have to state that the mathematicians in considering the notion of infinity tend toward two different directions. For the first group the infinite flows out of the finite, for them there exists infinity only because there is an unlimited number of limited possible things. For the others the infinite exists prior to the finite, the finite constituting a small sector of the infinite." [H. Poincaré: "Dernières pensées: Les mathématiques et la logique", Flammarion, Paris (1913) p. 144f]

"In order to obtain something absolutely nondenumerable, we would have to have either an absolutely nondenumerably infinite number of axioms or an axiom that could yield an absolutely nondenumerable number of first-order propositions. But this would in all cases lead to a circular introduction of the higher infinities; that is, *on an axiomatic basis higher infinities exist only in a relative sense*." [T. Skolem: "Some remarks on axiomatized set theory" (1922) quoted in J. van Heijenoort: "From Frege to Gödel – A source book in mathematical logic, 1879-1931", Harvard University Press, Cambridge (1967) p. 296]

Ernst Zermelo claims "that in contrast to the notion of natural number the field of analysis needs the existence of infinite sets: 'As a consequence, those who are really serious about rejection of the actual infinite in mathematics should stop at general set theory and the lower number theory and do without the whole modern analysis.' [...] Infinite domains 'can never be given empirically; they are set ideally and exist only in the sense of a Platonic idea' [...] in general they can only be defined axiomatically; any inductive or 'genetic' way is inadequate. [...] 'The infinite is neither physically nor psychologically given to us in the real world, it has to be comprehended and set as an idea in the Platonic sense'." [H.-D. Ebbinghaus: "Ernst Zermelo: An approach to his life and work", Springer (2007) pp. 64 & 153f]

"But in order to save the existence of 'infinite' sets we need yet the following axiom, the contents of which is essentially due to Mr. R. Dedekind. Axiom VII. The domain contains at least one set Z which contains the null-set as an element and has the property that every element a of it corresponds to another one of the form $\{a\}$, or which with every of its elements a contains also the corresponding set $\{a\}$ as an element." [E. Zermelo: "Untersuchungen über die Grundlagen der Mengenlehre I", Math. Ann. 65 (1908) p. 266f]

"The statement $\lim_{n\to\infty} 1/n = 0$ asserts nothing about infinity (as the ominous sign ∞ seems to suggest) but is just an abbreviation for the sentence: $1/n$ can be made to approach zero as closely as desired by sufficiently increasing the positive integer n. In contrast herewith the set of all integers is infinite (infinitely comprehensive) in a sense which is 'actual' (proper) and not only 'potential'. (It would, however, be a fundamental mistake to deem this set infinite because the integers 1, 2, 3, ..., n, ... increase infinitely, or better, indefinitely.)" [A.A. Fraenkel, A. Levy: "Abstract set theory", North Holland, Amsterdam (1976) p. 6]

"From the axiomatic viewpoint there is no other way for securing infinite sets but *postulating* them, and we shall express an appropriate axiom in several forms. While the first corresponds to Zermelo's original axiom of infinity, the second implicitly refers to von Neumann's method of introducing ordinal numbers." [A.A. Fraenkel, Y. Bar-Hillel, A. Levy: "Foundations of set theory", 2nd ed., Elsevier, Amsterdam (1973) p. 46]

"We introduce numbers for counting. This does not at all imply the infinity of numbers. For, in what way should we ever arrive at infinitely-many countable things? [...] In philosophical terminology we say that the infinite of the number sequence is only potential, i.e., existing only as a possibility." [P. Lorenzen: "Das Aktual-Unendliche in der Mathematik", Philosophia naturalis 4 (1957) p. 4f]

"Until then, no one envisioned the possibility that infinities come in different sizes, and moreover, mathematicians had no use for 'actual infinity'. The arguments using infinity, including the Differential Calculus of Newton and Leibniz, do not require the use of infinite sets. [...] Cantor observed that many infinite sets of numbers are countable: the set of all integers, the set of all rational numbers, and also the set of all algebraic numbers. Then he gave his ingenious diagonal argument that proves, by contradiction, that the set of all real numbers is *not* countable. A consequence of this is that there exists a multitude of transcendental numbers, even though the proof, by contradiction, does not produce a single specific example." [T. Jech: "Set theory", Stanford Encyclopedia of Philosophy (2002)]

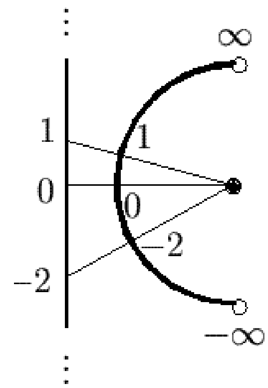
"Numerals constitute a *potential infinity*. Given any numeral, we can construct a new numeral by prefixing it with S. Now imagine this potential infinity to be completed. Imagine the inexhaustible process of constructing numerals somehow to have been finished, and call the result *the set of all numbers*, denoted by $\mathbb{N}$. Thus $\mathbb{N}$ is thought to be an *actual infinity* or a *completed infinity*. This is curious terminology, since the etymology of 'infinite' is 'not finished'." [E. Nelson: "Hilbert's mistake" (2007) p. 3]

According to (Gödel's) Platonism, objects of mathematics have the same status of reality as physical objects. "Views to the effect that Platonism is correct but only for certain relatively 'concrete' mathematical 'objects'. Other mathematical 'objects' are man made, and are not part of an external reality. Under such a view, what is to be made of the part of mathematics that lies outside the scope of Platonism? An obvious response is to reject it as utterly meaningless." [H.M. Friedman: "Philosophical problems in logic" (2002) p. 9]

"A *potential infinity* is a quantity which is finite but indefinitely large. For instance, when we enumerate the natural numbers as 0, 1, 2, ..., n, n+1, ..., the enumeration is finite at any point in time, but it grows indefinitely and without bound. [...] An *actual infinity* is a completed infinite totality. Examples: $\mathbb{N}$, $\mathbb{R}$, $C[0, 1]$, $L_2[0, 1]$, etc. Other examples: gods, devils, etc." [S.G. Simpson: "Potential versus actual infinity: insights from reverse mathematics" (2015)]

"*Potential infinity* refers to a procedure that gets closer and closer to, but never quite reaches, an infinite end. For instance, the sequence of numbers 1, 2, 3, 4, ... gets higher and higher, but it has no end; it never gets to infinity. Infinity is just an indication of a direction – it's 'somewhere off in the distance'. Chasing this kind of infinity is like chasing a rainbow or trying to sail to the edge of the world – you may think you see it in the distance, but when you get to where you thought it was, you see it is still further away. Geometrically, imagine an infinitely long straight line; then 'infinity' is off at the 'end' of the line. Analogous procedures are given by limits in calculus, whether they use infinity or not. For example, $\lim_{x\to0}(\sin x)/x = 1$. This means that when we choose values of x that are closer and closer to zero, but *never quite equal to zero*, then $(\sin x)/x$ gets closer and closer to one.

Completed infinity, or *actual infinity*, is an infinity that one actually reaches; the process is already done. For instance, let's put braces around that sequence mentioned earlier: {1, 2, 3, 4, ...}. With this notation, we are indicating the set of all positive integers. This is just *one* object, a set. But that set has infinitely many members. By that I don't mean that it has a large finite number of members and it keeps getting more members. Rather, I mean that it already *has* infinitely many members.



We can also indicate the completed infinity geometrically. For instance, the diagram at right shows a one-to-one correspondence between points on an infinitely long line and points on a semicircle. There are no points for plus or minus infinity on the line, but it is natural to attach those 'numbers' to the endpoints of the semicircle.

Isn't that 'cheating', to simply add numbers in this fashion? Not really; it just depends on what we want to use those numbers for. For instance, $f(x) = 1/(1 + x^2)$ is a continuous function defined for all real numbers x, and it also tends to a limit of 0 when x 'goes to' plus or minus infinity (in the sense of potential infinity, described earlier). Consequently, if we add those two 'numbers' to the real line, to get the so-called 'extended real line', and we equip that set with the same topology as that of the closed semicircle (i.e., the semicircle including the endpoints), then the function f is continuous everywhere on the extended real line." [E. Schechter: "Potential versus completed infinity: Its history and controversy" (5 Dec 2009)]

Here are some simple examples of potential infinity:

The sequence of increasing circumferences (or diameters, or areas) of circles is potentially infinite because the circumference of a circle can become arbitrarily long, but it cannot be actually infinite because then it would not belong to a circle. An infinite "circumference" would have curvature zero, i.e., no curvature, and it could not be distinguished what is the inner side and what is the outer side of the circle.

The length of periods of decimal representations of rational numbers is potentially infinite. The length is always finite although it has no upper bound. The decimal representation is equal to a geometric series, like $0.abcabcabc... = abc\cdot(10^{-3} + 10^{-6} + 10^{-9} + ...)$ which converges to the limit $10^{-3}\dfrac{abc}{1-10^{-3}} = \dfrac{abc}{999}$. A never repeating decimal sequence has an irrational limit.

An interval of natural numbers without any prime number is called a prime gap. The sequence of prime gaps assumes arbitrarily large intervals but it cannot become actually infinite. None of the numbers $n! + 2$, $n! + 3$, $n! + 4$, ..., $n! + n$ can be prime because $n! = 1\cdot2\cdot3\cdot...\cdot n$ contains 2, 3, ..., n as factors already. Therefore the set of gaps has no upper bound. It is potentially infinite. It is not actually infinite however, because there does not exist a gap with no closing prime number because there is no last prime number.

Finally, the most familiar example is this: The (magnitudes of) natural numbers are potentially infinite because, although there is no upper bound, there is no infinite (magnitude of a) natural number.

Based on the statements collected in this chapter we can characterize *the infinities of the set of countable cardinal number*s as follows:

Potential infinity: $\forall n\ \exists m$: $n < m$.

For every countable cardinal number n, there exists a greater countable cardinal number m.

Actual infinity: $\exists m\ \forall n$: $n \le m$.

There exists a countable cardinal number[1] $m = \aleph_0$ which is not less than any countable cardinal number n (or which is greater than every *other* countable cardinal number).

Briefly, in potential infinity every natural number has a successor, in actual infinity all natural numbers have a non-natural number ω, with $|\omega| = \aleph_0$, as their successor, namely "*an ordinal number* lim a_ν *which is following next upon all* a_ν *by magnitude*". [Cantor, p. 324]

Generally, we can give the following formal definitions for the two alternatives:

A collection C is potentially infinite (is a proper class) $\Leftrightarrow \forall A \subseteq C\ \exists B \subseteq C$: $A \subset B$.
A collection C is finite or actually infinite (is a ZF-set) $\Leftrightarrow \exists A \subseteq C$: $A = C$.

The distinction between potential infinity of analysis and actual infinity of set theory can best be understood by the sequence 0.1, 0.11, 0.111, ... of approximations to 1/9. The potentially infinite sequence is never complete, never reaching the limit. A gap of infinitely many missing terms of the sequence is always yawning before the limit. The actually infinite sequence 0.111... of all $\aleph_0$ indexed digits however contains more than any natural number of digits because $\forall n \in \mathbb{N}$: $n < \aleph_0$. No digit is missing between 0.1 and 1/9. The limit is determined by the sequence of digits.

In analysis of classical mathematics an infinite sequence of digits $0.d_1d_2d_3...$ is never considered complete and, therefore, cannot define a real number. But that is not required. Here the *limit* of the sequence is defined as a real number a such that the sequence (a_n) of rational partial sums or terms a_n approaches a as closely as desired. For every positive real number ε there exists an index n_ε such that for every index $n \ge n_\varepsilon$ the distance of the term a_n to the limit a is $|a - a_n| < \varepsilon$. For this definition it is not necessary that all terms exist. Therefore famous digit sequences like 3.14... or such with a short period like 0.111... can be used as an abbreviation to denote the limit. As a general rule, the term $a_3 = 0.d_1d_2d_3$ is used to express the formula $a = 0.d_1d_2d_3...$ This is a bit sloppy though because the correct formula reads $0.d_1d_2d_3... \to a$.

In actual infinity, on the other hand, the limit is the term following *next upon all* terms with finite indices. Since there the complete digit sequence $0.d_1d_2d_3...$ represents the sum of the complete sequence of partial sums, it cannot simultaneously denote the limit a in case of a strictly increasing or decreasing sequence. (For more detail cp. the section "Sequences and limits".)

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[1] The *numerical character* of all cardinal numbers is not only implied by the generic term "number" but also by their trichotomy properties: "*Let a and b be any two cardinal numbers, then we have either a = b or a < b or a > b.*" [Cantor, p. 285]

1.1 Cantor's original German terminology on infinite sets

The reader fluent in German may be interested in the subtleties of Cantor's terminology on actual infinity the finer distinctions of which are not easy to express in English. While Cantor early used "vollständig" and "vollendet" to express "complete" and "finished", the term "fertig", expressing "finished" too but being also somewhat reminiscent of "ready", for the first time appeared in a letter to Hilbert of 26 Sep 1897, where all its appearances had later been added to the letter.

But Cantor already knew that there are incomplete, i.e., potentially infinite sets like the set of all cardinal numbers. He called them "absolutely infinite". The details of this enigmatic notion are explained in section 1.2 (see also section 4.1. – Unfortunately it has turned out impossible to strictly separate Cantor's mathematical and religious arguments.)

1.1.1 Vollständig

"Wenn zwei wohldefinierte Mannigfaltigkeiten *M* und *N* sich eindeutig und **vollständig**, Element für Element, einander zuordnen lassen (was, wenn es auf eine Art möglich ist, immer auch noch auf viele andere Weisen geschehen kann), so möge für das Folgende die Ausdrucksweise gestattet sein, daß diese Mannigfaltigkeiten *gleiche Mächtigkeit* haben, oder auch, daß sie *äquivalent* sind." [Cantor, p. 119]

"*gegenseitig eindeutige und* ***vollständige*** *Korrespondenz*" [Cantor, p. 238]

"Die sämtlichen Punkte *l* unsrer Menge *L* sind also in gegenseitig eindeutige und **vollständige** Beziehung zu sämtlichen Punkten *f* der Menge *F* gebracht," [Cantor, p. 241]

"Zwei wohlgeordnete Mengen *M* und *N* heissen von *gleichem Typus* oder auch von *gleicher Anzahl*, wenn sie sich *gegenseitig* eindeutig und **vollständig** unter *beidseitiger Wahrung* der Rangfolge ihrer Elemente auf einander beziehen, abbilden lassen;" [G. Cantor, letter to R. Lipschitz (19 Nov 1883)]

"Zwei bestimmte Mengen M und M_1 nennen wir *äquivalent* (in Zeichen: $M \sim M_1$), wenn es möglich ist, dieselben gesetzmäßig, gegenseitig eindeutig und **vollständig**, Element für Element, einander zuzuordnen." [Cantor, p. 412]

"doch gibt es immer viele, im allgemeinen sogar unzählig viele Zuordnungsgesetze, durch welche zwei äquivalente Mengen in gegenseitig eindeutige und **vollständige** Beziehung zueinander gebracht werden können." [Cantor, p. 413]

"eine solche gegenseitig eindeutige und **vollständige** Korrespondenz hergestellt [...] *irgendeine* gegenseitig eindeutige und **vollständige** Zuordnung der beiden Mengen [...] auch eine gegenseitig eindeutige und **vollständige** Korrepondenz" [Cantor, p. 415]

"*Zwei n-fach geordnete Mengen M und N werden* '*ähnlich*' *genannt*, *wenn es möglich ist*, *sie gegenseitig eindeutig und* ***vollständig***, *Element für Element, einander so zuzuordnen*," [Cantor, p. 424]

1.1.2 Vollendet

"Zu dem Gedanken, das Unendlichgroße [...] auch in der bestimmten Form des **Vollendet**-unendlichen mathematisch durch Zahlen zu fixieren, bin ich fast wider meinen Willen, weil im Gegensatz zu mir wertgewordenen Traditionen, durch den Verlauf vieljähriger wissenschaftlicher Bemühungen und Versuche logisch gezwungen worden," [Cantor, p. 175]

"In den 'Grundlagen' formulire ich *denselben Protest*, indem ich an verschiedenen Stellen mich gegen die *Verwechslung* des *Uneigentlich-unendlichen* (so nenne ich das *veränderliche Endliche*) mit dem *Eigentlich-unendlichen* (so nenne ich das *bestimmte*, das ***vollendete*** Unendliche, oder auch das *Transfinite*, *Überendliche*) ausspreche. Das *Irrthümliche* in jener Gauss'schen Stelle besteht darin, dass er sagt, das **Vollendet**unendliche könne nicht Gegenstand mathematischer Betrachtungen werden; dieser Irrthum hängt mit dem andern Irrthum zusammen, dass er [...] das ***Vollendet**unendliche* mit dem *Absoluten, Göttlichen* identificirt, [...] Das **Vollendet**unendliche findet sich allerdings in gewissem Sinne in den Zahlen ω, $\omega + 1$, ..., ω^ω, ...; sie sind Zeichen für gewisse *Modi* des **Vollendet**unendlichen und weil das **Vollendet**unendliche in verschiedenen, von einander mit der äussersten Schärfe durch den *sogenannten* '*endlichen*, menschlichen Verstand' *unterscheidbaren* Modificationen auftreten kann, so sieht man hieraus deutlich *wie weit* man vom *Absoluten entfernt* ist, *obgleich* man das **Vollendet**unendliche *sehr wohl* fassen und sogar *mathematisch auffassen* kann." [G. Cantor, letter to R. Lipschitz (19 Nov 1883)]

"*da nun jeder Typus auch im letzteren Falle etwas in sich Bestimmtes*, ***vollendetes*** *ist*, *so gilt ein gleiches von der zu ihm gehörigen Zahl.* [...] 'Eigentlichunendlichem = Transfinitum = **Vollendet**unendlichem = *Unendlichseiendem* = kategorematice infinitum' [...] dieser Unterschied ändert aber *nichts* daran, daß ω als ebenso bestimmt und **vollendet** anzusehen ist, wie $\sqrt{2}$," [G. Cantor, letter to K. Laßwitz (15 Feb 1884). Cantor, p. 395]

"Wir wollen nun zu einer genaueren Untersuchung der *perfekten Mengen* übergehen. Da jede solche Punktmenge gewissermaßen *in sich begrenzt*, *abgeschlossen* und ***vollendet*** ist, so zeichnen sich die *perfekten Mengen* vor allen anderen Gebilden durch besondere Eigenschaften aus." [Cantor, p. 236]

1.1.3 Fertig

"Die Totalität aller Alefs ist nämlich eine solche, welche nicht als eine bestimmte, wohldefinirte ***fertige*** Menge aufgefaßt werden kann. [...] 'Wenn eine bestimmte wohldefinirte ***fertige*** Menge eine Cardinalzahl haben würde, die mit keinem der Alefs zusammenfiele, so müßte sie Theilmengen enthalten, deren Cardinalzahl *irgend* ein Alef ist, oder mit anderen Worten, die Menge müßte die Totalität aller Alefs in sich tragen.' Daraus ist leicht zu folgern, daß unter der eben genannten Voraussetzung (einer *best. Menge,* deren Cardinalzahl kein Alef wäre) auch die Totalität aller Alefs als eine best. wohldefinirte **fertige** Menge aufgefaßt werden könnte." [G. Cantor, letter to D. Hilbert (26 Sep 1897)]

"In meinen Untersuchungen habe ich, allgemein gesprochen, '**fertige** Mengen' im Auge und verstehe darunter solche, bei denen die Zusammenfassung aller Elemente zu einem Ganzen, zu *einem Ding für sich* möglich ist, *so daß eine* '***fertige** M.' eventuell selbst als Element einer andern Menge gedacht werden kann.* [...] Derartige Mengen, die die Bedingung '**fertig**' nicht erfüllen, nenne ich 'absolut unendliche' Mengen.

Nehmen wir einmal an, es könnten alle Alefs coexistieren, so führt uns dies zu einem Widerspruch. Denn alsdann würden alle Alefs, wenn wir sie nach ihrer Größe geordnet denken, eine *wohlgeordnete,* ***fertige*** *Menge M* bilden. Mit *jeder wohlgeordneten* ***fertigen*** *Menge M von* Alefs ist aber nach dem Bildungsgesetz der Alefs ein bestimmtes Alef gegeben, welches der Größe nach auf alle Individuen von *M nächstfolgt*.

Hier hätten wir also den Widerspruch eines Alefs, das größer wäre als *alle Alefs*, folglich auch größer als *es selbst*. Ich schließe also, daß *alle Alefs* nicht coexistent sind, nicht zu einem 'Ding für sich' zusammengefasst werden können, daß sie mit anderen Worten keine '**fertige** Menge' bilden.

Der Widerspruch erscheint mir so, als wenn wir von einer '*endlichen* Zahl' sprechen wollten, die größer wäre als '*alle endlichen* Zahlen'. Nur ist hier der Unterschied, daß alle endlichen Zahlen eine **fertige** Menge bilden, die nach oben von der kleinsten transfiniten Cardinalzahl $\aleph_0$ gewissermaßen begrenzt wird. Die *absolute Grenzenlosigkeit* der Menge aller Alefs erscheint als Grund der Unmöglichkeit, sie zu einem Ding für sich zusammenzufassen.

In dem von Ihnen vorgetragenen Beispiele wird aber die Menge aller Alefs als eine '**fertige** M.' vorausgesetzt und damit löst und erklärt sich der Widerspruch, auf den Sie durch Anwendung von Sätzen geführt werden, die nur für **fertige** Mengen bewiesen und gültig sind." [G. Cantor, letter to D. Hilbert (6 Oct 1898)]

"Aus der Definition: 'Unter einer ***fertigen*** *Menge* verstehe man jede Vielheit, bei welcher alle Elemente *ohne Widerspruch* als *zusammenseiend* und daher als *ein Ding für sich* gedacht werden können.' ergeben sich mancherlei Sätze, unter Anderm diese:
I 'Ist *M* eine **fert.** Menge, so ist auch jede Theilmenge von *M* eine **fert.** Menge.'
II 'Substituirt man in einer **fert.** M. an Stelle der Elemente **fertige** Mengen, so ist die hieraus resultirende Vielheit eine **fertige** M.'
III 'Ist von zwei aequivalenten Vielheiten die eine eine **fert.** M., so ist es auch die andere.'
IV 'Die Vielheit *aller Theilmengen* einer **fertigen** Menge *M* ist eine **fertige** Menge.' Denn alle Theilmengen von *M* sind 'zusammen' in *M* enthalten; der Umstand, daß sie sich theilweise decken, schadet hieran nichts.

Daß die 'abzählbaren' Vielheiten $\{\alpha_\nu\}$ **fertige** Mengen sind, scheint mir ein *axiomatisch sicherer* Satz zu sein, auf welchem die ganze Functionentheorie beruht. Dagegen scheint mir der Satz 'Das Linearcontinuum ist eine **fertige** Menge' ein *beweisbarer* Satz zu sein und zwar so: Das Linearcont. ist aequivalent der Menge $S = \{f(\nu)\}$ wo $f(\nu)$ die Werthe 0 oder 1 haben kann. [...] Ich behaupte also *S* ist eine '**fertige** Menge'. [...] Nach Satz IV ist aber die Vielheit aller Theilmengen von $\{\nu\}$ eine **fertige** Menge; dasselbe gilt also nach Satz III auch für *S* und für das Linearcontinuum.

Ebenso dürfte das Prädicat '**fertig**' für die Mengen $\aleph_1$, $\aleph_2$, ... *beweisbar* sein." [G. Cantor, letter to D. Hilbert (10 Oct 1898)]

"Unter Bezugnahme auf mein Schreiben v. 10$^{\text{ten}}$, stellt sich bei genauerer Erwägung heraus, daß der Beweis des Satzes IV keineswegs so leicht geht. Der Umstand, daß die *Elemente* der 'Vielheit aller *Theilmengen* einer **fertigen** Menge' sich theilweise decken, macht ihn illusorisch. In die Definition der **fert.** Menge wird die Voraussetzung des *Getrenntseins* resp. *Unabhängigseins* der Elemente als *wesentlich* aufzunehmen sein." [G. Cantor, letter to D. Hilbert (12 Oct 1898)]

"Ich habe mich jetzt daran gewöhnt, das was ich früher '**fertig**' genannt, durch den Ausdruck 'consistent' zu ersetzen;" [G. Cantor, letter to D. Hilbert (9 May 1899)]

"Die Totalität der Alefs lässt sich *nicht* als eine bestimmte **fertige** Menge auffassen." [G. Cantor, letter to A. Schönflies via D. Hilbert (28 Jun 1899)]

"Eine Vielheit kann nämlich so beschaffen sein, daß die Annahme eines 'Zusammenseins' *aller* ihrer Elemente auf einen Widerspruch führt, so daß es unmöglich ist, die Vielheit als eine Einheit, als 'ein **fertiges** Ding' aufzufassen. Solche Vielheiten nenne ich *absolut unendliche* oder *inkonsistente Vielheiten.*" [G. Cantor, letter to R. Dedekind (3 Aug 1899)]

"Zu *Elementen* einer Vielheit, können nur ***fertige*** *Dinge* genommen werden, nur *Mengen*, nicht aber *inconsistente Vielheiten*, in deren Wesen es liegt, daß sie nie als ***fertig*** und *actuell existirend* gedacht werden kann." [G. Cantor, letter to P. Jourdain (9 Jul 1904)]

"'Unter einer Menge verstehen wir jede Zusammenfassung von ... *zu einem Ganzen*', worin doch liegt, daß Vielheiten, denen das Gepräge des **fertigen** *Ganzen* oder der Dinglichkeit nicht nachgesagt werden kann, nicht als 'Mengen' *im eigentlichen Sinne des Wortes* anzusehen sind." [G. Cantor, letter to G. Chisholm-Young (9 Mar 1907)]

1.2 Cantor's notion of the absolute infinite

We find the first appearance of the term "absolute" in relation with the infinite in Cantor's paper on trigonometric series [G. Cantor: "Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen", Math. Annalen 5 (1872) pp. 123-132] where it simply denotes a potentially infinite extension of the number domain. "The notion of number as far as developed here bears the seed of an in itself necessary and absolutely infinite extension." [Cantor, p. 95]

Cantor uses the notion of absolute infinity to address God as well as the transfinite hierarchy. For instance in his paper "Grundlagen einer allgemeinen Mannigfaltigkeitslehre", Leipzig (1883) he writes that "the true infinite or absolute is God which cannot be determined."[1] Simultaneously he uses the term "absolute" already for the infinite hierarchy of infinities when he talks about "natural sections in the absolutely infinite sequence of real whole numbers."[2][3]

With respect to God "the absolute can only be acknowledged but never be recognized. [...] The absolutely infinite number sequence appears to me in some sense as an appropriate symbol of the absolute compared to which the infinity of the first number class [...] appears like a vanishing nothing. [...] The different cardinalities form an absolutely infinite sequence." [Cantor, p. 205]

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[1] "daß andrerseits das wahre Unendliche oder Absolute, welches Gott ist, keinerlei Determination gestattet. Was den letzteren Punkt anbetrifft, so stimme ich, wie es nicht anders sein kann, demselben völlig bei," [Cantor, p. 175].

[2] "so daß wir natürliche Abschnitte in der absolut unendlichen Folge der realen ganzen Zahlen erhalten, welche Abschnitte ich *Zahlenklassen* nenne." [Cantor, p. 167]

[3] Cantor addresses his transfinite numbers as "real whole numbers".

"As I see, you use the expression 'absolute' in the same sense as I use 'proper'[1]. *I* however use the word 'absolute' only for that which *cannot further be increased* or *completed*, in analogy to the 'absolute' in metaphysics. My proper infinite or, if you prefer, transfinite numbers ω, $\omega + 1$, ... are not 'absolute', because, although not being finite, they can be increased. The absolute however is incapable of being increased, and therefore also is inaccessible to us.

As far as I remember, Hegel does not have my *intermediate notion* of 'proper infinite' but distinguishes only between the 'bad infinite' which I call 'improper infinite' and the 'absolute infinite' which cannot be increased. Kant, I definitely know that, does not have a clue of my intermediate notion. He always makes the mistake to believe that the limits of the finite cannot be extended, which leads to a lot of false conclusions." [G. Cantor, letter to W. Wundt (5 Oct 1883)]

In a letter to Eneström Cantor confesses his belief in the actually infinite God outside of the world and accuses Gauss, because of his opinion (see Gauss in section V), of short-sightedness and of supporting the horror infinity which, according to Cantor, is a feature of the modern materialistic view of things.[2]

Cantor acknowledges "an 'Infinitum aeternum increatum sive Absolutum', which is related to God and his attributes," [G. Cantor, letter to J.B. Franzelin (22 Jan 1886). Cantor, p. 399] "A.-U. {{actual infinite}} that can be increased or *Transfinitum*. A.-U. that cannot be increased or *Absolutum*." [G. Cantor, letter to A. Eulenburg (28 Feb 1886). Cantor, p. 405]

Cantor deplores that the traditional view of the infinite, come down from Aristotle, distinguishes between potential infinity and the absolute infinite in God, but misses his tranfinitum which is between both.[3] "When I deal with the 'transfinite', the absolute infinite is not meant, which as actus purus and ens simplicissimum[4] neither can be increased nor decreased and only exists in Deo or rather *as* Deus optimus maximus." [G. Cantor, letter to I. Jeiler (13 Oct 1895)]

In later years Cantor often pointed out that there are absolute infinities which cannot be consistent (*fertige*) sets. But we have seen above that he knew about this fact in earlier years already.

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[1] eigentlich

[2] "*Deo extramundano aeterno omnipotenti sive natura naturante*, wo es das *Absolute* heißt, [...] Wenn aber aus einer berechtigten Abneigung gegen solche *illegitime* A. U. sich in breiten Schichten der Wissenschaft, unter dem Einflusse der modernen epikureisch-materialistischen Zeitrichtung, ein gewisser *Horror Infiniti* ausgebildet hat, der in dem erwähnten Schreiben von *Gauß* seinen klassischen Ausdruck und Rückhalt gefunden, so scheint mir die damit verbundene unkritische Ablehnung des *legitimen* A. U. kein geringeres Vergehen wider die Natur der Dinge zu sein, die man zu nehmen hat, wie sie sind, und es läßt sich dieses Verhalten auch als eine Art *Kurzsichtigkeit* auffassen, welche die Möglichkeit raubt, das A. U. zu sehen, obwohl es in seinem höchsten, absoluten Träger uns geschaffen hat und erhält und in seinen sekundären, transfiniten Formen uns allüberall umgibt und sogar unserm Geiste selbst innewohnt." [G. Cantor, letter to G. Eneström (4 Nov 1885). Cantor, pp. 372 & 374f]

[3] "liegt in der Hauptsache an der Mangelhaftigkeit der von Aristoteles (oder von noch älteren Philosophen) herrührenden Definition des Unendlichen, die entweder nur auf das potenziale Unendliche oder nur auf das absolut Vollkommene in Gott passen, die aber nicht auf dasjenige Unendliche genügend Rücksicht nehmen, welches ich Transfinitum nenne, also auf das, 'was zwar in sich constant und grösser als jedes Endliche, aber doch noch unbeschränkt vermehrbar und insofern begrenzt ist'," [G. Cantor, letter to A. Schmid (26 Mar 1887)]

[4] Pure act and simplest being are scholastic definitions of God.

"*Totalities* which cannot be comprehended by us as 'sets' (one example being the totality of all alephs as has been proven above) I have called 'absolutely infinite' totalities many years ago already and have distinguished them sharply from the *transfinite sets*." [G. Cantor, letter to D. Hilbert (26 Sep 1897)]

"Infinite sets however, the totality of which cannot be thought of as 'being together', as 'a thing in itself', as an 'αφωρισμενον', which therefore *in this totality* cannot be an object of further *mathematical* considerations, I call '*absolutely infinite* sets'; the 'set of all alephs' belongs to them." [G. Cantor, letter to D. Hilbert (2 Oct 1897)]

"Such sets, which do not satisfy the condition 'fertig' I call 'absolutely infinite sets'. [...] The *absolute boundlessness* of the set of all alephs appears as the reason for the impossibility, to combine them as a thing in itself." [G. Cantor, letter to D. Hilbert (6 Oct 1898)]

"Starting from the notion of a determined multitude (a system, an embodiment[1]) of things, I have recognized the necessity to distinguish two kinds of multitudes (I always mean *determined* multitudes). A multitude can have a constitution such that the assumption of the 'being together' of *all* its elements leads to a contradiction such that it is impossible to comprehend this multitude as a unit, as a 'finished thing'. Such multitudes I call *absolutely infinite* or *inconsistent multitudes*. As we easily convince ourselves the 'embodiment of all thinkable' is such a multitude. [...]

A. *The system Ω of all numbers is an inconsistent, an absolutely infinite multitude.* [...] As we convince ourselves the creation process of the alephs and of their corresponding number classes of the system Ω is *absolutely* unbounded.

B. *The system* ת *of all alephs* $\aleph_0, \aleph_1, ..., \aleph_\nu, ... \aleph_{\omega_0}, \aleph_{\omega_0+1}, ... \aleph_{\omega_1}, \aleph_{\omega_1+1}, ...$ *in their order of magnitude forms a sequence similar to the system Ω and therefore is also an inconsistent absolutely infinite sequence.*" [G. Cantor, letter to R. Dedekind (3 Aug 1899). Cantor, p. 443ff[2]]

"Consistent multitudes I call '*sets*'. Every multitude that is not 'consistent' I call 'inconsistent' or 'absolutely infinite'." [G. Cantor, letter to D. Hilbert (27 Jan 1900)]

"I distinguish two kinds of multitudes, *consistent* and *inconsistent* ones (the latter I also call absolutely infinite multitudes). [...] It is obvious that cardinal numbers befit only the consistent multitudes or sets. An inconsistent multitude has *no* cardinal number and *no* order type (just because of its inconsistency)." [G. Cantor, letter to P. Jourdain (4 Nov 1903)]

"What lies above all finite and transfinite is not a 'Genus'; it is the only completely individual unit within which all is contained, which comprises all, the 'absolute', incomprehensible to human intellect, therefore not subject to mathematics, the 'ens simplicissimum', the 'actus purissimus', which by many is called 'God'." [G. Cantor, letter to G. Chisholm-Young (20 Jun 1908)]

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[1] Inbegriff

[2] "the final section of {{Dedekind's}} correspondence with *Cantor* starts only in July 1899. This was the part from which extracts were published in the edition of *Cantor's* papers by *Zermelo* using the transcriptions made by *Cavaillès* [...] the standard of editing of the extracts is bad. [...] The collection begins with a letter from *Cantor* of 28 July 1899. lt is the most famous of them all [...] There does not exist a letter in this form." [I. Grattan-Guinness: "The rediscovery of the Cantor-Dedekind correspondence", Jahresbericht DMV 76 (1974) p. 126f]

II Transfinite set theory

Transfinite set theory is based on the possibility to define infinite sets M, N and to prove their equinumerosity by bijection, i.e., one-to-one mapping. This is denoted by $|M| = |N|$ or by $M \sim N$.

"By a 'manifold' or 'set' I understand in general every Many which can be thought as a One, i.e., every embodiment of definite elements which by a law can be connected to become a whole," [G. Cantor: "Grundlagen einer allgemeinen Mannigfaltigkeitslehre", Leipzig (1883). Cantor, p. 204]

"By a 'set' we understand every collection M of definite well-distinguished objects m of our visualization or our thinking (which are called the 'elements' of M) into a whole." [G. Cantor: "Beiträge zur Begründung der transfiniten Mengenlehre 1", Math. Annalen 46 (1895) § 1. Cantor, p. 282]

"If two well-defined manifolds, M and N, can be related completely, element by element, to each other [...], then for the following the expression may be permitted that these manifolds have the *same cardinality* or that they are *equivalent*."[1]

"Every well-defined set has a definite cardinality; two sets are ascribed the same cardinality if they mutually uniquely, element by element, can be mapped onto each other."[2]

"Two sets are called 'equivalent' if they mutually uniquely, element by element, can be mapped onto each other."[3]

This definition has not been changed since Cantor's days and opinions:

"Bijection: A transformation which is one-to-one and a surjection (i.e., 'onto')." [E.W. Weisstein: "Bijection", Wolfram MathWorld] Obviously here "one-to-one" means "element by element".

"In mathematics, a *bijection*, *bijective function* or *one-to-one correspondence* is a function between the elements of two sets, where each element of one set is paired with exactly one element of the other set, and each element of the other set is paired with exactly one element of the first set. There are no unpaired elements." ["Bijection", Wikipedia]

Whereas a bijection is obviously sufficient to prove the equinumerosity of finite sets, Cantor has expanded its domain to include infinite sets. The proof of equinumerosity by bijection between infinite sets, M and N, is justified by mathematical induction: If every element of set M can be

---

[1] "Wenn zwei wohldefinierte Mannigfaltigkeiten M und N sich eindeutig und vollständig, Element für Element, einander zuordnen lassen (was, wenn es auf eine Art möglich ist, immer auch noch auf viele andere Weisen geschehen kann), so möge für das Folgende die Ausdrucksweise gestattet sein, daß diese Mannigfaltigkeiten *gleiche Mächtigkeit* haben, oder auch, daß sie *äquivalent* sind." [Cantor, p. 119]

[2] "Jeder wohldefinierten Menge kommt danach eine bestimmte Mächtigkeit zu, wobei zwei Mengen dieselbe Mächtigkeit zugeschrieben wird, wenn sie sich gegenseitig eindeutig, Element für Element einander zuordnen lassen." [Cantor, p. 167]

[3] "Zwei Mengen werden hierbei 'äquivalent' genannt, wenn sie sich gegenseitig eindeutig, Element für Element, einander zuordnen lassen." [Cantor, pp. 380 & 441]

related to one and only one corresponding element of set N and vice versa, and if there is never an obstacle or halt[1] in this process of assignment, then both infinite sets are in bijection.

Since the relation of the elements m_k and n_k is tantamount to the relation of finite initial segments, we can state: If a bijection "$\leftrightarrow$" holds for all finite initial segments, then it holds for the whole sets:

$$\forall k \in \mathbb{N}: m_k \leftrightarrow n_k \Leftrightarrow \forall k \in \mathbb{N}: (m_1, m_2, ..., m_k) \leftrightarrow (n_1, n_2, ..., n_k) \Leftrightarrow M \sim N .$$

"*The equivalence of sets is the necessary and unmistakable criterion for the equality of their cardinal numbers.* [...] If now $M \sim N$, then this is based on a law of assigning, by which M and N are mutually uniquely related to each other; here let the element m of M be related to the element n of N." [Cantor, p. 283f]

"*As 'cardinality' or 'cardinal number' of M we denote the general notion which, aided by our active intellectual capacity, comes out of the set M by abstracting from the constitution of its different elements m and of the order in which they are given.*" [G. Cantor: "Beiträge zur Begründung der transfiniten Mengenlehre", Math. Annalen 46 (1895). Cantor, p. 282]

All well-ordered[2] sets can be compared. They have the same number if they, by preserving their order, can be uniquely mapped or *counted* onto each other.[3] "Therefore all sets are '*countable*' *in an extended sense*, in particular all '*continua*'." [G. Cantor, letter to R. Dedekind (3 Aug 1899)]

"'Infinite definitions' (that do not happen in finite time) are *non-things*. If König's theorem was *true*, according to which all 'finitely definable' real numbers form an embodiment of cardinality $\aleph_0$, this would imply that the whole number-continuum was countable, which is certainly *false*.

The question is: which error underlies the *alleged proof* of his wrong theorem?

The error (which also can be found in a note by a Mr. Richard in the last issue of the Acta Mathematica, which note Mr. Poincaré is keen to emphasize in the last issue of the Revue de Métaphysique et de Morale) is, as it appears to me, this one:

It is presupposed that the system $\{B\}$ of notions B, which possibly have to be used to define real number-individuals is finite or at most countably infinite.

This assumption *must* be *an error* because otherwise this would imply the *wrong theorem* 'the number-continuum has cardinality $\aleph_0$'.

Am I mistaken, or am I right?"
[G. Cantor, letter to D. Hilbert (8 Aug 1906)]

Note: To apply the notion of "cardinality" or "number" requires *different* elements, i.e., elements that are *distinct* and *can be distinguished* and well-ordered. Only *definable* elements can be uniquely related to each other. One-to-one requires to *distinguish* each "one".

---

[1] "und es erfährt daher der aus unsrer Regel resultierende Zuordnungsprozeß *keinen Stillstand*." [Cantor, p. 239]

[2] "Der Begriff der *wohlgeordneten Menge* weist sich als fundamental für die ganze Mannigfaltigkeitslehre aus." [Cantor, p. 169]

[3] "Dabei nenne ich zwei *wohlgeordnete* Mengen von demselben Typus und schreibe ihnen gleiche Anzahl zu, wenn sie sich unter Wahrung der festgesetzten Rangordnung ihrer Elemente *gegenseitig eindeutig* aufeinander abbilden, oder wie man sich gewöhnlich ausdrückt, aufeinander *abzählen* lassen." [G. Cantor, letter to W. Wundt (5 Oct 1883)]

2.1 Countable sets

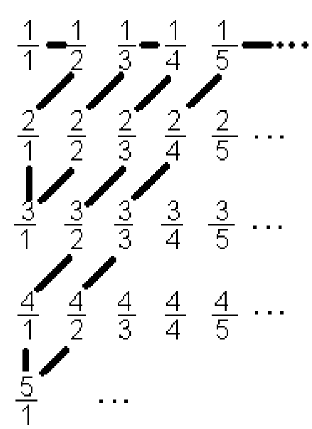
If a bijection between the set ℕ of natural numbers and a set M is *possible*, M is called "countably infinite" or briefly "countable". (Note: finite sets are usually called countable too.) This is written as $|M| = |\mathbb{N}| = \aleph_0$ or Card(M) = Card(ℕ). The elements of a countable set can be arranged in form of an infinite sequence $(m_n) = m_1, m_2, m_3, \ldots$.

So the set of all positive rational numbers is countable as was shown by Cantor [G. Cantor, letter to R. Lipschitz (19 Nov 1883)] by this sequence

1/1, 1/2, 2/1, 1/3, ~~2/2~~, 3/1, 1/4, 2/3, 3/2, 4/1, 1/5, ~~2/4~~, ~~3/3~~, ~~4/2~~, 5/1, 1/6, ...

where the sum s of numerator and denominator grows stepwise by 1, and for fixed sum the numerator grows stepwise by 1 from minimum 1 to maximum s - 1. If repeating values of fractions are eliminated, every positive *rational number* appears exactly once, if not, we get the sequence of all positive *fractions*. Inserting the corresponding negative fraction immediately before or after the positive one (and starting with zero) shows the countability of the set ℚ of all rational numbers, $|\mathbb{Q}| = \aleph_0$, or of all fractions, $|\{m/n \mid m \in \mathbb{Z}, n \in \mathbb{N}\}| = \aleph_0$.

It is also possible to write all positive fractions in an infinite matrix in order to string them together as is shown in the figure at right. Every fraction appears only once, every value appears infinitely often in the string. But this does not damage countability. The rational numbers are countable, independently of their multiplicity. If desired, all the repetitions can be eliminated by hand. This method, adopted from Cauchy, is sometimes called *Cantor's first diagonal method*.



There are many other methods to enumerate all fractions or to show an injective mapping into all integers ℤ which also proves their countability. When mapping the fraction $\pm m/n$ to the integer $\pm 2^m 3^n$ then this mapping is injective because of the uniqueness of prime factor decomposition.

The figure also shows that a set of $\aleph_0 \cdot \aleph_0$ elements is countable, i.e., it has the cardinal number $\aleph_0 \cdot \aleph_0 = \aleph_0^2 = \aleph_0$. From that it is easy to see that for every $n \in \mathbb{N}$: $\aleph_0^n = \aleph_0$.

It is even possible to enumerate the set 𝔸 of all algebraic numbers, i.e., all solutions of polynomial equations $\sum_{\nu=0}^{n} a_\nu x^\nu = 0$ with integer coefficients a_ν and finite degree n, i.e., $a_n \neq 0$. Ordering the polynomials as usual according to their degree, it is impossible to leave the first degree (and thus the rational numbers), because the coefficients a_0 already exhaust the whole set of integers. But when ordering the polynomials by their height H, i.e., the sum of their degree n and all absolute values of their coefficients a_ν

$$H = n + |a_0| + |a_1| + |a_2| + \ldots + |a_n|$$

then for every height H there is a finite number of polynomials. Every polynomial gets its place in the enumeration, and since every polynomial of degree n has at most n different roots, every root can be inserted in a sequence containing all of them. So the set of roots of all polynomials is countable. It is the set $\mathbb{A}$ of all algebraic numbers. [R. Dedekind, private note (29 Nov 1873). Cantor, p. 116)]

Height	Equations					
	Algebraic Numbers					
1	–					
2	$1x^1 = 0$					
	0					
3	$1x^2 = 0$	$2x^1 = 0$	$1x^1 \pm 1 = 0$			
	0	0	±1			
4	$1x^3 = 0$	$2x^2 = 0$	$3x^1 = 0$	$1x^2 \pm 1 = 0$	$2x^1 \pm 1 = 0$	$1x^1 \pm 2 = 0$
	0	0	0	±1 ±i	±1/2	±2

We note that it is obvious from the table that all algebraic numbers will appear infinitely often. If desired, all repetitions can be eliminated by hand (up to every chosen height).

König knew already in 1905 that the set of all finite definitions is countable. [Julius König: "Über die Grundlagen der Mengenlehre und das Kontinuumproblem", Math. Ann. 61 (1905) p. 157] Obviously it is possible to enumerate all finite strings of text. With no restriction to generality we can write every text using a binary alphabet {0, 1} (otherwise it could not be published via internet). Then the sequence of all finite expressions is 0, 1, 00, 01, 10, 11, 000, ...

Of course all infinite subsets of a countable set are infinite countable sets. [Cantor, p. 293] This is a special case of the Schröder-Bernstein Theorem [E. Borel: "Leçons sur la théorie des fonctions", Gauthier-Villars, Paris (1898) p. 103ff]: If $|M| \le |N|$ and $|N| \le |M|$, then $|M| = |N|$.

Proof: According to $M \sim N_1 \subset N$ and $N \sim M_1 \subset M$ we define disjoint subsets:

$M := A \cup B \cup C$ and $M_1 := A \cup B$ such that
$(A \cup B \cup C) \sim A := (A_1 \cup B_1 \cup C_1)$ where $A \sim A_1$ and $C \sim C_1$
$(A_1 \cup B_1 \cup C_1) \sim A_1 := (A_2 \cup B_2 \cup C_2)$ where $A_1 \sim A_2$ and $C_1 \sim C_2$
$(A_2 \cup B_2 \cup C_2) \sim A_2 := (A_3 \cup B_3 \cup C_3)$ where $A_2 \sim A_3$ and $C_2 \sim C_3$
and so on.

By definition $C \sim C_1 \sim C_2 \sim C_3 \sim ...$.
Define $D := A \cap A_1 \cap A_2 \cap A_3 \cap ...$.

Then $M = A \cup B \cup C = B \cup C \cup B_1 \cup C_1 \cup B_2 \cup C_2 \cup ... \cup D$ (*)
and $M_1 = A \cup B = B \cup C_1 \cup B_1 \cup C_2 \cup B_2 \cup C_3 \cup ... \cup D$.

All sets are disjoint. The sets directly beneath those of eq. (*) are equivalent to them. ■

This theorem has a remarkable history. Repeatedly stated (and claimed as proven) between 1882 [G. Cantor, letter to R. Dedekind (5 Nov 1882)][1] and 1895 [Cantor, p. 285] but never really proved by Cantor, this theorem is called after Ernst Schröder and Felix Bernstein, because both proved it. Alwin Korselt however discovered a flaw in Schröder's proof in 1902. Alas the Mathematische Annalen did not publish the correction before 1911. [A. Korselt: "Über einen Beweis des Äquivalenzsatzes", Math. Ann. 70 (1911) p. 294] Nevertheless it took some time until this correction received public attention. Ernst Zermelo noted in his edition of Cantor's collected works as late as in 1932: "The theorem, here mentioned without proof [...] has been proved only in 1896 by E. Schröder and 1897 by F. Bernstein.[2] Since then this 'equivalence-theorem' is considered of the highest importance in set theory." [Cantor, p. 209] We learn from this story that wrong proofs can survive in mathematics over many decades.

2.2 Uncountable set of transcendental numbers

The set $\mathbb{N}$ of all natural numbers as well as all its infinite subsets have cardinality $\aleph_0$. The set of all rational numbers and the set of all fractions and even the set of all algebraic numbers have cardinality $\aleph_0$. This recognition would not have been exciting and would not have stirred up so much ado, if $\aleph_0$ had remained the only infinite cardinal number. In 1874 however Cantor showed by his first uncountability proof that the set $\mathbb{R}$ of all real numbers is not countable but has a greater cardinality. This could only be understood because shortly before Joseph Liouville had shown the existence of transcendental numbers [J. Liouville: "Sur des classes trés étendues de quantités dont la valeur n'est ni rationnelle ni même réductible à des irrationnelles algébriques", Comptes Rendus Acad. Sci. Paris, 18 (1844) pp. 883-885 & 910-911]. Therefore Cantor's result was considered a non-constructive proof of the existence of transcendental numbers.

2.2.1 Transcendental numbers

Theorem   If α is an algebraic number of degree[3] n then the inequality

$$\left|\alpha - \frac{u}{v}\right| < \frac{1}{v^{n+1}} \qquad (*)$$

has only a finite set of solutions. See "History of the infinite", lesson 10.

Proof: The minimal polynomial $p(x)$ of an irrational number α has no rational zero. Otherwise $p(x)$ could be divided by the linear factor $(x - u/v)$ resulting in a polynomial of smaller degree with root α. Therefore $p(x) \neq 0$ for rational $x = u/v$ (with $u \in \mathbb{Z}$ and $v \in \mathbb{N}$).

---

[1] Cantor claims, in colloquial German, $M \subset M' \wedge M' \subset M'' \wedge M \leftrightarrow M'' \Rightarrow M \sim M' \sim M''$. This "intermediate-set theorem" is equivalent to the Schröder-Bernstein equivalence theorem.

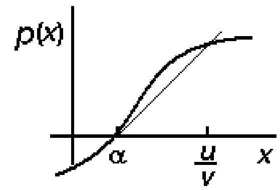
[2] It was proved, but not published by Dedekind in 1887 already. ["Satz von Cantor-Bernstein-Schröder", Wikipedia]

[3] The minimal polynomial with root α is of the form $p(x) = \sum_{k=0}^{n} a_k x^k$ with $a_k \in \mathbb{Z}$ and $a_n \neq 0$.

$$0 \neq \left| p\left(\frac{u}{v}\right) \right| = \left| a_0 + a_1 \frac{u}{v} + \ldots + a_n \frac{u^n}{v^n} \right| = \left| \frac{a_0 v^n + a_1 u v^{n-1} + \ldots + a_n u^n}{v^n} \right| \geq \frac{1}{v^n} \qquad (**)$$

because the numerator is a non-zero integer. We get (see figure)

$$\left| \frac{p\left(\frac{u}{v}\right) - 0}{\frac{u}{v} - \alpha} \right| = C$$



with a finite positive constant C, the absolute value of the slope of the secant.

Inserting this into (**) supplies

$$\frac{1}{v^n} \leq \left| p\left(\frac{u}{v}\right) \right| = \left| \alpha - \frac{u}{v} \right| \cdot C \,.$$

Together with the stated inequality (*) we obtain

$$\frac{1}{C \cdot v^n} \leq \left| \alpha - \frac{u}{v} \right| < \frac{1}{v^{n+1}} \,.$$

This restricts the possible natural denominators of fractions satisfying (*) to v < C. The stated inequality (*) can only be satisfied by fractions in the proximity of α which have denominators less than C. There can be only a finite number of such fractions. ■

In addition Liouville constructed a number

$$L = \sum_{k=1}^{\infty} \frac{1}{10^{k!}}$$

such that (*) has infinitely many solutions for every n.

For every $m \in \mathbb{N}$ there exist infinitely many rational numbers $L_n = \sum_{k=1}^{n+m} 10^{-k!}$ with $n \in \mathbb{N}$ and

$$L - L_n = \sum_{k=n+m+1}^{\infty} 10^{-k!} = \frac{1}{10^{(n+m+1)!}} + \frac{1}{10^{(n+m+2)!}} + \frac{1}{10^{(n+m+3)!}} + \ldots < \left(\frac{1}{10^{(n+m)!}} \right)^{n+1} = \frac{1}{10^{(n+m)!(n+1)}} = \frac{1}{v^{n+1}} \,.$$

$10^{-(n+m)!(n+1)}$ is always greater than the whole remaining series starting with $10^{-(n+m)!(n+m+1)}$. Therefore there are infinitely many solutions of Liouville's inequality for Liouville's number L.

2.2.2 Cantor's first uncountability proof

Consider an infinite sequence of different real numbers $(a_\nu) = a_1, a_2, a_3, \ldots$ which is given by any rule, then we can find in any open interval (α, β) a number η (and, hence, infinitely many of such numbers) which is not a member of the sequence (a_ν).

Take the first two members of sequence (a_ν) which fit into the given interval (α, β). They form the interval (α', β'). The first two members of sequence (a_ν) which fit into this interval (α', β') form the interval (α'', β'') and so on. The result is a sequence of nested intervals. Now there are only two possible cases.

Either the number of intervals is finite. Inside the last one $(\alpha^{(\nu)}, \beta^{(\nu)})$ there cannot be more than one member of the sequence. Any other number of this interval $(\alpha^{(\nu)}, \beta^{(\nu)})$ can be taken as η.

Or the number of intervals is infinite. Then both, the strictly increasing sequence $\alpha, \alpha', \alpha'', \ldots$ and the strictly decreasing sequence $\beta, \beta', \beta'', \ldots$ converge to different limits α^∞ and β^∞ or they converge to the same limit $\alpha^\infty = \beta^\infty$ (a case which always occurs in $\mathbb{R}$ if the sequence contains all rational numbers; due to the denumerability of $\mathbb{Q}$ this is possible). $\alpha^\infty = \beta^\infty = \eta$ is not a member of sequence (a_ν). If $\alpha^\infty < \beta^\infty$, then any number of $(\alpha^\infty, \beta^\infty)$ satisfies the theorem.

[G. Cantor: "Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen", Crelles Journal f. Mathematik **77** (1874) pp. 258-262. Cantor, p. 117]

So far Cantor's proof. It is an existence proof for transcendental numbers, since all algebraic numbers have been shown countable. We note that the "either"-case cannot occur if sequence (a_ν) contains at least all rational numbers because any interval $(\alpha^{(\nu)}, \beta^{(\nu)}) \subset \mathbb{Q}$ contains infinitely many and so at least two rational numbers forming the next interval $(\alpha^{(\nu+1)}, \beta^{(\nu+1)}) \subset \mathbb{Q}$.

2.2.3 Cantor's second uncountability proof

Much better known is Cantor's second uncountability proof, his diagonal argument, also denoted as *Cantor's second diagonal method*. It is assumed that all binary sequences can be enumerated. This assumption is then contradicted. If all real numbers can be represented by infinite binary sequences, this proof is tantamount to proving the non-denumerability of the set of real numbers. Cantor, in his original version [G. Cantor: "Über eine elementare Frage der Mannigfaltigkeits-lehre", Jahresbericht der DMV I (1890-91) pp. 75-78. Cantor, p. 278ff], applied binary sequences built from the letters *m* and *w*, probably taken from the German words männlich (male) and weiblich (female). No limits for these infinite sequences have been defined. So it only shows the uncountability of all infinite bit sequences. Nowadays the proof is usually applied to the set of real numbers of the unit interval (0, 1) represented by infinite sequences of decimal digits. The set of all real numbers of any given real interval cannot be written as a sequence.

For proof assume that the set of all real numbers $r_n = 0.a_{n1}a_{n2}a_{n3}...$ of the real interval (0, 1) has been enumerated, i.e., has been written as a sequence or list:

n	r_n
1	$0.a_{11}a_{12}a_{13}...$
2	$0.a_{21}a_{22}a_{23}...$
3	$0.a_{31}a_{32}a_{33}...$
...	...

If the diagonal digit a_{nn} of each real number r_n is replaced by

$$b_n \neq a_{nn} \quad \text{with} \quad 1 \leq b_n \leq 8$$

(in order to avoid problems with 0.0999... and 0.1000...)[1] we construct the *antidiagonal number*

$$d = 0.b_1b_2b_3...$$

belonging to the real interval (0, 1) but differing from any r_n of the list. This shows that this list and every such list must be incomplete. An example is the list

n	r_n
1	0.**0**00111199999...
2	0.1**2**3456789123...
3	0.55**5**555555555...
4	0.314**1**59265358...
5	0.1010**0**1000100...
...	...

with the diagonal sequence of digits 0.02510... and the antidiagonal number d = 0.13621... . (Here every digit a_{nn} differs from 8 and 9 and therefore could be replaced by $a_{nn} + 1$). Diagonal digits 8 and 9 could be replaced by 1, for instance.

By the bijection $1/r \leftrightarrow r$ for $1 \leq r < \infty$ it is proven that the cardinality of the unit interval is the same as that of the whole real line from 1 to infinity. $1/r \leftrightarrow (r - 1)$ for $1 \leq r < \infty$ proves that the unit interval has the same cardinality as the whole positive real axis.

[1] This precaution has been mentioned first by Felix Klein. [Felix Klein: "Vorträge über ausgewählte Fragen der Elementargeometrie", Teubner, Leipzig (1895) p. 42]

2.3 The halting problem

The so-called halting problem, going back to Alan Turing, is a variant of the diagonal argument. [Alan Turing: "On computable numbers, with an application to the Entscheidungsproblem", Proc. London Math. Soc., Ser. 2, 42 (1937) pp. 230-265] (The expression "halting problem" does not appear in Turing's paper.)

Turing defines the "computable" numbers "as the real numbers whose expressions as a decimal are calculable by finite means"; the digit sequences can be written down by a machine equipped with a program. This implies that the set of computable numbers is enumerable.

The machine $\mathcal{M}$ scans a square on a tape and writes or erases a symbol on that tape which then can be shifted one place to right or left. The action depends on the "*m*-configuration" q_k of the machine and the symbol S_j in the square just scanned. This *m*-configuration q_k, S_j, and all symbols written on the tape describe the complete configuration of the machine.

The machine can be circular. Then it writes never more than a finite sequence of bits. Otherwise the infinite sequence of bits, prepended by a decimal point, is a computable real number.

The description of the machine, expressed entirely by seven characters[1] $\{A\ C\ D\ L\ R\ N\ ;\}$, is called the standard description (S.D). It can be encoded as an integer, containing only digits from 1 to 7, the so-called description number (D.N). The machine whose D.N is n may be described as $\mathcal{M}(n)$. Every such S.D computes one and only one sequence or real number. Instead of different machines a universal machine $\mathcal{U}$ can be used. When fed with the S.D of machine $\mathcal{M}(n)$ (on the beginning of its tape) $\mathcal{U}$ computes the same real number as $\mathcal{M}(n)$.

If there is a complete sequence of computable sequences, then let a_n be the nth computable sequence, and let a_{nm} be the mth bit in a_n. Let d be the sequence which has $1 - a_{nn}$ as its nth bit (i.e., the complement 0 of 1 or 1 of 0). Since the antidiagonal sequence d is computable, there exists a number k such that $1 - a_{nn} = a_{kn}$ for all n. Putting $n = k$, we have $1 = 2a_{kk}$, i.e., 1 is even. This is impossible. Therefore a complete sequence of computable sequences cannot exist.

Turing concludes: The fallacy in this argument lies in the assumption that d is computable. It would be true if we could enumerate the computable sequences by finite means, but the problem of enumerating the sequences is equivalent to the problem of finding out whether a given number n is the D.N of a circle-free machine, and we have no general process for doing this in a finite number of steps. There cannot be any such general process. The simplest proof is that, if this general process exists, then there is a machine which computes d.

Finally Turing supplies another proof, showing that the sequence $(a_{nn})_{n\in\mathbb{N}}$, the complement of the antidiagonal number d, cannot be calculated.

---

[1] Letters D and A are used for *m*-configurations; for instance $q_3 = DAAA$. The symbols S_0 = blank, $S_1 = 0$, and $S_2 = 1$ are encoded by the letters D, DC, and DCC respectively. Further there are L, R, N for "shift to left", "shift to right", and "no shift", and finally the semicolon as the separator of commands.

Suppose there is a machine $\mathcal{T}$ which can test every S.D on circularity and can calculate the sequence (a_{nn}) from the real numbers produced by circle-free S.D. In the first N - 1 sections, among other things, the integers 1, 2, 3, ..., N - 1 have been written on the tape. The circular S.D have been sorted out as undefined by marking them with u. Let there remain $T(N - 1)$ description numbers D.N of S.D which yield computable numbers. In the Nth section $\mathcal{T}$ tests the number N. If N is the D.N of a circle-free S.D, then $T(N) = 1 + T(N - 1)$ and the first $n = T(N)$ digits of the corresponding computable sequence are calculated. The nth digit a_{nn} is written down. If N is not the D.N of a circle-free S.D, then the next number $N + 1$ is tested.

From the construction of $\mathcal{T}$ we can see that $\mathcal{T}$ is circle-free. Each section of the motion of $\mathcal{T}$ comes to an end after a finite number of steps. For, by assumption about $\mathcal{T}$, the decision is reached in a finite number of steps.

Now let K be the D.N of $\mathcal{T}$. Since $\mathcal{T}$ is circle free, the machine cannot print u and move on. On the other hand the $T(K)$th digit of the sequence, computed by the machine with K as its D.N, cannot be found because it has to be taken from a calculation that cannot be performed: the only advice given by the S.D of $\mathcal{T}$ is to *take* the calculated number. Therefore $\mathcal{T}$ is not circle free, and we have obtained a contradiction from the assumption that all computable numbers can be enumerated.

2.3.1 A brief account

Consider a complete enumeration of all programs $P(i)$. Assume there is a total computable function h such that

$$h(i, x) = \begin{cases} 1 \text{ if } P(i) \text{ halts on input } x \\ 0 \text{ else} \end{cases} .$$

In order to show that such a function cannot exist, consider an arbitrary total computable function f and a partial function g of f computed by a program e such that

$$g(i) = \begin{cases} 0 \quad \text{ if } f(i,i) = 0 \\ \text{otherwise loop forever} \end{cases} .$$

If $f(e, e) = 0$ then $g(e) = 0$. In this case $h(e, e) = 1$, because program e halts on input e.

If $f(e, e) \neq 0$ then $g(e)$ is undefined and does not halt. Therefore $h(e, e) = 0$.

In either case, f cannot be the same function as h. Since f was an arbitrary total computable function with two arguments, all such functions must differ from h.

["Halting problem", Wikipedia]

2.3.2 An uncomputable real number

Make a list of all possible computer programs. Order the programs by their size, and within those of the same size, order them alphabetically. The easiest thing to do is to include all the possible character strings that can be formed from the finite alphabet of the programming language, even though most of these will be syntactically invalid programs.

Here's how we define the uncomputable diagonal number $0 < r < 1$. Consider the kth program in our list. If it is syntactically invalid, or if the kth program never outputs a kth digit, or if the kth digit output by the kth program isn't a 3, pick 3 as the kth digit of r. Otherwise, if the kth digit output by the kth program is a 3, pick 4 as the kth digit of r.

This r cannot be computable, because its kth digit is different from the kth digit of the real number that is computed by the kth program, if there is one. Therefore there are uncomputable reals, real numbers that cannot be calculated digit by digit by any computer program.

[Gregory J. Chaitin: "How real are real numbers?", arXiv (2004)]

2.4 Uncountable power set of $\mathbb{N}$

The power set $\mathcal{P}(S)$ of S contains as its elements all subsets of S. A proof by Hessenberg [G. Hessenberg: "Grundbegriffe der Mengenlehre", offprint from Abhandlungen der Fries'schen Schule, Vol. I, no. 4, Vandenhoeck & Ruprecht, Göttingen (1906) § 24] shows that there is no bijection between $\mathbb{N}$ and its power set $\mathcal{P}(\mathbb{N})$. If $\mathbb{N}$ could be bijected with its power set $\mathcal{P}(\mathbb{N})$, then some $n \in \mathbb{N}$ could be mapped on subsets $s(n)$ not containing them. The subset M of all such numbers n, may it be empty, may it be $\mathbb{N}$, or may it be some other subset of $\mathbb{N}$,

$$M = \{n \mid n \to s(n) \land n \notin s(n)\}$$

belongs to $\mathcal{P}(\mathbb{N})$ as an element. But the set M together with the mapping s cannot exist. If M does not contain the element m which is mapped on it by s: $m \to M$, then m belongs to M, but exactly then M must not contain m: $m \in M \Rightarrow m \notin M \Rightarrow m \in M$ and so on.

This proof can easily be extended to show that the power set of a set always has larger cardinality than the set:

$$|\mathcal{P}(S)| > |S| .$$

For finite sets we find immediately

$$|\mathcal{P}(S)| = 2^{|S|} .$$

- The power set of the empty set $\emptyset = \{\ \}$ contains $2^0 = 1$ element $\{\emptyset\} = \{\{\ \}\}$.
- The power set of the singleton set $\{a\}$ contains $2^1 = 2$ elements $\{\{\ \}, \{a\}\}$.
- The power set of the set $\{a, b\}$ contains $2^2 = 4$ elements $\{\{\ \}, \{a\}, \{b\}, \{a, b\}\}$.

If we add one element c to the set $\{a, b\}$, then we double the number of its subsets because the former subsets remain as they were and, in addition, have to be taken with c added.

Representing all reals of the unit interval by infinite bit sequences like $t = 0.11101000110..._2$, we can construct a bijection between subsets s of $\mathbb{N}$ and infinite bit-sequences t mapped on s. We interpret the nth bit 1 of the sequence t as indicating that the natural number n belongs to the subset s whereas the nth bit 0 indicates that n does not belong to s. For example put

$\{1\} \leftrightarrow 0.1_2$
$\{2, 3, 4, ...\} \leftrightarrow 0.0111..._2$
$\{1, 3, 4\} \leftrightarrow 0.1011_2$
$\{\text{all even natural numbers}\} \leftrightarrow 0.010101..._2$
$\{1, 3, 4, 6, 8, 14, 17\} \leftrightarrow 0.10110101000001001_2$.

This bijection shows that $\mathbb{R}$ and $\mathcal{P}(\mathbb{N})$ have same cardinal number. Since $\mathbb{N}$ has $\aleph_0$ elements, $\mathcal{P}(\mathbb{N})$ has $|\mathcal{P}(\mathbb{N})| = 2^{|\mathbb{N}|} = 2^{\aleph_0}$ elements. This is called the cardinality of the continuum and is denoted also by C. In case we use decimal digits, we obtain $|\mathcal{P}(\mathbb{N})| = 10^{|\mathbb{N}|} = 10^{\aleph_0}$. This results in $2^{\aleph_0} = 10^{\aleph_0}$.

2.5 Higher infinite cardinal numbers

In his 1891 paper [Cantor, p. 280f] Cantor also proved that there are higher infinities. Try to set up a bijection between the points of the unit interval and the set F of all real functions $f(x)$

$y \leftrightarrow f_y(x)$

where $y \in [0, 1]$.

In order to get a picture assume functions like

$f_0(x) = x$
$f_{1/n}(x) = 4x \cdot n + 4711$ (for $n \in \mathbb{N}$)
$f_{2/3}(x) = 1234$ if $x \in \mathbb{Q}$, otherwise $1/x$
$f_{1/\sqrt{2}}(x) = \sin x + 1/\cos\sqrt{x}$
$f_L(x) = 1/x!$
$f_{1/\pi}(x) = e^{e^{\tan x}}$
$f_{e-\sqrt{3}}(x) = 0$ if $1/x$ is prime, otherwise $x^{15/8}$

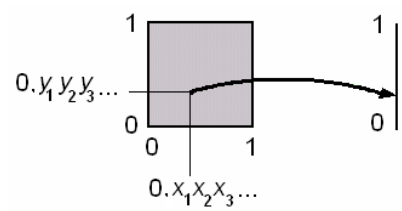
with $L = \Sigma 1/10^{\nu!} = 0.110001...$ being Liouville's number (cp. sect. 2.2.1). Consider the function $f_y(x)$, related to y, calculate its value at position $x = y$, namely $f_y(y)$, and increase it by 1, resulting in $f_y(y) + 1$. Combine all the values obtained in that way to create the function $g(y) = f_y(y) + 1$.

This is a function in the real interval [0, 1] which differs from every $f_y(x) \in F$ at least in one point, namely for $x = y$. So there is no bijection; the set F has a cardinality $|F| > C$. (That the set F of all real functions has *at least* cardinality C is shown by its subset of functions $f_y(x) = y$.)

Since $|\mathcal{P}(S)| = 2^{|S|}$ we get, by always taking the power set of an infinite set, an infinite sequence of infinitely increasing infinite cardinal numbers $\aleph_0$, $2^{\aleph_0}$, $2^{2^{\aleph_0}}$, $2^{2^{2^{\aleph_0}}}$, ... In contrast to the finite case however, 0, 1, 2, 3, 2^2, 5, 6, 7, 2^3, 9, 10, 11, 12, 13, 14, 15, 2^{2^2}, 17, ..., there are no cardinal numbers accessible in between (cp. section 3.2).

2.6 Cardinality of multidimensional continua

Cantor found yet another surprising result. He proved that the area of a square or the volume of an n-dimensional cube contains as many points as one of its edges, i.e. the real interval. In a letter to Dedekind he wrote "Je le vois, mais je ne le crois pas" {{I see it but I can't believe it}}. [G. Cantor, letter to R. Dedekind (29 Jun 1877)]



A bijection between the square $[0, 1]^2$ and the Interval [0, 1] can be constructed by merging the Cartesian coordinates $x = 0.x_1x_2x_3...$ and $y = 0.y_1y_2y_3...$ of a point of the square $(x \mid y) \in [0, 1]^2$ according to the scheme

$$(0.x_1x_2x_3... \mid 0.y_1y_2y_3...) \to 0.x_1y_1x_2y_2x_3y_3... .$$

The point $(x \mid y) = (0.111 \mid 0.222)$ of the unit square, for example, is mapped on the point 0.121212 of the unit interval. Of course, same can be accomplished with more dimensions.

This mapping has a disadvantage however. Of two coordinates like $x = 0.1000...$ and $x = 0.0999...$ only one must be used because the fusion with the other coordinate must yield only one result, either $0.1y_10y_20y_30y_4...$ or $0.0y_19y_29y_39y_4...$. An undoubtedly bijective mapping however can be accomplished with continued fractions. Another possibility, according to an idea of König, is the dissection of the digit sequences always till the next digit which differs from zero. An example is

$$(0.30005709... \mid 0.04685...) \leftrightarrow 0.3040005678095... .$$

The mapping between sets of different dimension is never continuous though; two very closely neighbouring points in one set can be mapped on very distant points in the other set. But that does not spoil the equicardinality of the sets.

The above example can obviously be extended to every number of dimensions by merging many coordinates or by repeating the process. Therefore the unit cube (and, as is easily demonstrated, also the unit sphere) contain as many points as the unit interval.

A bijection from (0, 1] to [0, ∞) can be accomplished by the mapping $1/r \leftrightarrow (r - 1)$ for $1 \le r < \infty$. Inversion of all three coordinate axes of the unit sphere yields a sphere with infinite radius. So the tiniest linear interval contains as many points as the infinite space.

2.7 Cardinal arithmetic

Arithmetic in the actual infinite obeys the following rules:

$$\aleph_0 = \aleph_0 + n = \aleph_0 + \aleph_0 = n\cdot\aleph_0 = \aleph_0\cdot\aleph_0 = \aleph_0{}^n \quad \text{for every } n \in \mathbb{N}\ .$$

But contrary to the potential infinite of calculus we have:

$$\aleph_0 < 2^{\aleph_0} = 10^{\aleph_0} = n^{\aleph_0} = \aleph_0{}^{\aleph_0} \quad \text{for every natural number } n > 1\ .$$

Using a convention introduced by Hausdorff [F. Hausdorff: "Grundzüge der Mengenlehre", Veit, Leipzig (1914); reprinted: Chelsea Publishing Company, New York (1965) p. 69] we put as an abbreviation $2^{\aleph_0} = \aleph$ and get

$$1 = 1^{\aleph_0} = 1^{\aleph}$$
$$\aleph = \aleph + n = \aleph + \aleph_0 = \aleph + \aleph = n\cdot\aleph = \aleph_0\cdot\aleph = \aleph\cdot\aleph = \aleph^n = \aleph^{\aleph_0} \quad \text{for every } n \in \mathbb{N}\ .$$

Larger is nothing before

$$2^{\aleph} = 10^{\aleph} = n^{\aleph} = \aleph_0{}^{\aleph} = \aleph^{\aleph} \quad \text{for every natural number } n > 1\ .$$

The equicardinality of different continua can also be shown by cardinal arithmetic. The plane has the same cardinality as the axis because of

$$\aleph_0 = \aleph_0 + \aleph_0 \quad\Rightarrow\quad 2^{\aleph_0} = 2^{\aleph_0+\aleph_0} = 2^{\aleph_0}\cdot 2^{\aleph_0} \quad\Rightarrow\quad \aleph\cdot\aleph = \aleph\ .$$

Instead of only two $\aleph$ we can also multiply countably many $\aleph$ with the same result.

The continuum hypothesis assumes that there is no cardinal number between $\aleph_0$ and $\aleph$. So $\aleph$ can be named $\aleph_1$. It is impossible to prove this or the contrary from the axioms of ZFC (cp. sect. 3.2).

An application is the addition of all natural numbers and other divergent infinite series. [G. Cantor, letter to G. Mittag-Leffler (10 Feb 1883) & (3 Mar 1883)][1]

$$1 + 2 + 3 + ... = \aleph_0$$
$$1 + 1/2 + 1/3 + ... = \aleph_0\ .$$

[1] Cantor wrote ω instead of $\aleph_0$ because in 1883 he had not yet invented cardinal numbers.

2.8 Inaccessible cardinal numbers

The cardinality $|\mathcal{P}(M)|$ of the power set $\mathcal{P}(M)$ of the set M is always greater than the cardinality $|M|$ of the set itself:

$$|\mathcal{P}(M)| = 2^{|M|} > |M| \ .$$

Gathering the results of an infinite process of power set construction and constructing a sequence

$$\mathcal{M} = M, \mathcal{P}(M), \mathcal{P}(\mathcal{P}(M)), \mathcal{P}(\mathcal{P}(\mathcal{P}(M))), ...$$

the limit or union[1] $\cup\mathcal{M}$ has a larger cardinality $|\cup\mathcal{M}|$ than every element of $\mathcal{M}$

$$|\cup\mathcal{M}| > |M|, 2^{|M|}, 2^{2^{|M|}}, ... \ .$$

This cardinality can be surpassed if we construct a sequence from $\cup\mathcal{M}$ and its power sets

$$\mathcal{S} = \cup\mathcal{M}, \mathcal{P}(\cup\mathcal{M}), \mathcal{P}(\mathcal{P}(\cup\mathcal{M})), ... \ .$$

The limit or union $|\cup\mathcal{S}|$ is again greater than all cardinalities considered so far – and so we can continue.

A cardinal number K is called *inaccessible* if it can neither be obtained by a sum nor by a product of less than K cardinal numbers $\kappa < \mathrm{K}$ nor by a power κ^λ of cardinal numbers $\kappa, \lambda < \mathrm{K}$. Examples are 0, 2 (not 1 which is $0^0 = 1$), and $\aleph_0$.

An infinite cardinal number R is called *regular*, if it is greater than every sum of less than R summands which all are less than R. An example is $\aleph_0$, because it cannot be written as a finite sum over finite summands. Another example is $\aleph_1$ since (with the axiom of choice, cp. 2.12.9) the union of a countable set of countable cardinal numbers is countable.

If a regular cardinal number $\mathrm{G} > \aleph_0$ always satisfies the condition

$$\mathrm{G} > X \ \Rightarrow \ \mathrm{G} > 2^X$$

then it is called *strongly inaccessible*.

If strongly inaccessible cardinal numbers exist, then already the smallest one G_0 must be very large, namely larger than $|\cup(\mathcal{S})|$ and all cardinal numbers constructible from it. And by means of power set and union as shown above, even larger cardinal numbers G_1, G_2, ... can be constructed. In the Zermelo-Fraenkel axiom system (cp. section 2.12) the existence of inaccessible cardinal numbers can neither be proved nor be disproved.

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[1] The union of two sets, $M_1 \cup M_2$, is the set containing all elements of M_1 and M_2. For the infinite union, henceforth denoted by $\cup$, the symbol is prepended to the general term: $\cup M_k = M_1 \cup M_2 \cup M_3 \cup ...$.

2.9 Ordinal numbers

A set is *well-ordered* if every non-empty subset contains a first element with respect to the order. The set $\mathbb{N}$ of all natural numbers is well-ordered, if the natural "smaller than" relation is applied as the order relation, because $1 < 2 < 3 < ...$. The sets (1, 1/2, 1/3, ...) and (*a*, *aa*, *aaa*, ...) are also well-ordered by obvious orderings. The set of all positive rational numbers can be well-ordered (cp. section 2.1) but is not well-ordered by size because there is no smallest rational number larger than a given rational number. Same is true for the algebraic numbers. Also the set of all integers can be well-ordered, for instance as 0, 1, -1, 2, -2, 3, -3, ..., but is not well-ordered by size because the whole set does not contain a first (smallest) element.

A well-ordered set that can be written as a sequence, in particular it has a first and no last element, has the order type ω. The ordered set $\mathbb{N}$ itself is an example: $\mathcal{O}\mathbb{N} = \mathcal{O}\{1, 2, 3, ...\} = \omega$. To indicate that order plays a role, sets are usually not included in curly brackets (because they are destroying any order) but in parentheses.[1] Then the order symbol $\mathcal{O}$ can be dropped and we write briefly $(1, 2, 3, ...) = \omega$. [Cantor, p. 299]

Order types of well-ordered sets like $\mathcal{O}\mathbb{N}$ are called *ordinal numbers*. All ordinal numbers, like all cardinal numbers, are in trichotomy with each other, i.e., for two ordinal numbers *a* and *b* always one and only one of the relations is satisfied: $a < b$ or $a = b$ or $a > b$. [Cantor, p. 321]

Pairwise combining the elements of two sets with ordinal number ω each, yields one set with ordinal number ω, i.e., an infinite sequence of pairs

$$(1, a, 2, b, 3, c, ...) = 2\cdot\omega = \omega \; .$$

Combining them as whole sets yields another ordinal number, namely two infinite sequences

$$(1, 2, 3, ..., a, b, c, ...) = \omega\cdot 2 \neq \omega \; .$$

Now an inner element without predecessor exists. $\omega\cdot 2 \neq 2\cdot\omega$ shows that ordinal multiplication is not commutative.

The ordinal numbers of (0, 1, 2, 3, ...) and (1, 2, 3, ..., 0) are also different. The second set has a last element, the first set does not: $(0, 1, 2, 3, ...) = 1 + \omega = \omega$ while $(1, 2, 3, ..., 0) = \omega + 1 \neq \omega$. All above sets have the cardinal number $|\mathbb{N}| = \aleph_0$ because the cardinal numbers of sets are not changed by reordering.

The set of all real numbers of the interval [0, 1] ordered by size or, as Cantor [Cantor, p. 310f] called it, the linear continuum $X = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$, is an *order type* but not an *ordinal number*. It is characterized by two properties. (1) In every ε-neighbourhood, however small, of a number of X there are uncountably many numbers of X. (2) X contains a countable set of rationals q such that between every two elements x_1 and x_2 of X there is at least one rational number q.

[1] In pure set theory ordered pairs (x, y) are expressed after Kuratowski by $\{x, \{x, y\}\}$. This method can be extended inductively to triples, $(x, y, z) = \{x, \{x, \{y, \{y, z\}\}\}\}$, quadruples, and so on.

2.10 Number classes

Two ordered sets are *similar* to each other, if there exists a bijection such that the order of the elements is preserved: $a \prec b \Leftrightarrow f(a) \prec f(b)$. [Cantor, p. 297] They have the same order type. That kind of bijection is called an isomorphism. *Similarity* implies *equivalence*, i.e., *equicardinality* or *equinumerosity*. To every order type γ of a well-ordered set, i.e., to every *ordinal number* γ there is a corresponding cardinal number. The cardinal numbers which correspond to the *transfinite numbers* of the sequence Ω of ordinal numbers were called "Alefs" by Cantor [Cantor, p. 292] (today we write "Alephs"), denoted by the first letter $\aleph$ of the Hebrew alphabet.

The system of all ordinal numbers γ belonging to one and the same cardinal number $\aleph(\gamma)$ is called "number class". In every number class there exists a smallest number γ_0, and there exists a number γ_1 outside, such that for every γ of this number class we have $\gamma_0 \leq \gamma < \gamma_1$.

"*The system* Ω *in its natural order constitutes a 'sequence'*. Adding 0 at the first place we get a sequence $\Omega' = 0, 1, 2, 3, ..., \omega, \omega + 1, ..., \gamma, ...$ {{Cantor wrote ω_0}} where we easily convince ourselves that *every* number γ appearing there is the *type* of the *sequence of all its preceding elements*. [...] *The system* Ω *of all numbers is an inconsistent, an absolutely infinite multitude.* [...] Every number class is a certain 'section' of the sequence Ω. [...] Certain numbers of the system Ω form *their own* number class *each*, these are the '*finite*' numbers 1, 2, 3, ..., ν, ... corresponding to the different 'finite' cardinal numbers". [Cantor, p. 445] These numbers are taken together as the *first number class*. All ordinal numbers α with cardinal number $\aleph_0$ belong to the second number class $Z(\aleph_0)$. They obey the condition $\omega \leq \alpha < \omega_1$ where ω with $|\omega| = \aleph_0$ is the smallest transfinite number and ω_1 is the smallest uncountable transfinite number, the cardinal number of which is not $\aleph_0$ but $\aleph_1$. Some ordinal numbers of the beginning of the sequence (with $k, m, n \in \mathbb{N}$) are:

$$0, 1, 2, 3, ..., \omega, \omega + 1, ..., \omega + k, ..., \omega + \omega\ (= \omega\cdot 2), \omega\cdot 2 + 1, ..., \omega\cdot k, ..., \omega\cdot k + m, ..., \omega\cdot\omega\ (= \omega^2), \omega^2 + 1, ..., \omega^2 + \omega, ..., \omega^2 + \omega\cdot k + m, ..., \omega^2\cdot 2, ..., \omega^2\cdot k + \omega\cdot m + n, ..., \omega^3 + \omega^2\cdot k + \omega\cdot m + n, ..., \omega^k, ..., \omega^\omega, \omega^\omega + 1, ..., \omega^\omega\cdot k, ..., \omega^{\omega+1}, \omega^{\omega+1} + 1, ..., \omega^{\omega\cdot k}, ..., \omega^{\omega^2}, ..., \omega^{\omega^\omega}, ..., \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}\ (= \varepsilon_0), \varepsilon_0 + 1, ..., \varepsilon_0^{\varepsilon_0}, ..., \varepsilon_0^{\varepsilon_0^{\varepsilon_0}}, ..., \varepsilon_0^{\varepsilon_0^{\varepsilon_0^{\cdot^{\cdot^{\cdot}}}}}\ (= \varepsilon_1), \varepsilon_1 + 1, ..., \varepsilon_1^{\varepsilon_1^{\varepsilon_1^{\cdot^{\cdot^{\cdot}}}}}\ (= \varepsilon_2), ..., \omega_1, ...$$

All the ordinal numbers 2^ω $(= \omega)$, ω^ω $(> \omega)$, ω^{ω^ω}, ε_0, ε_1, ... less than ω_1 belong to the second number class because they are countable. A representation of 2^ω is the set of all pairs of natural numbers. A model of ω^ω is the set of all finite sequences of natural numbers or, according to Hessenberg [G. Hessenberg: "Grundbegriffe der Mengenlehre", offprint from Abhandlungen der Fries'schen Schule, Vol. I, no. 4, Vandenhoeck & Ruprecht, Göttingen (1906) § 20], the ordering of the natural numbers by the number of prime factors and then by sizes of the factors.

$\varepsilon_0 = \lim(\omega, \omega^\omega, \omega^{\omega^\omega}, ...)$ has the property $\omega^{\varepsilon_0} = \varepsilon_0$; it is the first ordinal number which cannot be represented in finite notation, starting off with ω and using addition, multiplication and exponentiation; therefore a new name is necessary. Cantor called it the smallest giant of the second number class. [G. Cantor, letter to F. Goldscheider (11 Oct 1886)] Cantor wrote γ_1.

2.11 Cantor's construction of the natural numbers

Cantor was the first to construct the natural numbers on a set theoretic basis. [Cantor, p. 289f. G. Cantor: "Beiträge zur Begründung der transfiniten Mengenlehre 1", Math. Annalen 46 (1895) § 5] He promised to supply "the most natural, shortest, and strictest foundation of the finite number theory". To a single thing e_0, if we take it by the notion of a set $E_0 = (e_0)$, there corresponds as a cardinal number, that what we call "one". We have $1 = |E_0|$. Now combine another thing e_1 with E_0. The sum set be called E_1, such that $E_1 = (E_0, e_1) = (e_0, e_1)$. Note that Cantor does not yet correctly use the singleton $((e_0), e_1)$. The cardinal number $|E_1|$ of E_1 is called "two" and is denoted by $2 = |E_1|$. By adding new elements we obtain the sequence of sets $E_2 = (E_1, e_2)$, $E_3 = (E_2, e_3)$, ... which in an unbounded sequence successively supply the other so-called *finite cardinal numbers* denoted by 3, 4, 5, ... The auxiliary employment of the defined numbers as indices in the definition is justified by the fact that a number is used as an index only after it has been defined:

$$\nu = |E_{\nu-1}| \qquad E_\nu = (E_{\nu-1}, e_\nu) = (e_0, e_1, ..., e_\nu) \qquad |E_\nu| = |E_{\nu-1}| + 1 \ .$$

Every finite cardinal number (with the exception of 1) is the sum of its predecessor and 1. All terms of the unbounded sequence of all finite cardinal numbers 1, 2, 3, ..., ν, ... are different. Every number is larger than its predecessors and smaller than its successors. There is no number between ν and $\nu + 1$.

2.12 ZFC-axioms of set theory

Zermelo-Fraenkel (ZF) set theory, often including the axiom of choice (ZFC), is the only axiomatization of set theory that appears attractive also outside of the narrow world of set theorists. The axioms create the *Cumulative Hierarchy*: From a given set of sets (in fact only the empty set is necessary) other sets can be constructed and from those other sets without disturbing any of the predecessors. This creates a hierarchy by accumulating sets.

The details of axioms and notation change from author to author[1], but the nine axioms explained below are very frequently applied.

The axioms are neither independent of each other nor do they present the shortest formulation of the foundations. They have been selected by aspects of convenience. The axiom of separation for instance follows from the axiom of replacement together with the axiom of empty set. If A is a set, then the axiom of replacement guarantees the existence of the set B for the function f: $A \to B$, e.g. the set of squares by the function f: $n \to n^2$. The axiom of separation allows to separate the same set of squares from the set of natural numbers. The empty set follows from the axiom of infinity. Nevertheless often the additional axiom

[1] Often sets are distinguished by capital letters from elements denoted by lower case letters. Since in ZFC there are no "urelemente" but "everything is a set" and can be an element of another set, the following formal statements will contain only capital letters besides the logical symbols.

2.12.0 Axiom of empty set

is added: There exists an empty set

$$\exists X \, \forall Y \, (Y \notin X) \, .$$

The empty set is denoted by $\emptyset$ or by { } or by $\{X \mid X \neq X\}$.

2.12.1 Axiom of extensionality

The first axiom defines the extension of a set, its external properties. Every set is defined by its elements and by nothing else. This axiom is an attempt to formalize Cantor's definition of a set as "every collection of definite well-distinguished objects of our visualization or thinking".

If sets A and B have the same elements X, then $A = B$:

$$\forall A \, \forall B \, (\forall X \, (X \in A \Leftrightarrow X \in B) \Rightarrow A = B) \, .$$

Sets are completely defined by their elements. Lists or graphs are extensional objects in contrast to intensional objects like formulas or programs. The latter "explain" why an element belongs to a set. The former are arbitrary collections without any hint *why* an element has been chosen.

2.12.2 Axiom of separation

This axiom is also called axiom of subsets or axiom of restricted[1] comprehension: If the formula φ describes a predicate, i.e., a property (with parameter p), then there exists (for every parameter p) a set $B = \{X \in A \mid \varphi(X, p)\}$ that contains all $X \in A$ that have the property (described by) φ.

$$\forall A \, \forall p \, \exists B \, \forall X \, (X \in B \Leftrightarrow (X \in A \wedge \varphi(X, p))) \, .$$

As an example consider the set A of all cars X and the property φ of a car having a colour, for instance being red (p). The set B of red cars exists; it is a subset of A.

The axiom is also called an axiom schema because in first-order logic it is not possible to quantify over formulas or functions. Therefore every predicate φ requires its own axiom of separation.[2] In Zermelo's original work [E. Zermelo: "Untersuchungen über die Grundlagen der Mengenlehre I", Math. Ann. 65 (1908) 261-281] this had not yet been distinguished.

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[1] Restricted comprehension collects in B only the elements X of the given, i.e., already existing set A with predicate φ. Unrestricted comprehension collects all suitable objects satisfying the predicate φ. This has lead to Russell's paradox or better Russell's antinomy (cp. section 3.1.3).

[2] To put it in other words, every predicate has to be enumerated. This is possible because there are only countably many finite expressions which can serve as predicates.

2.12.3 Axiom of pairing

For any sets A and B there exists a set $S = \{A, B\}$ that contains A and B as elements and nothing else.

$$\forall A\ \forall B\ \exists S\ \forall X\ (X \in S \Leftrightarrow (X = A \lor X = B))\ .$$

Alternatively: Every pair of elements can be considered as a set. In case of $A = B$, the set containing the "pair" is called a singleton $\{A\}$.

2.12.4 Axiom of union

also called axiom of sum set. For any set A there exists the union $B = \bigcup A$ of all elements of A.

$$\forall A\ \exists B\ \forall Y\ (Y \in B \Leftrightarrow \exists X\ (Y \in X \land X \in A))\ .$$

Since every element of A is a set X, the elements Y of these sets can be put together into set B. They are elements of elements of A and elements of B.

2.12.5 Axiom of power set

For any set A there exists the set $B = \mathcal{P}(A)$ of *all* subsets of A.

$$\forall A\ \exists B\ \forall X\ (X \subseteq A \Leftrightarrow X \in B)\ .$$

All subsets of A are *elements* of its power set $\mathcal{P}(A)$.

2.12.6 Axiom of infinity

There exists an infinite set S. S contains the empty set $\emptyset$ and with X also the successor of X.

$\exists S\ (\emptyset \in S \land (X \in S \Rightarrow \{X\} \in S))$ [1] (Zermelo's version of successorship).

According to Zermelo his axiom of infinity goes back to Dedekind (see chapter I). Instead of Zermelo's successor-definition today usually von Neumann's version is applied because of its easy construction of the natural numbers as initial segments $n = \{0, 1, 2, ..., n - 1\} \subset \mathbb{N}$.

$\exists S\ (\emptyset \in S \land (X \in S \Rightarrow (X \cup \{X\}) \in S))$ (von Neumann's version of successorship).

---

[1] Usually the formulas $\exists S\ (\emptyset \in S \land \forall X\ (X \in S \Rightarrow \{X\} \in S))$ or $\exists S\ (\emptyset \in S \land \forall X\ (X \in S \Rightarrow (X \cup \{X\}) \in S))$ are given, but inserting the additional universal quantifier here is not necessary; $X \in S \Rightarrow P$ has the same truth value as $\forall X\ (X \in S \Rightarrow P)$ since the former is true if $X \notin S$ and, given the latter, also if $X \in S$, hence for all X. The rule that "for axioms" a universal quantifier has to be prepended does not apply here.

In both versions the minimal set S can be identified with the finite cardinal numbers, i.e., $\mathbb{N}_0$, the natural numbers including zero. Constructed from Zermelo's version we get

$$
\begin{aligned}
&0 = \{\,\} = \emptyset \\
&1 = \{\{\,\}\} = \{\emptyset\} = \{0\} \\
&2 = \{\{\{\,\}\}\} = \{\{\emptyset\}\} = \{\{0\}\} = \{1\} \\
&3 = \{\{\{\{\,\}\}\}\} = \{\{\{\emptyset\}\}\} = \{\{\{0\}\}\} = \{\{1\}\} = \{2\} \\
&\ldots\,.
\end{aligned}
$$

The cardinal numbers constructed from von Neumann's version are

$$
\begin{aligned}
&0 = \{\,\} = \emptyset \\
&1 = \{\,\} \cup \{\{\,\}\} = \emptyset \cup \{\emptyset\} = \{\emptyset\} = \{0\} \\
&2 = \{\,\} \cup \{\{\,\}\} \cup \{\{\,\} \cup \{\{\,\}\}\} = \emptyset \cup \{\emptyset\} \cup \{\emptyset \cup \{\emptyset\}\} = \{\emptyset, \{\emptyset\}\} = \{0, 1\} \\
&3 = \{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0, 1, 2\} \\
&\ldots\,.
\end{aligned}
$$

2.12.7 Axiom of replacement

If F is a function, then for any set A there exists a set $B = F[A] = \{F(X) \mid X \in A\}$.

$$\forall X\, \forall Y\, \forall Z\, (\varphi(X, Y) \land \varphi(X, Z) \Rightarrow Y = Z) \Rightarrow \forall A\, \exists B\, \forall Y\, (\exists X\, ((X \in A) \land \varphi(X, Y)) \Leftrightarrow Y \in B)\,.$$

The left-hand side shows that in the two-valued predicate φ for every X there is only one Y as is required in functions. The right-hand side shows that B is the image of A under φ.

Example: Let $\varphi(X, Y)$ be the function $Y = 2X$. If also $Z = 2X$, we have $Y = Z$. From this premise the existence of B follows, such that for every element Y of B an element X of A exists with the property $Y = 2X$ and vice versa.

The axiom of replacement facilitates the construction of sets which cannot be constructed by other axioms, for instance $\{\mathbb{N}, \mathcal{P}(\mathbb{N}), \mathcal{P}(\mathcal{P}(\mathbb{N})), \mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N}))), \ldots\}$ with the function $F(n) = \mathcal{P}^n(\mathbb{N})$. $\omega + \omega = \omega \cdot 2$ is already an example, because the axiom of infinity only guarantees the existence of ω. The axiom of replacement guarantees that the function $F(n) = \omega + n$ generates a set.

Like the axiom of separation the axiom of replacement is an axiom schema.

2.12.8 Axiom of foundation

or axiom of regularity: Every nonempty set has an $\in$-minimal element.

$$\forall S\, (S \neq \emptyset \Rightarrow \exists X \in S\colon (S \cap X) = \emptyset)\,.$$

No set can be an *element* of itself. As an example consider the set $\{X\}$ which has only the element X. $X \in \{X\}$ but $X \notin X$. This implies the difference between "contained as an element" and "contained as a subset". Always $X \subseteq X$ and $\emptyset \subseteq X$ but never $X \in X$ and rarely $\emptyset \in X$. Since the intersection of two sets A and B contains only all *elements* which are as well in A as in B, the intersection $\{X\} \cap X$ is empty, since, whatever elements X may have, there is no element in X which is also in $\{X\}$ as an element, i.e., as the only element X of $\{X\}$.

This axiom guarantees that every nonempty set S contains an element X which has no *element* in common with S. It excludes the formation of sets which contain themselves, in order to avoid Russell's antinomy (cp. section 3.1.3). There is no infinite sequence (S_k) of sets S_k such that each $S_{k+1} \in S_k$. There is no set that contains a set that contains a set that ... is the set. Such a set would not be contradicted by the other axioms. The set of all abstract notions is an abstract notion too, so it contains itself. "Since {{this axiom}} is not essential for mathematics, it cannot be regarded as fundamental by the traditional axiomatic attitude." [A.A. Fraenkel, Y. Bar-Hillel, A. Levy: "Foundations of set theory", 2nd ed., Elsevier, Amsterdam (1973) p. 89]

2.12.9 Axiom of choice

Every set T of nonempty sets X has a choice function F.

$$\forall T\, (\emptyset \notin T \Rightarrow (\exists F\colon T \to \bigcup T \wedge \forall X \in T\colon F(X) \in X))$$

The axiom of choice is the most controversial axiom of mathematics. It has been introduced, first in the form of so-called "γ-sets", in 1904. [E. Zermelo: "Beweis, daß jede Menge wohlgeordnet werden kann, Math. Ann. 59 (1904) pp. 514-516] In 1908 Zermelo specified: "Let T be a set of nonempty sets which have no common elements, then the union $\bigcup T$ contains at least one subset S which has with every element of T one and only one element in common." It is always possible to *choose* from every element of T one element and to union the chosen elements in a set S. [E. Zermelo: "Untersuchungen über die Grundlagen der Mengenlehre I", Math. Ann. 65 (1908) p. 266]

The axiom of choice is equivalent to the assertion that for every set of nonempty sets the Cartesian product is not empty. Let $\mathbb{I}$ be an index set. The Cartesian product of the set $\{X_i \mid i \in \mathbb{I}\}$ of nonempty sets X_i contains all ordered $\mathbb{I}$-tuples of elements of the sets X_i. If the Cartesian product is not empty (this could be the case for uncountable sets in the absence of the axiom of choice) every $\mathbb{I}$-tuple can serve as the choice funtion.

Example: Let $\mathbb{I} = \{1, 2\}$, $X_1 = \{a, b, c\}$, $X_2 = \{1, 2\}$. Then the Cartesian product is given by the pairs $\{(a, 1), (b, 1), (c, 1), (a, 2), (b, 2), (c, 2)\}$.

2	$a, 2$	$b, 2$	$c, 2$
1	$a, 1$	$b, 1$	$c, 1$
	a	b	c

Example: Let $\mathbb{I} = \mathbb{N}$, $X_i = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 0\}$ for all $i \in \mathbb{N}$. Then the Cartesian product is the set of all $\mathbb{N}$-tuples, i.e., all infinite digit sequences.

Example: Let $\mathbb{I} = \{1, 2, 3\}$, $X_1 = X_2 = X_3 = \mathbb{R}$. Then the Cartesian product is the set of all triples of real numbers, i.e., the Cartesian coordinates of the points of the space $\mathbb{R}^3$.

The axiom of choice is further equivalent to Zorn's Lemma: If a partially ordered set S has the property that every chain[1] has an upper bound in S, then the set S contains at least one maximal element[2]. Without this feature, every element would have another next element. This successorship would not stop at natural indices but would run through all ordinal numbers. Alas "all ordinal numbers" do not exist as a set. There is no set that could contain them all. Contradiction. (The axiom of choice comes into the play when choosing the successor, the other next element.)

These nine ZFC-axioms facilitate the elementary operations intersection, difference, and union of sets. According to the axiom of separation there exists for every set $A = \{X \mid X \in A\}$ a subset the elements of which simultaneously belong to set B, $\{X \mid X \in A \wedge X \in B\} = A \cap B$, and a subset the elements of which do not to belong to B, $\{X \mid X \in A \wedge X \notin B\} = A \setminus B$. The axiom of pairing makes of two sets A and B the set $\{A, B\}$. The axiom of union or sum set builds from these the set that contains all elements of the pair $\{X \mid X \in A \vee X \in B\} = A \cup B$.

It is remarkable that contrary to Cantor's naive set theory there is not explanation of what a set *is* but merely that sets exist and when two sets are the same set. The latter is accomplished by the axiom 1 of extensionality. The set of all negative natural numbers, the set of all prime numbers which are divisible by their square, and the set of all real roots of the equation $x^2 + 1 = 0$ are identical. According to the axiom of extensionality there is only one single empty set.[3]

After fixing the identity of sets in this way, axiom 2 of separation allows to diminish a set by forming a subset, and axioms 3, 4, and 5 show how to enlarge sets by pairing, summing, and power set operation. Axiom 2 is a restricted version of Cantor's definition of set and Gottlob Frege's axiom of unrestricted comprehension $M_\varphi = \{X \mid \varphi(X)\}$ which have lead to the Russell-antinomy (cp. section 3.1.3): X can have any definable property φ.

Two of the axioms are controversial: The axiom of infinity and the axiom of choice. The former, because it is assumed to establish the existence of an actually infinite set, the latter because it leads to some paradoxes or antinomies in case of uncountable sets (cp. chapters III and VI).

The refusal of different degrees of infinity cannot be contradicted, "the attitude of the (neo-) intuitionists that there do not exist altogether non-equivalent infinite sets is consistent, though almost suicidal for mathematics," [A.A. Fraenkel, A. Levy: "Abstract set theory", North Holland, Amsterdam (1976) p. 62] According to Hilbert the axiom of choice rests on a logical principle of general validity which already is indispensable for the first steps of mathematical concluding. [D.

[1] A chain is a totally ordered subset, i.e., besides transitivity $A \subseteq B \wedge B \subseteq C \Rightarrow A \subseteq C$ any two elements of the chain satisfy $A \subseteq B$ or $B \subseteq A$. An example is the chain of finite initial segments $(1, 2, 3, ..., n)$ of the sequence of natural numbers

[2] An element $M \in S$ is called maximal if there is no other element $X \in S$ such that it follows in order upon M.

[3] Therefore the habit to automatically exclude the empty set often appears as meaningless. For instance the statement *every non-empty set of real numbers can be well-ordered* is inappropriate, since there cannot be an empty set of real numbers. Every set of real numbers is not empty. Every empty set is empty of all elements including real numbers. To emphasize that it is empty of real numbers only is not useful let alone necessary.

Hilbert: "Die logischen Grundlagen der Mathematik", Math. Ann. 88 (1923) p. 152] "In the opinion of *Russell* and *Whitehead* these axioms are not logically provable and can be accepted or rejected as well, according to subjective discretion." [A. Fraenkel: "Einleitung in die Mengenlehre", 2nd ed., Springer, Berlin (1923) p. 182] Since the axiom of choice is logically independent of the other ZF-axioms, there is an analogy to Euclid's parallel axiom. Non-Cantorian set theories have become possible like non-Euclidean geometries.

2.13 Well-ordering theorem

The axiom of choice is further equivalent to the well-ordering theorem. The possibility of well-ordering of sets is essential because only well-ordered sets can be compared with respect to their ordinal number. Either both have the same ordinal number or one is an initial segment of the other. Zermelo produced in 1904 a *proof that every set can be well-ordered.*[1] [E. Zermelo: "Beweis, daß jede Menge wohlgeordnet werden kann", Math. Ann. 59 (1904) pp. 514-516]

Before that it was usual to argue as follows: From the set A to be well-ordered take by arbitrary choice an element and denote it as a_0, then from the set $A \setminus \{a_0\}$ an element a_1, then an element from the set $A \setminus \{a_0, a_1\}$ and so on. If the set $\{a_0, a_1, a_2, ...\}$ is not yet the complete set A, we can choose from $A \setminus \{a_0, a_1, a_2, ...\}$ an element a_ω, then an element $a_{\omega+1}$, and so on. This procedure must come to an end, because beyond the set W of ordinal numbers which are mapped on elements of A, there are greater numbers; these obviously cannot be mapped on elements of A.

This naiveté is reported as late as in 1914 by Felix Hausdorff, obviously without reservations because he remarks: "We cannot share most of the doubts which have been raised against this method." Hausdorff only deplores the undesired impression of a *temporal* process but confirms that the element a_ω is fully determined in the sense of transfinite induction (see section 2.14) and claims that every single action of choosing an element as well as their order has to be understood as timeless. "In order to support this timeless approach E. Zermelo has got the lucky idea to choose from the scratch from *every* non-empty subset A' of A one of its elements $a' = f(A')$, such that we do no longer have to wait until it is the turn of A' but for every set, whether or not it will come up, an element is available prae limine. The system of successive choices has been replaced by a system of simultaneous choices which in practical thinking of course is as unfeasible." [F. Hausdorff: "Grundzüge der Mengenlehre", Veit, Leipzig (1914); reprinted: Chelsea Publishing Company, New York (1965) p. 133f]

The first to point out that Cantor's method is blatantly wrong was Adolf Fraenkel. Zermelo called it a "well-known primitive attempt" [Cantor, p. 352]. Hausdorff remained convinced of its truth.

"With respect to such a procedure *Cantor* has called the well-ordering theorem a 'fundamental and momentous and by its general truth particularly remarkable law of thinking'; [...] But he has not given a *proof*. The above train of thought cannot be considered as a proper – not even halfway strict – proof, in particular because in no way it is shown that [...] the given set can really be *exhausted*. The inadmissibility of this method as a proof becomes obvious from the following:

[1] This is a direct translation. Zermelo does not claim that there *exists* a well-ordering of every set but he claims that *it can be done*!

It seems not only to establish the *possibility* of well-ordering but to give a *real way* how this can be accomplished for every set. That is contradicted by the fact [...] that the real *construction* of a well-ordering until today has not even been accomplished with certain simplest uncountable sets." [A. Fraenkel: "Einleitung in die Mengenlehre", 2nd ed., Springer, Berlin (1923) p. 141f]

Zermelo's proof is this [E. Zermelo: "Beweis, daß jede Menge wohlgeordnet werden kann, Math. Ann. 59 (1904) pp. 514-516]:

Let M be any set of cardinality $|M|$, the elements of which may be denoted by m. Let M' be any subset of cardinality $|M'|$ which contains at least one element m but may contain also all elements of M. The set of all subsets M' is called $\mathcal{M}$ (here Zermelo addresses the power set without the empty set). By M - M' the complement of M' in M is denoted.

To every M' may be attached an arbitrary element m'_1 which is a member of M'. It is called the distinguished element of M'. In this way we get a *covering* (Belegung) γ of the set $\mathcal{M}$ of a special kind. The number of coverings γ is equal to the product $\Pi m'$ over all subsets M' and therefore is not 0.

Example: Let $M = \{a, b\}$. Then there are $1 \cdot 1 \cdot 2 = 2$ coverings of $\mathcal{M} = \{\{a\}, \{b\}, \{a, b\}\}$, namely a, b, a and a, b, b.

Example: Let $M = \{a, b, 1, 2, 3, ...\}$ then $a, b, a, 1, 1, 1, ..., 2, 2, 2, ..., 3, 3, 3, ...$ is a possible covering. As distinguished elements of $\{a\}$, $\{b\}$, $\{a, b\}$ choose a, b, a respectively; for all subsets M' containing 1 this 1 is chosen as distinguished element, for all subsets not containing 1 but containing 2 this 2 is chosen; for all subsets not containing 1 and 2 but containing 3 this 3 is chosen and so on. Since the subsets M' are not empty, there is always an element that can be chosen as distinguished element.

Now we choose an arbitrary covering in order to derive a certain well-ordering γ of the elements of M.

Definition: A "γ-set" is every *well-ordered* set M_γ of *different* elements of M with the following property: If a is an arbitrary element of M_γ and A is the corresponding segment that consists of the preceding elements $x \prec a$ of M_γ, then a is always the distinguished element of M - A.

There are γ-sets in M. The distinguished element m_1 of $M' = M$ is a γ-set itself (here Zermelo does not yet distinguish between an element m_1 and a singleton $\{m_1\}$), and similarly is the (ordered) set $M_2 = (m_1, m_2)$, where m_2 is the distinguished element of M - m_1.

Example: The γ-set of $M = \{a, b, 1, 2, 3, ...\}$ is $M_\gamma = (a, b, 1, 2, 3, ...)$.

Zermelo concludes: If M'_γ and M''_γ are any two different γ-sets (which however belong to the fixed covering γ, chosen once and for all) then always one of them is identical with a segment of the other. In both cases m_1 is the distinguished element of M since the corresponding segment A does not contain an element: M - A = M. If there existed a *first* element m' of M'_γ that differed from the corresponding element m'' of M''_γ, the segments A' and A'' must be identical and with them also the complements M - A' and M - A'' and because of identical covering also their distinguished elements m' and m'' such that m' cannot differ from m''.

Finally every element of M appears in the γ-set because for every element there exists a subset M' that contains only this element.

Concluding, to every covering γ corresponds a well-ordering of the set M, although different coverings can result in the same well-ordering. So there must exist at least one well-ordering.

Two sets none of which is equivalent to an initial segment of the other cannot be compared by size. This case and therewith the possible incomparability is excluded by the well-ordering theorem which is a direct consequence of the axiom of choice. It cannot be proven without this axiom. Two well-ordered sets are always comparable. Either they are equivalent or one is equivalent to an initial segment of the other.

Without the axiom of choice it cannot even be excluded that the set of real numbers is a countable set of countable sets. "Feferman and Levy showed that one cannot prove that there is any non-denumerable set of real numbers which can be well ordered; [...] Moreover, they also showed that the statement that the set of all real numbers is the union of a denumerable set of denumerable sets cannot be refuted." [A.A. Fraenkel, Y. Bar-Hillel, A. Levy: "Foundations of set theory", 2nd ed., Elsevier, Amsterdam (1973) p. 62]

If a set is well-ordered then it contains no, by order, strictly decreasing infinite sequence, because such a sequence has no smallest element by definition. An example is the set $\mathbb{Z}$ of integers. In its natural order (..., -3, -2, -1, 0, 1, 2, 3, ...) it contains the reversed sequence (-n). Of course $\mathbb{Z}$ can be well-ordered, for instance as (0, -1, 1, -2, 2, -3, 3, ...), where also the subsequence of negative integers is ascending in order.

2.14 Transfinite induction

In order to prove that a property or theorem P is valid for every natural number, we use *mathematical induction*. First we have to show that P is true for 1. Then we have to show that if P is true for the natural number n (without fixing any natural number) then it is also true for the natural number $n + 1$. If both steps are valid, then P is true for 1 by the first step, P is true for 2 by choosing $n = 1$, P is true for 3 by choosing $n = 2$, and so on. Since we can reach every natural number by this unbounded sequence of steps, P is true for every natural number.

Example: Prove the formula $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

In the first step verify $\sum_{k=1}^{1} k = 1$.

In the second step prove that if the formula is valid for n, then it is also valid for $n + 1$ (= m).

$$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2} \Rightarrow \sum_{k=1}^{m} k = \frac{m(m+1)}{2} .$$

This technique has been extended as *transfinite induction* to transfinite ordinal numbers. It holds only for well-ordered sets, but by Zermelo's well-ordering theorem (cp. 2.13) every set can be well-ordered. A property P that is defined for all ordinal numbers can be proven, analogously to the method of mathematical induction on natural numbers, for all ordinal numbers although the ordinal numbers do not form a set (cp. chapter III). The property P is proven in three steps. First P must be shown to hold for 0. Second, from the truth of P for the ordinal number ξ it must be possible to conclude that P holds for the ordinal number $\xi + 1$. And third, the truth of P for *all ordinal numbers* ξ of a section must be shown to imply the truth of P for the limit ordinal number completing this section.

As an example consider Cantor's first application of transfinite induction in order to determine the power function in the second number class [G. Cantor: "Beiträge zur Begründung der transfiniten Mengenlehre 2", Math. Annalen 49 (1897) § 18. Cantor, p. 336ff]: Let ξ be a variable of the first or second number class including 0. Let $\gamma > 1$ be a constant.

Theorem There is only one completely determined function γ^ξ satisfying the conditions:

(1) $\gamma^0 = 1$.
(2) If $\xi' < \xi''$ then $\gamma^{\xi'} < \gamma^{\xi''}$.
(3) For every ξ: $\gamma^{\xi+1} = \gamma^\xi\gamma$.
(4) If (ξ_ν) is an arbitrary fundamental sequence, then (γ^{ξ_ν}) is a fundamental sequence too and for $\xi = \mathrm{Lim}_\nu\xi_\nu$ we have $\gamma^\xi = \mathrm{Lim}_\nu\gamma^{\xi_\nu}$.

Proof for the natural numbers. According to (1) and (3) we have $\gamma^1 = \gamma$, $\gamma^2 = \gamma\gamma$, $\gamma^3 = \gamma\gamma\gamma$, ... and because of $\gamma > 1$ it follows $\gamma^1 < \gamma^2 < \gamma^3 < ... < \gamma^\nu < \gamma^{\nu+1} < ...$. Therefore the function is completely determined for $\xi < \omega$.

Proof for the second number class: Assume that the theorem is true for all $\xi < \alpha$ where α is some number of the second number class. Then it holds also for $\xi \le \alpha$. If α has a predecessor α_1, then (3) supplies $\gamma^\alpha = \gamma^{\alpha_1}\gamma > \gamma^{\alpha_1}$. If α has no predecessor (being a limit ordinal like ω) and (α_ν) is a fundamental sequence with $\mathrm{Lim}_\nu\alpha_\nu = \alpha$, then it follows from (2) that (γ^{α_ν}) is a fundamental sequence too and from (4) that $\gamma^\alpha = \mathrm{Lim}_\nu\gamma^{\alpha_\nu}$. Considering another fundamental sequence (α'_ν) such that $\mathrm{Lim}_\nu\alpha'_\nu = \alpha$, then these fundamental sequences are corresponding, i.e., they have the same limit, such that $\gamma^\alpha = \mathrm{Lim}_\nu\gamma^{\alpha_\nu} = \mathrm{Lim}_\nu\gamma^{\alpha'_\nu}$ is uniquely determined.

In case $\alpha' < \alpha$ obviously $\gamma^{\alpha'} < \gamma^\alpha$. So conditions (2), (3), (4) are satisfied for $\xi \le \alpha$ too, and the theorem is valid for all values of ξ. If there were exceptions, then one of them was the smallest, call it α, such that the theorem was valid for all $\xi < \alpha$ but not for $\xi \le \alpha$, in contradiction with the proof.

2.15 Goodstein sequences

The expansion of a number like 13 in base 2 is

$$13 = 2^3 + 2^2 + 2^0 = 2^3 + 2^2 + 1\ .$$

Expressing the exponents larger than 2 also in base 2, we obtain the *pure expansion* in base 2:

$$13 = 2^{(2+1)} + 2^2 + 1\ .$$

The expansion in base 3 would already be pure:

$$13 = 3^2 + 3 + 1\ .$$

The Goodstein sequence

$$G(n) = n_1, n_2, n_3, ...$$

of a natural number n evolves when in the pure base-2 expansion every 2 is replaced by 3 and the resulting number is decreased by 1. Then in the pure base-3 expansion every 3 is replaced by 4 and the resulting number is decreased by 1. And so on. [R.L. Goodstein: "On the restricted ordinal theorem", Journal of Symbolic Logic 9,2 (1944) pp. 33-41]

Example: The first term of the sequence $G(2)$

$$n_1 = 2 = 2^1$$

supplies the second term

$$n_2 = 3^1 - 1 = 2$$

and the third term

$$n_3 = 2^1 - 1 = 1$$

because the base 3, to be replaced by 4 in the next step, is not existing. The fourth term is $n_4 = 0$ where the sequence ends by definition. The Goodstein sequence of 2 is

$$G(2) = 2, 2, 1, 0\ .$$

Example: The first term of the sequence $G(13)$

$$n_1 = 2^{(2+1)} + 2^2 + 1 = 13$$

supplies the second term

$$n_2 = 3^{(3+1)} + 3^3 + 1 - 1 = 3^{(3+1)} + 3^3 = 108 .$$

The next step nibbles at the right exponent

$$n_3 = 4^{(4+1)} + 4^4 - 1 = 4^{(4+1)} + 3\cdot 4^3 + 3\cdot 4^2 + 3\cdot 4^1 + 3 = 1279 .$$

After three further steps the 3 has been used up, and 3·8 becomes 2·8 + 7:

$$n_4 = 5^{(5+1)} + 3\cdot 5^3 + 3\cdot 5^2 + 3\cdot 5^1 + 2$$
$$n_5 = 6^{(6+1)} + 3\cdot 6^3 + 3\cdot 6^2 + 3\cdot 6^1 + 1$$
$$n_6 = 7^{(7+1)} + 3\cdot 7^3 + 3\cdot 7^2 + 3\cdot 7^1$$
$$n_7 = 8^{(8+1)} + 3\cdot 8^3 + 3\cdot 8^2 + 3\cdot 8^1 - 1 = 8^{(8+1)} + 3\cdot 8^3 + 3\cdot 8^2 + 2\cdot 8^1 + 7 .$$

Great numbers n create rapidly increasing sequences. Nevertheless Goodstein's theorem says that every sequence $G(n)$ will terminate at 0 after a finite number of steps. As soon as the base has become larger than the number, nothing remains to be replaced, and repeated subtraction of 1 pulls the sequence to 0.

For proof replace the base immediately by ω.

$$n_1 = 2^{(2+1)} + 2^2 + 1$$

then becomes

$$t_1 = \omega^{(\omega+1)} + \omega^\omega + 1 .$$

The sequence

$$G(t) = t_1, t_2, t_3, t_4, ...$$

of transfinite ordinal numbers, which is a *majorante* (i.e., an upper bound) of $G(n)$, is strictly monotonically decreasing because in every step 1 is subtracted whereas base ω, being already strictly greater than any natural number appearing in $G(n)$, is not increased.

$$t_2 = \omega^{(\omega+1)} + \omega^\omega$$

is followed by

$$t_3 = \omega^{(\omega+1)} + \omega^\omega - 1 .$$

How can we accomplish that? It is a general rule, forced by the well-foundedness (see 2.12.8) of the sequence of ordinal numbers, that every strictly decreasing sequence of ordinal numbers reaches its smallest element after a finite number of steps. Since limit ordinals have no direct predecessors, we have to jump down from them to some predecessor.

2.16 Set-theoretical limits of sequences of sets

A supremum of a sequence (S_n) of sets S_n contains all elements which are in the union of all sets of the sequence beginning with index n: $\bigcup_{k=n}^{\infty} S_k$. Limit superior is the intersection of all suprema, i.e., the smallest supremum

$$\text{LimSup}\, S_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} S_k \ .$$

An infimum of a sequence (S_n) of sets S_n contains all elements which are in the intersection of all sets of the sequence beginning with index n: $\bigcap_{k=n}^{\infty} S_k$. Limit inferior is the union of all infima, i.e., not less than any infimum

$$\text{LimInf}\, S_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} S_k \ .$$

A sequence (S_n) of sets S_n has a limit, Lim S_n for $n \to \infty$, if and only if

$$\text{LimSup}\, S_n = \text{LimInf}\, S_n = \text{Lim}\, S_n \ .$$

Example 1: $S_n = \{n\}$, Lim $\{n\} = \{\ \}$.
$\bigcup_{k=n}^{\infty} S_k = \{n, n+1, n+2, ...\}$. The intersection of all unions is empty since no natural number is in all unions but for every natural number there is a last union containing it.
$\bigcap_{k=n}^{\infty} S_k = \{n\} \cap \{n+1\} \cap \{n+2\} \cap ... = \{\ \}$ and so is the union of all intersections.

Example 2: $S_n = \{1/n\}$, Lim $\{1/n\} = \{\ \}$.
$\bigcup_{k=n}^{\infty} S_k = \{1/n, 1/(n+1), 1/(n+2), ...\}$. The intersection of all unions is empty since no unit fraction is in all unions but for every unit fraction there is a last union containing it.
$\bigcap_{k=n}^{\infty} S_k = \{1/n\} \cap \{1/(n+1)\} \cap \{1/(n+2)\} \cap ... = \{\ \}$ and so is the union of all intersections.

Example 3: $S_n = \{1, 2, 3, ..., n\}$, Lim $\{1, 2, 3, ..., n\} = \{1, 2, 3, ...\} = \mathbb{N}$.
$\bigcup_{k=n}^{\infty} S_k = \{1, 2, ..., n\} \cup \{1, 2, ..., n+1\} \cup \{1, 2, ..., n+2\} \cup ... = \mathbb{N}$. The intersection of all unions $\mathbb{N}$ is $\mathbb{N}$.
$\bigcap_{k=n}^{\infty} S_k = \{1, 2, ..., n\} \cap \{1, 2, ..., n+1\} \cap \{1, 2, ..., n+2\} \cap ... = \{1, 2, ..., n\}$. The union of all intersections is $\mathbb{N}$.

Example 4: $S_n = \{n, n+1, n+2, ..., 2n\}$, Lim $\{n, n+1, n+2, ..., 2n\} = \{\ \}$.
$\bigcup_{k=n}^{\infty} S_k = \{n, n+1, ..., 2n\} \cup \{n+1, n+2, ..., 2n+2\} \cup \{n+2, n+3, ..., 2n+4\} \cup ... = \{n, n+1, ...\}$. The intersection of all unions is $\{\ \}$.
$\bigcap_{k=n}^{\infty} S_k = \{n, n+1, ..., 2n\} \cap \{n+1, n+2, ..., 2n+2\} \cap \{n+2, n+3, ..., 2n+4\} \cap ... = \{\ \}$ and so is the union of all intersections.

Example 5: The sequence of real intervals $S_n = [0, 1\text{-}1/n]$ has the limit Lim $[0, 1\text{-}1/n] = [0, 1)$.
$\bigcup_{k=n}^{\infty} S_k = [0, 1\text{-}1/n] \cup [0, 1\text{-}1/(n+1)] \cup [0, 1\text{-}1/(n+2)] \cup ... = [0, 1)$. The intersection is $[0, 1)$.
$\bigcap_{k=n}^{\infty} S_k = [0, 1\text{-}1/n] \cap [0, 1\text{-}1/(n+1)] \cap [0, 1\text{-}1/(n+2)] \cap ... = [0, 1\text{-}1/n]$, yielding the union $[0, 1)$.
The point 1 is neither in intersections nor in unions.

Example 6: The sequence of real intervals $S_n = [\text{-}1/n, 0]$ has the limit point Lim $[\text{-}1/n, 0] = [0]$.
$\bigcup_{k=n}^{\infty} S_k = [\text{-}1/n, 0] \cup [\text{-}1/(n+1), 0] \cup [\text{-}1/(n+2), 0] \cup ... = [\text{-}1/n, 0]$. The intersection is $[0]$.
$\bigcap_{k=n}^{\infty} S_k = [\text{-}1/n, 0] \cap [\text{-}1/(n+1), 0] \cap [\text{-}1/(n+2), 0] \cap ... = [0]$, yielding the union $[0]$.

Example 7: The sequence of real intervals $S_n = [(\text{-}1)^n/n, 1]$ has no limit. LimSup $S_n \neq$ LimInf S_n.
$\bigcup_{k=n}^{\infty} S_k = [(\text{-}1)^n/n, 1] \cup [(\text{-}1)^{n+1}/(n+1), 1] \cup [(\text{-}1)^{n+2}/(n+2), 1] \cup ... = [\text{-}1/n, 1]$ or $[\text{-}1/(n+1), 1]$.
The intersection of all unions does not contain any unit fraction -1/n but 0, therefore it is $[0, 1]$.
$\bigcap_{k=n}^{\infty} S_k = [(\text{-}1)^n/n, 1] \cap [(\text{-}1)^{n+1}/(n+1), 1] \cap [(\text{-}1)^{n+2}/(n+2), 1] \cap ... = (0, 1]$. Since the intersections never contain 0 their union is $(0, 1] \neq [0, 1]$.

2.17 Partial models of ZFC

"Is there a model of ZFC? A consequence of Gödel's Second Incompleteness Theorem is that one cannot hope to *prove* the existence of a model of ZFC working just from the axioms of ZFC." [W.H. Woodin "The continuum hypothesis", Part I, Notices of the AMS (2001) p. 568]

A theory is a set of (first-order) axioms formulated in a language made up from symbols, strings, or words for relations, functions, and constants. (The language of set theory consists of the relations "equality, =" and "contained as an element, $\in$".) A model of the theory is a structure, i.e., a collection of objects with interpretations for each symbol that satisfy all axioms of the theory. A group like $(\mathbb{Z}, +)$ is a model of the group axioms, and a set like $\mathbb{N}$ is a model of the axiom of infinity. A model satisfying all axioms of ZFC, $\mathcal{M} \vDash$ ZFC, is unknown (a model of ZFC would prove the consistency of ZFC and *if shown in* ZFC would violate Gödel's incompleteness theorem) but we can construct partial models of ZFC, i.e., models that satisfy some of the axioms described in section 2.12; henceforth they are briefly called models of ZFC.

A model containing only well-founded sets, i.e., sets not being members of themselves, and the relation $\in$ of ordinary set membership is called a *standard model*. It is a submodel of the universe of all sets which is too large to be a set (cp. section 3.1). A model is called an *inner model* of ZFC if it contains all the ordinal numbers of the von Neumann universe V (cp. section 2.17.1) and has no sets beyond those in V.

"Well-founded sets are sets that are built up inductively from the empty set, using operations such as taking unions, subsets, powersets, etc. Thus the empty set { } is well-founded, as are {{ }} and the infinite set {{ }, {{ }}, {{{ }}}, ...}. They are called 'well-founded' because the nature of their inductive construction precludes any well-founded set from being a member of itself. We emphasize that if M is standard, then the elements of M are not amorphous 'atoms', as some of us envisage the elements of an abstract group to be, but are *sets*. Moreover, well-founded sets are *not* themselves built up from 'atoms'; it's 'sets all the way down'. [...] A standard model M of ZFC is *transitive* if every member of an element of M is also an element of M. (The term *transitive* is used because we can write the condition in the suggestive form '$x \in y$ and $y \in M$ implies $x \in M$'.) [...] a concept in V is *absolute* if it coincides with its counterpart in M. For example, 'the empty set', 'is a member of', 'is a subset of', 'is a bijection', and '$\aleph_0$' all turn out to be absolute for standard transitive models. On the other hand, 'is the powerset of' and 'uncountable' are *not* absolute. For a concept that is not absolute, we must distinguish carefully between the concept 'in the real world' (i.e., in V) and the concept in M." [Timothy Y. Chow: "A beginner's guide to forcing", arXiv (2008) p. 4ff]

2.17.1 The von Neumann universe V

The universe V is the class of hereditary well-founded sets, the transfinite hierarchy of sets. It was described in [John von Neumann: "Über eine Widerspruchsfreiheitsfrage in der axiomatischen Mengenlehre", Journal für die reine und angewandte Mathematik 160 (1929) pp. 227-241].

Every stage of the hierarchy contains the power set of the preceding stage, starting with the empty set: $V_0 = \emptyset$, $V_1 = \{\emptyset\}$, $V_2 = \{\emptyset, \{\emptyset\}\}$, $V_3 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$.

Since the power set of M contains $2^{|M|}$ elements (cp. section 2.4) V_4 contains 16, V_5 contains 2^{16}, and V_6 contains already $2^{2^{16}} = 2^{65536}$ elements, i.e., much more than there are atoms in the universe (cp. chapter VII), and therefore cannot be written down or distinguished.

The general definition is given by transfinite recursion

$$V_0 = \emptyset \qquad V_{\alpha+1} = \mathcal{P}(V_\alpha) \qquad V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$$

where α is an ordinal number and λ is a limit ordinal number. The index is also called rank of the stage. Since for $\alpha < \beta$, $V_\alpha \subset V_\beta$ the universe is also called cumulative hierarchy.

For every set S there is a first stage with $S \subseteq V_\alpha$ and $S \in V_{\alpha+1}$. This α is the rank of the set S. V_ω is the set of hereditarily finite sets, i.e., all its elements are finite. V_ω is a model of ZFC without the axiom of infinity. It contains only well-founded sets but not sets like, for instance, the set of abstract notions or the set of elephants.

2.17.2 Gödel's constructible universe L

Gödel shows that the axiom of choice (AC, cp. section 2.12.9) and the generalized continuum hypothesis (GCH, cp. section 3.2.4) are consistent with the usual axiom system of ZF (cp. section 2.12) if this is consistent. [Kurt Gödel: "The consistency of the continuum hypothesis", Princeton University Press, Princeton (1940), reprinted by Ishi Press, New York (2009)]

The primitive notions are *class* $\mathfrak{Cls}$, *set* $\mathfrak{M}$, and the diadic *element-of-relation* ε, which is applied to both sets and classes. [loc cit p. 3] Gödel uses the logical symbols

$$(X), (\exists X), \sim, \cdot, \vee, \supset, \equiv, =, (E!X)$$

denoting respectively: for all X, there is an X, not, and, or, implies, equivalence, identity, there is exactly one X. [loc cit p. 2] Gödel uses uppercase letters only for classes and lowercase letters for sets, no italics, of course, because the paper is typewritten, and lots of Greek and German letters. Elements of sets and classes are not separated by commas. Henceforth we will adopt modern notation and use italic capital letters for sets and classes.

Gödel uses the special axiom system Σ essentially taken from Bernays. [loc cit p. 7] The axiom of extensionality is assumed for sets and classes. A proper class (that is: not a set) cannot be an element of a set.

From the axiom of foundation $A \neq \emptyset \Rightarrow \exists X\colon (X \in A \wedge (X \cap A) = \emptyset)$ Gödel concludes $\neg(X \in X)$ and $\neg(X \in Y \wedge Y \in X)$. [loc cit p. 6]

Further we note the presence of the axiom of power set $\forall X\, \exists Y\colon U \subseteq X \Rightarrow U \in Y$. [loc cit p. 5]

In order to describe the structure of his *constructible model* Δ Gödel needs ordered tuples defined in the usual way [loc cit p. 4]

$$\langle X, Y\rangle = \{\{X\}, \{X, Y\}\}, \quad \langle X, Y, Z\rangle = \langle X, \langle Y, Z\rangle\rangle, \quad \langle X_1, X_2, ..., X_n\rangle = \langle X_1, \langle X_2, ..., X_n\rangle\rangle \, .$$

Although $X \neq \{X\} = \{X, X\}$, Gödel defines for the ordered set $X = \langle X\rangle$.

A *primitive propositional function* (ppf) is "a meaningful formula containing only variables, symbols for special classes $A_1, ..., A_k$, ε, and logical operators, and such that all bound variables are *set* variables." [loc cit p. 8] Gödel proves the *General Existence Theorem*: If $\varphi(X_1, X_2, ..., X_n)$ is a ppf containing no other variables than $X_1, X_2, ..., X_n$ then there exists a class A such that for any *sets* $X_1, X_2, ..., X_n$

$$\langle X_1, X_2, ..., X_n\rangle \in A \Leftrightarrow \varphi(X_1, X_2, ..., X_n) \, .$$

After defining ordinal numbers and cardinal numbers, Gödel explains the fundamental operations like pairset, difference, function from A to B, etc. for his *constructible model* Δ with the class L of *constructible sets*. The operation intersection is left out because $X \cap Y = X \setminus (X \setminus Y)$. [loc cit p. 35]

A class A is constructible if all its elements are constructible sets and if the intersection of A with any constructible set is a constructible set. [loc cit p. 38] Operations, notions, and special sets and classes defined for Σ can be *relativized*, i.e. applied to the model Δ (sets and classes appropriately marked by horizontal bars but leaving the logical symbols as they stand). The axioms of Σ hold for Δ. An object that exists in Δ as well as in Σ is called absolute. The relations $\in$, $\subseteq$, $\subset$ are absolute as well as the empty set $\emptyset$, the class L of constructible sets, ordered tuples and other functions of constructible sets. Only V itself and the power set are not absolute. [loc cit p. 42ff]

Summarizing, Gödel constructs his universe L by starting with the empty set too, but contrary to von Neumann extending the model on subsequent stages by the following rules: Only subsets of the previous stages α may be used that are definable by a first-order formula in the language of ZFC with parameters from the previous stage and quantifiers ranging over the previous stage

$$L_0 = \emptyset \qquad L_{\alpha+1} = \{X \subseteq L_\alpha \mid X \text{ definable in } L_\alpha\} \qquad L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$$

where α is an ordinal number and λ is a limit ordinal number. $\emptyset$ is constructible [loc cit p. 40] and therefore absolute [loc cit p. 44], but since by the axiom of infinity the model must contain the minimal inductive set ω we could have started from L_ω instead of L_0 as well.

Gödel's universe $L = \bigcup_\alpha L_\alpha$ where α extends over the class of all ordinals, is an inner model. It is part of every model of ZFC because only constructible sets are contained (as mentioned above L is absolute). It satisfies most axioms but not the power set axiom. Not all subsets of a set in L are in L (but only such that are in L). Some subsets of a set in L are missing. Otherwise the model would not be constructible by formulas, according to ZFC, also from "outside", i.e., in the mathematics of our world.

2.18 Forcing

The basic idea, due to Cohen [P.J. Cohen: "The independence of the continuum hypothesis". Proc. Nat. Acad. Sciences, USA 50 (1963) pp. 1143-1148 & 51 (1964) pp. 105-110], is to start from a (necessarily partial) model $\mathcal{M}$ of ZFC and to add a so-called generic set G such that the model $\mathcal{M}[G]$ satisfies some desired properties, for instance to violate the continuum hypothesis. In order to increase the legibility of the different texts collected in this section, we will write $\mathcal{M}[G]$ without exception, although some cited texts use a or X instead of G. This is in accordance with the full scale of modern treatments of this topic from [Tim Chow: "Forcing for dummies", sci.math.research (10 Mar 2001)] to [Nik Weaver: "Forcing for mathematicians", World scientific (2014)].

After Cohen's approach we will refer to an alternative and easier accessible version of forcing, developed by Scott, Solovay, and Vopenka, which proceeds via Boolean algebra. An example of forcing is concluding this section.

2.18.1 The discovery of forcing

Listening to Cohen's own narration of his discovery, with some abridgements and some additional explanations, may be a good outset. (Here $M(a)$ has been replaced by $\mathcal{M}[G]$.)

"So we are starting with a countable standard model $\mathcal{M}$, and we wish to adjoin new elements and still obtain a model. An important decision is that no new ordinals are to be created." Cohen adjoins elements already in $\mathcal{M}$, like sets of integers, and G. "I called such an element a 'generic' element." What set G may be added such that $\mathcal{M}[G]$ is still a model? Since the model is countable, there is a countable ordinal I larger than all ordinals of $\mathcal{M}$. "Since I is countable, it can be expressed as a relation on the integers and hence coded as a set G of integers. Now if by misfortune we try to adjoin this G to $\mathcal{M}$, the result cannot possibly be a model for ZF. For if it were, the ordinal I as coded by I would have to appear in $\mathcal{M}[G]$. However, we also made the rigid assumption that we were going to add *no new ordinals*. This is a contradiction, so that $\mathcal{M}[G]$ cannot be a model." It turns out that the set G cannot be determined completely. Since $\mathcal{M}[G]$ is constructed by transfinite induction on ordinals, the definition of what is meant by saying 'G is generic' must also be given by a transfinite induction. "The answer is this: the set G will not be determined completely, yet properties of G will be completely determined on the basis of very incomplete information about G." How can we find out whether a statement about G is true? There are properties which have to be defined, like the presence of a certain number in G, and properties which are true in general, mainly because they are implications like 'if G contains the empty set, then G is not empty' or 'if G contains ω then G is infinite'. "Now the definition of truth is obvious. It is done by induction on the number of quantifiers" reducing their number step by step. Elementary statements which cannot be possibly reduced must be taken as given. "An elementary statement (or forcing condition) is a finite number of statements of the form n in G, or n not in G, which are not contradictory. It is plausible to conjecture that, whatever the definition of truth is, it can be decided by our inductive definition from the knowledge of a finite number of elementary statements. This is the notion of *forcing*. If we denote the elementary conditions by P, we must now define the notion 'P forces a statement S'. The name, *forcing*, was chosen so as to draw the analogy with the usual concept of implication, but in a new sense."

In a countable model every question can be decided in sequential order, but "the enumeration would be done outside the model $\mathcal{M}$, and so one had to be sure that there was no contradiction in both working in and out of the model." Consider only statements S which have a *rank* bounded by some ordinal. "All our sets and variables are actually functions of the 'generic' set G. So in analogy with field theory, we are actually dealing with the space of all (rational) functions of G, not actual sets." An elementary statement P is a finite set of statements n in G, or not n in G (which are not contradictory) when these are contained in P. "Suppose a statement begins with 'there exists' a set x of rank less than α, such that $A(x)$ holds. If we have an example of a set (actually a function of G) such that P does force $A(x)$, clearly we have no choice (forced) to say that P forces 'there exists ...'. *Emphatically not*. For it may very well be that we shall later find an elementary condition which does force the existence of such an x. So we must treat the two quantifiers a bit differently. Now we must reexamine something about our elementary conditions. If P is such, we must allow the possibility that we shall later make further assumptions about the set G, which *must be consistent with P*. This means that we are using a natural partial ordering among these conditions. We say $P < Q$, if all the conditions of P are contained in Q. That is, Q is further along in determining the final G. This leads to a formal definition of forcing which I give here in a somewhat abbreviated form:

(a) P forces 'there exists x, $A(x)$' if, for some x with the required rank, P forces $A(x)$.

(b) P forces 'for all x, $A(x)$' if no $Q > P$ is such that Q forces the negation, i.e., for some y, Q forces not $A(y)$."

In order to construct G we need a complete sequence of P such that every statement S is forced by some P_k. This is possible since G is completely determined by statements n in G.

Cohen proved the lemma: "*For all P and S, there is an extension Q of P such that Q forces either S or Q forces not S*. [...] It follows that a complete sequence exists. Now if P_n is a complete sequence, for each integer k the statement k in G, or k not in G, must be forced by some

P_n. Thus it is easy to see that G is determined by P_n." Finally there is a truth lemma: "*Let P_n be a complete sequence. A statement S is true in $\mathcal{M}[G]$ if and only if some P_n forces S.*"

When trying to show that $\mathcal{M}[G]$ is a model one encounters basic differences with Gödel's result. If x is in $\mathcal{M}[G]$ it must be guaranteed, that all its subsets occur before some ordinal β of $\mathcal{M}$ since "we are dealing with ordinals of $\mathcal{M}$. Yet G is not in $\mathcal{M}$, so we cannot discuss x as a set, but only as designated by the ordinal α, a function of G. To work inside $\mathcal{M}$, we consider the set of P which forces a given set to lie in G or not lie in G. Because forcing is defined in $\mathcal{M}$, we can look at all possibilities of assigning sets of P, which force the members of x to lie in an arbitrary y. This set is the 'truth value' of the statement. So, in ordinary set theory a subset of x is determined by a two-valued function on the members of x. In our situation, a subset is determined by a function taking its values in the subset of the elementary conditions. *These values are all in the model* $\mathcal{M}$. Thus we can quantify over all possible truth values and, by a simple argument, show that any subset of x occurs before some ordinal β which is in $\mathcal{M}$. The other axioms are proved in essentially the same manner.

Now we have a method for constructing interesting new models. How do we know that G is a 'new' set not already contained in $\mathcal{M}$? A simple argument shows that, for any G' in $\mathcal{M}$ and P, we can force G to be not equal to G' by choosing any n, not already determined by P, and simply extending P by adding n in G, or n not in G, to prevent G from being equal to G'. In this way we see that G is not constructible, and so we have a model with a nonconstructible set of integers."

In order to violate the *continuum* hypothesis *distinct* sets of integers G_i are adjoined to $\mathcal{M}$ where $i < \aleph_2$. Alas, $\aleph_2$ belongs to $\mathcal{M}$, but the statement of CH is that the cardinality of the continuum is the *first* uncountable cardinal. It is therefore necessary to show that $\aleph_2$ in $\mathcal{M}$ is the *second* uncountable cardinal in the new model. However, "one can show that if two ordinals have different cardinality in $\mathcal{M}$, they will have different cardinality in the new model. There is an important fact about the elementary conditions which is responsible for it. This is the *countable chain condition*." A partially ordered set (abbreviated as poset) is said to satisfy this condition if every set of mutually incompatible elements (see section 2.18.2) is countable. The proof of this offers no particular difficulty. [Paul J. Cohen: "The discovery of forcing", Rocky Mountain Journal of Mathematics 32,4 (2002) pp. 1071-1100]

2.18.2 Boolean forcing

This section is based upon the following literature: [Kenneth Kunen: "Set theory – an introduction to independence proofs", Elsevier (2006). Thomas Jech: "The axiom of choice", North Holland (1973). Thomas Jech: "Set theory – the third millenium edition", Springer (2006). Rowan Jacobs: "Forcing" (2011). J. Donald Monk: "Lectures on set theory" (2017). Clive Newstead: "Boolean-valued models and forcing" (2012). Timothy Y. Chow: "A beginner's guide to forcing", arXiv (2008)] In order to have a common notation all sets will be denoted by capital letters and the model by $\mathcal{M}$.

A Boolean valued model has the advantage that no complete sequences need be chosen. It contains the desired new set which may be uncountable since this set is not constructed but only "approximated" by a countable poset. So the desired new set cannot be proved to be constructible and hence countable. We proceed as follows (the technical terms are explained further below):

- A countable poset is constructed to "approximate" the desired new (uncountable) set.
- Take a filter on this poset and take the union of its elements to construct the new set.
- Construct a Boolean ($\mathcal{B}$) valued model of ZFC containing the new set.
- The filter on the poset is used to form an ultrafilter on the Boolean algebra $\mathcal{B}$A.
- Convert with it the $\mathcal{B}$ valued model into a {0, 1}-valued model containing the new set.

We start with a transitive[1] model $\mathcal{M}$ inside the von Neumann universe or with the whole von Neumann universe V (cp. 2.17.1). Sometimes $\mathcal{M}$ is called the ground model. If X is a definable element of V, then $X^{\mathcal{M}}$ is the corresponding set in $\mathcal{M}$, i.e., X is an element of the underlying set M, $X \in M$, and $\mathcal{M}$ is a model of X.

A forcing order or *notion of forcing* is a triple $(P, \leq, 1)$ in $\mathcal{M}$ such that $\leq$ is reflexive and transitive (but not necessarily antisymmetric) on the nonempty poset P. 1 is the largest element: $\forall X \in P$: $X \leq 1$. The elements are called *forcing conditions*. If $X \leq Y$ then X is said to be stronger than Y; X is a function extending Y; X imposes more restrictions than Y, i.e., $X \wedge Y = X$. 1 imposes the restrictions defined by the empty set $\emptyset$, namely none. Two elements X, $Y \in P$ are *compatible* if there exists $Z \in P$ with $Z \leq X, Y$, i.e., Z is stronger than X and Y. A set of pairwise incompatible elements of P is called an *antichain*.

A subset $D \subseteq P$ is dense, if for every $X \in P$ there is $Y \in D$ with $Y \leq X$. For each $X \in P$ abbreviate all stronger forcing conditions by $P{\downarrow}X = \{Z \mid Z \leq X\}$.
A filter F on a poset P is a nonempty subset of P which

- is upwards closed, i.e., for all $X \in F$, if $X \leq Y$ (i.e., $X \wedge Y = X$) then $Y \in F$
- and if X, $Y \in F$ then exists $Z \in F$ extending both X, Y.

If this filter F intersects every dense set in the ground model $\mathcal{M}$, F is called $\mathcal{M}$-generic.
For any $X \in P$ there exists a filter F containing X.

A Boolean algebra is a poset M with a minimum element 0 and a maximum element 1, in which every element has a complement and every pair of elements has a least upper bound and a greatest lower bound. The $\mathcal{B}$ algebra has the operators OR, AND, NOT which act upon the elements of the set M. The following axioms hold for all X, Y, $Z \in M$:

X OR Y = Y OR X	X AND Y = Y AND X
X OR (Y OR Z) = (X OR Y) OR Z	X AND (Y AND Z) = (X AND Y) AND Z
X OR (Y AND Z) = (X OR Y) AND (X OR Z)	X AND (Y OR Z) = (X AND Y) OR (X AND Z)
X OR (X AND Y) = X	X AND (X OR Y) = X
X OR (NOT X) = 1	X AND (NOT X) = 0 .

The axioms at the right-hand side are the *dual forms*, obtained by exchanging OR and AND, 0 and 1 (as well as S and NOT S, but the latter sets have been renamed S in the axioms). Often instead of (M, OR, AND, NOT, 0, 1) algebraic, logical, or special operators are used such that the $\mathcal{B}$ algebra is represented by $(M, +, \cdot, -, 0, 1)$ or $(M, \vee, \wedge, \neg, 0, 1)$ or $(M, \sqcup, \sqcap, \sim, 0, 1)$, respectively. The prime example is the field of subsets of a given set $\mathcal{D}$: $(\mathcal{P}(\mathcal{D}), \cup, \cap, {}^c, \emptyset, \mathcal{D})$.

[1] $\mathcal{M}$ is a transitive model if its underlying set M is a transitive set, i.e., if $X \in M$ then $X \subset M$. Every member of an element of $\mathcal{M}$ is also an element of $\mathcal{M}$.

Here the equivalence of the dual forms like $X \cup Y = Z \Leftrightarrow X^c \cap Y^c = Z^c$ is obvious. The axioms stipulate then for all $X, Y, Z \in \mathcal{P}(\mathcal{D})$:

$X \cup Y = Y \cup X$	(A1)	$X \cap Y = Y \cap X$	(A1')
$X \cup (Y \cup Z) = (X \cup Y) \cup Z$	(A2)	$X \cap (Y \cap Z) = (X \cap Y) \cap Z$	(A2')
$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$	(A3)	$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$	(A3')
$X \cup (X \cap Y) = X$	(A4)	$X \cap (X \cup Y) = X$	(A4')
$X \cup X^c = \mathcal{D}$	(A5)	$X \cap X^c = \emptyset$.	(A5')

The axioms at the right-hand side are the dual forms, obtained by exchanging union $\cup$ and intersection $\cap$, empty set $\emptyset$ and domain $\mathcal{D}$ (as well as set S and its complement S^c, but the complements have been renamed S in the axioms).

Further we introduce a partial order $X \subseteq Y \Leftrightarrow X \cup Y = Y$ on $\mathcal{D}$ which is reflexive, $X \subseteq X$, transitive, $X \subseteq Y \wedge Y \subseteq Z \Rightarrow X \subseteq Z$, and antisymmetric, $X \subseteq Y \wedge Y \subseteq X \Rightarrow X = Y$. Since the dual form of $X \subseteq Y$ is $X^c \supseteq Y^c$, also $\subseteq$ and $\supseteq$ swap roles when constructing the dual expression.

Definition: Z is an upper bound for a set S of elements X if for all $X \in S$: $X \subseteq Z$.
Definition: Z is least upper bound for a set S if for all its upper bounds $Y \in \mathcal{D}$: $Z \subseteq Y$.
Definition: Z is a lower bound for a set S of elements X if for all $X \in S$: $Z \subseteq X$.
Definition: Z is greatest lower bound for a set S if for all its lower bounds $Y \in \mathcal{D}$: $Y \subseteq Z$.

By antisymmetry least upper bound and greatest lower bound are unique if they exist. Henceforth we call them *supremum* and *infimum*, respectively.
For the pair $\{X, Y\}$ the supremum is $X \cup Y$ and the infimum is $X \cap Y$.

Some obvious propositions that can easily be proved from the axioms include:

$$X \cup X = X = X \cap X$$
$$X \cup \emptyset = X \quad X \cap \mathcal{D} = X \quad X \cup \mathcal{D} = \mathcal{D} \quad X \cap \emptyset = \emptyset \quad \emptyset \subseteq X \subseteq \mathcal{D}$$
$$(X \cup Y = \mathcal{D} \wedge X \cap Y = \emptyset) \Leftrightarrow Y = X^c \Leftrightarrow Y^c = X .$$

Not all axioms are required, but for convenience the whole set is taken like the redundant axioms in ZFC (cp. section 2.12) or the unnecessary implication ($\Rightarrow$) which is often added to the set of logical operators. As an example we prove that (A3') is redundant.

$$\begin{aligned}
X \cap (Y \cup Z) &= (X \cap (X \cup Z)) \cap (Y \cup Z) && \text{by (A4')} \\
&= X \cap ((X \cup Z) \cap (Y \cup Z)) && \text{by (A2')} \\
&= X \cap ((Z \cup X) \cap (Z \cup Y)) && \text{by (A1')} \\
&= X \cap (Z \cup (X \cap Y)) && \text{by (A3)} \\
&= (X \cup (X \cap Y)) \cap (Z \cup (X \cap Y)) && \text{by (A4)} \\
&= ((X \cap Y) \cup X) \cap ((X \cap Y) \cup Z) && \text{by (A1)} \\
&= (X \cap Y) \cup (X \cap Z) . && \text{by (A3)}
\end{aligned}$$

Definition: If M is a subset of $\mathcal{P}(\mathcal{D})$, then $\Sigma M = \bigcup_{X_k \in M} X_k$ is the supremum and $\Pi M = \bigcap_{X_k \in M} X_k$ is the infimum of M. The $\mathcal{B}$ algebra is called *complete*, if Σ and Π always exist. Then, by duality,

$(\Sigma M)^c = \bigcap_{X_k \in M} X_k{}^c$ and $(\Pi M)^c = \bigcup_{X_k \in M} X_k{}^c$. Further, distribution can be extended to infinite sets too: $Y \cap \Sigma M = Y \cap \bigcup_{X_k \in M} X_k = \bigcup_{X_k \in M} (Y \cap X_k)$ and $Y \cup \Pi M = Y \cup \bigcap_{X_k \in M} X_k = \bigcap_{X_k \in M} (Y \cup X_k)$.

A topology $(\mathcal{D}, T)$ on a set $\mathcal{D}$ is a collection T of open subsets of $\mathcal{D}$ (elements of $\mathcal{P}(\mathcal{D})$) including $\emptyset$ and $\mathcal{D}$, such that *arbitrary* unions, and *finite*[1] intersections of elements of T are in T. $\mathcal{D}$ is called a topological space. The sets $\emptyset$ and $\mathcal{D}$ are always both open and closed.

The *interior* int(X) of a subset $X \subseteq \mathcal{D}$ is the union of all open sets contained in X. int(X) $\subseteq X$, and int($X \cap Y$) = int(X) $\cap$ int(Y) since every open set contained in the intersection is also contained in each of the intersecting sets. Obviously int(int(X)) = int(X), int($\emptyset$) = $\emptyset$, int($\mathcal{D}$) = $\mathcal{D}$.

Definition: A subset X of $\mathcal{D}$ is closed $\Leftrightarrow$ $\mathcal{D} \setminus X$ is open.
Finite unions[2] and *arbitrary* intersections of closed sets are closed.

The *closure* of X, denoted by cl(X), is the intersection of all closed sets containing X. Some obvious theorems are: cl(X) = $\mathcal{D} \setminus$ int($\mathcal{D} \setminus X$), int(X) = $\mathcal{D} \setminus$ cl($\mathcal{D} \setminus X$), $X \subseteq$ cl(X), cl(cl(X)) = cl(X), cl($\emptyset$) = $\emptyset$, cl($\mathcal{D}$) = $\mathcal{D}$, cl($X \cup Y$) = cl(X) $\cup$ cl(Y). If X is open, then $X \subseteq$ int(cl(X)).

Definition: X is *regular open* $\Leftrightarrow$ X = int(cl(X)). This means in particular that X does not contain isolated points which may disappear when forming the interior.

If X and Y are regular open, then ($X \cap Y$) is regular open. int(cl(X)) is regular open. If X is open, then int(cl(X)) is the smallest regular open set containing X.

Definition: Let RO($\mathcal{D}$) be the *collection of all regular open sets in $\mathcal{D}$*. Then the special operations $X \cup' Y$ = int(cl($X \cup Y$)) and $X^{c'}$ = int($\mathcal{D} \setminus X$) make (RO($\mathcal{D}$), $\cup'$, $\cap$, $^{c'}$, $\emptyset$, $\mathcal{D}$) a complete $\mathcal{B}$ algebra.

Theorem $T_P = \{X \subseteq P \mid \forall Z \in X\colon (P{\downarrow}Z) \subseteq X\}$ is a topology on P.
Proof: For every X, $\emptyset \subseteq X$, and $P = X$ is not excluded, so $\emptyset$ and P are in T_P.
For the union of any sets X, $Y \in T_P$, let $Z \in X \cup Y$, then $(P{\downarrow}Z) \subseteq X \cup Y$.
For the intersection assume X, $Y \in T_P$. If $Z \in X \cap Y$, then $Z \in X$, so $(P{\downarrow}Z) \subseteq X$, also if $Z \in Y$, so $(P{\downarrow}Z) \subseteq Y$, hence $(P{\downarrow}Z) \subseteq X \cap Y$. Concluding: $X, Y \in T_P \Rightarrow X \cup Y \in T_P \wedge X \cap Y \in T_P$.

Definition: A subset $X \subseteq P$ is dense below $Y \Leftrightarrow \forall Z \leq Y\colon \exists V \leq Z$ such that $V \in X$.

Definition of the mapping $e(P)$: Denote the complete $\mathcal{B}$ algebra of regular open sets in this topology by RO(P). For any $X \in P$, we define a mapping $e(X)$:= int(cl($P{\downarrow}X$)) that maps P into RO(P). Let (P, $\leq$, 1) be a forcing order and X, $Y \in P$, then $e(P)$ is dense in RO(P), i.e., for any nonzero $X \in$ RO(P) there is a $Y \in P$ such that $e(Y) \subseteq X$. If $X \leq Y$ then $e(X) \subseteq e(Y)$. The maps of incompatible X, $Y \in P$ have empty intersection: $e(X) \cap e(Y) = \emptyset$. For $e(X) \leq e(Y)$, X and Y are compatible, cl($P{\downarrow}X$) = $\{Y \mid X$ and Y are compatible$\}$, $e(X) = \{Y \mid \forall Z \leq Y$, Z and X are compatible$\}$.

[1] An infinite intersection of open intervals like ($-1/n$, $1/n$) can be a closed interval [0, 0] = [0].
[2] The infinite union of closed intervals like [$1/n$, 1] can be an open interval (0, 1]. The infinite intersection is the closed interval [1/1, 1] = [1].

Boolean-valued models $\mathcal{M}^{\mathcal{B}}$ The general idea is this. Let $\mathcal{B}$A be a complete $\mathcal{B}$ algebra. We map every sentence $X \in \mathcal{M}$ to some element $|[X]|$ of $\mathcal{B}$A. If we wish that X is true, we put $|[X]| = 1$, if we wish that X is false, we put $|[X]| = 0$. This assignment produces well-defined sets with a $\mathcal{B}$A containing only 0 and 1.

The mapping $X \to |[X]|$ has to satisfy the conditions

$$|[X \text{ OR } Y]| = |[X]| \vee |[Y]| \qquad |[X \text{ AND } Y]| = |[X]| \wedge |[Y]| \qquad |[\text{NOT } X]| = |[X]|^c \,.$$

If a sentence neither is determined to be true nor determined to be false, we put $0 < |[X]| < 1$. To turn $\mathcal{M}^{\mathcal{B}}$ into an actual model of ZFC with desired properties, we will take a suitable quotient of $\mathcal{M}^{\mathcal{B}}$ that eliminates the fuzziness. The $\mathcal{B}$-valued model $\mathcal{M}^{\mathcal{B}}$ is defined by transfinite recursion:

$$\mathcal{M}^{\mathcal{B}}{}_0 = \emptyset \qquad \mathcal{M}^{\mathcal{B}}{}_{\alpha+1} = [\mathcal{M}^{\mathcal{B}}{}_\alpha \to \mathcal{B}\mathrm{A}]\,^{1} \qquad \mathcal{M}^{\mathcal{B}}{}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{M}^{\mathcal{B}}{}_\alpha \qquad \mathcal{M}^{\mathcal{B}} = \bigcup_{\alpha\in\mathrm{Ord}} \mathcal{M}^{\mathcal{B}}{}_\alpha \,.$$

Very loosely speaking, it is intuitively helpful to think of elements of $\mathcal{M}^{\mathcal{B}}$ (which are functions) as being names for sets in another model. Then, for $X \in \mathcal{M}^{\mathcal{B}}$ and $Y \in \mathrm{dom}(X)$, $X(Y)$ is the probability that the set named Y is a member of the set named X.

For a formula $\phi(X_1, X_2, ..., X_n)$ with variables in $\mathcal{M}^{\mathcal{B}}$, its *Boolean value* $|[\phi(X_1, X_2, ..., X_n)]| \in \mathcal{B}$A is defined by recursion on the complexity of ϕ. The universal quantifier claims that all elements have a certain property and the existential quantifier says that at least one element has this property. Therefore, with $\bigvee M = \Sigma M$ and $\bigwedge M = \Pi M$:

$$|[\forall X\, \phi(X)]| = \bigwedge_{Y\in\mathcal{M}^{\mathcal{B}}} |[\phi(Y)]|$$
$$|[\exists X\, \phi(X)]| = \bigvee_{Y\in\mathcal{M}^{\mathcal{B}}} |[\phi(Y)]| \,.$$

The atomic expressions of equality and element-relation are defined in terms of each other

$$|[X \in Y]| = \bigvee_{Z\in\mathrm{dom}(Y)} (Y(Z) \wedge |[Z = X]|)$$
$$|[X = Y]| = \bigwedge_{Z\in\mathrm{dom}(X)} (X(Z) \Rightarrow |[Z \in Y]|) \wedge \bigwedge_{Z\in\mathrm{dom}(Y)} (Y(Z) \Rightarrow |[Z \in X]|) \,.$$

For the following statements it is intuitively helpful to think of the order $\leq$ in $\mathcal{B}$A as being the implication.

$$|[X = X]| = 1 \qquad X(Y) \leq |[Y \in X]| \qquad |[X = Y]| = |[Y = X]| \qquad |[X = Y]| \wedge |[Y = Z]| \leq |[X = Z]|$$
$$|[X = Z]| \wedge |[X \in Y]| \leq |[Z \in Y]| \qquad |[X = Z]| \wedge |[Y \in X]| \leq |[Y \in Z]|$$
$$|[X = Y]| \wedge |[\phi(X)]| \leq |[\phi(Y)]| \qquad |[(\exists Y \in X)\phi(Y)]| = \bigvee_{Y\in\mathrm{dom}(X)} \{X(Y) \wedge |[\phi(Y)]|\}$$
$$|[(\forall Y \in X)\phi(Y)]| = \bigwedge_{Y\in\mathrm{dom}(X)} \{X(Y) \Rightarrow |[\phi(Y)]|\} \,.$$

A formula $\phi(X_1, X_2, ..., X_n)$ with variables in $\mathcal{M}^{\mathcal{B}}$ is valid in $\mathcal{M}^{\mathcal{B}}$ if $|[\phi(X_1, X_2, ..., X_n)]| = 1$.

---

[1] $[X \to Y]$ denotes the set of functions f with $\mathrm{dom}(f) \subseteq X$ and $\mathrm{im}(f) \subseteq Y$.

Every axiom of the predicate calculus is valid in $\mathcal{M}^{\mathcal{B}}$, and if ϕ is obtained via a rule of inference applied to valid formulae, then ϕ is valid in $\mathcal{M}^{\mathcal{B}}$. Every axiom of ZFC is valid in $\mathcal{M}^{\mathcal{B}}$. In particular, by the previous remark, we also have that every sentence provable from the ZFC axioms is valid in $\mathcal{M}^{\mathcal{B}}$. And if $\|[\phi]\| = 1$ and ψ is a logical consequence of ϕ, then $\|[\psi]\| = 1$.

<u>Generic extensions</u> Now we step outside the model $\mathcal{M}$ and into the universe V. The goal is to extend $\mathcal{M}$ to a larger model $\mathcal{M}[G]$ obtained by adding to $\mathcal{M}$ a special subset G of our complete Boolean algebra $\mathcal{B}$A, and throwing in everything else that we need in order for this to be a model of ZFC (and nothing more).

A subset $G \subseteq \mathcal{B}$A is <u>an $\mathcal{M}$-generic ultrafilter</u> if

- G is an ultrafilter, i.e.
  - G is upwards-closed: for all $X \in G$, if $X \leq Y$ (i.e., $X \wedge Y = X$) then $Y \in G$.
  - G is closed under meets: for all $X, Y \in G$, $X \wedge Y \in G$.
  - If $X \in \mathcal{B}$A then either $X \in G$ or $\neg X \in G$, but not both (so, in particular $1 \in G$ and $0 \notin G$).
- G contains the meets of all its subsets which lie in $\mathcal{M}$, i.e., if $X \subseteq G$ and $X \in \mathcal{M}$ then $\bigwedge X \in G$.
  - So G intersects all subsets of $\mathcal{B}$A that are elements of $\mathcal{M}$.

A generic ultrafilter is precisely the special subset we wish to adjoin to $\mathcal{M}$. To construct $\mathcal{M}[G]$, we go via $\mathcal{M}^{\mathcal{B}}$. The subset G contains $\|[\phi]\|$ for every statement ϕ to hold in the new model of ZFC. For every X in $\mathcal{B}$A, G must contain either X or X^c. A subset of a poset P is *P-generic* if it intersects every dense subset D of P. D is dense in P if for every $X \in P$, there exists $Y \in D$ such that $Y \leq X$. The poset is $\mathcal{B}$A \ {0}, and the set G intersects every dense subset $D \subseteq \mathcal{B}$A \ {0} that is a member of $\mathcal{M}$. If $G \subseteq \mathcal{B}$A is $\mathcal{M}$-generic, then $\mathcal{M}[G]$ is isomorphic to a standard transitive model of ZFC that contains both $\mathcal{M}$ and G: $\mathcal{M} \subseteq \mathcal{M}[G]$ and $G \in \mathcal{M}[G]$. In the interesting case of countable models $\mathcal{M}$, the existence of $\mathcal{M}$-generic ultrafilters is easy to prove.

From a generic ultrafilter we can produce a {0, 1}-valued model $\mathcal{M}[G]$ through the following process. For all $X \in M^{\mathcal{B}}$, we define X^G by induction on rank: $\emptyset^G = \emptyset$, and $X^G = \{Y^G \mid X(Y) \in G\}$ (that is, values of $\mathcal{B}$A that are in G are taken to be true, and other values are taken to be false). Then, we define $\mathcal{M}[G] = \{X^G \mid X \in M^{\mathcal{B}}\}$. It follows[1]

$$\text{if } \|[\phi(X_1, X_2, \ldots, X_n)]\| \in G, \text{ then } \mathcal{M}[G] \vDash \phi(X_1{}^G, X_2{}^G, \ldots, X_n{}^G)\ .$$

<u>Generic Model Theorem</u> Let $\mathcal{M}$ be a transitive model of ZFC, and $(P, \leq, 1)$ a forcing order in $\mathcal{M}$. If $G \subseteq P$ is generic over P, there is a transitive model $\mathcal{M}[G]$ of ZFC with the following properties:

- $M[G] \vDash$ ZFC.
- $M \subseteq M[G]$, where M and $M[G]$ are the universes of $\mathcal{M}$ and $\mathcal{M}[G]$, respectively.
- $\mathrm{Ord}^{\mathcal{M}[G]} = \mathrm{Ord}^{\mathcal{M}}$ (there are no new ordinal numbers introduced in $\mathcal{M}[G]$).
- If $\mathcal{N}$ is a transitive model of ZF such that $M \subset N$ and $G \in N$, then $M[G] \subset N$.
- $G \in M[G]$.

[1] "$A \vDash B$" indicates that B is a logical consequence of A, i.e., the sentence B is true in all models of A. A satisfies B.

The Forcing Relation links statements about $\mathcal{M}[G]$ to elements of the forcing order $(P, \leq, 1)$. Given $e(P)$ the forcing relation is defined for $X \in P$ by X *forces* $\phi(N_1, N_2, ..., N_n)$ if and only if $e(X) \leq \|[\phi(N_1, N_2, ..., N_n)]\|$, or formalized[1]: $X \Vdash \phi(N_1, N_2, ..., N_n) \Leftrightarrow e(X) \leq \|[\phi(N_1, N_2, ..., N_n)]\|$.

Proof of the Generic Model Theorem: Let $(P, \leq, 1)$ be a forcing order in $\mathcal{M}$, with an $\mathcal{M}$-generic filter $G \subset P$. For every P-name $X \in M^P = M^{\mathcal{B}(P)}$, we define X^G inductively, such that $\emptyset^G = \emptyset$ and $X^G = \{Y^G \mid \exists Z \in G, e(Z) \leq X(Y)\}$. The universe of $\mathcal{M}[G]$ be $\{X^G \mid X \in M^P\}$. The generic filter G on P is used to define a generic ultrafilter H on $\mathcal{B}$A such that $H = \{Y \in \mathcal{B}\mathrm{A} \mid \exists X \in G, e(X) \leq Y\}$, i.e., H is the set of elements above elements of $e(P)$. H is $\mathcal{M}$-generic because it is generated from the $\mathcal{M}$-generic filter G, and it can be easily checked that $X^G = X^H$ for all $X \in M^{\mathcal{B}}$. Therefore $\mathcal{M}[G] = \mathcal{M}[H]$. The first four conditions of the theorem follow. To verify that $G \in \mathcal{M}[G]$ write G as $G = \{X \in P \mid e(X) \in H\}$, which gives $G \in \mathcal{M}[H]$ as desired.

The Forcing Theorem   Every statement that is true in the new model $\mathcal{M}[G]$ was forced by some element of the generic filter G. Let $(P, \leq, 1)$ be a forcing order in the ground model $\mathcal{M}$. If σ is a sentence in the language of set theory with P-names as parameters, then for all $G \subset P$ generic over $\mathcal{M}$, $\mathcal{M}[G] \vDash \sigma$ iff there is $X \in G$ such that $X \Vdash \sigma$.

Proof: Assume that $\mathcal{M}[G] \vDash \sigma$. Using the proof of the Generic Model Theorem, there is an ultrafilter H of $\mathcal{B}\mathrm{A}(P)$ such that $\mathcal{M}[G] = \mathcal{M}[H]$. Then $\mathcal{M}[H] \vDash \sigma$ implies $\|[\sigma]\| \in H$. By the construction of H, all elements of H are above some element of $e(P)$, so there is $X \in P$ such that $e(X) \leq \|[\sigma]\|$. By the definition of the forcing relation, we get that $X \Vdash \sigma$. Now assume that there exists $X \in P$ such that $X \Vdash \sigma$. Then $e(X) \leq \|[\sigma]\|$ by definition, so $\|[\sigma]\| \in H$, implying, as desired, that σ is a logical consequence of $\mathcal{M}[G]$: $\mathcal{M}[H] = \mathcal{M}[G] \vDash \sigma$.

2.18.3 An example

The following excerpt from [K. Hrbacek, T. Jech: "Introduction to set theory", 2nd ed., Marcel Dekker, New York (1984) p. 232ff] "reinforces" this topic by a detailed exemplary explanation of how the properties of the generic set G (which in the original version is called X) can be defined.

"$2^{\aleph_0} = \aleph_1$ is a distinct possibility (and becomes a provable fact if the Axiom of Constructibility is also assumed). In general, it is, therefore, necessary to *add* '*new*' *sets* to our universe in order to get a model for $2^{\aleph_0} > \aleph_1$.

We concentrate our attention on the task of adding just one 'new' set of natural numbers G. For the time being G is just a symbol devoid of content, a name for a set yet to be described. Let us see what one could say about it.

A key to the matter is a realization that one cannot expect to have *complete* information about G. If we found a property P which would tell us exactly which natural numbers belong to G, we could set $G = \{n \in \omega \mid P(n)\}$ and conclude on the basis of the Axiom Schema of Comprehension that G exists in our universe, and so is not a 'new' set. Cohen's basic idea was that

[1] The symbol $\Vdash$ denotes the forcing relation between elements X of a forcing notion P and statements ϕ with parameters (P-names) $N_1, N_2, ..., N_n$. $X \Vdash \phi$ means X forces ϕ to be the case in $\mathcal{M}[G]$ if X is in G.

partial descriptions of G are sufficient. He described the set G by a collection of 'approximations' in much the same way as irrational numbers can be approximated by rationals.

Specifically, we call finite sequences of zeros and ones *conditions*; for example, $\emptyset$, $\langle 1\rangle$, $\langle 1, 0, 1\rangle$, $\langle 1, 1, 0\rangle$, $\langle 1, 1, 0, 1\rangle$ are conditions. We view these conditions as providing partial information about G in the following sense: If the kth entry in a condition is 1, that condition determines that $k \in G$. If it is 0, the condition determines that $k \notin G$. For example, $\langle 1, 1, 0, 1\rangle$ determines that $0 \in G$, $1 \in G$, $2 \notin G$, $3 \in G$ (but does not determine, say, $4 \in G$ either way).

Next, it should be noted that adding one set G to the universe immediately gives rise to many other sets which were not in the universe originally, such as $\omega - G$, $\omega \times G$, G^2, $\mathcal{P}(G)$, etc. Each condition, by providing some information about G, enables us to make some conclusions also about these other sets, and about the whole expanded universe. Cohen writes $p \Vdash P$ (*p forces P*) to indicate that information provided by the condition p determines that the property P holds. For example, it is obvious that

$$\langle 1, 1, 0, 1\rangle \Vdash (5, 3) \in \omega \times G$$

(because, as we noted before, $\langle 1, 1, 0, 1\rangle \Vdash 3 \in G$, and $5 \in \omega$ is true) or

$$\langle 1, 1, 0, 1\rangle \Vdash (2, 3) \notin \mathcal{P}(G)$$

(because $\langle 1, 1, 0, 1\rangle \Vdash 2 \notin G$).

It should be noticed that conditions often clash: For example,

$$\langle 1, 1, 0\rangle \Vdash (0, 1) \in G^2,$$

while

$$\langle 1, 0, 1\rangle \Vdash (0, 1) \notin G^2."$$

The statement $p \Vdash P(G)$ is to be understood as *conditional*: *If* the set G is as described by the condition p, *then* G has the property P. $\langle 1, 1, 0\rangle \Vdash (0, 1) \in G^2$ means *if* $0 \in G$, $1 \in G$, $2 \notin G$, *then* $(0, 1) \in G^2$, while $\langle 1, 0, 1\rangle \Vdash (0, 1) \notin G^2$ means *if* $0 \in G$, $1 \notin G$, $2 \in G$, *then* $(0, 1) \notin G^2$. "If we knew the set G, we would of course be able to determine whether it is as described by the condition $\langle 1, 1, 0\rangle$ or by the condition $\langle 1, 0, 1\rangle$ or perhaps by another of the remaining six conditions of length three. Since we cannot know the set G, we never know which of these conditions is 'true'; thus we never are able to decide whether $(0, 1) \in G^2$ or not. Nevertheless, it turns out that there are a great many properties of G which can be decided, because they must hold *no matter what* conditions are 'true'. As an illustration, let us show that every condition forces that G is infinite. If not, there is a condition p and a natural number k such that $p \Vdash$ 'G has k elements'. For example, let $p = \langle 1, 0, 1\rangle$ and $k = 5$; we show that $\langle 1, 0, 1\rangle \Vdash$ 'G has five elements' is impossible. Consider the condition $q = \langle 1, 0, 1, 1, 1, 1, 1\rangle$. First of all, the condition q contains all information supplied by p: $0 \in G$, $1 \notin G$, $2 \in G$. If the conclusion that G has five elements could be derived from p, it could be derived from q also. But this is absurd because clearly $q \Vdash$ 'G has six elements'; namely, $q \Vdash 0 \in G$, $2 \in G$, $3 \in G$, $4 \in G$, $5 \in G$, $6 \in G$. The same type of argument leads to a contradiction for any p and k."

Every condition forces that G is a new set of natural numbers, not identical with any set A of the universe before adding G. If not, there is a condition p such that $p \Vdash G = A$. Assume that this condition is $p = \langle 1, 0, 1\rangle$ forcing $0 \in G$, $1 \notin G$, $2 \in G$. Now there remain two possibilities. If

$3 \in A$, let $q_1 = \langle 1, 0, 1, 0\rangle$. Since q_1 contains all information supplied by p, $q_1 \Vdash G = A$. But this is impossible, because $q_1 \Vdash 3 \notin G$, whereas $3 \in A$. If $3 \notin A$, let $q_2 = \langle 1, 0, 1, 1\rangle$. Again $q_2 \Vdash G = A$, but we get a contradiction from $q_2 \Vdash 3 \in G$ and $3 \notin A$. A similar argument works for any p.

"Let us now review what has been accomplished by Cohen's construction. The universe of set theory has been extended by adding to it a 'new', 'imaginary' set G (and various other sets which can be obtained from G by set-theoretic operations). Partial descriptions of G by conditions are available. These descriptions are not sufficient to decide whether a given natural number belongs to G or not, but allow us, nevertheless, to demonstrate certain statements about G, such as that G is infinite and differs from every set in the original universe. Cohen has established that the descriptions by conditions are sufficient to show the validity of all axioms of Zermelo–Fraenkel set theory with Choice in the extended universe.

Although adding one set of natural numbers to the universe does not increase the cardinality of the continuum, one can next take the extended universe and, by repeating the whole construction, add to it another 'new' set of natural numbers G'. If this procedure is iterated $\aleph_2$ times, the result is a model in which there are at least $\aleph_2$ sets of natural numbers, i.e., $2^{\aleph_0} \geq \aleph_2$. Alternatively, one can simply add $\aleph_2$ such sets at once by employing slightly modified conditions."

To summarize: A partial model $\mathcal{M}$ of ZFC has to be extended by a generic set G such that $\mathcal{M}[G]$ has the desired properties. Some facts about $\mathcal{M}[G]$ depend on the particular generic set G, while others are general facts that are true for all G. If, for example, we add an element X to extend a group, then always the inverse element X^{-1} is added too by the axiom of inverse element. This is a general property. That X obeys the axiom of closure, however, is an individual property.

2.19 Second-order logic

Logic takes as its particles sentences or statements A which have truth values, either false $(A) = 0$ or true $(A) = 1$, and connects them by conjunction ($\wedge$), disjunction ($\vee$), and negation ($\neg$) in order to form further, compound propositions. Logic is the set of rules how the truth-values of the resulting propositions have to be calculated. In the original, classical logic, developed by the ancient Greek, the sentences are called propositions.

Example: The two propositions "if you order a pizza" (P) and "you will get a salad free of charge" (S) are combined to obtain the proposition $\neg P \vee S$, i.e., an implication, often abbreviated as $P \Rightarrow S$. This implication is true unless $(P) = 1$ and $(S) = 0$. Note: If the premise is false, $(P) = 0$, the implication is always true, $(P \Rightarrow S) = 1$, but not necessarily the conclusion (S).

In first-order predicate logic (FOL), the sentences are called predicates. They concern statements about elements which belong to a given domain and are either true or false. A predicate can be interpreted as a function which maps a statement to the truth values 0 or 1. The main distinction from classical logic however is that statements can be equipped with two quantifiers. A statement with the universal quantifier $\forall$ claims truth for all elements of the domain $\mathcal{D}$ that are represented by the variable x in the expression $\forall x \in \mathcal{D}$. A statement with the existential quantifier $\exists$ claims that truth can be accomplished by inserting at least one[1] element of the domain in place of the variable x in the expression $\exists x \in \mathcal{D}$. Variables without quantifier are called *free variables*, variables with quantifier are called *bounded variables*: In the sentence "$\forall n \in \mathbb{N}$: $m \cdot n$ is even" the truth value depends on the choice of the free variable m but cannot depend on the bounded variable n because it is asserted for every possible element of the domain $\mathbb{N}$.

FOL is restricted to quantification over individuals, the elements of the domain, often a set, but not over predicates or functions which also cannot be taken as arguments for other predicates or functions. For instance FOL cannot quantify over all sets of real numbers, i.e., the power set $\mathcal{P}(\mathbb{R})$. (Since ZFC has been formulated in FOL and in ZFC everything is a set, this distinction becomes problematic.)

Second-order logic (SOL) quantifies over predicates or relations including functions. SOL cannot be reduced to FOL, according to current set theory, because FOL addresses at most a countably infinite set. By the theorem of Skolem (cp. 3.4 "The Löwenheim-Skolem paradox") there is a countable model that satisfies precisely the same first-order sentences about real numbers and sets of real numbers as the real set $\mathbb{R}$ of real numbers. In the countable model there cannot be all uncountably many subsets of $\mathbb{N}$ or $\mathbb{R}$ and the *least upper bound* property cannot be satisfied for every bounded subset of $\mathbb{R}$. So, if the real numbers are uncountable, then there must be many FOL models of $\mathbb{R}$. In SOL, however, there is merely one model (up to isomorphism).

SOL has a main disadvantage however. According to a result of Gödel's SOL does not admit a complete proof theory. So SOL is not logic, properly speaking. Quine calls it "set theory in sheep's clothing". He considers an expression which makes the choice of y depending on z too:

$$\forall z\, \exists w\, \forall x\, \exists y\, Fxyzw$$

"As a way of avoiding these unwanted dependences, the branching notation

$$\begin{matrix} \forall x \exists y \\ \forall z \exists w \end{matrix} Fxyzw \qquad (*)$$

suggests itself. [...] If we quantify over functions we can get (*) back into line thus:

$$\exists f\, \exists g\, \forall x\, \forall z\, Fx(fx)z(gz)\ .$$

But here we affirm the existence of abstract objects of a certain sort: functions. We leave logic and ascend into a mathematics of functions, which can be reduced to set theory but not to pure logic. [...] The logic of quantification in its unsupplemented form admits of complete proof procedures for validity. It also admits of complete proof procedures for inconsistency; for, to prove a schema inconsistent we have only to prove its negation valid. Now a remarkable fact [...] is that as soon as you branch out in the manner of (*) you get into a terrain that does not admit simultaneously of complete proof procedures for validity and inconsistency." [W.V. Quine: "Philosophy of logic", 2nd ed., Harvard Univ. Press (1986) p. 90f]

[1] Sometimes $\exists! x$ is used to express that there exist one and only one of the elements represented by x which makes the expression true.

Quine advanced the view that in predicate-language sentences the variable can be understood as a name denoting an object and hence can be quantified over. "The variables of quantification, 'something', 'nothing', 'everything', range over our whole ontology, whatever it may be; and we are convicted of a particular ontological presupposition if, and only if, the alleged presuppositum has to be reckoned among the entities over which our variables range in order to render one of our affirmations true.

We may say, for example, that some dogs are white, and not thereby commit ourselves to recognizing either doghood or whiteness as entities. 'Some dogs are white' says that some things that are dogs are white; and, in order that this statement be true, the things over which the bound variable 'something' ranges must include some white dogs, but need not include doghood or whiteness." [Willard V. Quine: "On what there is", The Review of Metaphysics 2,5 (1948) p. 32, reprinted in "From a logical point of view", Harvard University Press (1953)]

This reasoning has been criticized by Boolos. "It is of little significance whether second-order logic may bear the (honorific) label 'logic' or must bear 'set theory'. What matter, of course, are the reasons that can be given on either side. It seems to be commonly supposed that the arguments of Quine and others for not regarding second- (and higher-) order logic as logic are decisive, and it is against this view that I want to argue here." [George S. Boolos: "On second-order logic", The Journal of Philosophy 72,16 (1975) p. 509]

Boolos proves that sentences such as the classical Geach-Kaplan sentence "Some critics admire only each other" cannot be expressed in FOL using only the predicates occurring in the sentence itself. Further he favours "plural quantification" $\exists xx$ and $\forall xx$ meaning "there are some things represented by x" and "for any things represented by x". [George Boolos: "To be is to be a value of a variable (or to be some values of some variables)", Journal of Philosophy 81 (1984)] "He argues that it is simply a prejudice to insist that the plural locutions of natural language be paraphrased away. Instead he suggests that just as the singular quantifiers $\forall x$ and $\exists x$ get their legitimacy from the fact that they represent certain quantificational devices in natural language, so do their plural counterparts $\forall xx$ and $\exists xx$. For there can be no doubt that in natural language we use and understand the expressions 'for any things' and 'there are some things'. Since these quantifiers bind variables that take name (rather than predicate) position, they are *first-order* quantifiers, albeit *plural* ones." [Øystein Linnebo: "Plural quantification", The Stanford Encyclopedia of Philosophy (2017)]

"The First Order Logicians: the early, vast majority. These include Guisseppe Peano, C.S. Pierce, David Hilbert, Georg Cantor, Richard Dedekind, Skolem, Löwenheim, Zermelo, Fraenkel, Herbrand, the Bourbaki guys, Quine, Tarski, (early) Wittgenstein, etc. [...] Peano, Pierce, and Hilbert all developed First Order Logic roughly independently; this lends credence to the idea that FOL is a natural foundation for mathematics." [Matt W-D in "Is first order logic (FOL) the only fundamental logic?", Philosophy.StackExchange (29 Jul 2012)]

The disadvantage of being not logical at all may save SOL from being demasked as insufficient. In fact however SOL appears to be not indispensible in mathematics and sciences as will be seen in chapter VI by proving that there is nothing uncountable. So FOL will serve all mathematical and scientific purposes.

III Paradoxes and antinomies

A paradox is an astonishing result but not detrimental to the underlying theory. An antinomy is a contradiction that cannot be tolerated in a mathematical theory. Sometimes the judgements about what is a paradox and what is an antinomy are diverging (cp., e.g., Borel's statement about the axiom of choice in chapter V). In old set theory with unrestricted comprehension there appeared some contradictions which were considered and called paradoxes until their disastrous effect had been acknowledged. They have been removed by the axiom of restricted comprehension.

3.1 Antinomies and paradoxes of naive set theory

Set theory developed by Cantor, prior to the invention of the well-known axiom system ZFC by Zermelo and Fraenkel (and other systems like NBG by von Neumann, Bernays, and Gödel, or Quines' New Foundations NF) has been called naive set theory. Not based on axioms[1] but on the assumed reality of "nature" it was subject to many contradictions.

3.1.1 The Burali-Forti antinomy

The first antinomy has been published by Cesare Burali-Forti. In modern terms it reads: The set Ω of all ordinal numbers has (or is) an ordinal number itself and larger than every ordinal belonging to the set (and like every ordinal number it has a successor $\Omega + 1$). Therefore there is an ordinal number strictly larger than all ordinal numbers (which allegedly are contained in Ω). [C. Burali-Forti: "Una questione sui numeri transfiniti", Rendiconti del Circolo Matematico di Palermo 11 (1897) pp. 154-164; English translation: "A question on transfinite numbers" in Jean van Heijenoort: "From Frege to Gödel – A source book in mathematical logic, 1879-1931", Harvard Univ. Press, Cambridge (1967) pp. 104-112]

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[1] "*Hypotheses* are not at all mentioned in my arithmetical investigations about the finite and the transfinite, only reasons are given for what is really existing in nature. You on the other hand believe [...] that *also in arithmetic* hypotheses could be invented which is absolutely impossible. [...] As little as we can invent basic laws in arithmetic of finite numbers other than those known from time immemorial for the numbers 1, 2, 3, ..., so little a deviation from arithmetical basic truths is possible in the realm of the transfinite. 'Hypotheses' offending these basic truths are as false and contradictory as, e.g., the sentence 2 + 2 = 5 or a square circle. To see such hypotheses being used at the outset of an investigation suffices for me to know that this investigation must be wrong." [G. Cantor, letter to G. Veronese (17 Nov 1890)] Although Cantor here talks about "hypotheses" he is obviously meaning axioms: It is impossible to choose arbitrary axioms disobeying the natural laws of arithmetic.

Only much later, in a letter to Hilbert [G. Cantor, letter to D. Hilbert (27 Jan 1900)], Cantor discusses three kinds of axioms, namely (1) those of logic which mathematics shares with all other sciences, (2) the physical axioms, like those of geometry and mechanics, which are not necessary but can be replaced by others, and finally (3) the metaphysical axioms of arithmetic including the axiom of finite number theory "*Every finite multitude is consistent*" and the axiom of transfinite number theory "'Every multitude which a *signed aleph* $\aleph_\gamma$ belongs to (where γ is any ordinal number) is *consistent*.' In other words: *all signed alephs are real cardinal numbers*, being just as real as the finite cardinal numbers."

3.1.2 Cantor's set of all sets

Cantor independently knew of the inconsistency of the set of all ordinal numbers and the set of all sets. In a letter to Dedekind [G. Cantor, letter to R. Dedekind (3 Aug 1899)] he states that the set of all ordinal numbers does not exist. He calls it an incomplete or inconsistent multitude or an inconsistent system. Nowadays such inconsistent systems are called classes. "*The system* Ω *in its natural order of magnitude is a* '*sequence*'. Supplementing this sequence by the element 0, put in the first place, we get the sequence $\Omega' = 0, 1, 2, 3, ..., \omega_0, \omega_0 + 1, ... , \gamma, ...$ of which it is easy to see that *every* number γ is the *type* of the *sequence of all its predecessors* (including 0). (The sequence Ω has this property only for $\gamma \geq \omega_0 + 1$.) Ω' (and therefore also Ω) *cannot be a consistent* multitude; if Ω' was consistent, then to this well-ordered set a number δ would correspond which was larger than all numbers of the system Ω; but in the system Ω also δ appears because it contains *all* numbers including δ. That means δ would be larger than δ which is a contradiction. Therefore: *The system* Ω *of all numbers is an inconsistent, an absolutely infinite multitude*."

Then Cantor proves that also the system of all alephs is an inconsistent absolutely infinite set. He shows, based on the inconsistency of Ω, that no cardinal number corresponds to an inconsistent multitude. And he proves that every cardinality is an aleph. "Let us take a certain multitude V and presuppose that it does *not* correspond *to an aleph*, then we can conclude that V must be *inconsistent*. It is easily recognized that, under the made assumptions, the whole system Ω can be mapped into the multitude V, i.e., there must exist a partial multitude V' of V which is equivalent to the system Ω. V' is *inconsistent* because Ω is inconsistent. Therefore the same must be claimed of V. Hence every transfinite *consistent multitude*, every transfinite set must have a *determined aleph* as its cardinal number. *The system* ת *of all alephs is nothing else but the system of all transfinite cardinal numbers.* Hence, all sets are *in a generalized sense* '*countable*', in particular all '*continua*'. [...] 'If a and b are any cardinal numbers, then either $a = b$ or $a < b$ or $a > b$.' Because, as we have seen, the *alephs* have this character of magnitudes."

In short, the proof that the cardinality of the power set is always larger than that of the set forbids the existence of the set of all sets, because the set of all sets has to include its power set and thus has to have a larger cardinal number than itself.

3.1.3 The Russell antinomy

In 1901 Bertrand Russell devised the first antinomy that became widely known, the predicate *w*: "to be a predicate that cannot be predicated of itself"; in modern treatments replaced by the set S of all sets X that do not contain themselves as an element: $S = \{X \mid X \notin X\}$. With unrestricted comprehension and without the axiom of foundation it is possible to define a set containing itself as an element; for instance, the set of all abstract notions is also an abstract notion, hence contains itself as an element. Russell concluded an antinomy: $S \in S \Leftrightarrow S \notin S$. He mentioned this in 1902 in a letter to Frege [B. Russell, letter to G. Frege (16 Jun 1902), reprinted in Jean van Heijenoort: "From Frege to Gödel – A source book in mathematical logic, 1879-1931", Harvard Univ. Press, Cambridge (1967) p. 124f] who in 1879 had devised the axiom of (unrestricted) comprehension. The same antinomy had been found by Zermelo independently.

Russell constructed a lot of easily understandable parables. Best known is the male barber who shaves all men in his village who do not shave themselves. Who shaves the barber? If he shaves himself, he belongs to the set of men not to be shaved by the barber. If he does not shave himself, the barber has to shave him. Or consider the library catalogues (which in the past used to be thick books with pigskin cover) that do not contain themselves. If we have to make a general catalogue of all library catalogues that do not contain themselves, should this general catalogue contain itself? If yes, then no, and if no, then yes.

3.1.4 Richard's paradox

Richard's paradox was first stated in 1905 in a letter to the Revue générale des sciences pures et appliquées. The Principia Mathematica by Whitehead and Russell quote it together with six other paradoxes concerning the problem of self-reference. The paradox can be interpreted as an application of Cantor's diagonal argument. It inspired Kurt Gödel and Alan Turing to their famous works. Kurt Gödel considered his incompleteness theorem as analogous to Richard's paradox which in the original version runs as follows:

Consider a list of all permutations of the 26 letters of the French alphabet, i.e., a list of all words of length two, three, four letters, and so forth, and write them in alphabetical order into a table. Let us cross out all those which are no definitions of numbers. The remaining entries form the ordered set E of all numbers that can be expressed with a finite number of letters or words. This set is denumerable. Let p be the nth decimal of the nth number of the set E; we form a number N having zero for the integral part and $p + 1$ for the nth decimal, if p is not equal either to 8 or 9, and 1 if p is equal either to 8 or 9. This number N does not belong to the set E because it differs from any number of this set, namely from the nth number by the nth digit. But N has been defined by G, a finite number of words, namely those which are above written in red. It should therefore belong to the set E. That is a contradiction.

Richard then argues that this contradiction is only apparent. The collection G of letters has no meaning at the place where it appears in the table since it mentions the set E which has not yet been defined. Therefore there is no contradiction. Further, if, after N has been defined, its definiton G is inserted into the table, another diagonal number N' will result.

[Jules Richard: "Les principes des mathématiques et le problème des ensembles", Revue générale des sciences pures et appliquées 12 (1905) pp. 541-543; English translation: "The principles of mathematics and the problem of sets" in Jean van Heijenoort: "From Frege to Gödel – A source book in mathematical logic, 1879-1931", Harvard Univ. Press (1967) pp. 142-144]

The version given in Principia Mathematica by Whitehead and Russell is similar to Richard's original version, alas not quite as exact. Here only the digit 9 is replaced by the digit 0, such that identities[1] like 1.000... = 0.999... can spoil the result.

[1] In chapter VI we will see that no such identities exist, but they are erroneously assumed in set theory because otherwise it is impossible to interpret (the diagonal) digit sequences as irrational numbers.

3.1.5 König's paradox

This paradox was also published in 1905 by Julius (Gyula) König. Perhaps it had been written simultaneously with Richard's paper. "It is easy to show that *the finitely defined elements of the continuum determine a subset of the continuum that has cardinality* $\aleph_0$ [...] we are employing, lastly, the *logical antithesis* 'given an arbitrary element of the continuum, either it is finitely defined or this is not the case'." If the real numbers can be well-ordered, then there must be a first real number (according to this order) which cannot be defined by a finite number of words. But the first real number which cannot be defined by a finite number of words has just been defined by a finite number of words, namely those which are written in red. "The assumption that the continuum can be well-ordered has therefore lead to a contradiction." [Julius König: "Über die Grundlagen der Mengenlehre und das Kontinuumproblem", Math. Annalen 61 (1905) pp. 156-160; English translation: "On the foundations of set theory and the continuum problem" in Jean van Heijenoort: "From Frege to Gödel – A source book in mathematical logic, 1879-1931", Harvard Univ. Press, Cambridge (1967) pp. 146ff]

3.1.6 Berry's paradox

Berry's Paradox, first mentioned in the Principia Mathematica as fifth of seven paradoxes, is credited to G.G. Berry of the Bodleian Library. It uses the least integer not nameable in fewer than nineteen syllables; in fact, in English it denotes 111777. "But 'the least integer not nameable in fewer than nineteen syllables' is itself a name consisting of eighteen syllables; hence the least integer not nameable in fewer than nineteen syllables can be named in eighteen syllables, which is a contradiction. [Bertrand Russell: "Mathematical logic as based on the theory of types" (1908) in Jean van Heijenoort: "From Frege to Gödel – A source book in mathematical logic, 1879-1931", Harvard Univ. Press, Cambridge (1967) p. 153]

Berry's Paradox with letters instead of syllables is often related to the set of those natural numbers which can be defined by less than 100 (or any other large number of) letters. As the natural numbers are a well-ordered set, there must be the least number which cannot be defined by less than 100 letters. But this number was just defined by 65 letters including spatia.

Analogously, the smallest natural number without interesting properties acquires an interesting property by this very lack of any interesting properties.

The solution of Berry's paradox may be explained most easily in the second version. If all definitions with less than 100 letters already are given, then also the sequence of letters Z = "the least number which cannot be defined by less than 100 letters" does define a number. As an example, let a = 01, b = 02, c = 03 etc. By "example" we have defined the number *example* = 05240113161205 and also the sequence Z would already define a number. Only by the change of language from numeral to colloquial the apparent paradox occurs.

3.1.7 The Grelling-Nelson paradox

A word that describes its own property is an *autological* word. Examples are "old", a very old word, or "short", a very short word. A word that does not describe its own property is called *heterological*. Examples are "new" which is a very old word or "long" which is a very short word. To what class does the word "heterological" belong? If "heterological" is heterological, then it describes its property and is autological. But if it is autological, then it does not describe its property and is heterological. The dilemma is irresolvable. [K. Grelling, L. Nelson: "Bemerkungen zu den Paradoxien von Russell und Burali-Forti", Abhandlungen der Fries'schen Schule II, Göttingen (1908) pp. 300-334]

A related version is the following: The number of all finite definitions is countable. In lexical order we obtain a sequence of definitions D_1, D_2, D_3, ... Now, it may happen that a definition defines its own number. This would be the case if D_1 read "the smallest natural number". It may happen, that a definition does not describe its own number. This would be the case if D_2 read "the smallest natural number". Also the sentence "this definition does not describe its number" is a finite definition. Let it be D_n. Is n described by D_n? If yes, then no, and if no, then yes. The dilemma is irresolvable.

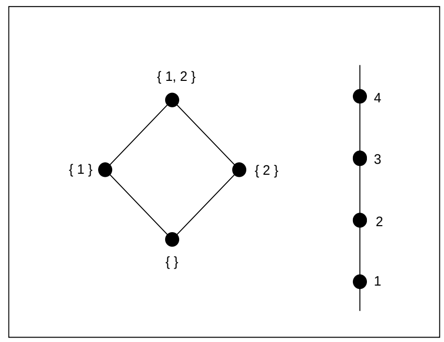
3.2 The undecidable continuum hypothesis

The most paradoxical result is certainly brought about by the fact that this theory, which nearly exclusively deals with different infinities, is unable to determine whether there is a cardinal number between $\aleph_0$ and $2^{\aleph_0}$. The so-called continuum hypothesis assumes that there is no further cardinal number in between. Cantor had tried to prove this and had often believed that he had accomplished it: "I show with absolute rigour that the cardinality of the second number class (II) is not only different from the cardinality of the first number class but that it is indeed the next higher cardinality; [G. Cantor: "Grundlagen einer allgemeinen Mannigfaltigkeitslehre", published by the author himself, Leipzig (1883)] In August 1884 he announced a proof in private correspondence. [G. Cantor, letter to G. Mittag-Leffler (26 Aug 1884)] But he had to withdraw it and instead announced a proof that the continuum had not the cardinality $2^{\aleph_0}$ of the second number class and no cardinality at all. [G. Cantor, letter to G. Mittag-Leffler (14 Nov 1884)] Sometimes his first mental breakdown is attributed to this problem. Meanwhile it has been established that the question cannot be decided. Under the assumption that ZFC is free of contradictions (otherwise everything could be proven) Kurt Gödel showed in 1938 that the continuum hypothesis cannot be disproved from the axioms of ZFC. [K. Gödel: "The consistency of the continuum-hypothesis", Princeton University Press, Princeton (1940)] And Paul Cohen showed in 1963 that it cannot be proved either. [P.J. Cohen: "The Independence of the continuum hypothesis", Proc. Nat. Acad. Sciences, USA 50 (1963) pp. 1143-1148 & 51 (1964) pp. 105-110] So the continuum hypothesis is independent of ZFC.

"There is no evidence that Cantor himself ever considered the possibility that the continuum hypothesis is unprovable and undecidable. Obviously formal investigations were far from his mind. For him mathematical theorems were theses about something being; he even was convinced that the cardinal numbers $\aleph_0$ and $\aleph$ were corresponding to realities in the physical world. We are afraid, he would not have enjoyed the 'solution' of his questions by the modern foundational researchers. [...] Obviously it was difficult for Cantor to express in hard mathematical language what he imagined. His 'definition' could appear rather questionable to a critical thinker like Kronecker." [Herbert Meschkowski: "Georg Cantor: Leben, Werk und Wirkung", 2nd ed., Bibl. Inst., Mannheim (1983) pp. 213 & 229]

Cantor was not the only one who thought that he had proved the continuum hypothesis. Also Hilbert did: "Not even the sketch of my proof of Cantor's continuum hypothesis has remained uncriticized. I would therefore like to make some comments on this proof." [E. Artin et al. (eds.): "D. Hilbert: Die Grundlagen der Mathematik" (1927). Abh. Math. Seminar Univ. Hamburg, Bd. 6, Teubner, Leipzig (1928) pp. 65-85; English translation in J. van Heijenoort: "From Frege to Gödel – A source book in mathematical logic, 1879-1931", Harvard Univ. Press, Cambridge (1967) p. 476]

In order to give an example for the notion of independence, consider that even in arithmetic trichotomy is not a general property. We can choose a structure which obeys trichotomy and a structure which does not, like the sequence of natural numbers and the power set of {1, 2}, respectively. [W. Mückenheim: "Mathematik für die ersten Semester", 4th ed., De Gruyter, Berlin (2015) p. 18] The same can happen in set theory with respect to the continuum hypothesis.



3.2.1 The continuum hypothesis cannot be refuted in ZFC.

In section 2.17.2 a partial model Δ (often denoted as *L*, according to its universe of sets) of ZF has been constructed which obeys most ZF-axioms (cp. section 2.12). It is used to show that the continuum hypothesis (CH) is satisfied there and to conclude (or better: to strongly support the belief) that CH would hold also in a true model of ZFC. For this purpose Gödel shows that the axiom of choice (AC) is valid in Δ. Since the model is constructible it is a matter of few lines to show this: "Now it remains only to be shown that the axiom of choice and the generalized continuum-hypothesis follow from $V = L$ and Σ. For the axiom of choice this is immediate since the relation [...], which singles out the element of least order in any non-vacuous constructible set, evidently satisfies {{AC}} if $V = L$." [Kurt Gödel: "The consistency of the continuum hypothesis", Princeton University Press (1940); reprinted by Ishi Press, New York (2009) p. 53]

"This model, roughly speaking, consists of all 'mathematically constructible' sets, where the term 'constructible' is to be understood in the semiintuitionistic sense which excludes impredicative procedures. [...] In particular, {{the generalized CH (see section 3.2.4)}} follows from the fact that all constructible sets of integers are obtained already for orders $< \omega_1$, all constructible sets of sets of integers for orders $< \omega_2$ and so on." [Kurt Gödel: "The consistency of the axiom of choice and of the generalized continuum-hypothesis", Proc. Nat. Acad. Sciences 24 (1938) p. 556]

"The next step in Gödel's proof is to show that the axiom of choice holds in L. The axiom of choice says in one form that given any set X, there is a function which assigns to each nonempty element y of X, an element of y. Now in the construction of L, each element is constructed at a *least* ordinal. Furthermore, at each ordinal there are only countably many formulas into which may be inserted particular constants that have already been constructed at a previous ordinal. [...] assuming we have well-ordered all the previously constructed sets, we obtain a well-ordering of whatever new sets are constructed at a given ordinal. Putting these altogether, we see that there is a definable well-ordering of L. This clearly gives a *definable* choice function by simply defining the choice function as the element which appears first in the well-ordering. At this point we have shown that 'models' exist in which the axiom of choice is true, and hence we know that it is impossible to prove AC false from the axioms of ZF." [Paul J. Cohen: "The discovery of forcing", Rocky Mountain Journal of Mathematics 32,4 (2002) p. 1084f]

A concise and instructive version of Kurt Gödel's argument is presented by Hrbacek and Jech: "Let now $X \subseteq \omega$. The Axiom of Constructibility guarantees that $X \in L_{\alpha+1}$ for some, possibly uncountable, ordinal α. This means that there is a property P such that $n \in X$ if and only if $P(n)$ holds in $(L_\alpha, \in)$. By the Skolem-Löwenheim Theorem, there is an *at most countable* set $B \subseteq L_\alpha$ such that $(B, \in)$ satisfies the same statements as $(L_\alpha, \in)$. In particular, $n \in X$ if and only if $P(n)$ holds in $(B, \in)$. Moreover, the fact that a structure is of the form $(L_\beta, \in)$ for some ordinal β can itself be expressed by a suitable statement, which holds in $(L_\alpha, \in)$ and thus also in $(B, \in)$. From all this, one can conclude that $(B, \in)$ is (isomorphic to) a structure of the form $(L_\beta, \in)$ for some, necessarily at most countable, ordinal β. Since X is definable in $(B, \in)$, we get $X \in L_{\beta+1}$.

We can conclude that every set of natural numbers is constructed at some at most countable stage, i.e., $\mathcal{P}(\omega) \subseteq \bigcup_{\beta<\omega_1} L_{\beta+1}$. To complete the proof $2^{\aleph_0} = \aleph_1$, we only need to show that the cardinality of the latter set is $\aleph_1$. This in turn follows if we show that L_γ is countable for all $\gamma < \omega_1$. Clearly $L^* = \omega$ is countable. The set L_1 consists of all subsets of L^* definable in $(L^*, \in)$, but there are only countably many possible definitions (each definition is a finite sequence of letters from a finite alphabet of some formalized language, together with a finite sequence of parameters from the countable set L^*), and therefore only countably many definable subsets of L^*. We conclude that L_1 is countable and then proceed by induction, using the same idea at all successor stages and the fact that a union of countably many countable sets is countable at limit stages." [K. Hrbacek, T. Jech: "Introduction to set theory", 2nd ed., Marcel Dekker, New York (1984) p. 232]

The consistency of CH can also be shown by forcing (cp. section 2.18). We assume that F be the poset of countably infinite partial functions f from ω_1 to $\mathbb{R}$. The sets $D_x = \{f \in F \mid x \in \text{dom}(f)\}$ and $D_r = \{f \in F \mid r \in \text{range}(f)\}$ are dense, so G intersects all of them. Thus, if G is generic over F, then $\bigcup G$ is a surjection from ω_1 to $\mathbb{R}$. So $\mathcal{M}[G]$ yields the set of real numbers of the model and shows that its cardinality is not larger than the ω_1 of the model: $\mathcal{M}[G] \vDash |\mathbb{R}^{\mathcal{M}}| \leq \omega_1{}^{\mathcal{M}}$. $M \subset M[G]$ implies $\omega_1{}^{\mathcal{M}} \leq \omega_1{}^{\mathcal{M}[G]}$. In order to show $\mathbb{R}^{\mathcal{M}} = \mathbb{R}^{\mathcal{M}[G]}$ let g be an element of $\mathbb{R}^{\mathcal{M}[G]}$ and g' be a name for that element. If $f \in F$ forces that g' is a function from ω to $\{0, 1\}$, then we can find a chain of f_n for all $n \in \omega$, such that $f_n \leq f_{n-1} \leq \ldots \leq f_0 \leq f$, and f_n forces $g(n)$ to have a particular value. This is a countable descending sequence of countable functions, so their union u is also a countable function, and is thus in F. u is, or at least contains, a function from ω to $\{0, 1\}$ which is equal to g, so g is in $\mathcal{M}$. Hence $\mathbb{R}^{\mathcal{M}} = \mathbb{R}^{\mathcal{M}[G]}$ as desired. Thus, $\mathcal{M}[G] \vDash |\mathbb{R}| \leq \omega_1$.

3.2.2 The continuum hypothesis cannot be proved in ZFC.

In order to construct a model of ZFC in which the continuum hypothesis is false ($\neg$CH), define the forcing order $P = \{p\colon \omega_2 \times \omega \to \{0, 1\} \mid p$ is a finite partial function$\}$ where p is stronger than q iff p extends q. If G is a generic filter on P, $f = \bigcup G$ will be a total function $f\colon \omega_2^{\mathcal{M}} \times \omega \to \{0, 1\}$ defining $\omega_2^{\mathcal{M}}$ Cohen generic reals (cp. section 2.18 "Forcing" and the literature listed there, in particular Rowan Jacobs: "Forcing").

In order for $\mathcal{M}[G]$ to model $\neg$CH, we must make sure that $\omega_2^{\mathcal{M}} = \omega_2^{\mathcal{M}[G]}$. The easiest way to show this is to verify that P satisfies the countable chain condition (abbreviated by ccc), defined as the condition that every antichain in P is countable.

Theorem If P satisfies the countable chain condition and G is an $\mathcal{M}$-generic filter of P, then for all limit ordinals α, the cofinalities[1] in $\mathcal{M}$ and $\mathcal{M}[G]$ are identical $\mathrm{cf}^{\mathcal{M}}(\alpha) = \mathrm{cf}^{\mathcal{M}[G]}(\alpha)$. As a result, for all cardinals κ, $\kappa^{\mathcal{M}} = \kappa^{\mathcal{M}[G]}$.

Proof: It is enough to show that if $\kappa^{\mathcal{M}}$ is a regular[2] cardinal, so is $\kappa^{\mathcal{M}[G]}$. So assume $\kappa^{\mathcal{M}}$ (henceforth denoted by κ) is regular, and let $\lambda < \kappa$. Let f' be a name, and $p \in P$ such that $p \Vdash f'$ is a function from λ to κ. For every ordinal $\alpha < \lambda$ define $A_\alpha = \{\beta < \kappa \mid \exists q < p, q \Vdash f'(\alpha) = \beta\}$. The set A_α contains every β that some q (extending p) forces to be a value for $f(\alpha)$. The set of witnesses $\{q_\beta$ such that $q_\beta \Vdash f'(\alpha) = \beta \mid \beta \in A_\alpha\}$ is an antichain, because if $\beta \neq \gamma$ then q_β and q_γ are not compatible. By ccc it is countable. So A_α must be countable too, for all $\alpha < \lambda$. As κ is regular, there is $\gamma < \kappa$ which is an upper bound to the set $\bigcup_{\alpha<\kappa} A_\alpha$. So for each $\alpha < \lambda$, $p \Vdash f'(\alpha) < \gamma$. Therefore $p \Vdash f'$ is bounded below κ for all $f' \in M^P$ and $p \in P$. Thus in $\mathcal{M}[G]$, for every $\lambda < \kappa$, $\mathrm{cf}(\kappa) > \lambda$. So $\mathrm{cf}(\kappa) = \kappa$, and κ is regular in $\mathcal{M}[G]$ as desired.

It remains to show that P is ccc and that the Cohen generic reals defined by P are distinct. For that sake we assume that $\mathcal{M}$ is a countable transitive model, and thus $f\colon \omega_2^{\mathcal{M}} \times \omega \to \{0, 1\}$ is countable in V, and we use the following lemma: Let P be a set of finite functions on a countable set C, where $p \leq q$ iff p extends q. Assume that if $p \cup q$ is a function then it is also in P. Then P has countable chain condition.

Then for all $\alpha < \omega_2$, define $f_\alpha(n) = f(\alpha, n)$, so f_α is the α^{th} function defined by G. For all α and n, this is defined, since the sets $D_{\alpha,n} = \{p \in P \mid (\alpha, n) \in \mathrm{dom}(p)\}$ are dense in P. Assume that $\alpha \neq \beta$. The set $D = \{p \in P \mid \exists n\colon p(\alpha, n) \neq p(\beta, n)\}$ is dense in P, so it intersects G. So $f_\alpha \neq f_\beta$. We can conclude that $\mathcal{M}[G] \vDash$ there are ω_2 distinct real numbers.

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[1] The cofinality $\mathrm{cf}(S)$ of a set S is the least possible order type of any ordering of its elements. For example, the cofinality of ω^2 is ω, because every countable limit ordinal has cofinality ω (cp. section 2.9). The minimum possible cardinality of a cofinal subset of S is referred to as the cofinality of S too.

[2] A regular cardinal number is equal to its own cofinality. It cannot be divided into smaller sets of smaller cardinalities. An example is $\aleph_0$ (cp. section 2.8).

3.2.3 The independence of the axiom of choice

If $\mathcal{M}$ is a model of the axiom of choice (AC), $\mathcal{M} \vDash$ AC, then also $\mathcal{M}[G] \vDash$ AC (cp. section 2.18 and see Rowan Jacobs: "Forcing"). But we can construct submodels which model ¬AC. For that sake we will add countably many Cohen generic reals to $\mathcal{M}$, then let $\mathcal{N}$ be the hereditarily ordinal-definable sets of $\mathcal{M}[G]$ and show that a well-ordering of our new reals is not hereditarily ordinal-definable, and thus not in $\mathcal{N}$. To easen this task, we assume that $\mathcal{M}$ is a model of $V = L$.

Let P be the poset of finite partial functions from $\omega \times \omega$ to $\{0, 1\}$. If G is a generic filter on P, let $a_k = \{n \in \omega \mid \exists p \in G, p(k, n) = 1\}$ and $A = \{a_k \mid k \in \omega\}$. A' and a'_k will be the names in $\mathcal{M}[G]$ for A and its elements. For every pair a'_k, a'_j and $p \in P$ there exists $q \supset p$ and $n \in \omega$ such that $q(k, n) = 1$ and $q(j, n) = 0$. So every $p \in P$ forces that all a'_k are distinct.

Let $N \subset M[G]$ be the class of all sets in $\mathcal{M}[G]$ that are hereditarily ordinal-definable with parameters in A – that is, the transitive closure of each element of $\mathcal{N}$ must be ordinal-definable. Thus, $\mathcal{N}$ is a submodel of $\mathcal{M}[G]$, and is a transitive model of ZFC. It remains to be shown that there is no well-ordering of A in $\mathcal{N}$.

Assume that $f\colon A \to Ord$ is one-to-one and ordinal-definable with parameters in A. Then there exists some finite sequence $s = (x_0, ..., x_k)$ such that f is ordinal-definable with parameters $x_0, ..., x_k, A$. For any $a \in A$ which is not some x_i, there is some φ such that $\mathcal{M}[G] \vDash a$ is the unique set such that $\varphi(a, \alpha_1, ..., \alpha_n, s, A)$, where α_i are ordinals.

Let $p_0 \Vdash \varphi(a', \alpha_1, ..., \alpha_n, s', A')$. We will show that there exists a name b' and $q \in P$ extending p_0 such that $q \Vdash a' \neq b'$ and $q \Vdash \varphi(b', \alpha_1, ..., \alpha_n, s', A')$. There are $i, i_0, ..., i_k$ and p_1 extending p_0 such that $p_1 \Vdash a' = a'_i$, and for all j, $p_1 \Vdash a'_{i_j} = x'_j$. Let j be a natural number distinct from i, such that for all m, $p_1(j, m)$ is not defined.

We define $\pi\colon \omega \to \omega$ as the transposition (i, j). This permutation induces an automorphism π of P, and thus of B and M^B. $\pi(a'_i) = a'_j$ and vice versa, and for all other n, $\pi(a'_n) = a'_n$. We also have $(\pi p_1)(i, m)$ not defined for all m, but p_1 and πp_1 agreeing at all other points. So p_1 and πp_1 are compatible. Let q extend both of them. We get $q \Vdash \varphi(a'_i, ...) \wedge \varphi(a'_j, ...) \wedge a'_i \neq a'_j$, contradicting the hypothesis that f is ordinal-definable with parameters in A. Thus, $\mathcal{N}$ is a model of a set that cannot be well-ordered. So, as desired $\mathcal{N}$ is a model of ZF where AC fails.

3.2.4 Further undecidable hypotheses

Gödel's First Incompleteness Theorem states that any axiom system sufficient to express elementary arithmetic cannot be both consistent and complete. If ZFC is consistent, then it cannot be complete. Then ZFC must have models which satisfy different statements. We have learned about the most prominent examples of incompleteness in the preceding part of this section.

Meanwhile plenty of independence results have been established by forcing although none being as disturbing as the undecidability of the simple continuum hypothesis $\aleph_1 = 2^{\aleph_0}$ which in case of real numbers can be expressed as "every uncountable set $S \subseteq \mathbb{R}$ has a bijective mapping on $\mathbb{R}$".

The classical examples include the generalized continuum hypothesis and Souslin's problem.

The generalized continuum hypothesis [Felix Hausdorff: "Grundzüge einer Theorie der geordneten Mengen", Mathematische Annalen 65 (1908) p. 494] states that for every transfinite cardinal number $\aleph_\alpha$ the next cardinal number is

$$\aleph_{\alpha+1} = 2^{\aleph_\alpha} .$$

Souslin in 1920 considered a complete linearly ordered set (R, <) without endpoints where every set of disjoint open intervals is at most countable. ["Problème de M.M. Souslin", ICM Bibliotheka Wirtualna Matematyki] The question whether (R, <) must be isomorphic to the real line cannot be answered in ZFC.

A bewildering facet of undecidability is that it is consistent that Gödel's explicit construction of a well-ordered subset of the real numbers is "all real numbers" (cp. 2.17.2 "Gödel's constructible universe L"), whereas Cohen's construction (cp. 3.2.2 "The continuum hypothesis cannot be proved in ZFC") shows that it is also consistent that this subset is not all real numbers.

All these problems become decidable when the axiom of constructibility is assumed; but this axiom is contradicting the power set axiom of ZFC – and nobody is authorized to modify it.

3.3 The paradox of Tristram Shandy

The belief in the possibility to finish infinite bijections raises the paradoxical result that Adolf A. Fraenkel explained by Laurence Sterne's novel "The life and opinions of Tristram Shandy, gentleman": "Well known is the story of Tristram Shandy who undertakes to write his biography, in fact so pedantically, that the description of each day takes him a full year. Of course he will never get ready if continuing that way. But if he lived infinitely long (for instance a 'countable infinity' of years [...]), then his biography would get 'ready', because, expressed more precisely, *every* day of his life, how late ever, finally would get its description because the year scheduled for this work would some time appear in his life." [A. Fraenkel: "Einleitung in die Mengenlehre", 3rd ed., Springer, Berlin (1928) p. 24] "If he is mortal he can never terminate; but did he live forever then no part of his biography would remain unwritten, for to each day of his life a year devoted to that day's description would correspond." [A.A. Fraenkel, A. Levy: "Abstract set theory", 4th ed., North Holland, Amsterdam (1976) p. 30]

To have an example with a simpler ratio consider Scrooge McDuck who per day earns 10 $ and spends 1 $. The dollar bills are enumerated by the natural numbers. McDuck receives and spends them in natural order. If he lived forever he would go bankrupt by the same argument. (Using coins he would get rich.)

3.4 The Löwenheim-Skolem paradox

Another paradox or antinomy (depending on the standpoint) has been published by Thoralf Skolem[1] based on previous work by Leopold Löwenheim[2]: Every first-order theory like ZFC, that has a model (i.e., that is consistent), also has a countable model. This appears as a hard blow to set theory, which mainly has gained interest because of the existence of uncountable sets.

3.4.1 Skolem's first proof

In 1920 Skolem introduced his "normal form[3]", showed that every satisfiable well formed formula of first order predicate calculus has a satisfiable Skolem normal form (and vice versa), and improved and generalized the proof of Löwenheim's theorem.

Theorem "*Every proposition in normal form either is a contradiction or is already satisfiable in a finite or denumerably infinite*[4] *domain.*" [loc cit p. 256]

Let us consider first the simplest possible case containing both quantifiers

$$\forall x\ \exists y\ U(x, y)$$

where $U(x, y)$ is a proposition constructed by means of conjunction ($\wedge$), disjunction ($\vee$), and negation ($\neg$) from relative coefficients having only x and y as arguments[5]. Then *by virtue of the axiom of choice* for every x a uniquely determined y can be chosen in such a way that $U(x, y)$ comes out true. (Remember we consider only satisfiable propositions.) Assume that for every x, x' is the image of x. Then for the values assigned to the relative symbols, the proposition $U(x, x')$ is true for every x. Calling O the domain, we can write this as

$$\forall x \in O\text{: } U(x, x') .$$

Let a be a particular individual of O. Then there exist certain classes X included in O that, first, contain a as an element ($X(a)$ is true) and, second, contain x', whenever they contain x.[6] Now let

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[1] [Th. Skolem: "Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze nebst einem Theoreme über dichte Mengen", Vidensskapsselskapets Skrifter I, Mat.-naturv. Klasse No. 4, Kristiania (1920), partially translated as "Logico-combinatorial investigations in the satisfiability or provability of mathematical propositions: A simplified proof of a theorem by L. Löwenheim and generalizations of the theorem" in J. van Heijenoort: "From Frege to Gödel – A source book in mathematical logic, 1879-1931", Harvard Univ. Press, Cambridge (1967) pp. 252-263]

[2] [Leopold Löwenheim: "On possibilities in the calculus of relatives" (1915) loc cit pp. 229-251]

[3] A *Skolem normal form* is a string of all universal quantifiers ($\forall$) followed by a string of all existential quantifiers ($\exists$) followed by a quantifier-free expression. (Skolem and Löwenheim wrote Π and Σ instead of $\forall$ and $\exists$.)

[4] "finite or denumerably infinite" will henceforth be abbreviated by "countable".

[5] Skolem writes U_{xy} and calls it *Aussage* (proposition) and the *xy Indizes* (subscripts). A binary *relative* U is a set of ordered pairs (x, y) for which U_{xy} holds.

[6] "$X(x')$ whenever $X(x)$ is true", or, in other words, "$\neg X(x) \vee X(x')$ is true for every x", briefly $X(x) \Rightarrow X(x')$.

X^0 be the logical (identical) product (i.e. the intersection) of all these classes. Then, according to Dedekind's theory of chains (see footnote 1 on p. 48), X^0 is a countable class. But it is clear further, that

$$\forall x \in X^0 \colon U(x, x')$$

must hold. Hence this proves that, if $\forall x\, \exists y\, U(x, y)$ is satisfied in a domain O, it is also satisfiable in a countable domain.

In order to prove the theorem in full generality Skolem inserts two lemmas.

Lemma 1. Let $R(x_1, ..., x_m, y_1, ..., y_n)$ be an $(m + n)$-ary relation which holds if for arbitrary given $x_1, ..., x_m$ in the domain O there is one and only one y_1, one and only one y_2, and so forth up to one and only one y_n. Let K be an arbitrary finite class, and let K_1 be the class of all those values of $y_1, ..., y_n$ that correspond to the various possible choices of $x_1, ..., x_m$ in K. Then $K \cup K_1$ is a finite class too.

This proposition is clearly true because if K consists of k objects, then there exist altogether k^m possible choices of $x_1, ..., x_m$, hence also k^m corresponding sequences $y_1, ..., y_n$. K_1 contains at most $k^m n$ elements and $K \cup K_1$ contains at most $k + k^m n$ objects.

Lemma 2. Let R be an $(m + n)$-ary relative with the property mentioned in lemma 1, and let Ξ be the logical product of all classes X that possess the two properties:

1) a is an element of X.

2) If $x_1, ..., x_m$ are arbitrarily chosen in X, then X also contains the objects $y_1, ..., y_n$ for which $R(x_1, ..., x_m, y_1, ..., y_n)$ holds.

Then Ξ is a countable class.

The class $K \cup K_1$ is defined as in lemma 1. Further $\{a\}$ is a singleton. Consider those classes of classes having two properties, namely containing $\{a\}$ as an element and if containing a class K, also containing $K \cup K_1$ as an element. The intersection A of these classes of classes is an ordinary Dedekind chain (see footnote 1 on p. 48) and consists of the classes $\{a\}, \{a\}', \{a\}'', ...$. The classes that are elements of A need not be mutually distinct; in any case A is a countable class of classes. In consequence of lemma 1 every element of A must be a finite class, for $\{a\}$ is finite, and if K is finite then also $K \cup K_1$ is finite. According to the definition of A the class of all finite classes must contain all of A, that is, every element of A is a finite class.

Then, by well-known set theoretic theorems, the sum $\bigcup A$ of all classes that are elements of A must be countable. This sum $\bigcup A$ is Ξ. First $\{a\}$ is included in Ξ as a subclass and with K also $K \cup K_1$ is a subclass. Therefore every element of A is a subclass of Ξ. Conversely Ξ must also be a subclass of $\bigcup A$. a is an element of $\bigcup A$, and if $x_1, ..., x_m$ are arbitrarily chosen in $\bigcup A$ then, since $\{a\}$ is a subclass of $\{a\}'$ and $\{a\}'$ is a subclass of $\{a\}''$ ..., there exists an element K of A that contains all $x_1, ..., x_m$. Then every y of $y_1, ..., y_n$ belongs to $K \cup K_1$, which is the successor element of K in A. Consequently all of the y belong to $\bigcup A$. According to the definition of Ξ, Ξ must then be a subclass of $\bigcup A$. So $\bigcup A = \Xi$, from which it follows that Ξ is countable.

It is now easy to prove the theorem in general. Assume the general first-order proposition

$$\forall x_1\ \forall x_2 \ldots \forall x_m\ \exists y_1\ \exists y_2 \ldots \exists y_n\ U(x_1, \ldots, x_m, y_1, \ldots, y_n)\ . \qquad (*)$$

By the axiom of choice we can then, for every choice of $x_1, \ldots, x_m$, imagine a definite sequence chosen from among the corresponding sequences $y_1, \ldots, y_n$ for which $U(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is true. Denote the corresponding sequence of y by $y_1(x_1, \ldots, x_m)$, $y_2(x_1, \ldots, x_m)$, ..., $y_n(x_1, \ldots, x_m)$. Then the proposition

$$U(x_1, \ldots, x_m, y_1(x_1, \ldots, x_m), y_2(x_1, \ldots, x_m), \ldots, y_n(x_1, \ldots, x_m)) \qquad (**)$$

holds for every choice of $x_1, \ldots, x_m$ and the relation has the properties considered in the lemmas. Hence, if we assume that a is a particular individual and that Ξ is the intersection of all classes X that contain a as an element and contain the y of the sequence $y_1, y_2, \ldots, y_n$ whenever they contain $x_1, \ldots, x_m$, then Ξ is countable and (**) holds for all possible choices of $x_1, \ldots, x_m$ in this Ξ.

Skolem then derives some generalizations of his theorem, for instance: If a proposition can be represented as a product of a denumerable set of first-order propositions, it is either a contradiction or is already satisfiable in a denumerable domain. But the main point has been made: There is no absolute uncountability.

3.4.2 Skolem's second proof

While Skolem's first proof made use of the axiom of choice and proved that a formula F of quantification theory that is satisfiable in a domain is also satisfiable in a countable subdomain, his second proof, delivered 1922 before the Fifth Congress of Scandinavian Mathematicians at Helsinki, avoided the axiom of choice and proved that every satisfiable formula, with some appropriate adaption of the predicate letters of F, is also satisfiable in the domain $\mathbb{N}$. This proof adhered closer than his first one to Löwenheim's original approach. [T. Skolem: "Einige Bemerkungen zur axiomatischen Begründung der Mengenlehre", Akademiska Bokhandeln, Helsinki (1923) pp. 217-232, reprinted as "Some remarks on axiomatized set theory" in J. van Heijenoort: "From Frege to Gödel – A source book in mathematical logic, 1879-1931", Harvard University Press, Cambridge (1967) pp. 290-301]

Among other points Skolem first discusses the fact that Zermelo's system is not sufficient to provide a foundation for ordinary set theory. Probably no one will find Zermelo's explanations of "definite proposition" satisfactory. Skolem gives a better explanation of "definite proposition", namely a finite expression constructed from the five logical operations conjunction, disjunction, negation, and universal and existential quantification. Further he gives another proof of Löwenheim's theorem: If a first-order proposition is satisfied in any domain at all, it is already satisfied in a countable domain. Skolem breaks this down to the set of natural numbers.

Let there be given an infinite sequence of first-order propositions $U_1, U_2, \ldots$ which are assumed to hold simultaneously. By a suitable choice of the class and relation symbols occuring in the propositions, they can be assumed to hold in the infinite sequence of positive integers.

Again Skolem uses the *normal form* given above (*). The proof proceeds then by way of an infinite sequence of steps. First let $x_1 = x_2 = ... = x_m = 1$. If U is consistent, then it must be possible to choose $y_1, ..., y_n$ among the numbers 1, 2, ..., $n + 1$ such that $U(1, 1, ..., 1, y_1, ..., y_n)$ is satisfied. The second step consists in choosing for $x_1, ..., x_m$ every permutation with repetition of the numbers 1, 2, ..., $n + 1$ (taken m at a time) with exception of the first permutation 1, 1, ..., 1. For at least one of the solutions obtained in the first step it must then be possible, for each of the $(n + 1)^m - 1$ permutations, to choose $y_1, ..., y_n$ among the numbers 1, 2, ..., $n + 1 + n((n + 1)^m - 1)$ in such a way that for each permutation $x_1, ..., x_m$ taken within the segment 1, 2, ..., $n + 1$, the proposition holds for a corresponding choice taken within 1, 2, ..., $n + 1 + n((n + 1)^m - 1)$. Thus from certain solutions gained in the first step we now obtain certain continuations, which constitute the solutions of the second step. It must be possible to continue the process in this way indefinitely if the given first-order proposition is consistent.

Then Skolem shows that it is possible to obtain a uniquely determined solution for the entire number sequence and applies his result to generalize Löwenheim's theorem in the case of Zermelo's axioms, which, as first-order propositions, can be enumerated according to their form. Skolem concludes: "*If Zermelo's axiom system, when made precise, is consistent, it must be possible to introduce an infinite sequence of symbols* 1, 2, 3, ... *in such a way that they form a domain B in which all of Zermelo's axioms hold provided these symbols are suitably grouped into pairs of the form* $a \in b$." [loc cit p. 295]

One of the symbols will be the null set (no other will stand in the $\in$-relation with that symbol). If a is one of the symbols, then $\{a\}$ is another. If M is a symbol, then $\mathcal{P}M$, $\bigcup M$, and $\bigcap M$ will be others.

In the discussion Skolem notes: "So far as I know, no one has called attention to this peculiar and apparently paradoxical state of affairs. By virtue of the axioms we can prove the existence of higher cardinalities, of higher number classes, and so forth. How can it be, then, that the entire domain B can already be enumerated by means of the finite positive integers? The explanation is not difficult to find. In the axiomatization, 'set' does not mean an arbitrarily defined collection; the sets are nothing but objects that are connected with one another through certain relations expressed by the axioms. Hence there is no contradiction at all if a set M of the domain B is nondenumerable in the sense of the axiomatization; for this means merely that *within B* there occurs no one-to-one mapping Φ of M onto Z_0 (Zermelo's number sequence). Nevertheless there exists the possibility of numbering all objects in B, and therefore also the elements of M, by means of the positive integers; of course, such an enumeration too is a collection of certain pairs, but this collection is not a 'set' (that is, it does not occur in the domain B). [...]

Even the notions 'finite', 'infinite', 'simply infinite sequence', and so forth turn out to be merely relative within axiomatic set theory." [loc cit p. 295]

We see that Skolem denied an antinomy because the definition of countability of a set in the model requires a bijection with the natural numbers within the model. If this bijection, also being a set, does not exist in the model, then the set is uncountable there, although it may be countable from outside. However, this implies that "*set*" *does not mean an arbitrarily defined collection*! It completely invalidates set theory as a theory of sets of real objects or of their names.

Nevertheless Skolem emphasized the relativism of uncountability. "In order to obtain something absolutely nondenumerable, we would have to have either an absolutely nondenumerably infinite number of axioms or an axiom that could yield an absolutely nondenumerable number of first-order propositions. But this would in all cases lead to a circular introduction of the higher infinities; that is, *on an axiomatic basis higher infinities exist only in a relative sense.*

With a suitable axiomatic basis, therefore, the theorems of set theory can be made to hold in a merely *verbal* sense, on the assumption, of course, that the axiomatization is consistent; but this rests merely upon the fact that the use of the *word* 'set' has been regulated in a suitable way. We shall always be able to define collections that are not called sets; if we were to call them sets, however, the theorems of set theory would cease to hold. [...]

Concluding remark: The most important result above is that set-theoretic notions are relative. [...] I believed that it was so clear that axiomatization in terms of sets was not a satisfactory ultimate foundation of mathematics that mathematicians would, for the most part, not be very much concerned with it. But in recent times I have seen to my surprise that so many mathematicians think that these axioms of set theory provide the ideal foundation for mathematics; therefore it seemed to me that the time had come to publish a critique." [loc cit pp. 296 & 300f]

3.4.3 Reception of Skolem's results

"The reception of Skolem's paradox illustrates the delay in the absorbing of new ideas in science. Fraenkel's influential 'Einleitung in die Mengenlehre' [1919] is a good example to trace the influence of Skolem. The second edition of 1923 mentions Skolem's paper, which had only just become available to Fraenkel, in a footnote. The paradox is referred to as 'a difficulty which has so far not yet been overcome'.

In the subsequent 'Zehn Vorlesungen über die Grundlegung der Mengenlehre' [1927], Skolem's paradox gets its own section, where it is discussed as a new, alarming attack at the axiomatic foundation of set theory. Fraenkel was not convinced of the correctness of the arguments of Skolem, he built in the *caveat* 'if the conclusions of Löwenheim and Skolem proceed without gaps and errors'. He did not see a solution to the paradox, but was inclined to see impredicativity as a possible source of the problem. The third edition of the 'Einleitung' [1928)] again questions the correctness of the Skolem argument. In spite of Skolem's crystal clear exposition, Fraenkel states:

'Since neither the books have at present been closed on the antinomy, nor on the significance and possible solution so far an agreement has been reached, we will restrict ourselves to a suggestive sketch.'

The remarks show that the role of logic in set theory was not quite clear to Fraenkel, no matter how much he admired Hilbert's proof theory. Apparently Skolem's arguments were beyond his expertise. [...]

Von Neumann straightforwardly acknowledged the relativity phenomenon, he ended his paper with the words:

'At present we can do no more than note that we have one more reason here to entertain reservations about set theory and that for the time being no way of rehabilitating this theory is known.'

Where the majority of the mathematicians followed Fraenkel's scepticism, and a few von Neumann's resignation, the first set theoretician to surpass Cantor in his own field, Ernst Zermelo, had decided that the Skolem paradox was a hoax. [...]

On October 4, 1937, under the title 'Relativism in set theory and the so-called Theorem of Skolem', he wrote down what he thought would be such a refutation. [...] The argument is clever, and it would probably have confused most readers, it certainly confused Zermelo himself. [...]

How could this happen? Surely, Zermelo would have been able to mathematically understand, say, Skolem's proof of the Löwenheim-Skolem Theorem. However, he seems to have been blocked to view this theorem as a purely mathematical statement and, instead, was caught in the special case of a first-order axiomatization of set theory, a system that – as we have seen – totally contradicted his understanding of set theory and strongly evoked his epistemological resistance." [D. van Dalen, H.-D. Ebbinghaus: "Zermelo and the Skolem paradox", Utrecht Research Institute for Philosophy, Logic Group, Preprint Series 183 (1998) pp. 3 & 10f]

"What about Zermelo? When faced with the existence of countable models of first-order set theory, his first reaction was not the natural one, namely to check Skolem's proof and evaluate the result – it was immediate rejection. Apparently, the motivation of ensuring 'the valuable parts of set theory' which had led his axiomatizations from 1908 and from the *Grenzzahlen* paper had not only meant allowing the deduction of important set-theoretic facts, but had included the goal of describing adequately the set-theoretic universe with its variety of infinite cardinalities. Now it was clear that Skolem's system, like perhaps his own, failed in this respect. Moreover, Skolem's method together with the epistemological consequences Skolem had drawn from his results, could mean a real danger for mathematics like that caused by the intuitionists: In his Warsaw notes W3 he had clearly stated that 'true mathematics is indispensably based on the assumption of infinite domains', among these domains, for instance, the uncountable continuum of the real numbers. Hence, following Skolem, 'already the problem of the power of the continuum loses its true meaning'.

Henceforth Zermelo's foundational work centred around the aim of overcoming Skolem's relativism and providing a framework in which to treat set theory and mathematics adequately. Baer speaks of a real *war* Zermelo had started [R. Baer, letter to E. Zermelo (27 May 1930)] [...] 'It is well known that inconsistent premises can prove anything one wants; however, even the strangest consequences that Skolem and others have drawn from their basic assumption, for instance the relativity of the notion of subset or equicardinality, still seem to be insufficient to raise doubts about a doctrine that, for various people, already won the power of a *dogma* that is beyond all criticism.' [E. Zermelo: 'Über Stufen der Quantifikation und die Logik des Unendlichen', Forschung und Fortschritte 8 (1932) p. 6f] [...] in the beginning Zermelo had doubts about the correctness of Skolem's proof {{then}} he tried right away to disprove the existence of countable models. [...] Starting with a countable model $\mathcal{M}$, he invoked various methods to obtain a contradiction by constructing sequences of subsets of ω in $\mathcal{M}$ of length uncountable in $\mathcal{M}$. [...] Besides his endeavour to refute the existence of countable models of set theory by providing a concrete inconsistency, Zermelo proceeded more systematically. The tone of the resulting papers – both the published and the unpublished ones – is harsh. [...] Zermelo regarded Skolem's position as a real danger for mathematics and, therefore, saw 'a particular duty' to fight against it. [...] His remedy consisted of infinitary languages. [...] Skolem had considered such a possibility, too, but had discarded it because of a vicious circle." [Heinz-Dieter Ebbinghaus: "Ernst Zermelo: An approach to his life and work", Springer (2007) p. 200ff]

Usually set theorists are inclined to play down the far reaching implications of Skolem's results. A typical treatment is given in the following: "A crucial counterexample is the powerset of $\aleph_0$, denoted by $2^{\aleph_0}$. Naively, one might suppose that the powerset axiom[1] of ZFC guarantees that $2^{\aleph_0}$ must be a member of any standard transitive model *M*. But let us look more closely at the precise statement of the powerset axiom. Given that $\aleph_0$ is in *M*, the powerset axiom guarantees the existence of *y* in *M* with the following property: For every *z* in *M*, $z \in y$ if and only if every *w* in *M* satisfying $w \in z$ also satisfies $w \in \aleph_0$. Now, does it follow that *y* is precisely the set of all subsets of $\aleph_0$?

No. First of all, it is not even immediately clear that *z* is a subset of $\aleph_0$; the axiom does not require that *every w* satisfying $w \in z$ also satisfies $w \in \aleph_0$; it requires only that *every w in M* satisfying $w \in z$ satisfies $w \in x$. However, under our assumption that *M* is *transitive*[2], every $w \in z$ is in fact in *M*, so indeed *z* is a subset of $\aleph_0$.

More importantly, though, *y* does not contain *every* subset of $\aleph_0$; it contains *only those subsets of x that are in M.* So if, for example, *M* happens to be *countable* (i.e., *M* contains only countably many elements), then *y* will be countable, and so a fortiori *y* cannot be equal to $2^{\aleph_0}$, since $2^{\aleph_0}$ is uncountable. The set *y*, which we might call the *powerset of* $\aleph_0$ *in M,* is not the same as the 'real' powerset of $\aleph_0$, a.k.a. $2^{\aleph_0}$; many subsets of $\aleph_0$ are 'missing' from *y*. This is a subtle and important point, [...]

More generally, one says that a concept in *V* is *absolute* if it coincides with its counterpart in *M*. For example, 'the empty set', 'is a member of', 'is a subset of', 'is a bijection', and '$\aleph_0$' all turn out to be absolute for standard transitive models. On the other hand, 'is the powerset of' and 'uncountable' are *not* absolute. For a concept that is not absolute, we must distinguish carefully between the concept 'in the real world' (i.e., in *V*) and the concept in *M*. A careful study of ZFC necessarily requires keeping track of exactly which concepts are absolute and which are not. [...] the majority of basic concepts are absolute, except for those associated with taking powersets and cardinalities," [Timothy Y. Chow: "A beginner's guide to forcing", arXiv (2008) p. 6f]

"Since the Skolem-Löwenheim 'paradox', namely, that a countable model of set theory exists which is representative of the stumbling blocks that a nonspecialist encounters, I would like to briefly indicate how it is proved. What we are looking for is a countable set *M* of sets, such that if we ignore all other sets in the universe, a statement in *M* is true precisely if the same statement is true in the true universe of all sets. After some preliminary manipulation, it is possible to show that all statements can be regarded as starting with a sequence of quantifiers, for all, there exists, etc. The set of all statements can be enumerated, say A_n. We go through the list and {{for}} every statement which begins with 'there exists' *and* is true in the universe, we pick out one set in the universe which makes it true. Since there are only countably many statements, we have chosen only countably many elements and we place them in *M*. Next we form all statements using these sets and again only look at those which begin with 'there exists'. If they are true in the universe,

[1] Every set *x* has a so-called power set $y = \mathcal{P}(x)$. This is expressed formally as $\forall x\ \exists y\ \forall z: z \in y \Leftrightarrow z \subseteq x$. Compare also section 2.12 "ZFC-Axioms of set theory".

[2] A standard model *M* of ZFC is *transitive* if every member of an element of *M* is also an element of *M*. (The term *transitive* is used because we can write the condition in the suggestive form '$x \in y$ and $y \in M$ implies $x \in M$'.) This is a natural condition to impose if we think of *M* as a universe consisting of 'all that there is'; in such a universe, sets 'should' be sets of things that already exist in the universe.

we pick out one set which makes them true and adjoin these to *M*. We repeat this process countably many times. The resulting collection of all sets so chosen is clearly countable. Now it is easy to see that the true statements of *M* are exactly the true statements in the universe. This is proved by induction on the number of quantifiers appearing at the beginning of the statement. If there are none, then the statement simply is composed of finitely many statements of the form '*x* is a member of *y*', connected by the Boolean operators. This clearly is true in *M* if and only if it is true in the universe. Now consider a statement with one quantifier. By considering its negation, if necessary, we may assume it begins with 'there exists'. Now our choosing process clearly guarantees that the statement is true in *M* if and only if it is true in the universe. The proof now proceeds by a simple induction on the number of quantifiers.

You may feel that this argument is too simple to be correct, but I assure you that this is the entire argument, needing only a very simple argument to show that one can always bring the quantifiers to the front of the statement. I might add that the underlying reason the argument is so simple is because it applies to any system whatsoever, as long as we have only finitely many predicates (even countably many will work the same way) so that the number of statements that can be formed is countable. This theorem is perhaps a typical example of how a fundamental result which has such wide application must of necessity be simple." [Paul J. Cohen: "The discovery of forcing", Rocky Mountain Journal of Mathematics 32,4 (2002) p. 1076f]

3.5 Vitali sets

The interval $[0, 1] = \{x \mid 0 \le x \le 1\}$ has measure 1. The set of rational numbers contained therein (and even in the whole set $\mathbb{R}$) has measure 0. The reason is that the rational numbers are countable. So we can construct a sequence (q_n) of all rational numbers. We can include the *n*th number into an interval of measure $\varepsilon/2^n$. Then all rational numbers are included in intervals of total measure ε. Since ε can be made arbitrarily small, smaller than every positive constant, the measure is smaller than every positive constant, that is the measure is $\mu = 0$.

Guiseppe Vitali proved the existence of unmeasurable sets of real numbers, i.e., violation of σ-additivity

$$\mu\left(\bigcup_{k\in\mathbb{N}} S_k\right) = \sum_{k\in\mathbb{N}} \mu(S_k)$$

by the following construction [G. Vitali: "Sul problema della misura dei gruppi di punti di una retta", Bologna, Tip. Gamberini e Parmeggiani (1905)]: If we add an irrational number *x* to every rational q_n, we get a set of only irrational numbers. (The distance $|q_\mu - q_\nu|$ between two rational numbers q_μ and q_ν is always a rational). Since there are uncountably many irrational numbers in [0, 1], we can construct, by axiom of choice, as many Vitali sets. The union of all these sets contains at least all real numbers of the interval [0, 1] and at most the real numbers of the interval [0, 2]. So the measure must be between 1 and 2. But the sum of infinitely many similar sets with fixed measure is either 0 or infinite. Therefore the Vitali sets cannot have any measure.

3.6 The Hausdorff sphere paradox

Felix Hausdorff shows in [F. Hausdorff: "Bemerkungen über den Inhalt von Punktmengen", Math. Annalen 75 (1914) pp. 428-433] that additivity of measure $\mu(\cup A_k) = \Sigma\mu(A_k)$ is violated already for finite sums of sets A_k.

First Hausdorff recapitulates Vitali's example (see section 3.5), applying it to a circle.

Consider a circle with radius $r = 1/2\pi$. All points with integer difference are identical: $x = x \pm n$. Let ξ be an irrational number. The countable set of points

$$P_x = \{..., x - 2\xi, x - \xi, x, x + \xi, x + 2\xi, ...\}$$

has no point in common with the similar set P_y unless $y = x \pm n\xi$ for some n. Then the sets are identical. Taking from each of the *different* sets one element, we build the set $A_0 = \{x, y, z, ...\}$. Turning the circle by $m\xi$, we obtain another set $A_m = \{x + m\xi, y + m\xi, z + m\xi, ...\}$ which has no point in common with the set A_0. In this way we can construct a countable set of congruent sets. The measure $\mu(A_m)$ of each one must be $\mu(A_m) \leq 1/n$ for every $n \in \mathbb{N}$, i.e., it must be $\mu(A_m) = 0$ whereas the sum of all measures is 1. This means the sets cannot be measured.

Since every set in the n-dimensional space can be expanded to a set in the n+1-dimensional space (by assigning the height 1 to an n-dimensional basis), this one-dimesional example shows that the measure-problem exists in the n-dimensional space for every $n \in \mathbb{N}$.

Then Hausdorff shows that additivity is even violated for finite numbers of n pairwise disjoint sets A_k. The formula

$$\mu(A_1 \cup A_2 \cup ... \cup A_n) = \mu(A_1) + \mu(A_2) + ... + \mu(A_n)$$

can be contradicted in spaces of three or more dimensions. For this sake the surface of the sphere is divided into three sets A, B, and C, such that A, B, and C, and $B \cup C$ are pairwise congruent.

Let φ be a rotation by π around a given axis and let ψ be a rotation by $2\pi/3$ around a different axis. Since φ^2 and ψ^3 yield the identity, i.e., the same result as one full or no rotation, we get the group G

$$I \mid \varphi, \psi, \psi^2 \mid \psi\varphi, \psi^2\varphi, \varphi\psi, \varphi\psi^2 \mid ... \qquad (G)$$

where the products of different numbers of factors have been separated by strokes[1]. φ, ψ, and ψ^2 are considered as different elements of the group. φ^2 and ψ^3 are not included because they are the empty word or unit I.

[1] Note that the order of the factors has been reversed with respect to the original paper by Hausdorff because we adhere to the matrix notation, such that the factor to be applied first appears on the right-hand side of the product.

The general formula of a product is one of the following four:

$$\alpha = \psi^{m_n}\varphi \ldots \psi^{m_2}\varphi\psi^{m_1}\varphi$$
$$\beta = \varphi\psi^{m_n}\varphi \ldots \psi^{m_2}\varphi\psi^{m_1}$$
$$\gamma = \varphi\psi^{m_n}\varphi \ldots \psi^{m_2}\varphi\psi^{m_1}\varphi$$
$$\delta = \psi^{m_n}\varphi \ldots \psi^{m_2}\varphi\psi^{m_1}$$

with $m_k = 1$ or 2. The insertion of φ^2 or ψ^3 into the expressions would not change them because φ with every exponent $m_k = 0 \bmod 2$ and ψ with every exponent $m_k = 0 \bmod 3$ produce the unit I.

Now Hausdorff shows that if the axes are suitably chosen all products of elements of the group G lead to results that differ from I.

Proof: If a product I is possible, then it can be assumed in the form of α because:

$$\beta = I \Rightarrow \varphi\beta\varphi = I \Rightarrow \varphi\varphi\alpha = \alpha = I$$
$$\gamma = I \Rightarrow \varphi\gamma\varphi = I \Rightarrow \varphi\varphi\delta\varphi\varphi = \delta = I$$
$$\delta = I \Rightarrow \psi^{m_1}\delta\psi^{3-m_1} = I \Rightarrow \psi^{m_1}\psi^{m_n}\varphi \ldots \psi^{m_2}\varphi\psi^{m_1}\psi^{3-m_1} = \psi^{m_1}\psi^{m_n}\varphi \ldots \psi^{m_2}\varphi = \alpha' = I$$

where α' has the factor ψ^1 or ψ^2 on the left-hand side and therefore is an α.

The ψ-axis is chosen in z-direction. The φ-axis is chosen in the x-z-plane with the angle $\vartheta/2$ between both axes. Then the matrix of ψ is obtained from the rotation matrix in z-direction

$$\begin{pmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} & 0 \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrix of ψ^2 is also obtained from the rotation matrix in z-direction

$$\begin{pmatrix} \cos\frac{4\pi}{3} & -\sin\frac{4\pi}{3} & 0 \\ \sin\frac{4\pi}{3} & \cos\frac{4\pi}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

by simply changing the signs of the terms $\frac{\sqrt{3}}{2}$.

The matrix of φ, a π-rotation around the axis defined by the unit vector $\overline{N^0} = \begin{pmatrix} \sin\frac{\vartheta}{2} \\ 0 \\ \cos\frac{\vartheta}{2} \end{pmatrix}$, is

$$\begin{pmatrix} \sin^2\frac{\vartheta}{2}(1-\cos\pi)+\cos\pi & 0 & \sin\frac{\vartheta}{2}\cos\frac{\vartheta}{2}(1-\cos\pi) \\ 0 & \cos\pi & 0 \\ \cos\frac{\vartheta}{2}\sin\frac{\vartheta}{2}(1-\cos\pi) & 0 & \cos^2\frac{\vartheta}{2}(1-\cos\pi)+\cos\pi \end{pmatrix} = \begin{pmatrix} -\cos\vartheta & 0 & \sin\vartheta \\ 0 & -1 & 0 \\ \sin\vartheta & 0 & \cos\vartheta \end{pmatrix}.$$

The matrix of the product $\psi\varphi$ is therefore

$$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} -\cos\vartheta & 0 & \sin\vartheta \\ 0 & -1 & 0 \\ \sin\vartheta & 0 & \cos\vartheta \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\cos\vartheta & \frac{\sqrt{3}}{2} & -\frac{1}{2}\sin\vartheta \\ -\frac{\sqrt{3}}{2}\cos\vartheta & \frac{1}{2} & \frac{\sqrt{3}}{2}\sin\vartheta \\ \sin\vartheta & 0 & \cos\vartheta \end{pmatrix}$$

and the matrix of the product $\psi^2\varphi$ is obtained by changing the signs of the expressions $\frac{\sqrt{3}}{2}$.

Now let α be a product of n double-factors $\psi\varphi$ or $\psi^2\varphi$ and let α' be a product of $n + 1$ such double-factors. Let α transform the point $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ into $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and let α' transform $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ into $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$. Then

$$\begin{pmatrix} \frac{1}{2}\cos\vartheta & \pm\frac{\sqrt{3}}{2} & -\frac{1}{2}\sin\vartheta \\ \mp\frac{\sqrt{3}}{2}\cos\vartheta & \frac{1}{2} & \pm\frac{\sqrt{3}}{2}\sin\vartheta \\ \sin\vartheta & 0 & \cos\vartheta \end{pmatrix} \bullet \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}.$$

Let us choose $\psi\varphi$ for the following (the choice of $\psi^2\varphi$ leads to a similar derivation)

$$\begin{pmatrix} \frac{1}{2}\cos\vartheta & \frac{\sqrt{3}}{2} & -\frac{1}{2}\sin\vartheta \\ -\frac{\sqrt{3}}{2}\cos\vartheta & \frac{1}{2} & \frac{\sqrt{3}}{2}\sin\vartheta \\ \sin\vartheta & 0 & \cos\vartheta \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\sin\vartheta \\ \frac{\sqrt{3}}{2}\sin\vartheta \\ \cos\vartheta \end{pmatrix}.$$

Applying again $\psi\varphi$ onto the result we get

$$\begin{pmatrix} \frac{1}{2}\cos\vartheta & \frac{\sqrt{3}}{2} & -\frac{1}{2}\sin\vartheta \\ -\frac{\sqrt{3}}{2}\cos\vartheta & \frac{1}{2} & \frac{\sqrt{3}}{2}\sin\vartheta \\ \sin\vartheta & 0 & \cos\vartheta \end{pmatrix} \bullet \begin{pmatrix} -\frac{1}{2}\sin\vartheta \\ \frac{\sqrt{3}}{2}\sin\vartheta \\ \cos\vartheta \end{pmatrix} = \begin{pmatrix} -\frac{1}{4}\cos\vartheta\sin\vartheta + \frac{3}{4}\sin\vartheta - \frac{1}{2}\sin\vartheta\cos\vartheta \\ \frac{\sqrt{3}}{4}\cos\vartheta\sin\vartheta + \frac{\sqrt{3}}{4}\sin\vartheta + \frac{\sqrt{3}}{2}\sin\vartheta\cos\vartheta \\ -\frac{1}{2}\sin\vartheta\sin\vartheta + \cos\vartheta\cos\vartheta \end{pmatrix} = \begin{pmatrix} \sin\vartheta\cdot\frac{3}{4}(1-\cos\vartheta) \\ \sin\vartheta\cdot\frac{\sqrt{3}}{4}(1+3\cos\vartheta) \\ \frac{3}{2}\cos^2\vartheta - \frac{1}{2} \end{pmatrix} .$$

After n transformations the resulting vector has the form $\begin{pmatrix} \sin\vartheta\cdot(a\cos^{n-1}\vartheta + ...) \\ \sin\vartheta\cdot(b\cos^{n-1}\vartheta + ...) \\ c\cos^n\vartheta + ... \end{pmatrix}$ where $\sin\vartheta$ is multiplied with a polynomial of degree n - 1 in $\cos\vartheta$. For n = 1 and 2 this is shown above. The general case is proved by mathematical induction. Application of another transformation ψφ yields

$$\begin{pmatrix} \frac{1}{2}\cos\vartheta & \frac{\sqrt{3}}{2} & -\frac{1}{2}\sin\vartheta \\ -\frac{\sqrt{3}}{2}\cos\vartheta & \frac{1}{2} & \frac{\sqrt{3}}{2}\sin\vartheta \\ \sin\vartheta & 0 & \cos\vartheta \end{pmatrix} \bullet \begin{pmatrix} \sin\vartheta\cdot(a\cos^{n-1}\vartheta + ...) \\ \sin\vartheta\cdot(b\cos^{n-1}\vartheta + ...) \\ c\cos^n\vartheta + ... \end{pmatrix} = \begin{pmatrix} \sin\vartheta\cdot(\frac{a-c}{2}\cos^n\vartheta + ...) \\ \sin\vartheta\cdot(\frac{\sqrt{3}(c-a)}{2}\cos^n\vartheta + ...) \\ (c-a)\cos^{n+1}\vartheta + ... \end{pmatrix}$$

where the smaller powers in the cos-polynomials have not been noted. (For $\psi^2\varphi$ only the sign of the y-term would change.) We see that the degrees of the cos-polynomials increase by 1.

The coefficients (c - a) of the z-term and $\frac{a-c}{2}$ of the x-term increase by the factor $\frac{3}{2}$ per step:

$$c' - a' = (c-a) - \frac{a-c}{2} = \frac{3}{2}(c-a) \quad .$$

Therefore after n transformations ψφ or $\psi^2\varphi$ the z-term is given by $\left(\frac{3}{2}\right)^{n-1}\cos^n\vartheta + ...$ which is not I except for a finite number of angles ϑ. By suitable choice of ϑ with the omission of countably many values the result I can be avoided such that the product α differs from I in all steps. Then the transformations of G, except I itself, have only two fixed points each on the surface of the unit sphere. Call the countable set of these points Q with $\mu(Q) = 0$. The remainder P of the surface S^2 of the sphere must have measure $\mu(S^2 \setminus Q) \equiv \mu(P) > 0$.

Every point $\overline{U} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ of P can be transformed by every element of G. This gives the countable set of different vectors or surface points, the G-orbit of $\overline{U}$

$$P_U = \{\overline{U}, \varphi\overline{U}, \psi\overline{U}, \psi^2\overline{U}, \psi\varphi\overline{U}, ...\} .$$

Choosing from every orbit P_U exactly one point (here the axiom of choice is required), we get

$M = \{\overline{U}, \overline{V}, \overline{W}, ...\}$

and the remainder P of the surface of the sphere is

$$P = M \cup \varphi M \cup \psi M \cup \psi^2 M \cup \psi\varphi M \cup ... .$$

Now it is possible to distribute these sets or transformations on two classes A, B, C such that

(1) of two transformations ρ, $\varphi\rho$ one belongs to A and the other to $B \cup C$;
(2) always one of three transformations ρ, $\psi\rho$, $\psi^2\rho$ belongs to A, B, C.

For proof assume that the products of n or less factors have been distributed already so that the conditions are satisfied. We call a product of n factors ψ_n if its last factor is ψ or ψ^2. We call a product of n factors φ_n if its last factor is φ. Every product of $n + 1$ factors then has one of the three forms $\varphi\psi_n$, $\psi\varphi_n$, $\psi^2\varphi_n$ ($\varphi\varphi_n$ would be the unit I, $\psi\psi_n$ would be $\psi^2\varphi_n$ or the unit I).

Consider a sphere with three equal surface elements A, B, C equally distributed around the z-axis. If $\rho_n = \varphi_n$ or ψ_n belongs to A, B, C, then $\psi\rho_n$ belongs to B, C, A and $\psi^2\rho_n$ belongs to C, A, B. For a sphere with two hemispheres, A and A', distributed around the $\overline{N^0}$ -axis, φ will always cause a switch between the hemispheres.

Now we define areas pointwise by the elements of G. First we distribute the factors equally between A and its complement A' by rotating around the $\overline{N^0}$ -axis:

A:	I	$\varphi\psi$	$\varphi\psi^2$	$\psi^2\varphi$	$\varphi\psi\varphi$	$\varphi\psi\varphi\psi$	$\varphi\psi^2\varphi\psi$	$\varphi\psi\varphi\psi^2$	$\varphi\psi^2\varphi\psi^2$	$\psi^2\varphi\psi^2\varphi$	...
A':	φ	ψ	ψ^2	$\varphi\psi^2\varphi$	$\psi\varphi$	$\psi\varphi\psi$	$\psi^2\varphi\psi$	$\psi\varphi\psi^2$	$\psi^2\varphi\psi^2$	$\varphi\psi^2\varphi\psi^2\varphi$	... .

Then we distribute the factors equally between A, B, and C by rotating around the z-axis, using for A exactly the same factors as above:

A:	I	$\varphi\psi$	$\varphi\psi^2$	$\psi^2\varphi$	$\varphi\psi\varphi$	$\varphi\psi\varphi\psi$	$\varphi\psi^2\varphi\psi$	$\varphi\psi\varphi\psi^2$	$\varphi\psi^2\varphi\psi^2$	$\psi^2\varphi\psi^2\varphi$	...
B:	ψ	$\psi\varphi\psi$	$\psi\varphi\psi^2$	φ	$\psi\varphi\psi\varphi$	$\psi\varphi\psi\varphi\psi$	$\psi\varphi\psi^2\varphi\psi$	$\psi\varphi\psi\varphi\psi^2$	$\psi\varphi\psi^2\varphi\psi^2$	$\varphi\psi^2\varphi$	...
C:	ψ^2	$\psi^2\varphi\psi$	$\psi^2\varphi\psi^2$	$\psi\varphi$	$\psi^2\varphi\psi\varphi$	$\psi^2\varphi\psi\varphi\psi$	$\psi^2\varphi\psi^2\varphi\psi$	$\psi^2\varphi\psi\varphi\psi^2$	$\psi^2\varphi\psi^2\varphi\psi^2$	$\psi\varphi\psi^2\varphi$	... .

Denoting the respective sets also by A, B, and C we get

$$A = M \cup \varphi\psi M \cup \varphi\psi^2 M \cup \psi^2\varphi M \cup ...$$
$$B = \psi M \cup \psi\varphi\psi M \cup \psi\varphi\psi^2 M \cup \varphi M \cup \; ...$$
$$C = \psi^2 M \cup \psi^2\varphi\psi M \cup \psi^2\varphi\psi^2 M \cup \psi\varphi M \cup ...$$

with

$$\varphi A = A' = P \setminus A, \quad \psi A = B, \;\; \psi^2 A = C .$$

The sets $P \setminus A$ and $B \cup C$ are congruent. A must have $\mu(A) = \mu(P)/3$ and $\mu(A) = \mu(P)/2$.

3.7 The Banach-Tarski paradox

The Banach–Tarski paradox [S. Banach, A. Tarski: "Sur la decomposition des ensembles de points en parties respectivement congruentes", Fundamenta Mathematicae 6 (1924) pp. 244-277], based on earlier work of Guiseppe Vitali (see sec. 3.5) and Felix Hausdorff (see sec. 3.6) on the *surface* of a sphere, states that there exists a decomposition of a *solid* sphere into a finite number of disjoint (non-measurable) parts, which can be reassembled to yield two spheres of same diameter such that no point of the original sphere is missing and no point is added. The minimum number of pieces is five [R. M. Robinson: "On the decomposition of spheres", Fund. Math. 34 (1947) pp. 246-260]. The decomposition does not work in one or two dimensions. The paradox needs the axiom of choice (cp. section 2.12) or an equivalent premise. It is not existing in pure ZF set theory. Therefore it has been taken as a contradiction of the axiom of choice (cp. for instance Borel's statement in chapter V).

The critical steps in the work of Banach and Tarski had already been furnished by Hausdorff. When composing a solid sphere by infinitely many surfaces of infinitesimal thickness, the result is established – according to the integral calculus and common sense. But with respect to the lack of common sense in this realm, we have to proceed more carefully. There is, for instance, the countable set of fixed points exempted from Hausdorff's sphere. Banach and Tarski were able to eliminate it. In the following we will prove the

Theorem If A and B are any two bounded subsets of $\mathbb{R}^3$ with non-empty interior then it is possible to partition A into finitely many pieces which can be rearranged to form B.

The standard text for this topic is [Stan Wagon: "The Banach-Tarski paradox", Cambridge University Press (1994) pp. 21-33]. Some visual impressions are given in [G. Tomkowicz, S. Wagon: "Visualizing paradoxical sets", Math. Intelligencer (2014)].

Paradoxical group Consider a group composed of two letters σ and τ, but unlike $\varphi = \varphi^{-1}$ in section 3.6, with the reversed letters, σ^{-1} and τ^{-1}, differing from σ and τ respectively, such that

$$I = \sigma\sigma^{-1} = \sigma^{-1}\sigma = \tau\tau^{-1} = \tau^{-1}\tau \,. \qquad (*)$$

Then we can concatenate the letters $\sigma, \sigma^{-1}, \tau, \tau^{-1}$ to generate finite words. If a word contains one of the pairs of (*), then this pair can be omitted without changing the result. The set W of words reduced in this way, where every reduced word differs from all other reduced words, is called a free group. The letters σ and τ are called its generators. This free group has a paradoxical decomposition; the set W of all words is the union of the empty word I and of four disjoint subsets of reduced words, where each word is beginning with one of the four letters:

$$W = W(\sigma) \cup W(\sigma^{-1}) \cup W(\tau) \cup W(\tau^{-1}) \cup \{I\} \,.$$

When prepending σ for another time, we remove one σ^{-1} from the words beginning with σ^{-1} and get all words that begin with other letters than σ (because no reduced word had begun with $\sigma^{-1}\sigma$)

$$\sigma W(\sigma^{-1}) = W(\sigma^{-1}) \cup W(\tau) \cup W(\tau^{-1}) \cup \{I\}$$

where $\{I\}$ is generated from the word σ^{-1}. Therefore we get the paradoxcial decompositions of the disjoint sets

$$W(\sigma) \cup W(\sigma^{-1}) \cup W(\tau) \cup W(\tau^{-1}) \cup \{I\} = W(\sigma) \cup \sigma W(\sigma^{-1}) .$$

In effect, four sets of cardinality $\aleph_0$ have become two sets[1]. Same holds for $\tau W(\tau^{-1})$.

Removing the holes The group of rotations can act on the set of all points of the surface S^2 of the sphere without the countable set Q of fixed points and thus induce a paradoxical decomposition of $S^2 \setminus Q$. This has already been shown in section 3.6. We will now proceed to remove the holes.

Definition: Two polygons in the plane are congruent by dissection if one of them can be decomposed into finitely many polygonal pieces that can be rearranged using isometries (and ignoring boundaries) to form the other polygon.

The set theoretic analogon of congruence may be stated in the context of group action.

Definition: Suppose G acts on X, and $A, B \subseteq X$. Then A and B are G-equidecomposable, $A \sim B$, if A and B can each be partitioned into the same finite number of G-congruent pieces. Formally

$$A = \bigcup_{k=1}^{n} A_k \quad \text{and} \quad B = \bigcup_{k=1}^{n} B_k \quad .$$

$A_k \cap A_j = \emptyset = B_k \cap B_j$ if $k < j \le n$, and there are $g_1, ..., g_n \in G$ such that, for each $k \le n$: $g_k A_k = B_k$.

Theorem If Q is a countable subset of S^2, then S^2 and $S^2 \setminus Q$ are equidecomposable by the group of rotations. (We apply two pieces using the axiom of choice, so they are not constructible).

One-dimensional example: The unit circle S^1 with the point $\begin{pmatrix}1\\0\end{pmatrix}$ removed, $S^1 \setminus \{\begin{pmatrix}1\\0\end{pmatrix}\}$ can be decomposed into two pieces, P and Q, such that after rotating Q by σ, $P \cup \sigma Q = S^1$.

Let Q be the set of all points of the circle with $\left\{\begin{pmatrix}\cos n\\ \sin n\end{pmatrix} \middle| \, n \in \mathbb{N}\right\}$. This set consists of infinitely many different points, because the circumference 2π is irrational. Let $P = (S^1 \setminus \{\begin{pmatrix}1\\0\end{pmatrix}\}) \setminus (Q)$. Rotation by $\sigma = \begin{pmatrix}\cos 1 & \sin 1\\ -\sin 1 & \cos 1\end{pmatrix}$ includes the point $\begin{pmatrix}1\\0\end{pmatrix}$ because $\sigma Q = \left\{\begin{pmatrix}\cos(n-1)\\ \sin(n-1)\end{pmatrix} \middle| \, n \in \mathbb{N}\right\}$. This

[1] Hausdorff in his decomposition used $\varphi = \varphi^{-1}$ and $\psi^2 = \psi^{-1}$. Thus $\psi W(\psi^{-1}) = W(\psi^{-1}) \cup W(\varphi) \cup \{I\}$ such that $W(\psi) \cup W(\psi^{-1}) \cup W(\varphi) \cup \{I\} = W(\psi) \cup \psi W(\psi^{-1})$. Therefore Hausdorff obtains only the reduction factor 2/3 instead of 1/2.

shifting-to-infinity technique shows how points can be absorbed. It is applied now to the set Q of fixed points remaining in the Hausdorff proof.

Proof: We seek a rotation ρ of the sphere such that the sets Q, ρQ, $\rho^2 Q$, ... are pairwise disjoint. This suffices since then, with $Q^* = \bigcup_{n\in\mathbb{N}_0} \rho^n Q$, $S^2 = Q^* \cup (S^2 \setminus Q^*) \sim \rho Q^* \cup (S^2 \setminus Q^*) = S^2 \setminus Q$. Let $\overline{N^0}$ be a vector such that $\{x \cdot \overline{N^0} \mid x \in \mathbb{R}\} \cap Q = \emptyset$. Let Ψ be the set of angles θ such that for some $n > 0$ and some $q \in Q$, ρq is also in Q, where ρ is the rotation about $\overline{N^0}$ through the angle $n\theta$. Since Ψ is countable there exists an angle $\varphi \notin \Psi$. Let σ be the corresponding rotation about $\overline{N^0}$, then for every $n > 0$: $\sigma^n Q \cap Q = \emptyset$, and whenever $0 \le m < n$ then $\sigma^m Q \cap \sigma^n Q = \emptyset$. So σ is as required. ■

The Banach-Tarski paradox is a corollary of this theorem: S^2 is paradoxical with respect to the group of rotations in the three-dimensional space. This also holds for any sphere centred at the origin, and any solid ball in $\mathbb{R}^3$, and for $\mathbb{R}^3$ itself.

Since none of the previous steps depends on the size of the sphere, spheres of any radius r admit paradoxical decompositions. We can use every radius $0 < r \le 1$ to prove the decomposition for the solid ball $B \setminus \{\bar{0}\}$. $\bar{0}$ can be absorbed by a rotation around an axis missing the origin. As usual a set $Q = \{\sigma^n(\bar{0}) \mid n \ge 0\}$ may be used to absorb it: $\sigma(Q) = Q \setminus \{\bar{0}\}$, so $B \sim B \setminus \{\bar{0}\}$.

In consequence this leads to the strong form of the Banach-Tarski paradox: If A and B are any two bounded subsets of $\mathbb{R}^3$, each having nonempty interior, then A and B are equidecomposable. For proof use solid balls and repeated duplication.

3.8 The Sierpinski-Mazurkiewicz paradox

While the choice of the rotations in the Banach-Tarski paradox requires the axiom of choice, the Sierpinski-Mazurkiewicz paradox gets by without. The involved sets are infinite though. [F.E. Su et al.: "Sierpinski-Mazurkiewicz paradox", Math Fun Facts. Jens Bossaert: "The Sierpinski-Mazurkiewicz paradox", Curiosa Mathematica]

Let $p(n)$ be a polynomial with nonnegative integer coefficients a_k. The value of the polynomial $p_n(x) = \sum_{k=0}^{n} a_k x^k$ at $x = e^i$ is a point in the complex plane. Because e^i is a transcendental number, i.e., never the root of a polynomial, each such value corresponds to a unique point in the complex plane.

Let P_0 be the set of all points of polynomials with constant zero

$$P_0 = \{p_n(e^i) \mid n \in \mathbb{N} \wedge a_0 = 0\}$$

and let P_+ be the set of all points of polynomials with positive integer constant

$$P_+ = \{p_n(e^i) \mid n \in \mathbb{N} \land a_0 > 0\} .$$

Then the set of all points together is

$$P = P_0 \cup P_+ .$$

If we add 1 to a polynomial of P_0 we get a polynomial of P_+ (because the constant is no longer zero). If we multiply a polynomial of P_+ by e^i we get a polynomial of P_0 (because the constant is now zero).

Vice versa this means, when shifting the set P_+ by a unit in the negative of the real axis, i.e., subtracting 1 from all points of P_+, we include all points P_0 such that $P_+ = P$. Further, when multiplying all points of P_0 by e^{-i}, i.e., when turning P_0 clockwise by 1 radian, we get $P_0 = P$.

All sets are countable and no choice has been required.

3.9 A simple decomposition

Decompose the set $\mathbb{Z}$ of all integers into A, the set of odd integers, and B, the set of even integers

$$\mathbb{Z} = A + B .$$

When the elements of B are divided (δ) by 2, then $\delta B = \mathbb{Z}$. When the elements of A are translated (τ) by one unit (in positive or negative direction) and then divided by 2, then $\delta\tau A = \mathbb{Z}$.

Same can be shown for other sets, for instance the set $\mathbb{N}$ of positive integers (then A must be translated by +1) or the set of non-negative integers (then A must be translated by -1).

[Rhett Butler: "Who is the original author of this simple paradoxical decomposition?", MathOverflow (8 Jan 2016)]

3.10 The Mirimanoff paradox

Every set of well-founded sets is well-founded i.e., it does not contain an infinite sequence (X_n) with $\forall n$: $X_{n+1} \in X_n$. Hence the collection of all well-founded sets is well-founded, and therefore a member of itself – hence it is not well-founded. But that makes it a set all of whose members are well-founded that is nevertheless not well-founded itself. This is a contradiction.

3.11 The Cantor set

"As an example of a perfect point set which in no interval, how small it ever may be, is overall dense, I offer the set of all real numbers given by the expression

$$z = \frac{c_1}{3} + \frac{c_2}{3^2} + \ldots + \frac{c_\nu}{3^\nu} + \ldots$$

where the coefficients c_ν have to take the values 0 and 2". [Cantor, p. 207]

This set has become famous as the Cantor set. Divide the unit interval into three equal parts and remove the central one such that only the closed intervals [0, 1/3] and [2/3, 3/3] remain. Do the same with the first and the last interval such that only [0, 1/9], [2/9, 3/9], [6/9, 7/9], [8/9, 9/9] remain. Continue by removing always the central open interval from every remaining interval.

and so on.

There remain only such numbers which can be written in ternary notation with digits 0 (first third) and 2 (last third) only, i.e., without digit 1. The 1 in 1/3 etc. can be circumvented by the period 222..., such that in ternary $1/3 = 0.1_t = 0.0222\ldots_t$. The endpoints are those numbers which have only periods of $000\ldots_t$ or $222\ldots_t$. The Cantor set, the intersection of all these sets, does not only contain endpoints of intervals. For example

$$1/4 = 2/9 + 2/9^2 + 2/9^3 + \ldots = 0.020202\ldots_t$$

is not an endpoint but an element of the Cantor set. The measure of the Cantor set is 0 because the measure of the removed parts is

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \ldots = \sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{3} \frac{1}{1-\frac{2}{3}} = 1 \ .$$

Since all digit sequences containing 0 and 2 are existing, the set is uncountable. Further it is closed since its complement is open.

Two-dimensional equivalents are the Sierpinski carpet and the Sierpinski triangle. Three-dimensional equivalents are the Menger sponge and the Sierpinski tetrahedron and the Sierpinski pyramid.

3.12 Well-ordering of undefinable elements

The set of all finite words, i.e., finite combinations of symbols, is countable. The sole or few multiple meanings of a word depend on the used language. But since every language has to be devised and stored in at least one memory, there are only finitely many languages. Therefore the list of all possible meanings of finite words in all languages is countable. Infinite words cannot be used in mathematical discourse in uncompressed form. But in compressed form they are finite words. This means that it is impossible to assign meaning to all elements of an uncountable set.

So there exist at least two elements of every uncountable set which cannot reasonably be put in an order. But according to the axiom of choice every set can be well-ordered. That means that *two elements which cannot be identified, distinguished, and put in an order can be identified, distinguished, and put in an order.*

This antinomy has only wormed its way into public awareness.

Cantor himself denied the existence of undefinable real numbers. "'Infinite definitions' (that do not happen in finite time) are *non-things*. If König's theorem was *true*, according to which all 'finitely definable' real numbers form an embodiment of cardinality $\aleph_0$, this would imply that the whole number-continuum was countable, which is certainly *false*. The question is: which error underlies the *alleged proof* of his wrong theorem?" [G. Cantor, letter to D. Hilbert (8 Aug 1906)]

Also Hessenberg had to cope with this problem. He proposed the following, obviously false, solution: "Taking into consideration that a finite representation means to assign a thing to a combination of symbols, we recognize that really every thing has a finite representation; at least I don't see why I should not assign to it a hitherto meaningless combination." [Gerhard Hessenberg; "Grundbegriffe der Mengenlehre", offprint from Abhandlungen der Fries'schen Schule, Vol. I, no. 4, Vandenhoeck & Ruprecht, Göttingen (1906) § 95] He overlooked that also meaningless combinations of symbols belong to the countable set of finite representations.

Finally Zermelo tried to circumvent the problem simultaneously to the Löwenheim-Skolem paradox by means of uncountable alphabets of infinitary languages (cp. section 3.4.3). In section "On uncountable aphabets" of chapter VI this approach will be discussed.

IV The environment of set theory

The educationally disadvantaged populace admires Adam Ries(e) as the greatest (and often as the only) German mathematician. For the educated class Carl Friedrich Gauß assumes this position. The mathematicians themselves however hold Georg Cantor in esteem as their greatest colleague. The highest award of the German Mathematical Union (DMV) carries Cantor's likeness and name, because he extended and enriched mathematics infinitely – as many believe. More biographical material has been collected about Georg Cantor, inventor of set theory and first president of the DMV, than about any other mathematician of the 19th century. From these tesserae we can obtain a vivid picture of his world view. In the present chapter some of the theological aspects which led to and seemingly supported his understanding of the infinite will be reproduced. Then Cantor's scientific attitude will be investigated and in section 4.6 the huge gap between his pretension and the realized applications of set theory will be shown. Further the environment of set theory will be elucidated by some typical quotes of his followers.

4.1 Cantor on theology

"it is clear that the theological considerations by which Cantor motivated his notion of the actual infinite, were metaphysical in nature." [A. Heyting: "Technique versus metaphysics in the calculus", in Imre Lakatos (ed.): "Problems in the philosophy of mathematics", North Holland, Amsterdam (1967) p. 43]

"Cantor is probably the last great exponent of the Newtonian attitude with respect to religion." [H. Meschkowski, W. Nilson: "Georg Cantor Briefe", Springer, Berlin (1991) p. 15]

"it was a certain satisfaction for me, how strange this may appear to you, to find in Exodus ch. XV, verse 18 at least something reminiscent of transfinite numbers, namely the text: 'The Lord rules in infinity (eternity) *and beyond*.' I think *this 'and beyond'* hints to the fact that ω is not the end but that something is existing *beyond*." [G. Cantor, letter to R. Lipschitz (19 Nov 1883)]

"Compare the concurring perception of the whole sequence of numbers as an actually infinite quantum by St Augustin (De civitate Dei. lib. XII, ch. 19) [...] While now St Augustin claims the total, intuitive perception of the set (ν), 'quodam ineffabili modo', a parte Dei, he simultaneously acknowledges this set *formally* as an actual infinite entity, as a *transfinitum*, and we are forced to follow him in this matter." [G. Cantor, letter to A. Eulenburg (28 Feb 1886)]

"It can be *absolutely* ascertained that St Thomas only with *great doubts* and *half-heartedly* adhered to the received opinion concerning the actually infinite numbers, going back to Aristotle. [...] Thomas' doctrine 'It can only be believed but it is not possible to have a proof that the world has begun' is known to appear *not only in that opusculo* but also [...] in many other places. *This doctrine however would be impossible* if the Aquinatus had thought that the theorem 'there are no actually infinite numbers' was proven. Because from this sentence (if it was true), it would *demonstrably* follow with *greatest evidence* that an infinite number of hours could not have passed before the present moment. The dogma of the begin of the world (a finite time ago) could not have been defended as a *pure dogma*." [G. Cantor, letter to C.F. Heman (2 Jun 1888)]

"Your understanding of the relation of the two propositions:

> I. 'The world *including* the time has begun before a *finite* time interval or, what is the same, the *duration* of the world *elapsed until now* (e.g., measured by hours) is *finite*.'

which is *true* and a *Christian dogma* and:

> II. 'There are no actually infinite numbers.'

which is *false* and *pagan* and therefore cannot be a Christian dogma –
I say you have not the correct idea about the relation of these two propositions. [...]

The truth of proposition I does *not at all* imply, as you seem to assume in your letter, the truth of proposition II. Because proposition I concerns the *concrete world of creation*; proposition II concerns the *ideal* domain of *numbers*; the latter could include the *actual infinite* without its necessarily being included in the former. [...]

The pagan wrong proposition II, even without possessing the property of being a dogma acknowledged by the church or ever having been in that possession, has, because of its dogma-*like* popularity, done unmeasurable damage to Christian religion and philosophy, and one cannot, in my opinion, *thank holy Thomas of Aquino too effusively* that he has *clearly* marked this proposition *as definitely doubtful*." [G. Cantor, letter to C.F. Heman (21 Jun 1888)]

For comparison: Thomas Aquinatus writes in his Summa Theologica I, q. 7, a. 4: "But no species of number is infinite; for every number is multitude measured by one. Hence it is impossible for there to be an actually infinite multitude, either absolute or accidental. Likewise multitude in nature is created; and everything created is comprehended under some clear intention of the Creator; for no agent acts aimlessly. Hence everything created must be comprehended in a certain number. Therefore it is impossible for an actually infinite multitude to exist, even accidentally. But a potentially infinite multitude is possible;" [Thomas Aquinas: "Summa"]

"All so-called proofs (and I hardly may have missed anyone) against the creational A. I. {{actual infinite}} prove *nothing* because they do not refer to the correct definition of the transfinite. The two, for their time and even today, strongest and profoundest arguments of St Thomas Aquinatus S. Th. I, q. 7, a. 4 [...] become invalid as soon as a principle of individuation, intention, and ordination of actually infinite numbers and sets has been found;" [G. Cantor, letter to A. Schmid (26 Mar 1887)]

"The teaching of the transfinite is *far from* shaking the fundaments of Thomas' doctrin. The time is not far, however, that my teaching *will turn out* to be a *really exterminating weapon* against all pantheism, positivism and materialism." [G. Cantor, letter to J. Hontheim (21 Dec 1893)]

"Metaphysics and theology, I will frankly confess it, have occupied my soul in such a degree that I cannot spare much time for my *first flame*. If my wishes of fifteen, yes even eight years ago had come true, then I had been appointed to a greater sphere of mathematical activity, for instance at the university of Berlin or Göttingen, and probably I would have not been doing worse there than Fuchs, Schwarz, Frobenius, Felix Klein, Heinrich Weber etc etc. However now I thank God, the all-wise and all-merciful, that he has denied my wishes forever, because so he has forced me, by deeper penetrating into theology, to serve Him and his holy Roman Catholic Church better than I could have done according to my probably weak mathematical talent when *exclusively* being occupied with mathematics." [G. Cantor, letter to C. Hermite (22 Jan 1894)]

"Allow me to remark that the reality and the absolute law of the integers appear to me to be *much stronger* than those of the world of sensations. And this being so has only one, very simple reason, namely that the integers separately as well as in their actually infinite totality are existing as eternal ideas in intellectu Divino in the highest degree of reality." [G. Cantor, letter to C. Hermite (30 Nov 1895)]

"With respect to the *third* question concerning the A. I. {{actual infinite}}, namely the A. I. in Deo aeterno omnipotenti seu in natura naturante (the last expression I have adopted from some great scholastics) I have no doubt that we agree again in its approval. The last A. I., i.e., the A. I. in Deo, I call the Absolute, as you will have noted in my little essay 'Grundlagen', and this falls completely out of number theory." [G. Cantor, letter to I. Carbonnelle (28 Nov 1885)]

"other, far more important reasons can be added which result from the absolute omnipotence of God and with respect to which every negation of the possibility of a 'Transfinitum seu Infinitum actuale creatum' appears like a violation of that attribute of God." [G. Cantor, letter to C. Gutberlet (24 Jan 1886)]

"I am having no doubts concerning the truth of the transfinite that I have recognized by help of God and have been studying in its diversity and unity for more than twenty years. Every year and nearly every day advances me in this science. I happen to know also a bit of several other sciences besides mathematics, and therefore I am able to compare theorems here and there with respect to their degree of objective certainty; I can say that I don't know of anything of the created nature with a *safer* or, if this expression is allowed, with *more certain* knowledge than of the theorems of transfinite theory of numbers and types. Therefore I am convinced that this theory one day will belong to the common property of *objective* science and will be confirmed in particular by *that theology* which is based upon the *holy bible*, *tradition* and the *natural disposition* of the human race – these *three* necessarily being in harmony with each other." [G. Cantor, letter to I. Jeiler (20 May 1888, Whitsun)]

"I am very glad to see from your friendly letter of 20 Oct. that meanwhile your qualms against the 'transfinitum' have disappeared. Some time I will write and send you a little essay where I want to show you in scholastic form in detail how my results can be defended against the well-known arguments and, above all, how by my system the foundations of Christian philosophy in all essentials remain unchanged, they are not shaken but rather become fixed, and how even their development in different directions can be promoted." [G. Cantor, letter to I. Jeiler (27 Oct 1895)]

"The speculation, in particular the mathematical one, occupies only a small part of my time, after having overcome the original main difficulties with respect to the transfinite. I devote the speculation to *theology* and 'good works'. [...] You can be sure that I will adhere to the standpoint, mentioned to Don Zoel Garcia de Galdeano, and contribute according to my power to initiate healthier states in Spain, as I have been trying for years for Italy and France by means of my relations with the mathematicians there. [...] The institution university requires the rather peaceful collaboration of the *four* faculties; and if, caused by the reformation, this relation in many catholic countries first has been shaken and then completely dissolved everything has to be done (with care and cleverness, of course) to re-establish step by step this only natural state. [...] You would also do a 'good work' if you would push some of your younger, metaphysically interested patres to occasionally visit me for a short time and discuss privatissime with me about the actual infinite (this 'quaestio multis molestissima de infinita multitudine' as Card. *Franzelin* calls it in his Tr. de Deo uno sec. nat. Thes. XLI). You can be sure that the point of view taken by the majority of your patres (but also by the majority of catholic theologians) in the long term is *completely untenable*. [...] The only reason I have to be grateful to this Langbehn {{German author of the bestseller 'Rembrandt'}} is his hint [...] to his observation that my head and face allegedly resemble the holy *Ignatius of Loyola*; whose spiritual exercises I have been knowing and reading for many years. Perhaps this has had an influence on my looks. But perhaps this comparison is as much *nonsense* and as silly as most comparisons in his 'Rembrandt'. [...] By the way, should my agitation in Spain become successful, I would be *very grateful* for a *clever cooperation* of your patres in kindred spirit." [G. Cantor, letter to A. Baumgartner (27 Dec 1893)]

"Monsignore, allow me to present you the included (galley proofs of a) little essay of which I will send you some copies [...] as soon as preprints will be completed. I would be glad if my attempt, contained therein, to properly distinguish between the *three* main questions with respect to the *actual infinite* could be scrutinized thoroughly by Christian-Catholic philosophers." [G. Cantor, letter to J.B. Franzelin (17 Dec 1885)]

"Presently I have little occasion to deal with metaphysical discussions; but I confess that in my opinion that what the author calls the 'Transfinitum in natura naturata', cannot be defended and in a certain sense, which however the author does not seem to assert, would include the error of pantheism." [J.B. Franzelin, letter to G. Cantor (25 Dec 1885)]

"Accordingly I distinguish an 'Infinitum aeternum sive Absolutum' that refers to God and his attributes, and an 'Infinitum creatum sive Transfinitum' that is stated wherever in the Natura creata an actual infinite has to be postulated, like, *for example*, with respect to the, according to my firm conviction, *actually infinite number* of created individuals, in the universe as already on our earth and, most probably, even in each extended part of the space, however small it may be. Here I agree completely with *Leibniz*. [...]

I know that this teaching of the 'Infinitum creatum' is objected, if not by all yet by most doctors of the church, and contrary opinions are given in particular by the great St Thomas Aquinatus in his Summa theol. I, q. 7. a. 4. But the reasons that have imposed themselves on me in this question and, I can say against my will, rather captivated me during 20 years of research [...] are stronger than everything contrary that I have heard, although I have scrutinized it to a large extent. Further I believe that the words of the Holy Bible like Sap. c. 11, v. 21: 'Omnia in pondere, numero et mensura disposuisti' which have been assumed to contradict infinite numbers, do not have this meaning; for given the case, actually infinite 'powers', i.e., cardinal numbers, and a. i. numbers {{Cantor uses 'Anzahlen' as numbers of well-ordered sets}}, i.e., ordinal numbers [...] existed, as I think to have proved, which like the finite numbers obey firm laws given by God, so clearly also these transfinite numbers would be covered by that holy remark – and it cannot be used, in my opinion, against actually infinite numbers to avoid a circular argument.

But it can be proved in different ways that an 'Infinitum creatum' has to be assumed. [...]

One of the proofs starts from the notion of God and infers first from the highest perfection of the Supreme Being the possibility of the creation of a Transfinitum ordinatum, then from God's loving kindness and glory the necessity of an actually created Transfinitum.

Another proof shows a posteriori that the assumption of a 'Transfinitum in natura naturata' facilitates a better, because more complete, explanation of the phenomena, in particular of the organisms and psychological phenomena than the contrary hypothesis. {{This has never been further elaborated by Cantor.}} [...]

I believe that pantheism, perhaps only by means of my conception of the things, over time can be overcome completely. [...] Materialism and related ideas seem to me to belong to the evils which, just because they belong to the scientifically most untenable and easiest refutable, the human race in its temporal existence will never completely be released of." [G. Cantor, letter to J.B. Franzelin (22 Jan 1886)] In this letter the terminus "cardinal-number" appears for the first time.

"In your valued letter to me you say first quite rightly (provided that your notion of the transfinitum is not only compatible with religion but also *true*, what I do not judge), 'one of the proofs starts from the notion of God and infers first from the highest perfection of the Supreme Being the possibility of the creation of a Transfinitum ordinatum.' Assuming that your transfinitum *actuale* in itself contains no contradiction, your conclusion of the *possibility of the creation* of a transfinitum out of the notion of God's omnipotence is quite right. But to my regret you go on and infer from his 'loving kindness and glory *the necessity* of an actually created Transfinitum'. Just because God himself is the absolute infinite good and the absolute glory, which good and which glory nothing can be added to and nothing can be missing, the necessity of some creation, whatever it might be, is a contradiction, [...]

I am not able to continue the correspondence about your philosophical opinions because of my many occupations which direct me to quite another field. You might excuse if I will not react on your possible replies, which however, as far as they will be related to your system, I beg you to refrain from." [J.B. Franzelin, letter to G. Cantor (26 Jan 1886)]

"Your Eminence, I thank you very much indeed for the clarifications given in your kind letter of 26 January which I agree to with full conviction, because in the short hint in my letter of 22 January I did not opine to talk about an objective, metaphysical necessity of the act of creation, which God, the *absolutely free* had been subject to, but I only wanted to point to a certain subjective necessity *for us*, to infer from God's loving kindness and glory an actually *done* (*not* a parte Dei *to be done*) creation, not only of a *Finitum* ordinatum but also of a *Transfinitum* ordinatum." [G. Cantor, letter to J.B. Franzelin (29 Jan 1886)]

I would be most delighted if my works {{on transfinite cardinal numbers and transfinite order types}} would be for the benefit of the Christian philosophy which is next to my heart, namely the 'philosophia perennis' {{perpetual, eternal philosophy}}. This would only then be thinkable and possible, if they would be scrutinized by the old, meanwhile by His Holiness Leo XIII so beautifully restored, revived school." [G. Cantor, letter to T. Esser (19 Dec 1895)]

"Attempts that I have made many years ago and repeatedly recently, to win members of the German province of S. J. {{Societas Jesu}} for a confidential scientific correspondence about the actual infinite, have been *without any success* although many of them have been knowing and possessing my works for more than ten years, whereas the late Cardinal J.B. Franzelin S. J. very plainly has been pointing to the importance of this question for theology and philosophy in his letters directed to me just 10 years ago." [G. Cantor, letter to T. Esser (25 Dec 1895)]

"The general set theory [...] definitely belongs to metaphysics. You can easily convince yourself when examining the categories of cardinal numbers and the order type, these basic notions of set theory, on the degree of their generality.

[...] and the fact that my presently written work is issued in mathematical journals does not modify the metaphysical contents and character of this work.

[...] By me Christian philosophy is for the first time confronted with the true teachings of the infinite in its beginnings. I know quite firmly and for sure, that my teachings will be accepted. The question is only, whether this will happen soon or after my death. But I am completely calm about this alternative. It doesn't touch my poor soul which, however, dear Pater, I recommend to your and yours pious prayer." [G. Cantor, letter to T. Esser (1/15 Feb 1896)]

"Since he {{Cantor}}, because of his bold endeavour, had been attacked from all sides, he tried to get support from me, the only one who, as he believed, agreed with his opinions. Since he was noble minded, he did not share the contempt of the disbelieving science against the Christian philosophers. And it was not only pure poverty which led him to me, but, as he said, he had a Catholic-friendly attitude because his mother was Catholic. He inquired with me about the teachings of the scholastics with respect to this question. I could point him in particular to St Augustin and to P. {{father}} Franzelin, the later cardinal. This highly esteemed teacher of mine defended the actually infinite set in the cognition of God, supported by the explicit teaching of St Augustin, and it was he, who induced that writing of mine and who calmed me during the violent attacks with the argument that I only had repeated the teaching of St Augustin. *Cantor* himself then addressed the cardinal and reported his statements, without revealing his name, in an essay of the 'Zeitschrift für Philosophie und philosophische Kritik'." [C. Gutberlet: Philos. Jahrbuch der Görres-Gesellschaft 32 (1919) p. 364ff]

Cantor "tried to actively intervene in education policy in favour of Catholicism. He wished that chairs of philosophy in Germany should not be occupied by professors who supported Darwinism or atheism." He tried to reach this aim with cunning and even under circumventing the official appointment procedure, always without success though. [H. Meschkowski, W. Nilson: "Georg Cantor Briefe", Springer, Berlin (1991) pp. 13 & 374 & 380]

"From the Catholic point of view we have to be happy that you got rid of Prof. Riehl [...] and one can only wish that he will not be replaced by a kindred spirit. Because this sort of men is able to cause much damage, as you have experienced with Riehl over many years. [...] The theologians at Kiel may convince themselves what they have got and may look how they can live with him. Further we cannot know whether divine providence places just such radical people in Protestant universities in order to accelerate the undermining and decay of Protestantism. Would we be interested to hinder that? Not at all!

[...] The government of the grand duke of Baden should be informed *in a private way* (by your friend, the member of parliament) of the fact that a pupil and friend of Prof. Riehl (Dr. Förster) has been sentenced because of lèse-majesté; this should be a reason to meet the candidate recommended by Prof. Riehl with greatest suspicion. In the senate it may be preferable not to touch this point. [...]

As I have been told by a personally known brother monk of the author, P. Esser is momentarily in Rome as a co-worker of an extremely important commission, appointed by the Holy Father in order to revise the Index," [G. Cantor, letter to F.X. Heiner (31 Dec 1895)] This "extremely important" commission renewed the index of prohibited books, i.e., those books which a devout Catholic was not allowed to read.

"I do highly appreciate that the pretended scientific appearance has been snatched away from Haeckel's shameless attacks against Christianity in front of the widest audience. The noble shyness toward hearty polemics (in other circles so usual!) had to give way with respect to such wretchedness." [G. Cantor, letter to F. Loofs (24 Feb 1900)]

"You have, as far as I know, 10 universities in Spain. Theology however is excluded and is only taught in seminaries. The former constitution of universities, which included theology, is better in my opinion. That holds for *Spain* as well as for *France*, where the same exclusion has been introduced. I do not only care about a non-hostile attitude of the other sciences towards the ancestral theology, but I believe that also theology can only stand to gain from a close relation to the other faculties." [G. Cantor, letter to Don Zoel Garcia de Gáldeano (1893), quoted in a letter from G. Cantor to A. Baumgartner (15 Dec 1893)]

"Especially bold in this matter, with regard to his time, appears *Rod. Arriaga* S. J. {{Rodrigo de Arriaga (1592-1667)}}. (I mention here that the teaching of the creational actual infinite (what I call transfinitum) by *Rod. Arriaga* has *not at all* been founded *free of contradiction*; same is true for the Minime *Em. Maignan* {{Emanual Maignan (1601-1676)}}. Both I have only become acquainted with a long time after I had completed my theory internally and had cleared it. Both of them are lacking the *correct notions* of transfinite *cardinal numbers* {{Mächtigkeiten}} and the transfinite *order types* and *ordinal numbers*, just that tool which helps to make *the whole theory faultless*.); but also *Suarez* S. J. {{Francisco Suarez (1548-1617)}} is not so disconnected from my position as it might appear. [...] With respect to my high esteem and admiration of your religious order I could not win more encouragement from any party to continue with my work than from you and yours." [G. Cantor, letter to J. Hontheim (21 Dec 1893)] So the roots of set theory reach back into times which saw Bruno and Galilei sentenced as heretics.

"[...] it is certain for instance of *Leibniz* that he has assumed a creational infinite in different relations as *really existing*. [...] On the other hand *Leibniz* has as little as his predecessors and successors recognized the actually infinite, i.e., transfinite *numbers* and *order types*; he even refutes their possibility. [...]

You say [...] that you have problems with the notion of the transfinite because you cannot give up the theorem that the possibility of *addition* implies the presence of a *potential*. But it has *not* been asserted by me that a *transfinitum* be *only* act, rather the transfinite *in the sense in which it is multipliable* is *potency*; only the *absolute* is *actus purus* or rather *actus purissimus*. [...]

Whereas the emphasized (that the principle 'totum majus est sua parte' is wrong *in a certain sense*) with respect to the *substantial forms* is acknowledged in general (the soul of a living organism, for instance, in its essential being remains always *the same* during the growing or decreasing of the body) one seems to believe that this does not refer to the *accidental* forms too. This *prejudice* originates from the observation that, as I just stressed, observation has been restricted to only *finite* sets which *always obey* the principle 'totum majus est sua parte' with respect to the cardinal number forms belonging to them; without further investigation, but also *without any justification*, its validity *in the noted sense* has been carried over to *infinite* sets too, and there is no reason to be surprised about the contradictions resulting from such an *utterly wrong premise*." [G. Cantor, letter to I. Jeiler (20 May 1888, Whitsun)]

"To use scholastic terms: Something that can be multiplied is *in potentia* to this further actus, hence something *potential*; it belongs to that notion. Then your transfintum could only be some *subsection* of the usually taught *potential* infinite." [I. Jeiler, letter to G. Cantor (22 Jun 1890)]

"The results, which I have arrived at, are as follows:

Such a transfinitum, whether it is thought of in concreto or in abstracto, is free of contradiction, therefore possible and creatable by God, as well as a finitum. [...]

All special modes of the transfinite have been existing forever as ideas in intellectu divino. [...]

If you express this fact by saying: 'every transfinite is in potentia to another actus and in this respect is a potential', so I have no objection. Because actus purus is only God; but every creational, in the sense mentioned by you, is being in potentia to another actus.

Nevertheless the transfinite cannot be considered as a subsection of what is usually called 'potential infinite'. Because the latter is *not* (like every individual transfinite and in general everything due to an 'idea divina') *determined in itself*, *fixed*, and *unchangeable*, but a *finite in the process of change*, having in each of its actual states a *finite size*; like, for instance, the *time* elapsed after the beginning of the world, which, measured in some time-unit, for instance a year, is *finite* in every moment, but *always growing beyond all finite limits, without ever becoming really infinitely large*." [G. Cantor, letter to I. Jeiler (13 Oct 1895)]

"In the first half of the last century a curious attempt has been made by the famous Frenchman Fontenelle (in the book 'Eléments de la Géometrie de l'infini', Paris 1727), to introduce actually infinite numbers; this attempt however has failed and has brought the author some mockery, not quite undeserved, from the mathematicians who were active in the 18th century and in the first quarter of the present century; the present generation does not know about it. Fontenelle's attempt was doomed to failure because his infinite numbers brought with them a flagrant contradiction; it has been easy to show this contradiction, and that has been done best by R. P. Gerdil. But if d'Alembert, Lagrange, and Cauchy believed that the dormant idea of the transfinite had been stroken deadly by that for all times, then this error appears by far greater than that of Fontenelle and the *more grave* because Fontenelle in the most humble way confesses to be a layman in mathematics whereas those three not only have been professionals but *really great* mathematicians. [...]

The R. P. Ign. Carbonelle, in his beautiful essay 'Les confins de la science et de la philosophie, 3e ed. t. I, cap. 4', has tried to save Gerdil's proof for a temporal beginning of the world by very astutely and scholarly defending the proposition: 'Le nombre actuellement infini n'est pas absurd', but adding the hard, merciless, and dissonant afterthought: 'mais il est essentiellement indéterminé'. Perhaps he would have refrained from that afterthought if he had known at his time my works already, which from the beginning, for meanwhile nearly twenty years, have been concerned with ways of individuation, specification and ordination of the actual infinite in natura creata. But the mathematical proof for the temporal beginning of the creation, undertaken by R. P. Carbonelle, stands and falls with just this afterthought.

Finally, with respect to the *third* thesis of your esteemed letter I fully agree with you when you say with Nicolaus of Cusa that 'in God all is God' as well as that 'the cognition of God objectively cannot recognize the incommensurable as commensurable or the irrational as rational, because the divine omniscience as well as the divine omnipotence cannot give rise to the impossible.' [...]

If it is said here that a *mathematical* proof of the beginning of the world in finite time cannot be given, then the emphasis is on the word 'mathematical', and only in that respect my opinion is in agreement with St Thomas. On the other hand, just *based upon the true teaching of the transfinite*, a mixed mathematical *metaphysical* proof of the theorem might well be possible, and in so far I differ from St Thomas, who holds the opinion: S. Th. q. 46, a. 2 concl. '*Only by belief we know that the universe did not always exist, and that cannot be checked by proof on its genuineness*'." [G. Cantor, letter to A. Schmid (26 Mar 1887)]

"If one has recognized something of the truth, then one *knows* to be in possession of this truth and one feels (even if saying like me 'non quaero ab hominibus gloriam' {{I do not demand glory from mankind}}) sort of duty, as far and as long as power reaches, to tell it to others. Under this aspect you, Reverend Father, will kindly forgive that in the following I will in greater detail support and amplify what I said in my recent messages." [G. Cantor, draft of a letter to A. Schmid (18 Apr 1887)]

"*The fact of the act. infinitely large numbers is so little a reason for the possibility of an a parte ante infinite duration of the world that, on the contrary, by means of the theory of transfinite numbers the necessity of a beginning of motion and time in finite distance from the present can be proven.*

The detailed grounds of this theorem I will postpone to another opportunity because I would not like to weigh down the merry beginning of your holidays with mathematical and metaphysical considerations." {{Cantor never supported his claim.}} [G. Cantor, letter to A. Schmid (5 Aug 1887)]

"After having recently looked through your paper 'Institutiones philosophicae' I have got the impression that I do not much deviate in the most important metaphysical questions concerning the philosophy of Saint Thomas Aquinatus, which you, Reverend Father, support so masterly and enlightening, and that those points, where a difference could be stated, are such in which modifying the teaching of the great philosopher might be allowed and perhaps even be desirable." [G. Cantor, letter to M. Liberatore (7 Feb 1886)]

"In religious questions and relations my opinion is *not a denominational* one because *I am not member of any existing organized church*. My religion is that revealed by the Triune one and only God himself, and my theology is founded on God's word and work, and I am *admiring* as my teachers *mainly* the Apostolic Fathers, the Church Fathers, and the most respected teachers of the Church of the *first 15 centuries* after Christ (i.e., the time preceding the church-revolution of the 16th century)." [G. Cantor, letter to Constance Pott (7 Mar 1896)]

"I have never been assuming a 'Genus supremum' of the actual infinite. On the contrary I have proven strictly that a 'Genus supremum' of the actual infinite does not exist. That which is higher than all finite and transfinite is not a 'Genus', it is the only absolutely individual unit, in which all is contained, which comprehends all, the 'Absolute', for the human intellect incomprehensible, therefore not being subject to mathematics, unmeasurable, the 'ens simplicissimum', the 'actus purissimus', which by many is called 'God'." [G. Cantor, letter to Grace Chisholm-Young (20 Jun 1908)]

"Our Holiest Father, Pope LEO XIII

With regard to the integrating apostolic letters of your Holiness, in particular that published on April 14, 1895 that you have sent to the English people, I have held it necessary to remind all Christians, in particular the adherents of the Anglican Church, of the creed of Francis Bacon 'the fine specimen of his century and his nation, adorning literature and being its adornment'.

Permit, Greatest Pontifex, that I dedicate to you *seven* specimen of a new edition of that little work and that I include three volumes of the works of Francis Bacon.

I implore you, Beatissime Pater, to condescent to accept those 10 little gifts, which I dare to offer you in order to be a token of my deference and of my love to your Holiness and to the Holy Roman Catholic Church.

Your Holiness most humble and most obsequious servant
Georg Cantor
mathematician."

[G. Cantor, letter to Pope Leo XIII (13 Feb 1896)] An answer of the Pope is not known.

In 1905 Cantor published on is own expense the essay "Ex oriente lux" where he opposed the dogma of virgin birth. "During his studies in a psychiatric clinic he had come to the conclusion that Joseph of Arimathia was the natural father of Jesus" [H. Meschkowski, W. Nilson : "Georg Cantor Briefe", Springer, Berlin (1991) p. 444] The essay concludes with a refusal of Catholicism: "It *remains* until the end of all days resting on an *unshakeable rock, Christ himself*: the *invisible* church, which *He* has founded. *He is the Lord who does not need a governor on earth.*" ["Ex oriente lux. Gespräche eines Meisters mit seinem Schüler über wesentliche Puncte des urkundlichen Christenthums. Berichtet vom Schüler selbst." (1905) {{Light from the east. Talks of a master with his pupil about the essential points of the original Christianity. Reported by the pupil himself.}}]

"Recently Mr. *Bernstein* has committed the new carelessness, to try to show in the mathem. Annalen that 'there are sets existing which cannot be well-ordered'. I have not the time to look for the error in his proof but I am firmly convinced that such an error exists.

Hopefully time and opportunity will come soon to frankly express my full opinion about all those immature attempts. [...]

The fundament of my opinion about redemption is that Jesus is the predicted Messiah of the Jews and as such in his human nature is a *real descendant* of David. This we know absolutely sure from himself and as such he has been considered by all his apostles after his resurrection. From this point I arrive, as you have seen, based on the New Testament, at the distinction of *two* Josephs, the royal *Joseph* and *physical father* of Christ and the breadwinner Joseph. [...]

Concerning the resurrection of Christ (about which you have inquired me), this has been attested best and most comprehensively by the writings of the New Testament. I firmly believe it as a fact and do not brood over the 'how' of it." [G. Cantor, letter to P. Jourdain (3 May 1905)]

"That Jesus Christ was the natural son of Joseph of Arimathea was one of the obsessions which Cantor adopted in his later life, although he published nothing more on it after 'Ex Oriente Lux'." [Ivor Grattan-Guinness: "The correspondence between Georg Cantor and Philip Jourdain", Jahresbericht der Deutschen Mathematiker-Vereinigung 73 (1971) p. 127]

As an ironical footnote it should be mentioned that St Augustin, Cantor's prima facie source of knowledge about the infinite set of numbers in the divine sphere, did not like the tables. In his confessions [Aurelius Augustinus: "Confessions", 1, 13, 22] he confessed that the chant "unum et unum duo, duo et duo quattuor" sounded ugly to him.

4.2 Cantor on sciences

Cantor devised set theory for application to reality and planned to write a paper about set theory and its applications: "The *third* part contains the applications of set theory to the *natural sciences*: physics, chemistry, mineralogy, botany, zoology, anthropology, biology, physiology, medicine etc. It is what the Englishmen call 'natural philosophy'. Added to that are applications to the so-called 'humanities', which, in my opinion, have to be conceived as natural sciences too, *because also the 'mind' belongs to nature*." [G. Cantor, letter to D. Hilbert (20 Sep 1912)]

Cantor explained his impetus for devising set theory to Mittag-Leffler: "Further I am busy with scrutinizing the applications of set theory to the physiology of organisms, [...] I have been occupied for 14 years with these ideas of a closer exploration of the basic nature of all organic; they are the true reason why I have undertaken the painstaking and hardly rewarding business of investigating point sets, and all the time never lost sight of it, not for a moment. Further I am interested, purely theoretically, in the nature of the states and what belongs to them, because I have my opinions on that topic which later may become formulated mathematically; the striking impression that you perhaps may obtain will disappear, when you consider that also the state in some sense represents an organic being." [G. Cantor, letter to G. Mittag-Leffler (22 Sep 1884)]

"I expect great benefits from the general *theory of types* in all respects. It constitutes an important and great part of *pure set theory* (Théorie des ensembles), i.e., also of *pure mathematics*, because in my opinion the latter is nothing else but *pure set theory*. [...] By *applied set theory* I understand what usually is called *physical science* or *cosmology*; to this realm all so-called *natural sciences* are belonging, those concerning the *anorganic* as well as the *organic* world. [...] For *mathematical physics* the *theory of types* is particularly important because the latter theory is a powerful and sharp tool for the *discovery* and the *intellectual construction* of the so-called *matter*. Related to this is the applicability of *the theory of types* in *chemistry*. [...] The application of *mathematical type theory* on study and research in the domain of the *organic* seems to me of very special interest " [G. Cantor, letter to G. Mittag-Leffler (18 Nov 1884)]

"This has created my desire to replace the mechanical explanation of nature by a more complete one, which I would call in opposition to the former an 'organic' one. However it could be satisfactory to me only if the conventional notions would be replaced by new and improved notions which with respect to mathematical determination and accessibility do not fall behind the former or rather present ones." [G. Cantor, letter to W. Wundt (4 Mar 1883)]

"Therefore Wundt errs if he believes that the transfinitum has no physical meaning, but only the infinitum; strictly speaking the opposite is true because the 'improper infinite' is only an auxiliary and relative notion." [G. Cantor, letter to K. Laßwitz (15 Feb 1884)]

"The A. I. {{actual infinite}} in abstracto and in concreto, however, where I call it transfinitum, are not only subject of an extended number theory but also, as I hope to show, of an advanced natural science and physics." [G. Cantor, letter to I. Carbonnelle (28 Nov 1885)]

Cantor says that he has no safer knowledge of anything in nature than of his transfinite set theory. "Therefore I am convinced that this theory one day will belong to the common property of *objective* science" [G. Cantor, letter to I. Jeiler (20 May 1888, Whitsun)]

4.2.1 Temporal origin of the world

Cantor frequently expressed his opinion that the world has a beginning and his claim that he could prove that. But he never gave a proof and he hardly has foreseen the evidence in favour of the big bang. He was a creationist undoubtedly.

"With respect to the creation of the world and its temporal beginning I completely agree with you Reverend Father but I also agree with St Thomas Aq., who contests in his Opusc. de aeternitate mundi the mathematical provability of this theorem (that a temporal beginning of the world has to be assumed), [...] If it is said here that a *mathematical* proof of the beginning of the world in finite time cannot be given, then the emphasis is on the word 'mathematical' and only in that respect my opinion is in agreement with St Thomas. On the other hand, just *based upon the true teaching of the transfinite*, a mixed mathematical *metaphysical* proof of the theorem might well be possible. In so far I differ from St Thomas, who holds the opinion: '*Only by belief we know that the universe did not always exist, and that cannot be checked by proof on its genuineness*.'" [G. Cantor, letter to A. Schmid (26 Mar 1887)]

"I definitely agree with you, Reverend Father, in the assumption of a temporal beginning of the world. I have always considered the contrary dogma of present natural sciences as violating good reason in highest degree;" [G. Cantor, letter to A. Schmidt (5 Aug 1887)]

"*This doctrine* {{of a temporal beginning}} *would be impossible*, if the Aquin. had considered the theorem 'there are no act. infinite numbers' as proven. Because from this theorem (if it was true) it would follow *demonstrably* with *greatest evidence* that an infinite number of hours could not have passed before this moment. The dogma of a beginning of the world (a finite time ago) could not have been defended as *bare credo*." [G. Cantor, letter to C.F. Heman (2 Jun 1888)]

"I do not only maintain with all Christian philosophers the temporal beginning of the creation, I also claim like you that this truth can be proven by rational reasons. [...] The foundation of actually infinitely great or, as I call them, transfinite numbers, does not entail that we have to refrain from the rational proof of the beginning of the world." [G. Cantor, letter to J. Hontheim (21 Dec 1893)]

"for instance, the *time elapsed* since the beginning of the world, which, measured in some time-unit, for instance a year, is *finite* in every moment, *but always growing beyond all finite limits*, *without ever becoming really infinitely large*." [G. Cantor, letter to I. Jeiler (13 Oct 1895)]

"With great interest I have studied your essay: 'The teachings of holy Thomas of Aquino about the possibility of a creation without beginning'. It was very satisfying for me to see the position of holy Thomas concerning actual infinity be discussed from such a profound expert and to learn that I had correctly understood holy Thomas in this point and related questions, in particular that his arguments against the actual infinite in creatis or against the possibility of actually infinitely great numbers has, *for himself*, not the meaning of a demonstratio, quae usquequaque de necessitate concludit leading to metaphysical certainty, but was in his own eyes only probable to a certain degree." [G. Cantor, letter to T. Esser (5 Dec 1895)]

4.2.2 Physical space

In a paper about continuity preserving manifolds Cantor proved that the manifold $\mathbb{R}^n$ (with $n \geq 2$) remains continuous if the set of points with purely algebraic coordinates is taken off. According to Cantor's interpretation this is *a peculiar property of countable sets*. (For a correction see section "On continuity-preserving manifolds" in chapter VI.)

"In this manner it has been shown that two points N and N' of the domain A, which remains after subtracting the overall dense countable point set (M) from the {{initial}} domain, can be connected by a continuous line l' constructed of a finite number of circular arcs which with all their points belong to the domain A, i.e., do not contain any point of (M). [...]

Connected with these theorems are considerations concerning the constitution of the three-dimensional space which the real world has to be based upon in order to explain the phenomena appearing therein. It is known that the space is assumed as *universally continuous* because of forms appearing there and in particular because of the movement occuring there. The latter property consists, according to the simultaneous and independent investigations by Dedekind (cp. the little essay: Stetigkeit und irrationale Zahlen by R. Dedekind, Braunschweig 1872) and by the author (Mathem. Annalen Vol. V, pp. 127 and 128), in nothing else but that every point the coordinates x, y, z of which are given with respect to an orthogonal coordinate system by *any* definite real, rational or irrational, numbers are thought of as *really belonging to the space*; in general this is not necessitated by an inherent force but has to be considered as a free act of our intellectual construction. The *hypothesis of continuity of space* is therefore nothing else but the rather arbitrary assumption of the complete, mutually unique correspondence between the three-dimensional *purely arithmetic continuum* (x, y, z) and the space which the world of phenomena is based upon." [G. Cantor: "Über unendliche lineare Punktmannichfaltig-keiten" (3), Math. Ann. 20 (1882) pp. 113-121. Cantor, p. 156]

"You are quite right in that you deny the real background of Gauß-Riemann-Lobatchewsky's spaces but accept them as 'logical postulates'." [G. Cantor, letter to W. Wundt (5 Oct 1883)]

"I refer you to what I have found in *Math. Annalen* Vol. XX pp. 118-121, that in the space filled with *body matter* (since I assume the body matter being of *first* cardinality) for the *ether* (the matter of *second* cardinality) there is an *enormous* space remaining for *continuous* movement, such that all phenomena of *transparency of bodies* as well as those of *radiating heat*, the *electric* and *magnetic induction* and *distribution* appear to get a *natural* basis *free of contradictions*." [G. Cantor, letter to G. Mittag-Leffler (16 Nov 1884)]

"The paper by Pohle about the objective meaning of the infinitely small contains quite nice and comprehensive reflections. But he errs in the assumption that the infinitely small be necessary as an actually integrating or constituent element for the explanation of the continuum or as a foundation of the infinitesimal calculus. I agree with him concerning the objective importance of the infinitely small, but not the infinitely small as far as it is something actual, being infinitely small, rather only something potential, becoming infinitely small. As an *element* of the continuum the infinitely small is not only unusable but even unthinkable or impossible as I can strictly prove." [G. Cantor, letter to C. Gutberlet (1 May 1888)]

4.2.3 Created creatures and atoms of matter and ether

"If we consider the epitome of all organic cells in our universe, which to all directions is infinitely extending, at a given time, then this epitome certainly consists of infinitely many individuals; therefore we can ask about the cardinality of this set, and I am able to strictly prove that this cardinality is the *first* one, i.e., not a higher one." [G. Cantor, letter to W. Wundt (16 Oct 1883)] Cantor never delivered a proof of this thesis.

"The proper infinite, as we encounter it, e.g., in well-defind point sets or in the constitution of bodies of point-like atoms (here I do not mean the chemical-physical (*Democritian*) atoms, because I do not believe that they exist, neither as a notion nor in reality – whatever useful might have been accomplished, up to a certain limit, by this fiction) has found its most decided defender however in an extremely sharp-minded philosopher and mathematician of our century, in *Bernard Bolzano*," [G. Cantor: "Grundlagen einer allgemeinen Mannigfaltigkeitslehre", Leipzig (1883). Cantor, p. 179]

"Therefore and since the present Sunday provides me with some free hours I will briefly tell you my opinions about the *constitution of matter.*

Like *Boscovich, Cauchy, Ampère, Wilh. Weber, Faraday,* and many others I think that the last elements have *no extent*, that means, speaking geometrically, they are purely pointlike. Willingly I accept the usual expressions *centres of force* or *material points*. You see that I already here deviate from that *atomism* which assumes the last elements to have *extent* but to be *not divisible by any forces*. This is the opinion which is *common* today in *chemistry* and *prevailing* in *physics*. I will call this *atomism* the *chemical atomism*.

Although *those* authors differ with me in the just mentioned respect from the *chemical* atomism, they maintain another form of atomism which I temporarily will call, for the sake of brevity, *point atomism*.

But strictly speaking I am not an adherent of *point atomism* either, although *for me too* the last elements are indestructible centres of force. I think that I have to refuse the *chemical* as well as the *point atomism*, the latter at least in its present form.

Nevertheless I am not an unconditional advocate of the hypothesis of continuity, at least not in the vague form in which it hitherto has been developed by some philosophers.

I believe *with the point atomists* that for an explanation of the *anorganic* and, *up to a certain limit*, also the *organic natural phenomena two classes* of *created and, after having been created, separate, indestructible, singular elements with no extent and equipped with forces,* which I will also call *atoms*, are necessary and sufficient. Those of the first class I will call *body atoms*, those of the other class I will call *ether atoms*.

I go on to believe also, and that is the *first* point where I rise over the *point atomism*, that the *totality* of *body atoms* is of the *first cardinality*, the *totality* of *ether atoms* is of the *second cardinality* – and this is my *first hypothesis*.

In a short letter I cannot explain all reasons which in my opinion are supporting this hypothesis. I refer you to what I have found in *Math. Annalen* Vol. XX pp. 118-121, that in the space filled with *body matter* (since I assume the body matter being of *first* cardinality) for the *ether* (the matter of *second* cardinality) there is an *enormous* space remaining for *continuous* movement, such that all phenomena of *transparency of bodies* as well as those of *radiating heat*, the *electric* and *magnetic induction* and *distribution* appear to get a *natural* basis *free of contradictions*.

Just as little I want to talk today about the forces which have to be ascribed to the two different elements. Only one point I have to add to the above, simultaneously constituting the apparent *second difference* with the *point atomism*.

I believe that in the state of equilibrium, because of the mutual attraction and repulsion exerted by the elements upon each other[1] and because of the innumerable *grades* of this attraction, both the *body matter for itself* can only exist in form of a *geometric homogeneous point set dense in itself* (of first order or cardinality) and the *ether for itself* can only exist in form of a *geometric homogeneous point set dense in itself* (of second order or cardinality[2]). And this is my *second* hypothesis. This second hypothesis concerns only the *static* phenomena. In *general* however the following decompositions are relevant: $P \equiv rP + i_1P$, $Q \equiv rQ + i_1Q + i_2Q$, where P is the matter of first class {{body atoms}}. Here in general all parts have *physical* meaning. From my investigations in Acta math. Vol. IV pp. 388-390 and Math. Annalen Vol. XXIII pp. 473-479 it follows that here the notion of *volume* does not vanish." [G. Cantor, letter to G. Mittag-Leffler (16 Nov 1884)]

"in this respect I have developed the *hypothesis*, years ago already, that the *cardinality* of the *body matter* is what I call in my investigations the *first* cardinality, the *cardinality* of the *ether matter*, on the other hand, is the s*econd* one." [G. Cantor: "Ueber verschiedene Theoreme aus der Theorie der Punktmengen in einem *n*-fach ausgedehnten stetigen Raume G_n. Zweite Mitteilung.", Acta Mathematica Vol. 7 (1885) pp. 105-124. Cantor, p. 276]

"Accordingly I distinguish an 'Infinitum aeternum sive Absolutum' that refers to God and his attributes, and an 'Infinitum creatum sive Transfinitum' that has to be applied wherever in the Natura creata an actual infinite is observed, like, *for example*, with respect to the, according to my firm conviction, *actually infinite number* of created individuals, in the universe as already on our earth and, most probably, even in each extended part of the space, however small it may be, wherein I agree completely with *Leibniz*. [...]

Another proof shows a posteriori that the assumption of a 'Transfinitum in natura naturata' delivers a better, because more complete, explanation of the phenomena, in particular of the organisms and psychological phenomena than the contrary hypothesis." [G. Cantor, letter to J.B. Franzelin (22 Jan 1886)]

4.2.4 Energy and matter

"Should I also mention a point where I do not quite agree with you, so it is your unreserved confidence in the modern so-called law of energy conservation. I do not at all wish to doubt the teaching of the equivalence of the different natural forces transforming into each other as far as this has been sufficiently verified by experiment.

That which I have serious reservations against is the elevation of the asserted law into the rank of a metaphysical principle which is claimed to govern the recognition of so important theorems as the immortality of the soul as well as further the completely unjustified extension

[1] Here Meschkowski, Nilson (1991) deviate from Meschkowski (1983) and Purkert, Ilgauds (1987) and give: "I believe that because of the mutual attraction exerted by *similar* elements upon each other".

[2] Here Meschkowski, Nilson (1991) deviate from Meschkowski (1983) and Purkert, Ilgauds (1987) who give: "of first and second order or cardinality".

and application of the theorem of conservation of energy onto the whole world system which the gentlemen Thomson, v. Helmholtz, Clausius, and comrades like to do, who link phantastic speculations to it which in my opinion are without any value." [G. Cantor, letter to C. Gutberlet (1 May 1888)]

"the matter, as far as it is able to generate light, is just called 'ether'. It is not meant that this ether is different in character from the other matter; they differ only in their degree of density. [...] I have been pursuing for many years my special thoughts with regard to the explanation of light phenomena, in connection with the 'Théorie des ensembles'" [G. Cantor, letter to G. Mittag-Leffler (5 Oct 1883)]

"In close connection with the above results {{concerning point sets}} are mathematical-physical groundworks which I have been pursuing for several years without interruption. I have never been quite satisfied with even the most splendid results of mathematical analysis in physics because the hypotheses that they are based upon seemed to me partly contradictory and partly not clear and determined enough, and early already I recognized as the reason of this lack, that about the constitution of matter, the ponderable as well as the imponderable, the so-called ether, the truth had not yet been found anywhere. [...] The result of my investigations, which are not at all purely speculative but take into account experience and observation too, is that {{instead of atomism and continuity hypothesis}} a third hypothesis is conceivable, a name for which I have not yet found, a hypothesis which somehow lies between both, but is distinguished with respect to them by great simplicity and naturality and in particular precision; it participates in the advantages of the two others but seems to be free of their disadvantages and contradictions." [G. Cantor, letter to G. Mittag-Leffler (20/28 Oct 1884)]

4.2.5 Philosophy

"General set theory [...] definitely belongs to metaphysics." [G. Cantor, letter to T. Esser (1/15 Feb 1896)]

"Most likely I will quit the mathematical lectures here completely after some semesters because the teaching of the courses required for the education of mathematics teachers, like calculus, anal. geometry, and mechanics, does no longer appeal to me in the long run; I will give philosophical lectures instead" [G. Cantor, letter to G. Mittag-Leffler (20/28 Oct 1884)]

Cantor "began during the previous semester to lecture on Leibniz' philosophy. In the beginning he had 25 students. Then, little by little, the audience melted together, first to 4, then to 3, then to 2, finally to one. Cantor held out nevertheless and continued to lecture. But, alas! One fine day the last of the Mohicans came, somewhat troubled, and thanked the professor very much but explained that he had so many other things to do so he could not longer manage to follow the professor's lectures. Then Cantor, to his wife's unspeakable joy, gave the solemn promise never to lecture on philosophy again." [Sonja Kowalewskaja, letter to G. Mittag-Leffler (21 May 1885)]

"although I appreciate the *mathematical*, *physical*, and *astronomical* merits of Newton, I cannot but believe that he has *done much harm* to philosophy indirectly and that the main idea of his *metaphysics* is completely wrong. In the latter respect I think that his great rival *Leibniz* came infinitely closer to the truth." [G. Cantor, letter to É. Blanc (22 May 1887)]

"I have to draw your attention to two facts: 1st to the untearable ties that connect metaphysics and theology, in that the latter is the lodestar according to which the former is adjusting itself. [...] There follows a threefold: a) that in metaphysical discussions it is sometimes inevitable to have theology join in; b) that every real progress in metaphysics strengthens or multiplies the tools of theology; it may even happen that human reason (of course subjugating to the infallible decisions of the church) may obtain deeper and richer symbolic insights with respect to the mysteries of the religion than has been expected or foreseen before. [...]

Every expansion of our insight into the realm of the creational-possible therefore must lead to an expanded knowledge of God." [G. Cantor, letter to T. Esser (1/15 Feb 1896)]

"I am Baconion in the Bacon-Shakespeare question and I am *quite an adversary* of *Old Kant*, who, in my eyes has done much harm and mischief to philosophy, even to mankind; as you easily see by the most perverted development of metaphysics in Germany in all that followed him, as in Fichte, Schelling, Hegel, Herbart, Schopenhauer, Hartmann, Nietzsche, etc. etc. on to this very day. I never could understand that and why such reasonable and enabled peoples as the Italiens, the English and the French are, could follow yonder *sophistical philistine*, who was *so bad a mathematician*.

And now it is that in just this abominable mummy, as Kant is, Monsieur Poincaré felt quite enamoured, if he is not bewitched by him. So I understand quite well the opposition of Mons. Poincaré, by which I felt myself honoured, so he never had in his mind to honour me, as I am sure. If he perhaps expect, that I will answer him for defending myself, he is certainly in great a mistake." [G. Cantor, letter to B. Russell (19 Sep 1911) {{original English by Cantor}}]

"I have only recently found an occasion to get a more precise image of the so-called Nietzschean philosophy (a counterpart of Haeckel's monistic evolutionary philosophy). Because of its stylistic appeal it finds an uncritical *recognition* in Germany, which seems to me *highly alarming* with respect to its perverse contents and the Herostratic-antichristian motives." [G. Cantor, letter to F. Loofs (24 Feb 1900)]

"Without a little bit of metaphysics in my opinion no exact science can be founded. Therefore please excuse the few words that I have to say in the introduction about this in modern times so frowned upon doctrin. Metaphysics, as I understand it, is the teaching of the *being* or, what is the same, of that which is *there*, i.e., existing, that means: of the world how it is as such itself and not how it appears to us. All we can perceive with our senses and imagine with our abstract thinking is *not-being* and therefore is at most a trace of the being itself." [G. Cantor, unpublished text, written about 1913]

4.3 Cantor on the ease of his theory

At least in five of his preserved letters (and that is merely a small fraction of his complete correspondence) Cantor emphasizes that very little mathematical knowledge is required to understand his theory:

"To understand the basic idea of the teaching of the transfinite no scholarly education in newer mathematics is required; this could even be a hindrance because in the so-called infinitesimal analysis the potential infinite has been pushed to the fore and lead to the opinion, even of the heroes, as if they with their 'differentials' and 'integrals' mastered the heights of knowledge and skill. Strictly speaking however, the potential infinite is always unthinkable without the foundational A. I. (which only most of those gentlemen will not or can not account for). So, if you expect to get an 'expertly' competent judgement in the current question from those circles you might find your expectations disappointed. The only forum here is the {{Empress}} Reason who does not acknowledge any reputation of privileged, scholarly, academical guilds. *She* persists and rules – we humans come and go." [G. Cantor, draft of a letter to A. Schmid (18 Apr 1887)]

"Should you have time to read my papers, you might find that very little previous mathematical knowledge is required for the understanding." [G. Cantor, letter to C.F. Heman (28 Jul 1887)]

"With respect to the question of the actual infinite in creatis I repeat first of all, what I wrote you one year ago: A scholarly preparation in mathematics is not at all necessary for the understanding of my relevant papers but a careful study of the latter is sufficient. Everyone, and in particular the trained philosopher, is able to scrutinize the teaching of the transfinite and to convince himself from its correctness and truth." [G. Cantor, letter to C.F. Heman (2 Jun 1888)]

"The comprehension of the elements of the teaching of the transfinite does not require any scholarly preparation in the newer mathematics. That could rather be disadvantageous than useful for this purpose because the modern mathematicians, in their majority, [...] have got into a victory flush which lets them degenerate into materialistic one-sidedness and makes them *blind* for any objective-metaphysical recognition and therefore also for the foundations of their *own* science." [G. Cantor, letter to I. Jeiler (20 May 1888, Whitsun)]

"I mention that great previous knowledge *of mathematics* is *not required* to understand my teachings, but only thorough philosophical knowledge, as it is learned best and most beautifully at your institution." [G. Cantor, letter to A. Baumgartner (27 Dec 1893)]

And finally, Cantor gives a general advice: "As a philosopher you do well, in my opinion, to be very sceptical against mathematical authorities in all mathematical-philosophical questions, in memory of *Pascal's* true saying: 'Il est rare, que les géomètres soient fins, et que les fins soient géomètres.'" {{It is rare that the mathematicians are sharp-witted and that the sharp-witted are mathematicians.}} [G. Cantor, letter to A. Schmid (8 May 1887)]

4.4 The rising of the empty set

Bernard Bolzano, the inventor of the notion set (Menge) in mathematics would not have named a nothing an empty set. In German the word "Menge" has the meaning of many or great quantity. Often we find in German texts the expression "große (great or large) Menge", rarely the expression "kleine (small) Menge". Therefore Bolzano apologizes for using this word in case of sets having only two elements: "Allow me to call also a collection containing only two parts a set." [B. Bolzano: "Einleitung zur Grössenlehre", J. Berg (ed.), Friedrich Frommann Verlag, Stuttgart (1975) p. 152]

Also Richard Dedekind discarded the empty set. But he accepted the singleton, i.e., the non-empty set of less than two elements: "For the uniformity of the wording it is useful to permit also the special case that a system S consists of a *single* (of one and only one) element a, i.e., that the thing a is element of S but every thing different from a is not an element of S. The empty system, however, which does not contain any element, shall be excluded completely for certain reasons, although it may be convenient for other investigations to fabricate such." [R. Dedekind: "Was sind und was sollen die Zahlen?" Vieweg, Braunschweig (1887), 2nd ed. (1893) p. 2]

Bertrand Russell considered an empty class as not existing: "An *existent* class is a class having at least one member." [Bertrand Russell: "On some difficulties in the theory of transfinite numbers and order types", Proc. London Math. Soc. (2) 4 (1906) p. 47]

Gottlob Frege shared his opinion: "If, according to our previous use of the word, a class consists of things, is a collection, a collective union of them, then it must disappear when these things disappear. If we burn down all the trees of a forest, then we burn down the forest. Thus an empty class cannot exist." [G. Frege: "Kleine Schriften", I. Agelelli (ed.), 2nd ed., Olms, Hildesheim (1990) p. 195]

Georg Cantor mentioned the empty set with some reservations and only once in all his work: "Further it is useful to have a symbol expressing the absence of points. We choose for that sake the letter O; $P \equiv O$ means that the set P does *not* contain *any single point*. So it is, strictly speaking, not existing as such." [Cantor, p. 146]

And even Ernst Zermelo who made the "Axiom II. There is an (improper) set, the '*null-set*' 0 which does not contain any elements" [E. Zermelo: "Untersuchungen über die Grundlagen der Mengenlehre I", Mathematische Annalen 65 (1908) p. 263], this same author himself said in private correspondence: "It is not a genuine set and was introduced by me only for formal reasons." [E. Zermelo, letter to A. Fraenkel (1 Mar 1921)] "I increasingly doubt the justifiability of the 'null set'. Perhaps one can dispense with it by restricting the axiom of separation in a suitable way. Indeed, it serves only the purpose of formal simplification." [E. Zermelo, letter to A. Fraenkel (9 May 1921)]

So it is all the more courageous that Zermelo based his number system completely on the empty set: { } = 0, {{ }} = 1, {{{ }}} = 2, and so on. He knew that there is only one empty set. But many ways to create the empty set can be devised, like the empty set of numbers, the empty set of bananas, the uncountably many empty sets of all real singletons, and the empty set of all these empty sets. Is it the emptiest set? Anyhow, "zero things" means "no things". So we can safely say (pun intended): Nothing is named the empty set.

4.5 Gems from the surroundings of set theory

4.5.1 Proof of God

A being exists which reconciles all positive properties in itself. That has been proven by the legendary logician Kurt Gödel by means of a complicated formula. Two scientists have scrutinized this proof of God – and have approved it. "The existence of God can in future be assumed to be a proven logical theorem." [T. Hürter: "Mathematiker bestätigen Gottesbeweis", SPIEGEL ONLINE, Wissenschaft (9 Sep 2013)] "Gödel's ontological proof has been analysed for the first-time with an unprecedented degree of detail and formality with the help of higher-order theorem provers." [Christoph Benzmüller, Bruno Woltzenlogel Paleo: "Formalization, mechanization and automation of Gödel's proof of God's existence", arXiv (2013)] This is a very lucid example of the advantage of the technique of formalizing and the safety, reliability, dependability, and trustworthiness gained by checking theorems by means of theorem provers.

In this context it may be of interest to note Cantor's answer to the question: "Couldn't God, after having produced an infinite set, e.g, of stones or angels, produce further angels? *Of course He can do this*, has to be answered here. If he {{Durandus de Sancto Porciano}} further derives from that: That means that the angels produced first have not been infinitely many, then this conclusion is *utterly wrong* because the supposed set of produced angels is a *transfinitum* capable of increase and decrease." [G. Cantor, letter to I. Jeiler (20 May 1888, Whitsun)]

"These *angels* cannot influence our fate; nevertheless, the more often we ask for their help, the more lucky is our lot." [Franz Ludescher: "Engel" (Oct 2010)]

"Great cardinal numbers can be helpful even there where they are not really used." [Ralf Schindler: "Sind große Kardinalzahlen entbehrlich?" (23 Apr 2010) p. 24]

"Gödel's second incompleteness theorem says that the consistency of mathematics cannot be proven mathematically. Therefore we can never exclude that there are two contradictory statements both of which can be proved correctly. (We believe firmly and unshakeably that two such statements are *not* existing)." [Manfred Burghardt: "Notes of a lesson by Peter Koepke", Bonn (1996) p. 3]

And this is the victory that has overcome the world: our faith. [Anonymous: "The holy bible", 1 John 5:4]

4.5.2 Odd and even transfinite ordinals

"Yes, the first infinite ordinal ω is even, but other infinite ordinals, such as $\omega + 1$ or $\omega^\omega + 5$ are odd ordinals. The cardinality of the set of even natural numbers is ω which is even. In fact, under AC, all infinite cardinals are even." [J.D. Hamkins, MathOverflow, Q 21457 (15 Apr 2010)][1]

[1] Unfortunately, meanwhile not even rudiments of this question are available any longer.

4.5.3 Definability and countability of real numbers

A preprint by J.D. Hamkins et al. contains the following phrases, starting smugly: "One occasionally hears the argument – let us call it the math-tea argument, for perhaps it is heard at a good math tea – that there must be real numbers that we cannot describe or define, because there are only countably many definitions, but uncountably many reals. Does it withstand scrutiny? [...]

Question 1. *Is it consistent with the axioms of set theory that every real is definable in the language of set theory without parameters?*

The answer is Yes. Indeed, much more is true: if the ZFC axioms of set theory are consistent, then there are models of ZFC in which every object, including every real number, every function on the reals, every set of reals, every topological space, every ordinal and so on, is uniquely definable without parameters.[1] Inside such a universe, the math-tea argument comes ultimately to a false conclusion. [...]

In a pointwise definable model, every object can be specified as the unique object with some first-order property. In such models, all objects are discernible;" [J.D. Hamkins et al.: "Pointwise definable models of set theory", arXiv (2012)]

Usually it is claimed that every given real number can be distinguished in a finite process from every other given real number. That is true – but most real numbers cannot *be given*. Neverthelss forcing even proves that all sets, including ℝ, can be made countable: "Namely, the method of forcing shows that every set that exists in any model of set theory can become countable in another larger model of set theory, the forcing extension obtained by collapsing the cardinality of that set to ω. Thus, the concept of countability loses its absolute meaning; whether a set is countable or not depends on the set theoretic background. So if X is any set, then in some forcing extension of the universe, the set X becomes countable." [J.D. Hamkins in "Is every model of ZF countable 'seen from the outside'?", MathOverflow (21 Jan 2010)] The first statement is certainly true since there is no model of set theory. The readers may make their own judgement about the scientific value of forcing.

4.5.4 Real damage

"Wolfgang Mückenheim is probably one of the most dangerous cranks out there. He has a professorship at the University {{of Applied Sciences}} of Augsburg, Germany, where he is teaching physics and mathematics!! Currently {{having started in 2003}}, he is teaching a lecture called 'History of the Infinite'. This man does real damage." [Michael Greinecker in "Nominalist foundations of mathematics", tea.MathOverflow (1 May 2012)]

A curse uttered by an anonymous matheologian culminated in the words: "No punishment, within legal boundaries {{death penalty, if in suitable states of the USA?}}, would be too severe for you for your wrongdoings. [...] Rest assured that my contact, the senior German civil servant who refused to believe this fiasco was going on, is being copied into these threads. I sincerely hope there are severe repercussions. Those exposed to this type of 'education', assuming they are, or their guardians if they are minors, have every right to seek legal remedies in the civil courts against the perpetrator(s)." [Port563 in "What is a real number", sci.math (9 May 2014)]

[1] So, by contraposition we easily find that the ZFC axioms are inconsistent.

4.5.5 Questions hastily deleted in MathOverflow

MathOverflow is "a forum to ask and answer research level mathematics questions". The overwhelming majority of users there are mainstream mathematicians, who anxiously suppress any heretic idea. The following questions have been deleted – most very soon – obviously because the participants are not able to answer them in a convincing or satisfying way. This is not a surprise because they deny that infinite sequences do never end; they are unable to understand the infinite but project their ideas about finite, exhaustible sets onto infinite sequences and sets. The reader may judge whether or not these questions have mathematical content.

"Let (s_n) be a sequence of sets $s_n = \{n+1, n+2, ..., 2n\}$ of natural numbers 1, 2, 3, ...

There is a limit of the sequence of sets, namely the empty set { }, showing that no number n remains in the sequence.

There is a limit of the sequence of cardinal numbers, namely $\aleph_0$. What does this limit mean?" [Bacarra: "What is the meaning of the limit of a sequence of cardinal numbers?", MathOverflow (27 Jun 2014)]

"Let (s_n) be the sequence of sets $s_n = \{n\}$ of natural numbers 1, 2, 3, ... Then the limit is the empty set { }. The sequences of sets $t_n = \{n^1\}$ or $u_n = \{n^n\}$ or $v_n = \{n/(n+1)\}$ have also empty limit sets. This shows that no natural number will remain in all sets of the sequence. All natural numbers will be exhausted.

But this simple argument appears to fail in cases like $w_n = \{1_n\}$ (where n is only an index) or $x_n = \{n/n\}$ or $y_n = \{n^0\}$? Of course all limits are {1}. But why don't the natural numbers get exhausted in these cases?" [Bacarra: "Why do the natural numbers get exhausted in some limits but not in all?", MathOverflow (27 Jun 2014)]

A typical comment, characteristic of the research level of the research-level-researchers researching there: "I don't understand your question, but I'm pretty sure it's not about mathematical research, which is what the MO website is for." [Gerry Myerson, loc cit]

"The limit of the sequence (s_n) with $s_n = \{n\}$ is the empty set. This means, among others, that there is no natural number $n \in \mathbb{N}$, that remains in all terms of the sequence.

The ordered character of the natural numbers allows us to understand this sequence as a supertask, transferring the complete set $\mathbb{N}$ from a reservoir A to a reservoir Z. Every single transfer of a natural number during the supertask can be represented by a term of the sequence and vice versa.

However, if we introduce an intermediate reservoir M and define that every transfer has to pass via M, and further, that a number n may leave M only after the number $n+1$ has been inserted into M, then we have the same limit, i.e., the whole set $\mathbb{N}$ will finally be in Z although this can be excluded by the definition of the supertask. How can this contradiction be solved?" [Illegal border: "Minimal super task", MathOverflow (10 Jul 2014)]

"How can the difference between ordinal and cardinal exponentiation be expressed when only multiplication is allowed, if $a^n = a \cdot a \cdot a \cdot ... \cdot a$?" [Charbotew: "How can the difference between ordinal and cardinal exponentiation be expressed when only multiplication is allowed?", MathOverflow (7 Oct 2018)]

"The following three principles seem to be incompatible:

(1) Cantor's axiom: Every real number determines a point of the real line.

(2) The set of rational numbers is countable and therefore can be included in intervals of arbitrary small total measure.

(3) For every pair (x, y) of different irrational numbers $x < y$ there is a rational number q with $x < q < y$.

To visualize the problem assume a covering of all rational numbers of the real axis by closed intervals with irrational endpoints like $[q_n\text{-}10^{-n}\cdot\sqrt{2}, q_n+10^{-n}\cdot\sqrt{2}]$. It has measure less than 1 and could be made as small as desired. The remaining space does not contain rational numbers. As it is included into the irrational endpoints it cannot contain irrational numbers either. This result is required by (3) in case (1) and (2) hold, but it is obviously not true.

Therefore my question: How can these three principles be reconciled?"
[Hans Hinterseher: "How can the following three principles be reconciled?" (29 Nov 2018)]

"Borel proved that a countable set of intervals of total length 1 removed from the real axis, leaves an uncountable number of separated points in the complement. The points must be separated because the set of intervals could contain all rational numbers, but irrational numbers cannot exist side by side. We can replace the set of intervals by their midpoints without reducing the complement or changing the structure of separation. And the example does not depend on the rational character of points, since we could translate the whole system by adding π. Then we get the theorem (where the lengthy expression 'interval or degenerate interval or point' has been abbreviated by 'atom'):

(1) Removing $\aleph_0$ atoms separates $2^{\aleph_0}$ atoms in the complement.

By symmetry however (removing what in (1) has remained) we get also:

(2) Removing $2^{\aleph_0}$ atoms separates $\aleph_0$ atoms in the complement.

But if we stop in (2) after having removed $\aleph_0$ atoms, we have, according to (1), separated more atoms in the complement, namely $2^{\aleph_0}$, than after finishing (2). What is the correct answer to this apparent paradox?" [Ganzhinterseher: "Can symmetry considerations help in topology?" MathOverflow (5 Jan 2019)]

"I am looking for a countable model of ZF reduced to only the axiom of infinity and the axiom of powerset. It is sufficient to find a subset of ω that is provably not in the model.

Clarification: I am not interested in any subset of ω that cannot be in a countable model. Such a subset does not exist. I am interested in a proposal of a countable model that does not contain at least one subset of ω." [Brahibam: "What subset of ω can be missing in a countable model?", MathOverflow (21 Jan 2019)]

4.5.6 Suppression of the distinction between infinities

"But potential infinity is almost forgotten now. In the ZFC set theory mindset, people tend not to even remember that distinction. They just think infinity means actual infinity and that's all there is to it." [S. Simpson quoted in N. Wolchover: "Dispute over infinity divides mathematicians", Scientific American (3 Dec 2013)]

As mentioned in chapter I already [P.L. Clark in "Physicists can be wrong", tea.MathOverflow (2 Jul 2010)] terms like completed and potential infinity are not part of the modern vernacular.

A typical opinion is this one: "The notion of actual vs. potential infinity is simply not required in modern mathematics." [D. Christensen in "How absurd will things get before things change", sci.math (2 Dec 2015)]

In set theory however actual infinity is urgently needed, for instance in form of the complete digit sequence of Cantor's antidiagonal number that uniquely defines this real number. A potentially infinite sequence could never accomplish this because in every case only a finite number of digits is defined whereas always most, i.e., infinitely many, digits are undefined. In other instances only the potentially infinite sequence can be tolerated. In fact nothing more is given by the axiom of infinity. The only successful approach consists in forgetting about the difference.

4.5.7 On the continuum hypothesis

- Cantor believed in the Continuum Hypothesis (CH).
- Gödel proved: (ZFC is consistent) ⇒ (ZFC + CH is consistent).
- Cohen proved: (ZFC is consistent) ⇒ (ZFC + ¬CH is consistent).

Nevertheless both seem to distrust the axioms and pursue their own opinions:

"Only someone who (like the intuitionist) denies that the concepts and axioms of classical set theory have any meaning (or any well-defined meaning) could be satisfied with such a solution, not someone who believes them to describe some well-determined reality. For in this reality Cantor's conjecture must be either true or false, and its undecidability from the axioms known today can only mean that these axioms do not contain a complete description of this reality; [...] not one plausible proposition is known which would imply the continuum hypothesis. Therefore one may on good reason suspect that the role of the continuum problem in set theory will be this, that it will finally lead to the discovery of new axioms which will make it possible to disprove Cantor's conjecture." [Kurt Gödel: "What is Cantor's continuum problem?", The American Mathematical Monthly 54,9 (1947) pp. 520 & 524]

"A point of view which the author feels may eventually come to be accepted is that CH is obviously false. The main reason one accepts the axiom of infinity is probably that we feel it absurd to think that the process of adding only one set at a time can exhaust the entire universe. Similarly with the higher axioms of infinity. Now $\aleph_1$ is the cardinality of the set of countable ordinals, and this is merely a special and the simplest way of generating a higher cardinal. The set C [the continuum] is, in contrast, generated by a totally new and more powerful principle, namely the power set axiom. It is unreasonable to expect that any description of a larger cardinal which attempts to build up that cardinal from ideas deriving from the replacement axiom can ever reach C. Thus C is greater than $\aleph_n$, $\aleph_\omega$, $\aleph_a$, where $a = \aleph_\omega$, etc. This point of view regards C as an incredibly rich set given to us by one bold new axiom, which can never be approached by any piecemeal process of construction. Perhaps later generations will see the problem more clearly and express themselves more eloquently." [P. Cohen: "Set theory and the continuum hypothesis" Dover Publications (2008) p. 151]

4.6 "Applications" of set theory

In his paper on Grenzzahlen [E. Zermelo: "Über Grenzzahlen und Mengenbereiche", Fund. Math 16 (1930) pp. 29-47] Zermelo railed against the "scientific reactionaries and antimathematicians", and he enlightened his contemporaries about the "enormous importance and the unlimited applicability of set theory". On the other hand in modern times doubts have been raised: With respect to pure and applied mathematics "it is clear that sooner or later there will be a question about why society should pay money to people who are engaged in things that do not have any practical applications". [Vladimir Voevodsky in "Интервью Владимира Воеводского" (1 Jul 2012), translated by John Baez] Let us scrutinize then the unlimited applicability of set theory.

4.6.1 Test of set theory by its impact on sciences

"I think that it is even possible that axioms could be tested by their impact on fields outside of mathematics like physics. It may sound like an outrageous speculation and admittedly we do not have any concrete example of such a possible impact, but in the next section we shall give an example {{cp. section 4.6.7 of the present chapter}}where the set theory we use may have some relevance to the mathematical environment in which a physical theory is embedded." [Menachem Magidor: "Some set theories are more equal", ResearchGate.net (2015)]

4.6.2 Transfinite conclusions in relativity and quantum theory

"Have you realized what a richness of multitudes of transfinite conclusions and calculations of the most difficult and painstaking kind is immanent for instance in relativity theory and similarly in quantum theory. And nature acts precisely according to these results. The beam of the fixed star, the mercury {{perihelion rotation}} and the most entangled spectra here on earth and in the distance of 100000 light years? And all that should be pure chance?" [D. Hilbert, letter to O. Becker (autumn 1930?) published in Volker Peckhaus: "Becker und Zermelo"]

4.6.3 Dark energy density and fractal-Cantorian space-time

From time to time there appear books or articles proposing the application of transfinite set theory to the scientific domain, preferably quantum theory and cosmology, on levels that escape every present experimental verification; see Mohamed S. El Naschie: "From highly structured E-infinity rings and transfinite maximally symmetric manifolds to the dark energy density of the cosmos", Advances in Pure Mathematics 4 (Dec 2014) pp. 641-648 or Jerzy Król: "Model and set-theoretic aspects of exotic smoothness structures on R^4", arXiv (2016).

"M.S. El Naschie [...] began to work on his Cantorian version of fractal space-time. He showed that the *n*-dimensional triadic Cantor set has the same Hausdorff dimension as the dimension of a random inverse golden mean Sierpinski space to the power *n*-1. [...] The author is indebted to the many members of the fractal-Cantorian space-time community." [L. Marek-Crnjac: "A short history of fractal-Cantorian space-time", Chaos, Solitons and Fractals 41 (2009) pp. 2697-2705]

4.6.4 String set theory

"In the present paper I would like to develop a different point of views on the continuum. [...] As a background this point-set theoretic concept is influenced by individualism in modern civilization. 19th and 20th centuries are the centuries of individualism, and the individualism played an important role in the revelation of people and high advancement of science and technology. Historically individualism came from liberalism, which in turn came from Reform of Religion by earlier Protestants, and the fundamental roots can even go upstream to Apostle Paul. Anyway by historical reason Protestantism performed an important role to the development of civilization. It is marvellous, if it is taken into consideration that religion is conservative in nature, that Protestantism contributed the advancement of science that sometimes contradicts against Bible. (This is caused because Protestantism abandoned to be a religion.) [...] As a counterpart of point-set theory string-set theory is proposed. It is asserted that the string-set is the essence of continuum in one aspect [...] And importance of introducing string-set theoretical point of view not only to make mathematics useful but also correct crippled modern civilization." [Akihiko Takizawa: "String set theory" (2002) link expired]

4.6.5 Holographic virtual universe?

"An uncountable number of string-like vibrations creating subatomic and then atomic particles forming a nearly substance-less multiplex of universes! Particles of so little actual substance and with so much space between them that an electromagnetic field surrounds all objects and matter to keep everything from flowing through everything else. We never actually touch anything or anyone, only our fields rub against each other and all the things we come into contact with!" [T.D. Spoon: "String theory: The control mechanism of creation? Holographic virtual universe?", Alternative Reality News (2011) link expired]

4.6.6 The Casimir effect

A popular hobbyhorse of advocates of uncountability in physics is the Casimir effect. "The progress of the real photon is delayed as it travels through this quantum vacuum 'crystal', where it meets uncountable numbers of electrically charged virtual particles." [Tom Ostoma, Mike Trushyk: "The light velocity Casimir effect", arXiv (1999)]

"Just to mention to which extent the point about 'counting' is subtle. If we trust the 'number of modes argument', on the one hand we have a slab of size *L* corresponding to an infinite but countable set of modes and on the other hand we have two semi-infinite spaces corresponding to an infinite and uncountable set of modes. The difference between the two should be infinite and that's about it end of the story ...." [gatsu in "Casimir effect as an entropic force" (1 Oct 2013)]

"Alex Filippenko discusses the Casimer effect as an example of virtual particles which might also explain dark energy. He gives a hand waving argument which attempts to explain the effect as being caused by an uncountable number of virtual particle waves of arbitrary length outside the two parallel plates with a countable number of standing waves inside the plates." [Ricky Jimenez: "Cantor's uncountability theory explains Casimer effect?", sci.physics (6 Apr 2011)]

4.6.7 Entangled states in quantum mechanics: The EPR-paradox

In 1935 Einstein, Podolsky, and Rosen (EPR) [1] proved by means of a Gedankenexperiment "that the description of reality as given by the wave function is not complete". Their approach has been transformed by D. Bohm [2] to the case of two entangled spin-1/2-particles in a common spin-0 state. Then J.S. Bell [3] has shown that the existence of "local hidden variables" would result in violating quantum mechanical results of correlated polarization measurements. He considers the product of measurement results performed in directions $\boldsymbol{a}$ and $\boldsymbol{b}$. If the single result s depends on a "hidden variable" λ, then the product of the correlated results $s_1(\boldsymbol{a}, \lambda)$ and $s_2(\boldsymbol{b}, \lambda)$ yields an expectation value $E(\boldsymbol{a}, \boldsymbol{b})$. Corresponding expressions for a third direction $\boldsymbol{c}$ complete the famous Bell-inequality (with $h/2\pi = 1$ and $|\boldsymbol{a}| = |\boldsymbol{b}| = |\boldsymbol{c}| = 1$):

$$|E(\boldsymbol{a}, \boldsymbol{b}) - E(\boldsymbol{a}, \boldsymbol{c})| \leq E(\boldsymbol{b}, \boldsymbol{c}) + 1/4 \, .$$

It is violated by the quantum theoretical expectation values $\langle s_1(\boldsymbol{a}) \cdot s_2(\boldsymbol{b}) \rangle = -\cos(\boldsymbol{a}, \boldsymbol{b})$ for instance when $\boldsymbol{b}$ bisects the angle $2\pi/3$ formed by $\boldsymbol{a}$ and $\boldsymbol{c}$.

After a lot of work done by many authors to establish an experimentally verifiable version of Bell's inequality the quantum mechanical predictions have been confirmed by several experiments (for a review see [4]). Since Bell's proof is based on a local theory, the experiments prove the existence of superluminal interactions (or of absolute determinism, i.e., the universe is operating like the frames of a movie). They are not in contradiction with relativity though, because the interaction cannot be used to transmit information with superluminal velocity. Nevertheless the existence of such spooky action at a distance is not very satisfactory physics.

Therefore attempts have been made by I. Pitowsky [5], introducing non-measurable sets (where it may happen that almost all objects are red and almost all objects are small, but no object is small and red) and by myself [6], introducing negative probabilities, originally considered by Dirac, Bartlett, Wigner and others [7], to circumvent Bell's result which is based on a measurable and positive semidefinite probability distribution of the local hidden variables like the direction of a total spin vector $\boldsymbol{S}$ with $|\boldsymbol{S}| = \sqrt{3}/2$ (where $h/2\pi = 1$). When integrating the probability functions $w_+(\boldsymbol{a}, \boldsymbol{S}) = 1/2 + \boldsymbol{a} \cdot \boldsymbol{S}$ for "spin up" and $w_-(\boldsymbol{a}, \boldsymbol{S}) = 1/2 - \boldsymbol{a} \cdot \boldsymbol{S}$ for "spin down" over the whole sphere the quantum theoretical expectation values are precisely reproduced. Further the functions satisfy $w_+ (\boldsymbol{a}, \boldsymbol{S}) + w_-(\boldsymbol{a}, \boldsymbol{S}) = 1$. But ranging from $(1 - \sqrt{3})/2$ to $(1 + \sqrt{3})/2$ they assume negative values.

[1] A. Einstein, B. Podolsky, N. Rosen: "Can quantum-mechanical description of physical reality be considered complete?", Phys. Rev. 47 (1035) pp. 777-780.
[2] D. Bohm: "Quantum theory", Dover, New York (1951) p. 614.
[3] J.S. Bell: "On the Einstein-Podolsky-Rosen paradox", Physics 1,3 (1964) pp. 195-200.
[4] W. Mückenheim: "Das EPR-Paradoxon und die Unbestimmtheit der Realität", Physikalische Blätter 39 (1983) pp. 331-336.
[5] I. Pitowsky: "Resolution of the EPR and Bell paradoxes", Phys. Rev. Lett. 48 (1982) pp. 1299-1302 & "Deterministic model of spin and statistics", Phys. Rev. D 27, pp. 2316-2326.
[6] W. Mückenheim: "A resolution of the Einstein-Podolsky-Rosen paradox", Lett. Nuovo Cim. 35 (1982) 300-304.
[7] W. Mückenheim et al.: "A review of extended probabilities", Phys. Rep. 133 (1986) pp. 337-401.

"When a physical theory is stated in terms of mathematical concepts like real numbers, Hilbert spaces, manifolds etc. it implicitly adapts all the mathematical facts which are accepted by the Mathematicians to be valid for these concepts. If the mathematical 'truths' may depend of the foundation of Set Theory then it is possible, at least in principle, that whether a given physical theory implies a particular physically meaningful statement may depend on the foundational framework in which the implicitly assumed Mathematics is embedded.

This may seem far fetched and it is very likely that physically consequences of a physical theory will never depend on the set theoretical foundation of the mathematical reasoning that accompanied the theory but the point of this paper is that this is still a definite possibility. [...]

I. Pitowsky used Continuum Hypothesis to construct hidden variable models for spin-1/2 and spin-1 particles in quantum mechanics. His functions are not measurable and are therefore not directly subject to Bell-type no-go theorems [...] Also, under the same assumption Pitowsky constructed a function that almost violates the no-go theorem of Kochen and Specker [...] such that for every vector $x \in S^2$ there are at most countably many exceptions [...]. We prove that no such function exists in some model of the Zermelo-Fraenkel set theory with the Axiom of Choice, ZFC, confirming a conjecture of Pitowsky. While this independence result probably does not have physical interpretation, it gives some weight to the conjecture that one could decide between different set theories on the basis of their scientific consequences. [...]

We prove [...] that if there exists a σ-additive extension of the Lebesgue measure to the power-set of the reals then Pitowsky models do not exist." [Ilijas Farah, Menachem Magidor: "Independence of the existence of Pitowsky spin models", arXiv (2012) p. 1f]

"In an attempt to demonstrate that local hidden variables are mathematically possible, Pitowsky constructed 'spin-1/2 functions' [...] {{This}} construction uses the Continuum Hypothesis. Farah and Magidor took this as an indication that at some stage physics might give arguments for or against adopting specific new axioms of set theory. We would rather argue that it supports the opposing view, i.e., the widespread intuition 'if you need a non-measurable function, it is physically irrelevant'. [...]

Pitowsky used the Continuum Hypothesis to construct a spin-1/2 function model. Pitowsky suggested that the existence of such a function might not follow from the usual axioms of set theory alone, which has recently been confirmed by Farah and Magidor (2012). In the same paper, as well as in (Magidor, 2012), it has been argued that the spin-1/2 model is an indication that physical considerations might provide input on which new axioms should be adopted for set theory.

We do not share this opinion: [...] Pitowsky uses the Axiom of Choice for this model, but we think even that is unnecessary [...]

But in the end, Pitowsky probability turns out to be just a variant of super-determinism. Accordingly, the models are obviously consistent, but physically not relevant. (And doubly so: super-determinism is physically unfeasible, and hidden variables are pointless within super-determinism.)

So we come to quite the opposite conclusion as Farah and Magidor: Instead of indicating connections between physics and set theory, Pitowsky's attempts of hidden variables rather seem to reaffirm the old intuition: 'if nontrivial set theory, non-constructive mathematics or a non-measurable set is used in an essential way, it cannot be physically relevant'." [Jakob Kellner: "Pitowsky's Kolmogorovian models and super-determinism", arXiv (2016)]

4.6.8 Could any physical mathematics be independent of ZFC?

"One remark that Penelope Maddy makes several times in *Naturalism in Mathematics*, is that if the indispensability argument was really important in justifying mathematics, then set theorists should be looking to debates over quantum gravity to settle questions of new axioms. Since this doesn't seem to be happening, she infers that the indispensability argument can't play the role Quine and Putnam (and perhaps her earlier book?) argued that it does. [...] I don't know much about the details, but from what I understand, physicists have conjectured some deep and interesting connections between seemingly disparate areas of mathematics, in order to explain (or predict?) particular physical phenomena. These connections have rarely been rigorously proved, but they have stimulated mathematical research both in pursuing the analogies and attempting to prove them. Although the mathematicians often find the physicists' work frustratingly imprecise and non-rigorous, once the analogies and connections have been suggested by physicists, mathematicians get very interested as well.

If hypothetically, one of these connections was to turn out to be independent of ZFC, I could imagine that there would at least be a certain camp among mathematicians that would take this as evidence for whatever large cardinal (or other) principle was needed to prove the connection. Set theorists themselves haven't paid too much attention to these issues, because the interesting connections are in mathematical areas traditionally considered quite distant from set theory. Instead, they have traditionally looked at intra-set-theoretic considerations to justify large cardinals. But if it became plausible that some of these other debates would turn out to be connected, I'm sure they would start paying attention to the physics research, contrary to what Maddy suggests." [Kenny Easwaran: "Set theory and string theory", Antimeta, wordpress.com (29 Oct 2006)]

And a Greg answered: "This strikes me as a Very Good Point. I guess the next relevant question to ask is: is it at all reasonable (or even conceivable) that any mathematical claim that these physicists are making could end up being independent of ZFC?" [loc cit (30 Oct 2006)]

Of course. *Every* physical result is independent of ZFC.

4.6.9 Fine structure of the Saturn rings

"One of the remarkable observations made by the Voyager 2 probe was of the extremely fine structure of the Saturn ring system. [...] The Voyager 1 and 2 provided startling images that the rings themselves are composed of thousands of thinner ringlets each of which has a clear boundary separating it from its neighbours.

This structure of rings built of finer rings has some of the properties of a Cantor set. The classical Cantor set is constructed by taking a line one unit long, and erasing its central third. This process is repeated on the remaining line segments, until only a banded line of points remains." [H. Takayasu: "Fractals in the physical sciences", Manchester University Press (1990) p. 36]

"Mandelbrot conjectures that radial cross-sections of Saturn's rings are fat Cantor sets." [NN: "Fractal folds", users.math.yale.edu]

Are the rings made of anticountable super matter? The idea is not new however:

4.6.10 A map of England

"A map of England, contained within England, is to represent, down to the minutest detail, every contour and marking, natural or artificial, that occurs upon the surface of England. [...] For the map, in order to be complete, according to the rule given, will have to contain, as a part of itself, a representation of its own contour and contents. In order that this representation should be constructed, the representation itself will have to contain once more, as a part of itself, a representation of its own contour and contents; and this representation, in order to be exact, will have once more to contain an image of itself; and so on without limit. We should now, indeed, have to suppose the space occupied by our perfect map to be infinitely divisible, even if not a *continuum*. [...] That such an endless variety of maps within maps could not physically be constructed by men, and that ideally such a map, if viewed as a finished construction, would involve us in all the problems about the infinite divisibility of matter and of space, I freely recognize." [Josiah Royce: "The world and the individual", MacMillan, London (1901) p. 504f]

4.6.11 Fuzzy and rough set theory

"Fuzzy Set Research in Production Management

1. Job Shop Scheduling
2. Quality Management
3. Project Scheduling
4. Facility Location and Layout
5. Aggregate Planning
6. Production and Inventory Planning
7. Forecasting"

[A.L. Guiffrida, R. Nagi: "Fuzzy set theory applications in production management research"]

"Rough set theory has an overlap with many other theories dealing with imperfect knowledge, e.g., evidence theory, fuzzy sets, Bayesian inference and others. [...] Let us start our considerations from a very simple tutorial example concerning churn modeling in telecommunications [...]. In the table condition attributes describing client profile are: *In* – incoming calls, *Out* – outgoing calls within the same operator, *Change* – outgoing calls to other mobile operator, the decision attribute describing the consequence is *Churn* and *N* is the number of similar cases." [Zdzisław Pawlak: "Rough set theory and its applications", Journal of Telecommunications and Information Technology (3/2002) pp. 7-10]

Unfortunately the Fuzzy- and Rough Set Theories are lacking any transfinitude. So they are off topic here. Further practical applications of set theory are unknown to the author.

4.6.12 The uses of set theory in mathematics

"1. *The ideal of compact operators*

The purely analytic question 'Is the ideal of compact operators on Hilbert space the sum of two properly smaller ideals?' is equivalent to purely set-theoretic combinatorics.

2. *A characterization of free groups*

The proof that 'an Abelian group is free if and only if it has a discrete norm' exploits the use of model theory within set theory.

3. *The fundamental group*

The proof that 'the fundamental group of a nice space is either finitely generated or has cardinality of the first uncountable cardinal' uses methods related to consistency results.

4. *The Hawaiian Earring*

Questions in strong homology theory are related to consistency results and the continuum hypothesis.

5. *A Banach space with few operators*

An example of a nonseparable Banach space where every linear operator is a scalar multiplication plus an operator with separable range is connected to set theory through infinite combinatorics on the first uncountable ordinal.

6. *The free left-distributive algebra on one generator*

Questions on free left-distributive algebras on one generator are connected to large cardinal theory.

I have presented a few theorems of mainstream mathematics that have been proved by set-theoretic techniques." [Judith Roitman: "The uses of set theory", The Mathematical Intelligencer 14,1 (1992) pp. 63-69]

So we can be sure that all these theorems are uncertain.

4.6.13 Wiles' original proof of FLT

"Even if a particular problem can be solved in principle in a weaker system, it is many times the case that the first time the proof is discovered or the more natural and simpler proof is discovered in a stronger system. An illustrative case is the story of Wiles' proof of Fermat last theorem. The original proof used Grothendieck's universes, hence formally it assumed the existence of inaccessible cardinals. As everybody expected they can be eliminated but the point is that {{when}} Wiles constructed his proof it came naturally for him to make the assumption that formally moved him away from *ZFC*. The interesting twist is that when I talked to several number theorists about the project of getting the proof in a weaker system like *ZFC* or P*A* they were not interested! The assumption of the existence of Grothendieck's universes (hence the assumption of the existence of unboundedly many inaccessible cardinals) seems to them such a natural extension of *ZFC* that having a proof of this Π_1^0 statement in this theory looks like good enough ground for believing the truth of the theorem and an attempt to eliminate the use of the stronger axioms looks to them like an unnecessary logicians' finicking." [Menachem Magidor: "Some set theories are more equal", ResearchGate (2015) p. 6f]

"In the case of Wiles, the structures involved are all finite. With some stretching of interpretation, maybe something is countably infinite." [Harvey Friedman: "Report from expert", FOM (6 Apr 1999)]

"Wiles' article 'Modular elliptic curves and Fermat's last theorem' uses Grothendieck duality over fields," [Colin McLarty: "Expert error", FOM (7 Apr 1999)]

4.6.14 A list of "practical applications" of set theory

"I'm asking about the practicality of the knowledge of the properties of infinite sets, and their cardinality. [...] Seeing as how there was so much resistance to infinite sets at the beginning, even among mathematicians, I wonder has the math of infinite sets be 'proven worthwhile' by having a practical application outside of mathematics, so that no one can say it's just some imaginative games?" [user2020: "What practical applications does set theory have?", MathOverflow (2010)]

The answers are sobering. For example set theory is needed for topology, topology is needed for differential geometry, differential geometry is needed for general relativity.

"Topology changes infinite sums from nonsense into legitimate mathematical objects because it allows us to talk about convergence." [Harry Gindi, loc cit] Limits of series have been meaningful before set theory – and will be after. (Unfortunalety actual or *completed* infinity of set theory supports the wrong impression that limits of series are "infinite sums".)

"There are many uses of infinite sets and their properties. [...] Transfinite induction covers all possible ways in which one could show that a program terminates, while the ordinal numbers are used to express how complex the proof of termination is (the bigger the number, the more complicated it is to see that the program will actually terminate)." [Andrej Bauer, loc cit] I'd like to see a programmer who checks his program by transfinite induction!

The other answers are revolving around the useful symbolics (which is a virtue of *finite* set theory), grouping objects with a common property, measure theory, and transcendental numbers.

"You need set theory to have measure theory and you need measure theory to have the analysis required to support, for example, Fourier series. Really, most of what is going on in real analysis (and hence in calculus) depends on having a predictable understanding of how infinite sums, sequences, and sets behave. So, elementary set theory and the ideas about infinite sets in particular are crucial for all kinds of 'practical' math." [S. Donovan, loc cit] Since the days of A.A. Fraenkel the assertion has been maintained that transfinite set theory is useful and indispensable as the basis of mathematics. With the same justification the sale of alcohol at gasoline stations could be claimed to be the basis of social harmony.

The reader will search in vain any application of set theory in practical life. This is in accordance with Hilbert's statement: "The infinite is nowhere realized; it is neither present in nature nor admissible as the foundation of our rational thinking." [D. Hilbert: "Über das Unendliche", Math. Annalen 95 (1925) p. 190]

But a set theorist will always find a way to defend his preconceived dogma: "I think that any language and framework which helps promote clear thinking and reasoning in mathematics is practical and pragmatic – just not in the limited way that people might be interpreting those words." [Todd Trimble, deleted at loc cit]

Or is there, after all, a practical application yet – even in everyday language? "The major division of English nouns is into 'countable' and 'uncountable'." [EnglishClub: "Countable nouns"]

V Scepticism about transfinite set theory

The present chapter is not destined to show or discuss arguments against transfinite set theory. This would go beyond its scope; plenty of arguments, including many mentioned by authors cited here, will be presented in chapter VI. Chapter V is destined merely to show that not all mathematicians, philosophers, and scientists agree with the necessity or even the existence of actual infinity, with infinite decimal sequences having a definite meaning, with transfinite set theory and its current interpretations. Here I want to show that there exists a considerable minority challenging these properties of transfinite set theory, and further, that even its strongest advocates now and then have uttered doubts. Not necessary to mention that all quoted clerics agree to the actual infinity of God. The quotes have been ordered alphabetically by the names of their authors with no regard to their mathematical status, because the comprehension of Cantor's ideas does not require advanced mathematical skills as Cantor himself repeatedly emphasized, cp. section 4.3.

Abdelmalek Abdesselam [...] no mathematical physics should use the metaphysics of the AC, [A. Abdesselam in "Does the axiom of choice appear to be 'true' in the context of physics?", Physics.StackExchange (5 Aug 2015)]

Wilhelm F. Ackermann The reviewer however cannot follow the author when he speaks of the possibility of a more than countable set of primitive symbols since such a system of names cannot exist. [W. Ackermann: "Review of Leon Henkin: 'The completeness of the first-order functional calculus'", J. Symbolic Logic 15,1 (1950) p. 68]

Pietro Dell'Acqua We reconsider Cantor's diagonal argument for the existence of uncountable sets from a different point of view. After reformulating well-known theoretical results in new terms, we show that, contrary to what stated by Cantor, they do not imply uncountability. [P. Dell'Acqua: "A note on Cantor's diagonal argument", ResearchGate (Jan 2016)]

James Ada Cantor stated that we can have infinite possibilities within the finite. What he really was studying was the infinite possibilities within the only number. However my argument is we can only have infinite outside of finite. [J. Ada: "Georg Cantor was wrong about infinity", TED Conversations Archives (2012)]

Mark Adkins Cantor's diagonal proof of the existence of hierarchies of infinities is a flawed argument based upon a simple logical error. [M. Adkins: "Cantor's perpetual fallacy", sci.math (23 Nov 1999)]

Jean-Baptiste le Rond d'Alembert When understood well once, the supposition that one has made of infinitely small quantities will be felt to be only for abridging and simplifying reasoning. [...] It is not a matter, as we say ordinarily, of infinitely small quantities in the *differential* calculus, but, uniquely, a matter of the limits of finite quantities. And so the metaphysics of infinity and infinitely small quantities each larger or smaller, is totally useless to the *differential* calculus. The term *infinitely small* only makes us ready to abbreviate its expression. [J.-B. le Rond d'Alembert: "Differential calculus", The Encyclopedia of Diderot & d'Alembert, Vol. 4 (1754) pp. 985-988]

Bhupinder Singh Anand We can neither conclude that Cantor's diagonal argument determines an uncountable Dedekind real number, nor conclude from it that the cardinality of the Dedekind real numbers necessarily differs from that of the Dedekind (Peano) natural numbers. [B.S. Anand: "Three beliefs that lend illusory legitimacy to Cantor's diagonal argument", arXiv (2003)]

Anonymous Like phrenology and astrology, Cantor's mathematics will eventually be recognized for what it really is – utter nonsense. Every field has its charlatans, and mathematics is no exception. [NN: "Nonsense, nonsense, nonsense!", Wikipedia, Talk: Cantor's diagonal argument / Archive 1]

Thomas Aquinas In the *Summa Theologiae*, then, the actually infinite turns out to be the kind of thing that by definition cannot exist, for anything which can exist can be surpassed, either in magnitude or in number. [Joseph William Yarbrough, III: "Philip the chancellor, Bonaventure of Bagnoregio, and Thomas Aquinas on the eternity of the world", Dissertation, Cornell University (May 2011) p. 141]

Aristotle What is continuous is divided ad infinitum, but there is no infinite in the direction of increase. [...] Our account does not rob the mathematicians of their science, by disproving the actual existence of the infinite in the direction of increase, in the sense of the untraversable. In point of fact they do not need the infinite and do not use it. They postulate only that the finite straight line may be produced as far as they wish. It is possible to have divided in the same ratio as the largest quantity another magnitude of any size you like. Hence, for the purposes of proof, it will make no difference to them to have such an infinite instead, while its existence will be in the sphere of real magnitudes. [...] It remains to dispose of the arguments which are supposed to support the view that the infinite exists not only potentially but as a separate thing. [Aristotle: "Physics, Book III", Part 7-8 (350 BC)]

Vladimir I. Arnold Mathematics is a part of physics. Physics is an experimental science, a part of natural science. Mathematics is the part of physics where experiments are cheap. [V.I. Arnold: "On teaching mathematics" (1997)]

Richard Arthur In any case where there is a requirement of a *recursive connection* between any pair of the things numbered, the Cantorian conception of the infinite will not be valid. This is because the set $\mathbb{N}$ of natural numbers ordered by the relation > (is 'greater than') is recursively connected if and only if every number is finite. If limit ordinals (Cantor's ω, ω^2 etc.) are included, recursive connectedness fails. [R. Arthur: "Leibniz and Cantor on the actual infinite" (2001) p. 4]

Averroes Every actual number is something actually numbered and that which is actually numbered must be either even or odd, and that which is even or odd must necessarily be finite. [H.A. Wolfson: "Crescas' critique of Aristotle", Harvard Univ. Press, Cambridge (1929) p. 223]

Arnon Avron answers these questions: Do you agree that the continuum hypothesis is a meaningful statement that has a definite truth value, even if we do not know what it is? "No." Do you agree that the axiom which states the existence of an inaccessible cardinal is a meaningful statement that has a definite truth value, even if we do not know what it is? "No." [A. Avron in "Ten questions about intuitionism", intuitionism.org (2005)]

John Baez Mathematicians crave consensus. If any sort of argument is of the sort that it only convinces 50% of mathematicians, we'll either say it's "not mathematics", or discuss, polish and/or demolish the argument until it convinces either 99% of mathematicians or just 1%. (Example: Cantor's proofs.) If someone doesn't play the game according to the usual rules, we'll make up a new game and say they're playing that game instead, thus eliminating potential controversy. (Example: intuitionistic mathematics.) Finally, we reward people who quickly admit their errors, instead of fighting on endlessly. We say they're smart, not wimps. (Example: Edward Nelson.) People who fight on endlessly are labelled crackpots and excluded from the community. (Examples: too numerous to list here.) [J. Baez in "The (in)consistency of PA and consensus in mathematics", M-Phi (7 Oct 2011)]

René Louis Baire As you know, I share Borel's opinion {{about Zermelo's note}} in general, and if I depart from it, it is to go further than he does {{see Émile Borel in this chapter}}. [...] In particular, when a set is given (we agree to say, for example, that we are given the set of sequences of positive integers), *I consider it false to regard the subsets of this set as given.* I refuse, *a fortiori*, to attach any meaning to the act of supposing that a choice has been made in every subset of a set. [...] In order to say, then, that one has established that every set can be put in the form of a well-ordered set, the meaning of these words must be extended in an extraordinary way and, I would add, a fallacious one. [...] For me, progress in this matter would consist in delimiting the domain of the definable. And, despite appearances, in the last analysis everything must be reduced to the finite. [R.L. Baire, letter to J. Hadamard (1905)]

Ajaib Singh Banyal Unboundedness of any Set, Space or any thing is termed as Infinite; if some one succeeds in finding in its bounds, then it is not infinite and it becomes finite, so infinity means beyond our reach hypothetically, which could be attemped to reach, without success and definitely it is an endless process. [A.S. Banyal in "Does actual infinity exist?", ResearchGate (8 Aug 2012)]

Dick Batchelor Thus, all we can conclude from the diagonalization argument is that there is no maximal set of real numbers; this does not necessarily require that the set of real numbers has more members than a countable set. [D. Batchelor in "More Cantor", sci.math (28 Jul 1999)]

Alastair Bateman It seems to me that the proof given that 1 = 0.999... is flawed and cannot therefore, as I see it, be used by anyone to draw deep philosophical deductions about number groups that rely on it. [A. Bateman in "False proofs" (2006)]

Andrej Bauer The success of set theory has lead many to believe that it provides an unshakeable foundation for mathematics. It does not, at least not the mystical kind that some would like to have. It provides a unifying language and framework for mathematicians, which in itself is a small miracle. Always remember that practically all classical mathematics was invented *before* modern logic and set theory. [A. Bauer in "Set theory and model theory", MathOverflow (30 Apr 2010)]

Roger Bear When a unary (i, ii, iii, ...) system is used to count the elements of an arbitrary (infinite) set, there are no diagonals, which immediately proves Cantor's 2nd diagonal argument wrong as math is always free in chosing whatever number (representation) system. [R. Bear: "Cantor's 2nd diagonal argument proven wrong", sci.math (13 Nov 2010)]

Edouard Belaga Ultimately, all modern transfinite set theory represents only a well designed fantasy founded on Zermelo's axiomatic, the fantasy which pushes to their limits the rich constructionist faculties of this system. All adaptations of these fantasies to even very modest aspects of the Continuum realities remain absolutely unsatisfactory. [E. Belaga: "From traditional set theory – that of Cantor, Hilbert, Goedel, Cohen – to its necessary quantum extension", IHES/M/11/18 (2011) p. 24]

Jean Paul van Bendegem The third point is that under these conditions it is straightforward to show that the procedure "Give me any numeral *n* you can imagine, I will give you the next one" has to break down at a certain point. [...] What is being asked is to imagine a numeral so huge that it cannot be imagined. [J.P. van Bendegem: "Why the largest number imaginable is still a finite number", Logique et Analyse 165-166 (1999) p. 119]

Nico Benschop Cantor's diagonal procedure necessarily *cannot* start with a complete list, because the word "diagonal" means there is a square table, [...] So we *know* beforehand that whatever table you take, assuming you are going to apply Cantor's diagonal procedure, it is *not* complete – just as any finite but square table of k-bit binary strings necessarily has k numbers in it: no more, no less; while we know there are $2^k > k$ (for any $k > 0$) such numbers. [N. Benschop in "Why Cantor was wrong", sci.math (20 Jul 1999)]

Paul Bernays [...] it is not an exaggeration to say that platonism reigns today in mathematics.

But on the other hand, we see that this tendency has been criticized in principle since its first appearance and has given rise to many discussions. This criticism was reinforced by the paradoxes discovered in set theory, even though these antinomies refute only extreme platonism. [...] Several mathematicians and philosophers interpret the methods of platonism in the sense of conceptual realism, postulating the existence of a world of ideal objects containing all the objects and relations of mathematics. It is this absolute platonism which has been shown untenable by the antinomies, particularly by those surrounding the Russell-Zermelo paradox. [...]

The essential importance of these antinomies is to bring out the impossibility of combining the following two things: the idea of the totality of all mathematical objects and the general concepts of set and function; for the totality itself would form a domain of elements for sets, and arguments and values for functions. We must therefore give up absolute platonism.

[...] The first step is to replace by constructive concepts the concepts of a set, a sequence, or a function, which I have called quasi-combinatorial. The idea of an infinity of independent determinations is rejected. One emphasizes that an infinite sequence or a decimal fraction can be given only by an arithmetical law, and one regards the continuum as a set of elements defined by such laws. [...]

Nonetheless, if we pursue the thought that each real number is defined by an arithmetical law, the idea of the totality of real numbers is no longer indispensable, and the axiom of choice is not at all evident. [...]

Let us proceed to the second step of the elimination. It consists in renouncing the idea of the totality of integers. This point of view was first defended by Kronecker and then developed systematically by Brouwer. [P. Bernays: " Platonism in mathematics" (1935) p. 5ff]

Errett A. Bishop Brouwer's criticisms of classical mathematics were concerned with what I shall refer to as "the debasement of meaning". [...] rejection of common sense in favour of formalism; debasement of meaning by wilful refusal to accomodate certain aspects of reality; [...] The codification of insight is commendable only to the extent that the resulting methodology is not elevated to dogma and thereby allowed to impede the formation of new insight. [...]

(A) Mathematics is common sense.

(B) Do not ask whether a statement is true until you know what it means.

(C) A proof is any completely convincing argument.

(D) Meaningful distinctions deserve to be maintained.

[E.A. Bishop: "Schizophrenia in contemporary mathematics", Amer. Math. Soc. Colloquium Lecture, Seventy-eighth summer meeting, University of Montana, Missoula, Montana (1973)]

Bishop has attracted a small band of followers. He argues, as Brouwer did, that much of standard mathematics is a meaningless game; [...] Most mathematicians respond to his work with indifference or hostility. [...] From his point of view, classical mathematics is a jumble of myth and reality. He prefers to do without the myth. From his point of view, it is classical mathematics that appears as an aberration; constructivism is just the refusal to participate in the acceptance of a myth. [P.J. Davis, R. Hersh, E.A. Marchisotto: "The mathematical experience", Birkhäuser, Boston (1995) p. 416]

Eckard Blumschein This study also profits from endless public discussions and a brochure by Wolfgang Mückenheim: History of the Infinite (in German), Augsburg 2004 [...]

- It is not allowed to use the quality "infinite" as a quantitative measure.
- The definition of infinite numbers cannot be justified.
- Numbers beyond the infinite are nonsensical.

[E. Blumschein: "Cantors Kontinuum, eine Bestandsaufnahme", de.sci.physik (4 Apr 2005)]

Nico du Bois Why should diagonalisation only be applied on the interval (0,1)? Cantorians always do it to explain diagonalisation. That is because it is easy and it seemingly leads to no contradictions. Try diagonalisation on $\mathbb{R}$ and you will have a problem. [N. du Bois in "Why Cantor was wrong", sci.math (19 Jul 1999)]

Bernard Bolzano From the only reason that two sets, *A* and *B*, are related to each other such that for every part *a* being in *A* we can choose, by a certain rule, a part *b* being in *B* to the end that all pairs (*a* + *b*) formed in this way contain every thing contained in *A* or *B*, and each thing only once – only from this reason it is, as we see, not at all allowed to conclude *that these two sets, if they are infinite*, with respect to the multitude of their parts (disregarding all their differences) *are equal to each other*; [...] Two finite sets, if they are of a constitution such that we can find to every thing *a* of the first set a thing *b* of the second set and pair them, to the end that no thing is remaining in one set without a corresponding thing in the other set, and no thing exists that appears in two or more pairs, have always equal multitude. It appears as if this property should persist for infinite sets too. It appears so, I say; but by closer inspection we see that it is not at all necessary because the reason of this for finite sets lies just in their finiteness and therefore lapses for infinite sets. [...] This conclusion obviously gets invalid as soon as the set of things in *A* is *infinite* because now not only our *counting* will never get to a last thing in *A*, but by definition of an infinite set, there is no *last* thing in *A*, i.e., how many we ever may have named, there are always others to be named. And although there is no lack of things in set *B* too, to be combined to new and new pairs with those of *A*, the reason lapses to conclude that the multitudes of both sets are equal. [B. Bolzano: "Paradoxien des Unendlichen", Reclam, Leipzig (1851) §§ 21 & 22]

Bonaventure The first premise: it is impossible to add to the infinite. This premise is known *per se* because everything which receives an addition becomes larger, "but nothing is larger than the infinite". [...] The second premise: it is impossible to order an infinite series. (Literally, "it is impossible that an infinite number of things be ordered".) [J.W. Yarbrough, III: "Philip the chancellor, Bonaventure of Bagnoregio, and Thomas Aquinas on the eternity of the world", Dissertation, Cornell University (May 2011) p. 157f]

George Boolos The difficulty we are confronted with is that ZFC makes a claim we find implausible. To say we can't criticize ZFC since ZFC is our theory of sets is obviously to beg the question whether we ought to adopt it despite claims about cardinality that we might regard as exorbitant. [G. Boolos: "Must we believe in set theory?" in R. Jeffrey (ed.): "Logic, logic, and logic", Harvard University Press (1998) pp. 120-132]

Article 2 contains Boolos' defense of Fraenkel's, in contrast to Zermelo's, position that first-order but not second-order logic is applicable to set theory. Boolos criticizes the view of Charles Parsons (and D. A. Martin) that it makes sense to use second-order quantifiers when first-order quantifiers range over entities that do not form a set. Boolos' answer to the title of article 8, "Must We Believe in Set Theory?" is 'no': the phenomenological argument (due to Gödel) does not imply that the axioms of set theory correspond to something real, and the indispensability argument (due to Carnap) that mathematics is required by our best physical theory, is dismissed as "rubbish". [G. Mar: "Review of George Boolos: 'Logic, logic, and logic', Harvard University Press (1998)", Essays in Philosophy 1,2 (Jun 2000)]

Émile Borel I prefer not to write alephs. [...] One may wonder what is the real value of these arguments that I do not regard as absolutely valid but that still lead ultimately to effective results. In fact, it seems that if they were completely devoid of value they could not lead to anything since they would be meaningless collections of words. This, I believe, would be too harsh. They have a value analogous to certain theories in mathematical physics, through which we do not claim to express reality but rather to have a guide that aids us, by analogy, in predicting new phenomena, which must then be verified. It would require considerable research to learn what is the real and precise sense that can be attributed to arguments of this sort. Such research would be useless, or at least it would require more effort than it would be worth. [É. Borel, letter to J. Hadamard (1905)]

Borel declared that the Banach-Tarski paradox amounts to an inconsistency proof of the Axiom of Choice. [É. Borel: "Les paradoxes de l'infini" 3rd ed., Paris (1946) p. 210]

So, in Borel's view, most reals, with probability one, are mathematical fantasies, because there is no way to specify them uniquely. [G. Chaitin: "How real are real numbers?", arXiv (2004)]

Jorge Luis Borges There is a concept which is the corruptor and the seducer of the others. I do not speak of Evil, whose limited empire is ethics; I speak of the infinite. [J.L. Borges: "Los avatares de la tortuga", Sur 63 (Dec 1939) p. 18]

Andrew Boucher Cantor and modern logicians would have us think that their theories are somehow talking about infinite numbers, and that the number of $\mathbb{R}$ is greater than the number of $\mathbb{N}$. But their reasoning is fallacious. [A. Boucher: "Cantor and infinite size", sci.math (22 May 1999)]

Ross Brady, Penelope Rush As a long-time university teacher of formal logic and philosophy of mathematics, the first author has come across a number of students over the years who have cast some doubt on the validity of Cantor's Diagonal Argument. [...] Unwittingly, I have always given the standard response that the conclusion is inescapable, [...] What we aim to show in this paper is that there is also an important point to the student's concerns about Cantor's Diagonal Argument, thus making amends to these students. [R. Brady, P. Rush: "What is wrong with Cantor's diagonal argument?", Logique et Analyse 202 (2008) p. 185f]

Percy W. Bridgman The ordinary diagonal Verfahren I believe to involve a patent confusion of the program and object aspects of the decimal fraction, which must be apparent to any who imagines himself actually carrying out the operations demanded in the proof. In fact, I find it difficult to understand how such a situation should have been capable of persisting in mathematics. Doubtless the confusion is bound up with the notions of existence; the decimal fractions are supposed to "exist" whether they can be actually produced and exhibited or not. But from the operational point of view all such notions of "existence" must be judged to be obscured with a thick metaphysical haze, and to be absolutely meaningless from the point of view of those restricted operations which can be allowed in mathematical inquiry. [...]

One can obviously say that all the rules for writing down nonterminating decimals formulatable by the entire human race up to any epoch in the future must be denumerable [...] I do not know what it means to talk of numbers existing independent of the rules by what they are determined; operationally there is nothing corresponding to the concept. If it means anything to talk about the existence of numbers, then there must be operations for determining whether alleged numbers exist or not, and in testing the existence of a number how shall it be identified except by means of the rules? [...] From the operational point of view a transcendental is determined by a program or procedure of some sort; Mengenlehre has nothing to add to the situation. And this, as far as my elementary reading goes, exhausts the contributions which Mengenlehre has made in other fields. [P.W. Bridgman: "A physicist's second reaction to Mengenlehre", Scripta Mathematica 2 (1934) p. 225ff]

Luitzen E.J. Brouwer Cantor's second number class does not exist. [L.E.J. Brouwer: "Over de grondslagen der wiskunde", Thesis, Univ. Amsterdam (1907) Statement XIII]

[...] we can create in mathematics nothing else but ending sequences, and further, on the ground of the clearly conceived "and so on", the order type ω, but only consisting of *equal elements*, so that we can never imagine the *arbitrary* infinite binary fractions as finished, [L.E.J. Brouwer: "Over de grondslagen der wiskunde", Thesis, Univ. Amsterdam (1907) p. 142f]

The belief in the universal validity of the principle of the excluded third in mathematics is considered by the intuitionists as a phenomenon of the history of civilization of the same kind as the former belief in the rationality of π, or in the rotation of the firmament about the earth. The intuitionist tries to explain the long duration of the reign of this dogma by two facts: firstly that within an arbitrarily given domain of mathematical entities the non-contradictority of the principle for a single assertion is easily recognized; secondly that in studying an extensive group of simple every-day phenomena of the exterior world, careful application of the whole of classical logic was never found to lead to error. [L.E.J. Brouwer: "Lectures on intuitionism – Historical introduction and fundamental notions" (1951), Cambridge University Press (1981)]

Brouwer, in his dissertation, refutes the well-ordering theorem by pointing out that in the case of the continuum most of the elements are unknown, and hence cannot be ordered individually – "So this matter also turns out to be illusory." (Thesis p. 153) Examples of (according to Brouwer) meaningless word play are the second number class and the higher power sets. [D. van Dalen: "Mystic, geometer, and intuitionist: The life of L.E.J. Brouwer", Oxford Univ. Press (2002)]

Han de Bruijn I can accept very well that, for example, the natural numbers are "impossible of completion", hence so to speak "infinite". But I can not accept that, for example, there are "as many" natural numbers as there are even numbers. IMHO, there are twice as much. Simple. Please, don't explain why I'm "wrong". I can reproduce the official "proof" entirely by myself. An argument of my own holds approximately for every finite set of natural numbers. However, the bigger the set the better. Taking the "limit", there are exactly twice as much natural as there are even numbers. [H. de Bruijn: "Natural philosophy" (2015) link expired]

Jed Brunozzi Firstly, saying the limit of $\Sigma(1/k^2) = \pi^2/6$ is incorrect. It pretends (incorrectly) that we have formed an algebraic statement of equality. It should read $\Sigma(1/k^2) \rightarrow \pi^2/6$. Secondly, because of such statements, we want to call these irrational quantities (like π) numbers and incorrectly try to put them on the number line. Finally, this lures us into saying the number line is filled with irrational 'numbers' because we have convinced ourselves quantities like π are numbers, when in fact they are not. This I believe is at least one big problem with the misuse of the concept of infinity. It gives us a picture of numbers that is incorrect. [J. Brunozzi in "The law of logical honesty and the end of infinity", YouTube (23 Apr 2016)]

Helmut Büch Representations of numbers can only be finite representations of numbers, existing in the real world, be it in the brain, in a computer, on paper, or elsewhere. [H. Büch in "Das Kalenderblatt 100618", de.sci.mathematik (23 Jun 2010)]

Otávio Bueno Platonists often emphasize that it is because mathematical objects, relations, and structures exist that mathematics is ultimately objective. [...] This move, however, does not go through. It is unclear that the existence of mathematical objects, relations, and structures does any work to support the objectivity of mathematics. After all, if mathematical objects, relations, and structures turn out not to exist, it is unclear that anything would change in mathematical practice (Azzouni 1994). Mathematicians would continue to do their work in precisely the same way as they currently do: proposing, articulating, and refining mathematical definitions and principles, and drawing consequences from them. The actual existence of mathematical objects is largely irrelevant for that. [O. Bueno: "Relativism in set theory and mathematics", Wiley Online Library (2011) p. 560]

Georg Cantor "Infinite definitions" (that do not happen in finite time) are *non-things* {{*Undinge*}}. If König's theorem was *true*, according to which all "finitely definable" real numbers form an embodiment of cardinality $\aleph_0$, this would imply that the whole continuum was countable, which is certainly *false*. [G. Cantor, letter to D. Hilbert (8 Aug 1906)]

Paola Cattabriga In this article it is shown that, defining the relative complement of the self-referring statement, Cantor's power set theorem cannot be derived. Moreover, it is given a refutation of the first proof, the so-called Cantor's diagonal argument. [P. Cattabriga: "Beyond uncountable", arXiv (2006)]

Augustin-Louis Cauchy We say that a variable quantity becomes *infinitely small* when its numerical value decreases indefinitely in such a way as to converge toward the limit zero. [...] We say that a variable quantity becomes *infinitely large* when its numerical value increases indefinitely in such a way as to converge toward the limit ∞. [A.-L. Cauchy: "Cours d'analyse de l'Ecole Royale Polytechnique", Paris (1821) p. 26f]

We cannot admit the assumption of an infinite number, of beings or of objects coexisting, without being trapped by manifest contradictions. [A. Cauchy "Sept lecons de physique générale", Gauthier-Villars, Paris (1868) p. 23]

Ben Cawaling Even granting arguendo Georg Cantor's tenet of actual or completed infinity, his diagonalization argument is untenable [...] [B. Cawaling: "Small flaw", Wikipedia, Talk: Cantor's diagonal argument / Archive 1]

Paul A.C. Chang I totally agree with WM. Applied Math is an indispensable part of various engineering disciplines because its application and usefulness in predictive models has been validated against real-world conditions again and again and again. If, for argument's sake, PA was proven inconsistent, then math merely becomes a defacto natural science like biology or chemistry, in the sense that the "validity" of math no longer stems from axioms, but rather validation against real world conditions and observations. [P.A.C. Chang in "The inconsistency of arithmetic", The n-Category Café (2 Oct 2011)]

Adrian Chira It is generally claimed that axiomatic set theory (such as Zermelo-Fraenkel set theory (ZFC)) avoids Curry's paradox by replacing the axiom of unrestricted comprehension by the standard axioms of axiomatic set theory. [...] It is here suggested however that the axioms don't succeed in avoiding the paradox. [A. Chira: "Curry's non-paradox and its false definition", viXra (2016)]

Jon Cogburn The proud announcement of the conquering of infinity is fatuous at best. Cantorian diagonalization rather showed (among other things) that we have no idea what we're talking about when we talk about infinity. [J. Cogburn in "Cantor's theorem and its discontents", Philosophical Percolations (3 Sep 2015)]

Paul J. Cohen However, in all honesty, I must say that one must essentially forget that all proofs are eventually transcribed in this formal language. In order to think productively, one must use all the intuitive and informal methods of reasoning at one's disposal. [...] The early years of the twentieth century were marked by a good deal of polemics among prominent mathematicians about the foundations of mathematics. These were greatly concerned with methods of proof, and particular formalizations of mathematics. It seemed that various people thought that this was a matter of great interest, to show how various branches of conventional mathematics could be reduced to particular formal systems [...] Thus, Russell and Whitehead, probably influenced by what appeared to be the very real threat of contradictions, developed painstakingly in their very long work, *Principia Mathematica*, a theory of "types" and then did much of basic mathematics in their particular formal system. The result is of course totally unreadable, and in my opinion, of very little interest. Similarly, I think most mathematicians, as distinct from philosophers, will not find much interest in the various polemical publications of even prominent mathematicians. My personal opinion is that this is a kind of "religious debate". One can state one's belief but, with rare exceptions, there are few cases of conversion. [...] The only reality we truly comprehend is that of our own experience. [P.J. Cohen: "The discovery of forcing", Rocky Mountain Journal of Mathematics 32,4 (2002) pp. 1078 & 1080 & 1099]

Thomas Colignatus The notion of a limit in ℝ cannot be defined independently from the construction of ℝ itself. Occam's razor eliminates Cantor's transfinites. [T. Colignatus: "Contra Cantor pro Occam", viXra (2015) p. 1]

Alexandre Costa-Leite answers these questions: Do you agree that the continuum hypothesis is a meaningful statement that has a definite truth value, even if we do not know what it is? "No." Do you agree that the axiom which states the existence of an inaccessible cardinal is a meaningful statement that has a definite truth value, even if we do not know what it is? "No." [A. Costa-Leite in "Ten questions about intuitionism", intuitionism.org (2006)]

William L. Craig [...] an actually infinite number of things cannot exist. For if an actually infinite number of things could exist, this would spawn all sorts of absurdities. [W.L. Craig: "The existence of God and the beginning of the universe", Truth: A Journal of Modern Thought 3 (1991) pp. 85-96]

Brian L. Crissey Turing's proof of the insolubility of the Halting Problem prevents unpredictable irrationals from being ordered, which prevents their being real, if they exist at all. Thus the cardinality of the reals is the same as that of the integers. Transfinite mathematics is discredited and may be relegated to history. Much rewriting of mathematical texts lies ahead. [B. Crissey: "Unreal irrationals: Turing halts Cantor", MAA-SE at Elon University (2010)]

Dirk van Dalen, Heinz-Dieter Ebbinghaus On October 4, 1937 Zermelo [...] gives a refutation of "Skolem's paradox", i.e., the fact that Zermelo-Fraenkel set theory – guaranteeing the existence of uncountably many sets – has a countable model. Compared with what he wished to disprove, the argument fails. [...] In his first letter to Gödel of September 21, 1931 he had written that the Skolem paradox rested on the erroneous assumption that every mathematically definable notion should be expressible by a finite combination of signs, whereas a reasonable metamathematics would only be possible after this "finitistic prejudice" would have been overcome, "a task I have made my particular duty". [...] The remarks show that the role of logic in set theory was not quite clear to Fraenkel, no matter how much he admired Hilbert's proof theory. Apparently Skolem's arguments were beyond his expertise. [D. van Dalen, H.-D. Ebbinghaus: "Zermelo and the Skolem Paradox", The Bulletin of Symbolic Logic 6,2 (Jun 2000) pp. 145 & 148]

George Dance Whatever number you could give, in the unary representation, you would leave out as many natural numbers as there are in ℕ itself. Whatever number anyone could possibly represent in that way, call it x, must be finite; assuming ℕ is infinite, the set $X = \{x+1, x+2, x+3, ...\}$ is infinite; X has the same cardinality as ℕ, and therefore the same number of elements as ℕ. [G. Dance in "Cantor's 'proof' of transfinite sets", sci.math (20 Mar 2003)]

David van Dantzig Whether a natural number be defined according to Peano, Whitehead and Russell and Hilbert as a sequence of printed signs (e. g. primes, affixed to a zero) or, according to Brouwer, as a sequence of elementary mental acts, in both cases it is required that each individual sequence can be recognized and two different ones can be distinguished. If – as it is usually done both by formalists, logicists and intuitionists – one assumes that by such a procedure in a limited time arbitrarily large natural numbers could be constructed, this would imply the rejection of at least one of the fundamental statements of modern physics (quantum theory, finiteness of the universe, necessity of at least one quantum jump for every mental act). Modern physics implies an upper limit, by far surpassed by $10^{10^{10}}$ for numbers which actually can be constructed in this way. Weakening the requirement of actual constructibility by demanding only that one can *imagine* that the construction could actually be performed – or, perhaps one should say rather, that one can *imagine* that one *could* imagine it – means imagining that one would live in a different world, with different physical constants, which might replace the above mentioned upper limit by a higher one, without anyhow solving the fundamental difficulty. [D. van Dantzig: "Is 10^10^10 a finite number?", Dialectica 9 (1955) p. 273]

David K. Davis If Cantor's work is invalid, modern mathematics goes up in smoke. The investment is too great – if something's wrong we'll just change logic. [D.K. Davis in "Cantor's transfinite numbers", sci.math (31 Oct 1996)]

Richard Dedekind The numbers are free creations of the human mind. They serve as a means to easen and to sharpen the perception of the differences of things. [R. Dedekind: "Was sind und was sollen die Zahlen?", Vieweg, Braunschweig (1888) preface]

Nathaniel Deeth I have shown how a complete countable ∞ list of all describable real numbers can be constructed. [N. Deeth: "A flaw in Cantor's diagonal method", sci.math (13 Jul 1999)]

Kevin Delaney A second negative effect of transfinite theory is that by positioning transfinite theory as the basis of arithmetic and logic, transfinite theorists have accomplished the unintended goal of basing mathematics on mystical concepts: namely, the completed infinity. Such a result is highly attractive to the mystical mind. Again, classical logicians would abhor the idea of basing reasoning on mystical concepts, but the mystical mind adores it ... gaining further acceptance for transfinite theory. [...] Leopold Kronecker is often criticized for his constructivist stands, however, computing devices are limited to the mechanical logics championed by Kronecker. [K. Delaney: "Curing the Disease"]

Alois Dempf The problem of the genuine infinity cannot be settled without the *notion of spirit*. It leads to the entire set of questions of personal-spiritual weltanschauung of faith and of religious objects or at least of its main subject the notion of God, and at the other hand to the atomistic mechanism. [A. Dempf: "Das Unendliche in der mittelalterlichen Metaphysik und in der Kantischen Dialektik", Veröffentlichungen des Katholischen Institutes für Philosophie, Albertus-Magnus-Akademie zu Köln, Band II, Heft 1, Aschendorff, Münster (1926) p. 8]

Stephen R. Diamond 1. Infinitesimal quantities can't exist; 2. If actual infinities can exist, actual infinitesimals must exist; 3. Therefore, actual infinities can't exist. [S.R. Diamond: "Infinitesimals: Another argument against actual infinite sets", Juridical Coherence (25 Jan 2013)]

Wilhelm Dieck In case of sets of only one element of our visualization or our thinking we can absolutely not talk about a collection since there is only one element. Further there is nothing well-distinguished and collected. The second kind of these extremely odd sets, the null set, is defined as having no element. "It is not a proper set but shall be regarded (in an improper sense) as a set." A set that actually isn't a set but shall be regarded as a set. Isn't that a contradictory notion? [W. Dieck: "Die Paradoxien der Mengenlehre", Annalen der Philosophie und philosophischen Kritik 5,1 (Dec 1925) p. 45]

William Dilworth The illusion that Cantor has each time come up with a "new number" involves a misreading of the decimal expressions he uses. By a straightforward inspection of decimals, as well as by a general professional consensus, many scalar numbers cannot be expressed exactly in decimal form. He who fails, through a lack of rigor, to remember the limitations of the decimal system, may imagine that he sees in Cantor's truncated decimal forms the "objective real numbers"; he slides into Cantor's subtle mistake. [...] "Yes Sir", the head of the mathematics department of a Univ. of Illinois section said matter-of-fact to my face, "The integers will become exhausted". Believe it or not, Georg Cantor made these remarkable claims stick with the world's mathematicians of his time, and they stick unto this day. The effects of the Cantorian grip on the professional mind have to be experienced to be believed. [...] So why have the "endless decimal expansions" (or binary expansions for that matter) been raised to such a status as to be equated with scalar numbers themselves? The answer, I believe, is now clear. Because the open admission that closed forms for scalar numbers, which forms can obviously be ordered and counted, are available in other expansion systems but not in the decimal or binary type, would lead rather quickly to the exposure of the number-theory fallacies in the Cantorian diagonal argument and the deductions made from it. [W. Dilworth: "A correction in set theory", Wisconsin Academy of Sciences, Arts and Letters 62 (1974) pp. 206 & 212f & 216]

Norbert Domeisen Concerning the usefulness: We find for almost infinitely many problems of practice at least one solution by finite mathematics. We are looking in vain however for any practical problem that could be solved with infinite mathematics. [N. Domeisen: "Texte zur Philosophie und Geschichte" (2016)]

If the mapping, choice, or projection cannot be finished, it does not yield a result but only the tautological statement that it is without end, endless. Aborting the process does not yield a result but trivially the state at abortion. Herewith Cantor's proof of the existence of uncountable infinite sets and of different transfinite cardinal numbers by the diagonal procedure has been refuted; Hilbert's paradise is lost. [N. Domeisen: "Der Zauber Cantors oder das verlorene Paradies"]

Peter G. Doyle, John Horton Conway *What's wrong with the axiom of choice?* Part of our aversion to using the axiom of choice stems from our view that it is probably not 'true'. [...] Cohen's result implies that there is no hope of describing a definite choice function that will work for 'all' non-empty sets of real numbers, at least as long as you remain within the world of standard Zermelo-Fraenkel set theory. And if you can't describe such a function, or even prove that it exists without using some relative of the axiom of choice, what makes you so sure there is such a thing? Not that we believe there really are any such things as infinite sets, or that the Zermelo-Fraenkel axioms for set theory are necessarily even *consistent*. Indeed, we're somewhat doubtful whether large natural numbers (like 80^{5000}, or even 2^{200}) exist in any very real sense, and we're secretly hoping that Nelson will succeed in his program for proving that the usual axioms of arithmetic – and hence also of set theory – are inconsistent. All the more reason, then, for us to stick with methods which, because of their concrete, combinatorial nature, are likely to survive the possible collapse of set theory as we know it today. [P.G. Doyle, J.H. Conway: "Division by three", arXiv (2006)]

Michael Dummett In intuitionistic mathematics, all infinity is potential infinity: there is no completed infinite. [...] Rather, the thesis that there is no completed infinity means, simply, that to grasp an infinite structure is to grasp the process which generates it, that to refer to such a structure is to refer to that process, and that to recognize the structure as being infinite is to recognize that the process will not terminate. [...] The platonistic conception of an infinite structure as something which may be regarded both extensionally, that is, as the outcome of the process, and as a whole, that is, as if the process were completed, thus rests on a straightforward contradiction: an infinite process is spoken of as if it were merely a particularly long finite one. [...] But, since mathematical objects have no effect upon us save through our thought-processes, the conception of an analogous means of determining the truth-value of a statement involving quantification over an infinite mathematical totality is an absurdity. [M. Dummet: "Elements of intuitionism" 2nd ed., Clarendon Press, Oxford (2000) p. 40ff]

Russell Easterly Is it reasonable to assume infinity exists? I can think of some pretty unreasonable finite numbers. [...] Why couldn't there be some incredibly huge natural number big enough to encode everything we could ever hope to know about the universe? [...] Assume I am talking about ordinals. Things we count with. Like fingers and toes. Can we assume we never run out of toes? [R. Easterly: "How big is infinity?", sci.math (26 Aug 2006)]

Heinz-Dieter Ebbinghaus Fraenkel was not sure about the correctness of the proof of the Löwenheim-Skolem theorem, and he seems to have had difficulties in analysing the role of logic with sufficient rigour to understand Skolem's paradox {{while}} von Neumann instantly recognized the importance of the results, but he reacted with scepticism about the possibility of overcoming the weakness of axiomatizations they reveal. [H.-D. Ebbinghaus: "Ernst Zermelo: An approach to his life and work", Springer (2007) p. 200]

Edgar E. Escultura Therefore, Cantor's diagonal method is flawed. With the failure of Cantor's diagonal method, the *continuum hypothesis* also collapses. [E.E. Escultura: "The resolution of the great 20th century debate in the foundations of mathematics", Advances in Pure Mathematics 6 (2016) p. 147]

Ilijas Farah, Menachem Magidor The new aspect that came up after Cohen's independence proofs of 1963 is that mathematical problems that were considered to be central to the particular discipline were shown to be undecided. So the very notion of mathematical truth was shaken. [I. Farah, M. Magidor: "Independence of the existence of Pitowsky spin models", arXiv (2012) p. 1]

Solomon Feferman No set-theoretically definable well-ordering of the continuum can be proved to exist from the Zermelo-Fraenkel axioms together with the axiom of choice and the generalized continuum hypothesis. [S. Feferman: "Some applications of the notions of forcing and generic sets", Talk at the Int. Symposium on the Theory of Models, Berkeley (1963)]

In his concluding chapters, Feferman uses tools from the special part of logic called proof theory to explain how the vast part if not all of scientifically applicable mathematics can be justified on the basis of purely arithmetical principles. At least to that extent, the question raised in two of the essays of the volume, "Is Cantor Necessary?", is answered with a resounding "no". [S. Feferman, "In the light of logic", Oxford Univ. Press (1998) jacket flap]

The actual infinite is not required for the mathematics of the physical world. [S. Feferman: "Infinity in mathematics: Is Cantor necessary?" in "In the light of logic", Oxford Univ. Press (1998) p. 30]

I am convinced that the platonism which underlies Cantorian set theory is utterly unsatisfactory as a philosophy of our subject despite the apparent coherence of current set-theoretical conceptions and methods. To echo Weyl, *platonism is the medieval metaphysics of mathematics*; surely we can do better. [S. Feferman: "Infinity in mathematics: Is Cantor necessary? (Conclusion)" in "In the light of logic", Oxford Univ. Press (1998) p. 248]

Feferman shows in his article "Why a little bit goes a long way" on the basis of a number of case studies that the mathematics currently required for scientific applications can all be carried out in an axiomatic system whose basic justification does not require the actual infinite. [S. Feferman: private communication]

I came to the conclusion some years ago that CH {{continuum hypothesis}} is an *inherently vague* problem [...]. This was based partly on the results from the metatheory of set theory showing that CH is independent of all remotely plausible axioms extending ZFC, including all large cardinal axioms that have been proposed so far. In fact it is consistent with all such axioms (if they are consistent at all) that the cardinal number of the continuum can be "anything it ought to be", i.e. anything which is not excluded by König's theorem. The other basis for my view is philosophical: I believe there is no independent platonic reality that gives determinate meaning to the language of set theory in general, and to the supposed totality of arbitrary subsets of the natural numbers in particular, and hence not to its cardinal number. Incidentally, the mathematical community seems implicitly to have come to the same conclusion: it is not among the seven Millennium Prize Problems established in the year 2000 by the Clay Mathematics Institute, for which the awards are $1,000,000 each; and this despite the fact that it was the lead challenge in the famous list of unsolved mathematical problems proposed by Hilbert in the year 1900, and one of the few that still remains open. [S. Feferman: "Philosophy of mathematics: 5 questions", Academia (2007) p. 12]

Walter Felscher Concerning the application of transfinite numbers in other mathematical disciplines, the great expectations originally put on set theory have been fulfilled only in few special cases. [W. Felscher: "Naive Mengen und abstrakte Zahlen III", Bibliographisches Institut, Mannheim (1979) p. 25]

Jailton C. Ferreira *A proof that the set of real numbers is denumerable is given.* **[...]** Each real number of the interval [0, 1] can be represented by an infinite path in a given binary tree. [J.C. Ferreira: "The cardinality of the set of real numbers", arXiv (2013) p. 1]

José Ferreirós Because of the way in which Cantor combined romanticism with mathematics, modern methods with a vindication of rationalist metaphysics and theology, recent scientific trends with Platonism and an emphasis on the soul; it is not far fetched to label his orientation a *reactionary* modernism. Paraphrasing Thomas Mann, one might say that it was "a highly mathematical romanticism". [J. Ferreirós: "Paradise recovered? Some thoughts on Mengenlehre and modernism" (2008)]

Ludwig Fischer The uncountable set of real numbers from which the diagonal number can be constructed is therefore – how big it ever might be chosen – only a subset of an even bigger countable set of real numbers. The proof of the existence of a diagonal number of such an origin can in no way prove the uncountability of the set of all real numbers. [L. Fischer: "Die unabzählbare Menge", Meiner, Leipzig (1942)]

Reinhard Fischer I've shown already that you cannot make this conclusion {{1 - 0.999... = 0}}, since a "point" 0.999… is not defined on the number line. [netzweltler in "The common mistake", sci.math (10 Oct 2016)]

Peter Fletcher *What's wrong with set theory?* I have posed the fundamental question in the philosophy of mathematics as 'what does mathematics mean and how can we know for sure that it is true?' Before developing my answer to this question, I must explain why I do not accept the orthodox answer, namely that mathematics is the study of sets and that our knowledge of mathematics is derived from our set-theoretic intuition using classical logic. In this chapter I shall argue that 'set-theoretic intuition', as formalized in the Zermelo-Fraenkel axioms with the axiom of choice (ZFC), is conceptually incoherent. In the following chapter I shall argue that infinite quantifiers, the distinctive feature of classical logic, are meaningless. [P. Fletcher: "Truth, proof and infinity", Springer (1998) p. 13]

It might be argued, therefore, that the reasons that led to the acceptance of actual infinity at the end of the nineteenth century have now been superseded. Potential infinity works just as well. [P. Fletcher: "Infinity" in "Philosophy of logic", Dale Jaquette (ed.), Elsevier (2007) p. 564]

Henry A. Flynt No matter how much the content of mathematics exploits paradox, mathematicians express dedication to policing their doctrine against inconsistency. Mathematicians do not welcome those who attempt inconsistency proofs of favored theories. [...] I will propose that the main factor in the establishment of "truth" in mathematics is professional procedure and discipline. [...]

Truth is negotiated on the basis of manipulation of import by distorting interpretations. Interpretation takes the form of discarding traditional intentions concerning mathematical structure: the privileged position of Euclidian geometry; the invariance of dimension; the association between integer and magnitude; uniqueness of the natural number series; etc.

From time to time, results are discovered which patently embarrass the conventional wisdom, or controvert popular tenets. [The Gödel theorems.] Then follows a political manipulation, to distort the unwanted result by interpretation so that it is seen to "enhance" the popular tenet rather than to controvert it. [...]

Even if my sense of the situation is right, the appearance of such a professionally compelling proof would be more a matter of packaging and selling than anything else. [...] The biggest hurdle such an attempted proof faces is professional discipline. Whether inconsistency proofs are recognized to have occurred is subject to entirely "political" manipulation.

[...] So, even though, for example, the Hausdorff-Banach-Tarski paradox has been called the most paradoxical result of the twentieth century, classical mathematicians have to convince themselves that it is *natural*, because it is a consequence of the Axiom of Choice, which classical mathematicians are determined to uphold, because the Axiom of Choice is required for important theorems which classical mathematicians regard as intuitively natural. [H. Flynt: "Is mathematics a scientific discipline?" (1996)]

Thomas Forster The explanation is that – for people who want to think of foundational issues as resolved – it {{set theory}} provides an excuse for them not to think about foundational issues any longer. It's a bit like the rôle of the Church in Mediæval Europe: it keeps a lid on things that really need lids. [...] Mathematics doesn't need foundations – at least not of the kind that Set Theory was ever supposed to be providing – and the idea that Set Theory had been providing them annoyed a lot of people and did Set Theory much harm politically. [T. Forster: "The axioms of set theory" (2009)]

C. Fortgens For me the word infinite always has meant "without a definite bound". It never was a number. So you could not have infinite many natural numbers in a basket (or a set), because infinity is not a number. [C. Fortgens in "The law of logical honesty and the end of infinity", YouTube (23 Apr 2016)]

Nicolas de la Foz For instance, if we admit the one-to-one correspondence between $\mathbb{N}$ and $\mathbb{R}$, then we are confronting two different kinds of infinity, the potential infinite represented by the asymptotic approximation of the naturals to the infinite (never reaching it), and the actual infinite represented by the set of all real numbers; so that, a complete bijection between both sets is not possible. [N. de la Foz: "Cantor + infinite = problems", sci.math (28 Dec 2003)]

Adolf Abraham Fraenkel, Y. Bar-Hillel, A. Levy Feferman and Levy showed that one cannot prove that there is any non-denumerable set of real numbers which can be well ordered. Moreover, they also showed that the statement that the set of all real numbers is the union of a denumerable set of denumerable sets cannot be refuted. [A.A. Fraenkel, Y. Bar-Hillel, A. Levy: "Foundations of set theory" 2nd ed., North Holland, Amsterdam (1984) p. 62]

Edward Fredkin Digital Philosophy (DP) is a new way of thinking about the fundamental workings of processes in nature. DP is an atomic theory carried to a logical extreme where all quantities in nature are finite and discrete. This means that, theoretically, any quantity can be represented exactly by an integer. Further, DP implies that nature harbors no infinities, infinitesimals, continuities, or locally determined random variables. [E. Fredkin: "What is digital philosophy?", Digital Philosophy]

Pascoal Freitas Infinite decimals just show us our decimal notation isn't perfect since we *can't* write down any measure using the decimal notation or the more general rational number notation. [P. Freitas in "Difficulties with real numbers as infinite decimals I", YouTube (3 May 2012)]

Harvey Friedman What if Wiles proved FLT using Universes. Would Wiles get credit for having established the truth of FLT? The answer is that he would definitely get credit for having established the truth of FLT. However, there would be considerable interest paid to the problem of removing Universes from the proof. In fact, in making the proof as "normal" as we now know it can be made. Thus in a sense, "a proper class of strongly inaccessible cardinals" – what Grothendieck used in his Universe work – is a kind of gray area for mathematics. [...] In my opinion, the Editorial Board and/or the referee would require that the use of Universes be explicitly mentioned. In any case, the acceptance of Universes does show that it is possible to get the mathematical community to accept at least some large cardinal hypotheses. [H. Friedman "Report from expert", FOM (6 Apr 1999)]

Jacob Friedrich Fries Also Schulze's proof {{of Euclid's parallel axiom, using infinite angular areas}} is not sound because he takes the infinite as a completed whole which is a contradiction. [G. Schubring: "Das mathematisch Unendliche bei J.F. Fries" in G. König (ed.): "Konzepte des mathematisch Unendlichen im 19. Jahrhundert", Vandenhoeck & Ruprecht, Göttingen (1990) p. 157]

An infinite largeness or smallness never must be considered as a given whole. [J.F. Fries: "Die mathematische Naturphilosophie, nach philosophischer Methode bearbeitet", Heidelberg (1822) in G. König, L. Geldsetzer (eds.): "Jakob Friedrich Fries – Sämtliche Werke", Vol. 13, Aalen (1979) p. 254]

Orrin Frink It seems to me that the author, besides producing an interesting book, has made a good case for the contention that if we accept Brouwerism, we can get along theoretically without the notion of an infinite set, whether or not that notion is meaningless, as the author maintains. [O. Frink: "Review of Felix Kaufmann: 'Das Unendliche in der Mathematik und seine Ausschaltung', Deuticke, Leipzig (1930)", Bull. Amer. Math. Soc. 37,3 (1931) p. 149f]

George S. Fullerton When two lines are infinite, we have no point to measure from, and no point to measure to, and no measurement – therefore no comparison – is possible. [...] *The terms longer, shorter, and equal, can, therefore, have no meaning as applied to infinite lines.* They can be used only in speaking of the finite. We cannot, then, say that one infinite is greater or less than another, and just as little can we say that all infinites are equal; for any such proposition, however possible in words, is impossible in thought, and is an attempt to join contradictory notions. [G.S. Fullerton: "The conception of the infinite and the solutions of the mathematical antinomies: A study in psychological analysis", Lippincott, Philadelphia (1887) p. 22f]

John Gabriel Prof. Mueckenheim shares many revealing quotes from the very people who gave you set theory and real analysis, [...] His book should be used as a standard text book [...] But it is much more than just quotes. Mueckenheim explains the contradictions and absurdities in such a simple way that even an undergraduate will understand. Download it now, while it's free. [J. Gabriel: "To remove the scales from your eyes, read Prof. W. Mueckenheim's book on the transfinites. It's an eye-opener!", sci.math (6 Dec 2015)]

This video reveals a proof by Prof. W. Mueckenheim that demonstrates clearly the flaws in set theory with respect to infinite sets. The proof is so simple that a high school student can understand. [J. Gabriel: "Academic ignorance and stupidity – part 28", YouTube (27 Jul 2016)]

The proof by WM is a stroke of genius. [...] His disproof of infinite set theory is truly ingenious. WM uses the very same bogus and ill-formed mainstream concepts to show how set theory breaks without even one trying too hard. [J. Gabriel: "A proof that infinite set theory is flawed – by Prof. W. Mueckenheim", sci.math (28 & 31 Jul 2016)]

Galileo Galilei I consider that the attributes of greater, lesser, and equal do not suit infinities, of which it cannot be said that one is greater, or less than, or equal to, another. [G. Galilei: "Discorsi e Dimostrazioni Matematiche Intorno a Due Nuove Scienze", Univ. Wisconsin Press, Madison (1974) p. 40]

Carl Friedrich Gauß {{Concerning a proof of Schumacher's for the angular sum of 180° in triangles with two infinite sides}} I protest firstly against the use of an infinite magnitude as a *completed* one, which never has been allowed in mathematics. The infinite is only a mode of speaking, when we in principle talk about limits which are approached by certain ratios as closely as desired whereas others are allowed to grow without reservation." [C.F. Gauß, letter to H.C. Schumacher (12 Jul 1831)]

Murray Gell-Mann Pure mathematics and science are finally being reunited and, mercifully, the Bourbaki plague is dying out. [M. Gell-Mann: "Nature conformable to herself", Bulletin of the Santa Fe Institute 7 (1992) p. 7, link expired]

Giacinto Sigismondo Gerdil Essay about a mathematical proof against the eternal existence of matter and movement, derived from the proven impossibility of an actually infinite sequence of terms, whether being permanent or successive. [G.S. Gerdil: "Essai d'une démonstration mathématique contre l'existence éternelle de la matière et du mouvement, déduite de l'impossibilité démonstrée d'une suite actuellement infinie de termes, soit permanents, soit successifs", Opere edite et inedite, Vol. IV, Roma (1806) p. 261]

Laurent Germain In this paper, I show that the cardinality of the set of real numbers is the same as the set of integers. I show also that there is only one dimension for infinite sets, ℵ. [L. Germain: "The continuum is countable: Infinity is unique", arXiv (2008)]

Narayan Ghosh Not clear how the 'actual' and 'infinity' comes together. For any Mathematical, Physical or Cosmological concept of infinity perhaps no one can add the word actual. [N. Ghosh in "Does actual infinity exist?", ResearchGate (7 Jan 2013) link expired]

Josiah Willard Gibbs The pure mathematician can do what he pleases, but the applied mathematician must be at least partially sane. [J.W. Gibbs, quoted in Morris Kline: "Mathematics: The loss of certainty", Oxford University Press (1982) p. 285]

Olaf Gladis In A^* there are all possible combinations of arbitrary length of the elements of the alphabet. Therefore also ℝ is a subset of A^*. Since A^* however is countably infinite, also ℝ must be countably infinite. [O. Gladis: "Beweis für die Abzählbarkeit von R", de.sci.mathematik (10 Oct 2008)]

Kurt Gödel The true reason for the incompleteness that is inherent in all formal systems of mathematics lies in the fact that the generation of higher and higher types can be continued into the transfinite whereas every formal system contains at most countably many. This will be shown in part II of this paper. {{Part II never appeared.}} In fact we can show that the undecidable statements presented here always become decidable by adjunction of suitable higher types (e.g., adding the type ω to system *P*). Same holds for the axiom system of set theory. [K. Gödel: "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I", Monatshefte für Mathematik und Physik 38 (1931) p. 191]

But the situation becomes entirely different if we regard the properties as *generated* by our definitions. For it is certainly a vicious circle to generate an object by reference to a totality in which this very object is supposed to be present already. [...] The result of the preceding discussion is that our axioms, if interpreted as meaningful statements, necessarily presuppose a kind of Platonism, which cannot satisfy any critical mind and which does not even produce the conviction that they are consistent. [K. Gödel: "The present situation in the foundations of mathematics" (1933) in S. Feferman et al. (eds.): "Kurt Gödel, collected works, Vol. III, unpublished essays and lectures", Oxford Univ. Press, (1995) p. 50]

Only someone who (like the intuitionist) denies that the concepts and axioms of classical set theory have any meaning (or any well-defined meaning) could be satisfied with such a solution, not someone who believes them to describe some well-determined reality. For in this reality Cantor's conjecture must be either true or false, and its undecidability from the axioms as known today can only mean that these axioms do not contain a complete description of this reality; [...] not one plausible proposition is known which would imply the continuum hypothesis. Therefore one may on good reason suspect that the role of the continuum problem in set theory will be this, that it will finally lead to the discovery of new axioms which will make it possible to disprove Cantor's conjecture. [K. Gödel: "What is Cantor's continuum problem?", The American Mathematical Monthly 54,9 (1947) p. 520 & 524]

Jehanger Grami, Angela Grami In the present paper Cantor's proof will be refuted and the cardinality of the real numbers will be estimated by a constructive procedure directly and without an enumeration. [J. Grami, A. Grami: "Die reellen Zahlen sind abzählbar" (2010)]

Edward Grattan A one-to-one mapping (i.e. a bijection) between natural numbers, and real numbers between 0 & 1, is constructed; the mapping formula is simple, direct, and easy to calculate and work with. The traditional Cantor diagonal argument is then traced through: but at each step we use the mapping formula to show that the number generated thus far, is present in our one-to-one mapping; thus contradicting the traditional conclusion of said diagonal argument. [E. Grattan: "A one-to-one mapping from the natural numbers to the real numbers" (2012)]

Edward Green Obviously *all* irrational numbers cannot be so specified, or they in fact would be countable. So most irrational numbers are poor lost souls which not only have non-repeating decimal representations, but can't even be named in any meaningful way – they are unknowable. [E. Green in "Shannon defeats Cantor = single infinity type", sci.math (10 Dec 2003)]

S. Green Kronecker was right ... there is no mathematics here. [S. Green in "Proof – There are more real numbers than natural numbers", YouTube (29 May 2009) link expired]

Paul Guldin In my opinion no geometer will grant Cavalieri that the surface is, and could, in geometrical language be called "all the lines of such a figure"; never in fact can several lines, or all the lines, be called surfaces; for, the multitude of lines, however great that might be, cannot compose even the smallest surface. [P. Guldin: "Centrobaryca seu de centro gravitatis trium specierum quantitatis continuae", Vol. 4 (1641)]

Francisco Gutierrez Any string of 0s and 1s can be represented in this tree. Since the tree is infinite we can represent any infinite string, even if the string was constructed through diagonalization there will always be a path in the tree for that string. [...] This shows that we can map the natural numbers to the real numbers and therefore the real numbers are countable. [F. Gutierrez: "Was Cantor wrong?", Medium (2013)]

Jeremy Gwiazda There is no number of natural numbers. [J. Gwiazda: "Infinite numbers are large finite numbers", PhilPapers (2011)]

Theodore Hailperin This paper investigates the ontological presuppositions of quantifier logic. It is seen that the actual infinite, although present in the usual completeness proofs, is not needed for a proper semantic foundation. [T. Hailperin: "Herbrand semantics, the potential infinite, and ontology-free logic", History and Philosophy of Logic 13,1 (1992) p. 69]

Casper Storm Hansen It is first argued that the reasons for being skeptical towards actual infinity are so strong that mathematics should not be based on it; it is much more unclear what is in the "Cantorian paradise" than normally assumed, and supertasks (including a new one presented here) imply absurd consequences of actual infinity. Instead we will have to make do with mental constructions and potential infinity. [C.S. Hansen: "Constructivism without verificationism", Doctoral Thesis, University of Aberdeen (2014)p p. V]

C.G. Axel Harnack Whilst using here the new expressions for different classes of infinite numbers, introduced by Mr. *Cantor*, I'd like to remark that I cannot accept his general distinction between the "improper infinite" and the "proper infinite". [...] The summation of the terms of a convergent infinite series for example cannot be finished, but the resulting magnitude can be nameable. By continuing to operate with this magnitude, for instance adding further summands, we get the opportunity to talk about a definite continuation of the infinite process or to treat the notion of infinitely many summands as closed in itself. [...] {{J. Bernoulli}} advocates the necessity to fix the infinite as a completed magnitude. His mode of expression, there exists the infinitely large and the infinitely small, however is hardly admissible. Because the internal contradictions begin then and only then when one tries to attach to these limit notions a reality existing by itself in our thinking or in the outer world. [A. Harnack: "Ueber den Inhalt von Punktmengen", Math. Annalen 25 (1885) p. 244f]

Felix Hausdorff We have to violate the sacred axiom "totum parte majus". [F. Hausdorff: "Grundzüge der Mengenlehre", Veit, Leipzig (1914) p. 48]

Tristan Haze Of course, to try to refute Cantor's theorem is folly – the theorem certainly holds. But I think what the cranks who do try this are dimly feeling is something I feel too, only I have the good sense to realize that it lies not in the mathematics itself being false, but our interpretation of it being idiotic. [T. Haze: "Cantor's theorem and its discontents", Philosophical Percolations (9 Feb 2015)]

Georg Wilhelm Friedrich Hegel The *infinite quantum*, as *infinitely big* or *infinitely small*, itself is an infinite progress. It is quantum as a big or small and is simultaneously not-being of the quantum. The infinitely big and infinitely small are therefore pictures of the imagination which on closer inspection turn out as vain mist and shadows. [G.W.F. Hegel: "Wissenschaft der Logik", Holzinger, Berlin (2013) p. 276]

Eric C.R. Hehner [...] by itself it {{{Cantor's diagonal argument}} says nothing about the sizes of sets. Set size measurement and comparison, like all mathematics, should be chosen to fit the needs of an application domain. Cantor's countability relation is not a useful way to compare set sizes. [E.C.R. Hehner: "The size of a set" (2013)]

Martin Heidegger *Mathematics*, which is seemingly the most rigorous and most firmly constructed of the sciences, has reached a crisis in its 'foundations'. In the controversy between the formalists and the intuitionists, the issue is one of obtaining and securing the primary way of access to what are supposedly the objects of this science. [M. Heidegger: "Being and time", Harper & Row, New York (1962) p. 29f]

Arend Heijting Brouwer was right that intuitionistic mathematics is the only form of mathematics which has a perfectly clear interpretation [...] and it is desirable that as much of mathematics as possible will be made constructive. [A. Heijting: "Address to Professor A. Robinson", Nieuw Arch. Wisk. (3) 21 (1973) p. 135]

Geoffrey Hellman What sense are we to make of the idea that certain sets, say highly inaccessible ones, are merely possible but not actual, that they *might have existed* but in fact do not? [...] "Which ones then *are* actual?", one may well ask. Where is the line between actual and merely possible to be drawn? [G. Hellman: "Maximality vs. extendability: Reflections on structuralism and set theory" (2002) p. 24]

Charles Hermite The impression that Cantor's memoirs make on us is distressing. Reading them seems, to all of us, to be a genuine torture ... . While recognizing that he has opened up a new field of research, none of us is tempted to pursue it. For us it has been impossible to find, among the results that can be understood, a single one having *current interest*. The correspondence between the points of a line and a surface leaves us absolutely indifferent and we think that this result, as long as no one has deduced anything from it, stems from such arbitrary methods that the author would have done better to withhold it and wait. [C. Hermite, letter to G. Mittag-Leffler (1883)]

I never met a mathematician who to a higher degree than Hermite has been a realist in the sense of Plato, and yet I can claim that I never met a more decided opponent of the Cantorian ideas. This is the more a seeming contradiction, as he himself stated frankly: I am an opponent of Cantor because I am a realist. [H. Poincaré: "Letzte Gedanken: Die Mathematik und die Logik", Akademische Verlagsgesellschaft, Leipzig (1913) p. 162f]

Gerhard Hessenberg Also the infinitely many digits of a decimal fraction can successively be determined in an arbitrary way [...]. But a statement about the number defined by that process, which concerned only that number, would be possible only after completion of the process – and this process cannot be completed. [...] We can now generalize the postulate of the choice of type ω by requiring: *For every choice that is possible it shall be possible to iterate it according to type* ω. If I can choose an element, then I can also choose ω elements. If I can choose ω elements, then I can also choose ω times ω elements, i.e., a set of type ω^2. Analogously I can choose ω^3, ω^4, ... and according to my principle also ω^ω. I can continue this actually without arriving at an end, unless the set gets exhausted. In that case the chosen set is identical with the given one, but now it is well-ordered. If I do not arrive at an end, the snag will appear: Then to *every* ordinal number an element of M is assigned, and therefore the set M possesses a subset which is similar to W. It is really paradox, to furnish this procedure, which does not reach an end, eventually with an end. But this paradox is founded in the notion of the set W, and it is clear from the outset that with the possibility of a subset of M being equivalent to W the gates are wide open for illogicality. [G.

Hessenberg: "Grundbegriffe der Mengenlehre", offprint from Abhandlungen der Fries'schen Schule, Vol. I, no. 4, Vandenhoeck & Ruprecht, Göttingen (1906) §§ 103 & 104]

Craig Hicks The non intuitive (and also not useful from an engineering perspective) feature of Ross' analysis is the "jump" in the limit to 0, with no convergence. [C. Hicks in "At each step of a limiting infinite process, put 10 balls in an urn and remove one at random. How many balls are left?", CrossValidated.StackExchange (21 Dec 2017)]

David Hilbert *R. Dedekind* has clearly recognized the mathematical difficulties in founding the notion of number and has delivered in an outmost sharp-minded way a construction of the theory of whole numbers for the first time. But I would call his method a *transcendental* one in so far as he derives the proof of the existence of the infinite in a way the basic idea of which, although being utilized by philosophers, I cannot recognize as possible and save because of the inevitable contradiction caused by the required notion of the totality of all things. [D. Hilbert: "Über die Grundlagen der Logik und der Arithmetik", A. Krazer (ed.): Verh. III. Intern. Math.-Kongr. in Heidelberg (1904), Teubner, Leipzig (1905) p. 175]

[...] when I found it, I thought in the beginning that it causes invincible problems for set theory that would finally lead to the latter's eventual failure; now I firmly believe, however, that everything essential can be kept after a revision of the foundations, as always in science up to now. I have not published this contradiction. [D. Hilbert: "Logische Principien des mathematischen Denkens" (1905), lecture notes by E. Hellinger, Library of the Mathematics Seminar of the University of Göttingen, p. 204]

Finally we will remember our original topic and draw the conclusion of all our reflections. The total result then is this: The infinite is nowhere realized; it is neither present in nature nor admissible as the foundation of our rational thinking – a remarkable harmony between being and thinking. [D. Hilbert: "Über das Unendliche", Math. Annalen 95 (1925) p. 190]

Thomas Hobbes Hobbes claimed that anything we conceive of must have a finite magnitude. Consequently, when we say that something is infinite, all we can justifiably mean is that "we are not able to conceive the ends, and bounds of the thing named; having no conception of the thing, but of our own inability". [...] Hence, actual infinity being ruled out, in Hobbes' mathematical universe, every figure at every stage of production is finite. [P. Mancosu: "Philosophy of mathematics and mathematical practice in the seventeenth century", Oxford University Press, Oxford (1996) p. 146]

Igor Hrncic Unfortunately, Cantor was wrong. His notion of transfinite bijection is flawed. Cantor introduced this notion of transfinite bijection as the additional axiom, even though without even realising this. [I. Hrncic: "The infinitesimal error", viXra (2017)]

Richard L. Hudson That a list can be both complete and endless is a contradiction. [R.L. Hudson: "Cantor diagonal argument – reexamined", viXra (2018) link expired]

Aleksandar Ignjatovic Some say that infinite processes can be completed, others say they definitely cannot be. A third group says depends on the situation. That group then may defer from one concrete situation to the next. Another group says everything is allowed which brings us to the right results (and those results happen to be based on that group's preference). Some make a difference between potential and actual infinity, others disregard this difference completely. So it is pretty chaotic out there. [...] My own preferences are that infinite processes cannot be completed under any circumstances and that potential infinity should be used instead of actual. For example, $\sqrt{2}$ can be thought of as, instead of having infinitely many decimals, in fact having as many as we want or need but finitely many. Pretty much like we practically use it, but in theory too. [A. Ignjatovic in "The law of logical honesty and the end of infinity", YouTube (23 Apr 2016)]

Aleric Inglewood In my eyes, a number either exists or not. If it exists it may or may not be possible to describe it using decimals, but it should be possible to describe it, or there is no proof that it exists. For example the square-root of 2 exists because it is the solution to "*x* squared equals two", that is larger than zero. [A. Inglewood in "Difficulties with real numbers as infinite decimals I", YouTube (3 May 2012)]

I know you don't believe that infinite is a meaningful concept, and neither do I (never did, not since I first heard about it). [A. Inglewood in "Problems with limits and Cauchy sequences", YouTube (23 May 2012)]

David Isles [...] the "natural numbers" of today are not the same as the "natural numbers" of yesterday. Although the possibility of such denotational shifts remains inconceivable to most mathematicians, it seems to be more compatible with the history both of the cultural growth (and of growth in individuals) of the number concept than is the traditional, Platonic picture of an unchanging, timeless, and notation-independent sequence of numbers. [D. Isles: "What evidence is there that 2^{65536} is a natural number?", Notre Dame Journal of Formal Logic 33,4 (1992) p. 478]

Mike Jacobs If the theorem assumes that there are an infinite number of individual elements of matter and we know this is not the case for real matter then it is not a paradox. It is just a waste of time. I read in the book by Gibbons "Shrodingers Kittens etc etc" where he spends a few pages spouting out that there are profound physical consequences of the BTT {{Banach-Tarski Theorem}} in quantum theory. That is too incredible to be true. [...] It is possible to prove anything if you assume false statements. Life is too short to worry about hypothetical mathematical constructs. [M. Jacobs in "Banach Tarski paradox", sci.math (6 Nov 1998)]

Jimmy James I'm no mathematician but it seems to me that the issue here is that there is no end to the process of adding 10 balls and removing one. More than anything this seems like a great example of how mathematics is not reality. [J. James in "At each step of a limiting infinite process, put 10 balls in an urn and remove one at random. How many balls are left?", CrossValidated.StackExchange (30 Nov 2017) link expired]

Colin James III We map the argument for Cantor's diagonal argument into the Meth8 modal logic model checker. The two main equations as rendered are not tautologous. Hence Cantor's diagonal argument is not supported. [C. James III: "Refutation of Cantor's diagonal argument", viXra (2017)]

Edwin T. Jaynes Infinite-set paradoxing has become a morbid infection that is today spreading in a way that threatens the very life of probability theory, and it requires immediate surgical removal. [...] we sail under the banner of Gauss, Kronecker, and Poincaré rather than Cantor, Hilbert, and Bourbaki. [...] we have to recognize that there are no really trustworthy standards of rigor in a mathematics that has embraced the theory of infinite sets. [...] We are, like Poincaré and Weyl, puzzled by how mathematicians can accept and publish such results {{as the Hausdorff sphere paradox}}; why do they not see in this a blatant contradiction which invalidates the reasoning they are using? [...] my belief in the existence of a state of knowledge which considers congruent sets on a sphere equally probable, is vastly stronger than my belief in the soundness of the reasoning which led to the Hausdorff result. Presumably, the Hausdorff sphere paradox and the Russell Barber paradox have similar explanations: one is defining weird sets with self-contradictory properties, so, of course, from that mess it will be possible to deduce any absurd proposition we please. [...] For now, it is the responsibility of those who specialize in infinite-set theory to put their own house in order before trying to export their product to other fields. Until this is accomplished, those of us who work in probability theory or any other area of applied mathematics have a right to demand that this disease, for which we are not responsible, be quarantined and kept out of our field. In this view, too, we are not alone; and indeed have the support of many non-Bourbakist mathematicians. [E.T. Jaynes: "Probability theory: The logic of science", edited by G.L. Bretthorst, Cambridge Univ. Press (2003) pp. XXII & XXVII & 672f]

Joseph F. Johnson The universe uses real numbers, but only uses sets which are measurable, is, I think, the moral to be drawn from the Banach-Tarski paradox. Anyway, functional analysis only uses measurable functions ... . [J.F. Johnson in "Does the Banach-Tarski paradox contradict our understanding of nature?", Physics.StackExchange (10 Apr 2012)]

Pravin K. Johri, Alisha A. Johri There aren't infinite (actual) infinities. Not even one! [P.K. Johri, A.A. Johri: "The flaw in mathematics" (2016)]

John Jones A number occurs in an application in which the number is generated. So the largest number is the largest number in the application. Numbers are not transferable between applications. Therefore, the largest number in the count 1, 2, 3, 4, 5, 6. is 6. The largest number is six. If you think that that answer is a joke it is because it does not flatter your inflated conception of a conceptually impoverished mathematics. You will not be able to disprove this. Let your set theorists try. They will not succeed. [J. Jones in "How big is infinity?", sci.math (24 Aug 2006)]

Peter P. Jones It is clear that for as many digits of *r* as are fully specified, we can form a corresponding mirror natural *n*. [P.P. Jones: "Contra Cantor's diagonal argument", ResearchGate (2017)]

A simple application of combinatorics with set theory suggests that the real numbers are countable. [P.P. Jones: "P(N) and countability", ResearchGate (2017)]

Hans Joss Cantor's proof is about dogmatic stubbornly-religious trust. [H. Joss in "Cantors Diagonalbeweis widerlegt", de.sci.mathematik (12 Mar 2005)]

Dieter Jungmann The definition of equicardinality of infinite sets is locically doubtful. Its consistency never has been proved but is presumed. [D. Jungmann in "Fragen an Dieter Jungmann", de.sci.mathematik (26 Jan 2001)]

Herman Jurjus But aren't there people out there who can imagine 'potentially' infinite sets, but not 'actual' infinite sets? To those people, would Cantor's argument make sense? [H. Jurjus in "Problem with Cantor's diagonal argument", sci.math (14 Feb 2002)]

Theodore Kaczynski The entire structure of pure mathematics is a monstrous swindle, simply a game, a reckless prank. You may well ask: "Are there no renegades to reveal the truth?" Yes, of course. But the facts are so incredible that no one takes them seriously. So the secret is in no danger. [Wikiquote]

László Kalmár I think, second order categoricity results are *deceiving*: they serve only to puzzle ordinary mathematicians who do not know enough logic to distinguish between first order and second order methods. One can say humorously, while first order reasonings are convenient for proving true mathematical theorems, second order reasonings are convenient for proving false metamathematical theorems. [L. Kalmár: "On the role of second order theories" in I. Lakatos (ed.): "Problems in the philosophy of mathematics", North Holland, Amsterdam (1967) p. 104]

Akihiro Kanamori, Menachem Magidor The adaptation of strong axioms of infinity is thus a theological venture, involving basic questions of belief concerning what is true about the universe. [A. Kanamori, M. Magidor: "The evolution of large cardinal axioms in set theory" in G.H. Müller und D.S. Scott (eds.): "Higher set theory", Springer, Berlin (1978) p. 104]

Immanuel Kant The true (transcendental) notion of the infinite is that the successive synthesis of the unit in traversing of a quantum never can be finished. [...] Because the notion of the totality itself is in this case the idea of a completed synthesis of the parts, and this completion, and therefore also its notion, is impossible. [I. Kant: "Critik der reinen Vernunft", Hartknoch, Riga (1781) chapter 77]

Jean-Michel Kantor One central agent of the connection between mathematics and religion is the concept of infinity (but it is not the only one!). [...] until the Cantorian parthenogenesis between mathematics and religion [...] A second theme running through many chapters of the book is the search for a global vision uniting mathematics and religion. [J.-M. Kantor: "Review of Teun Koetsier and Luc Bergmans (eds.): 'Mathematics and the Divine. A Historical Study', Elsevier, Amsterdam (2005)", The Mathematical Intelligencer 30,4 (2008) p. 70f]

Bassam King Karzeddin The endless digits numbers (without being all as zeros) are not accepted in maths if they are on the left of the decimal notation, so must be the case on the right direction of the decimal notation, both are divergent in two opposite directions, one to the large infinite, the other towards the tiny infinitum. [B.K. Karzeddin in "Only an absolute moron makes statements such as these ...", sci.math (8 Oct 2016)]

Felix Kaufmann If you understand – contrary to the results of our investigations – the diagonal procedure as a proof of the existence of uncountable transfinite domains, the question rises how the Löwenheim-Skolem antinomy is compatible with the diagonal procedure. Fraenkel, who investigates this question, arrives at the result that Skolem's proof does not consider non-predicative aspects. Without non-predicativity it would in fact be impossible to leave the countable domain. But if we let not blind us by pretentious speech but look at the linguistic meanings themselves, then we recognize that non-predicative "notions" are meaningless. By a "non-predicative notion" we understand "quite generally the construction of two notions such that one is used in the definition of the other". [...] This means that the object denoted by z_1 is totally – or partially – defined by the object meant by z_2; this object however is defined by the object defined by z_1. Circular "definitions" however, define, as far as they are circular, nothing. [...] The rise to higher cardinalities than $\aleph_0$ is excluded. It follows in particular that a meaning cannot be attached to the notion of the set of all decimal fractions, the "number continuum". So the continuum problem disappears. [F. Kaufmann: "Das Unendliche in der Mathematik und seine Ausschaltung", Wissenschaftliche Buchgesellschaft Darmstadt (1968) pp. 163 & 168]

Jakob Kellner Of course we know that, e.g., the Axiom of Choice is required for many mathematical theorems (such as: every vector space has a basis), which in turn can be applied in physics. However, on closer inspection it turns out that for all concrete instances that are used, the axiom of choice is not required. The same applies even for the existence of an infinite set: One can use a very constructive, "finitary" form of mathematics that is perfectly sufficient for physics. [...] the old intuition: "if nontrivial set theory, non-constructive mathematics or a non-measurable set is used in an essential way, it cannot be physically relevant". [J. Kellner: "Pitowsky's Kolmogorovian models and super-determinism", arXiv (2016) p. 11f]

Joseph Kempenstein I have found a spectacular contradiction in the analysis based upon ZFC, which once and for all should teach the Cantor disciples here otherwise. [J. Kempenstein: "Klarer Widerspruch in der mengenbasierten Analysis", de.sci.mathematik (3 Feb 2006)]

Peter Kepp The second diagonal argument is a circular argument, the conclusions are not valid. [P. Kepp: "Logik des Formalismus, Teil V", www.mathe-neu.de (2017)]

Mohammad Shafiq Khan It is a settled issue in philosophy that infinity cannot exist. [M.S. Khan in D. Chow: "No end in sight: Debating the existence of infinity", Live Science (20 Sep 2013) link expired]

Torsten Kildal I don't believe in a "definite" infinite. Infinite is an indefinite magnitude. [T. Kildal in "B-Baum", de.sci.mathematik (10 Jun 2005)]

Minseong Kim This paper exposes a contradiction in the Zermelo-Fraenkel set theory with the axiom of choice (ZFC). [M. Kim: "Inconsistency of the Zermelo-Fraenkel set theory with the axiom of choice and its effects on the computational complexity", arXiv (2016)]

Laurence Kirby I argue for the use of the adjunction operator (adding a single new element to an existing set) as a basis for building a finitary set theory. [L Kirby: "Finitary set theory", Notre Dame Journal of Formal Logic 50,3 (2009) p. 227]

Morris Kline The crises and conflicts over what sound mathematics is have also discouraged the application of mathematical methodology to many areas of our culture [...] The hope of finding objective, infallible laws and standards has faded. The Age of Reason is gone. [...] What is characteristic of pure mathematics is its irrelevance to immediate or potential applications. [...] Should one design a bridge using theory involving infinite sets or the axiom of choice? Might not the bridge collapse? [...] Mathematics has been shorn of its truth; it is not an independent, secure, solidly grounded body of knowledge. [M. Kline: "Mathematics: The loss of certainty", Oxford University Press (1980) pp. 7 & 285 & 351f]

Teun Koetsier, Luc Bergmans Mathematics in its relation with the divine has played a special role in the course of history. [...] Mathematics is abstract and it often seems absolute, universal, eternal and pure. More than other kinds of knowledge it possesses characteristics that we associate with the divine. [T. Koetsier, L. Bergmans: "Mathematics and the divine: A historical study", Elsevier (2005) p. 4]

Andrej Nikolaevič Kolmogorov [...] objects whose existence is postulated by this axiom {{Zermelo's axiom of choice}} appeared to be not only useless but sometimes destructive to the simplicity and rigorousness of crucial mathematical theories. [A.N. Kolmogorov: "Modern debates on the nature of mathematics", Nauchae Slovo 6 (1929) pp. 41-54]

Kalevi Kolttonen In this article I will suggest an alternative definition of cardinality that I regard as more natural than Cantor's definition. Using the alternative definition, the cardinality of $\mathbb{N}$ and $\mathbb{R}$ is actually proven to be equal. [K. Kolttonen: "Alternative definition of cardinality" (18 Oct 2018)]

Julius König It is easy to show that the finitely defined elements of the continuum determine a subset of the continuum that has cardinal number $\aleph_0$. [...] The assumption that the continuum can be well-ordered therefore has lead to a contradiction. [J. König: "Über die Grundlagen der Mengenlehre und das Kontinuumproblem", Math. Ann. 61 (1905) pp. 156-160]

Nancy Kopell, Gabriel Stolzenberg Today the "official" position is still that mathematics is not about anything. Yet, for those who are not content merely to play a game, the need for meaning is as real as was the need for rigor. [N. Kopell, G. Stolzenberg: "Commentary on Bishop's talk", Historia Math. 2,4 (1975) p. 521]

András Kornai This paper takes the first steps in developing a theory of "explicit finitism" which puts explicit limits on the size of finite objects. [...] We introduce the subset J of the real numbers that is the central mathematical object emerging from considerations of explicit finitism, and take the first steps in studying its properties. [A. Kornai: "Explicit finitism", International Journal of Theoretical Physics 42,2 (2003) p. 301]

Kazuhiko Kotani Finally, the diagonal argument is shown to be inapplicable to the sequence of the potentially infinite number of potentially infinite binary fractions, which contains all n-bit binary fractions in the interval [0,1) for any n. [K. Kotani: "A refutation of the diagonal argument", Open Journal of Philosophy 6 (2016) p. 282]

Gerhard Kowalewski Cantor considered the sequence of his alephs as "something holy", as "the steps that lead to the throne of God". [G. Kowalewski: "Bestand und Wandel", München (1950) p. 201]

Vivek Krishna It seems like the original argument is relying on the assumption that list of natural numbers ends somewhere and we come up with a new decimal number to disprove one-one correspondence. [V. Krishna: "I don't understand the concept of different sizes of infinity", Math.StackExchange (30 Oct 2016)]

Leopold Kronecker Often it has been said that mathematics should start with definitions. The mathematical theorems should be deduced from the definitions and the postulated principles. But definitions themselves are an impossibility, as Kirchhoff used to say, because every definition needs notions which have to be defined themselves, and so on. We cannot, like Hegel's philosophy does, develop the being from the nothing. [...] The whole mathematics is there to be applied. [...] Mathematics is a natural science – not better, not more complete, and not simpler the phenomena can be described than mathematically. [...] If nothing else but the digits up to a certain position are known of decimal fractions, then not even two of them can be added. [...] Again and again it is confused whether a decimal fraction is known up to a certain digit or whether its formula $\Sigma_{n=1}^{\infty} f(n)/10^n$ is known, i.e., $f(n)$ for every n. [J. Boniface, N. Schappacher (eds.): "'Über den Begriff der Zahl in der Mathematik', Kronecker's last lecture, Berlin (1891)", Revue d'histoire des mathématiques 7 (2001) pp. 225f & 251 & 252f & 268f]

I believe that we will succeed one day to "arithmetize" the whole contents of all the mathematical theories, that is to base it solely on the concept of number in its stricter sense, i.e., to get rid of the added modifications and extensions (namely the irrational and continuous magnitudes) which mainly have been caused by the applications on geometry and mechanics. [L. Kronecker in K. Hensel (ed.): "Leopold Kroneckers Werke" III, Teubner, Leipzig (1895-1931) p. 253]

I don't know what predominates in Cantor's theory – philosophy or theology, but I am sure that there is no mathematics there. {{I could not verify this quote, but Cantor authenticates it in a letter to G. Mittag-Leffler of 9 Sep 1883:}} "Kronecker, who visited me at the beginning of July, declared with the friendliest smile that he had much correspondence about my last paper with Hermite in order to demonstrate to him that all that was only 'humbug'."

Lucas B. Kruijswijk At the moment you say that infinite sets are of the same size when there exists a bijection, then you already introduce some of the choices Cantor made. But those are choices, not proofs. [...] there are at least two choices you can make: a) You may refer to ℝ even when there is no logical system that can list all its elements (this is Cantor's choice). b) You must be aware that ℝ is never complete in your logical system (alternative to Cantor). I don't think choice b is very attractive, but to my opinion it is a way you can try to go. However, I always questioned if saying that |ℕ| < |ℝ| has any more meaning than saying that irrational numbers are green. [...] I think the people that don't like Cantor have the greatest problem with the fact that whatever system you use, ℝ contains numbers which can not be expressed. At least I can say about it, it is a fair concern. [L.B. Kruijswijk in "Shannon defeats Cantor = single infinity type", sci.math (11 & 12 Dec 2003)]

Peng Kuan The diagonal number created by Cantor's diagonal argument is the "new guest" in the "hotel" of the list which has infinitely many members. So, it can be fitted in the list. [P. Kuan: "Cardinality of the set of binary-expressed real numbers", Academia (2015) p. 3]

Gustavo Lacerda answers these questions: Do you agree that the continuum hypothesis is a meaningful statement that has a definite truth value, even if we do not know what it is? "No." Do you agree that the axiom which states the existence of an inaccessible cardinal is a meaningful statement that has a definite truth value, even if we do not know what it is? "No." [G. Lacerda in "Ten questions about intuitionism", intuitionism.org (2005)]

Joseph-Louis de Lagrange It is well known that higher mathematics continually uses infinitely large and infinitely small quantities. Nevertheless, geometers, and even the ancient analysts, have carefully avoided everything which approaches the infinite; and some great modern analysts hold that the terms of the expression "infinite magnitude" contradict one another. The Academy hopes, therefore, that it can be explained how so many true theorems have been deduced from a contradictory supposition, and that a principle can be delineated which is sure, clear – in a word, truly mathematical – which can appropriately be substituted for "the infinite". [J.-L. Lagrange: "Prix proposes par l'Academie Royale", Nouveaux memoires de l'Academie Royale des Sciences et Belles Lettres de Berlin, Vol. 15, Decker, Berlin (1784) pp. 12-14]

Detlef Laugwitz [...] according to our view it is meaningless to talk about the set of all points of the continuum. [...] Numbers facilitate counting, measuring, and calculating. [...] The numbers belong to the realm of thinking, the continuum belongs to the realm of visualizing. I repeat, for me the continuum is not identical with the set $\mathbb{R}$. [...] the logical irrationality results from transforming the infinitely increasing number of digits of the potential infinite into an actual infinite and identifying $\sqrt{2}$ with the never ending decimal representation 1.4142... [D. Laugwitz: "Zahlen und Kontinuum", Bibl. Inst., Zürich (1986) p. 13ff]

Shaughan Lavine A system of finite mathematics is proposed that has all of the power of classical mathematics. [...] The finite mathematics of sets is comprehensible and usable on its own terms, without appeal to any form of the infinite. [S. Lavine: "Finite mathematics", Synthese 103,3 (1995) p. 389]

Henri Léon Lebesgue {{Fréchet and Lebesgue refused to present Tarski's theorem: (For all infinite sets X there exists a bijection of X to $X×X$) ⇒ (Axiom of Choice).}} Fréchet wrote that an implication between two well known propositions is not a new result. Lebesgue wrote that an implication between two false propositions is of no interest. [J. Mycielski: "A system of axioms of set theory for the rationalists", Notices of the AMS 53,2 (2006) p. 209]

Antonio León It seems reasonable that Plato were platonic in Plato's times, but is certainly surprising the persistence of that primitive way of thinking in the community of contemporary mathematicians [...] But for those of us who believe in the organic nature of our brains and in its abilities of perceiving and knowing modelled through more than 3600 million years of organic evolution, platonism has no longer sense. And neither self-reference nor the actual infinity may survive away from the platonic scenario. [A. Leon Sanchez: "Extending Cantor's paradox", arXiv (2008) p. 16f]

Michael Lew Let x be the number of balls that have been removed and y be the number of balls remaining. After each cycle $y = 9x$. [...] There will be infinitely many balls in the urn at 12 PM. [M. Lew in "At each step of a limiting infinite process, put 10 balls in an urn and remove one at random. How many balls are left?", CrossValidated.StackExchange (24 Nov 2017)]

Keith A. Lewis 1. If a number is indescribable and unrepresentable, it is irrelevant. The set of relevant numbers has cardinality equal to the naturals. 2. No continuum has been discovered in physics – everything seems to change in finite units called quanta. That's the real world. 3. The Dedekind cut paradox. Rationals on [0,1] are totally ordered and countable, yet there are an uncountable number of cuts based on this total ordering. [K.A. Lewis in "Objections to Cantor's theory (Wikipedia article)", sci.math (19 Jul 2005)]

John Locke The infinity of numbers, to the end of whose addition every one perceives there is no approach, easily appears to any one that reflects on it. But, how clear soever this idea of the infinity of number be, there is nothing yet more evident than the absurdity of the actual idea of an infinite number. [J. Locke: "An essay concerning human understanding" Chapter XVII: Of infinity (1690)]

Paul Lorenzen During the renaissance, in particular with Bruno, the actual infinite is carried over from God to the world. The finite world models of present science show clearly, how the superiority of the idea of an actual infinite has come to an end with classical (modern) physics. In this light the inclusion of the actual infinite into mathematics, which explicitly began during the end of the last century with G. Cantor, appears disconcerting. In the intellectual framework of our century – in particular when considering existential philosophy – the actual infinite appears really as an anachronism. [p. 3] We introduce the numbers for counting. This does not at all imply the infinity of numbers. For, in what way should we ever arrive at infinitely many countable things? [...] (1) Start with I. (2) When *x* is reached, supplement *x*I. These rules [...] supply a constructive definition of numbers (namely their scheme of construction). Now we can immediately say that according to these rules infinitely many numbers are possible: [...] One has to be aware of the fact that here only the *possibility* is asserted – and this is just secured by the rule itself. [...] To assert however that infinitely many such numbers really were existing, i.e., had really been constructed according to these rules – that would be wrong of course. [...] In arithmetic [...] there does not exist a motive to introduce the actual infinite. The surprising appearance of an actual infinite in modern mathematics therefore can only be understood by taking geometry into consideration. [p. 4f] A finite decimal fraction can be written out, an infinite one can never be written. To talk about a sequence of infinitely many digits is therefore – if it is not nonsense at all – at least a hazardous business. [p. 9] Similar to the elimination of the infinitely small at that time {{in the 19th century when the naive notion of continuity was replaced by the ε-δ-definition}} now the infinitely great (more precisely the actual infinite) has to be shown dispensable. The motivating force of this reform is not based on a haughty purism but on the wish to restore the absolute safety and reliability of mathematics that presently is in danger to be lost in the set theoretic contradictions or their circumvention by rather arbitrary set theoretic formalisms. [p. 11] [P. Lorenzen: "Das Aktual-Unendliche in der Mathematik", Philosophia naturalis 4 (1957) pp. 3-11]

Harvey Lubin The concept of "infinity" is anti-science, and the use of infinity as a constant by mathematicians and scientists, whether they know it or not, plays into the hands of advocates of religions and other forms of the supernatural, who also describe the objects of their beliefs as infinite. [...] Unfortunately, the recognition of infinity in science is also an excellent justification for zealots to convince others that, since science accepts infinity as a reality, deities that are just as infinite and unknowable are also reality. [H. Lubin in "To settle infinity dispute, a new law of logic", Quanta Magazine (20 Jul 2014)]

Laureano Luna As I see it, WM has clearly made the point that there is a pairing procedure able to produce a one-one application between nodes and paths *as* the construction of the binary tree goes on. [...] This method leaves behind no unmatched trajectory. So, if the binary tree is the outcome in the limit of a constructive procedure, the pairing procedure will end up in a bijection between the set of all nodes and the set of all paths. As a consequence, the binary tree as a representation of the set of all infinite binary sequences is not constructive. [LauLuna: "WM on the binary tree", sci.logic (19 Oct 2008)]

Laureano Luna, William Taylor Cantor's proof that the powerset of the set of all natural numbers is uncountable yields a version of Richard's paradox when restricted to the full definable universe, that is, to the universe containing all objects that can be defined not just in one formal language but by means of the full expressive power of natural language: this universe seems to be countable on one account and uncountable on another. [L. Luna, W. Taylor: "Cantor's proof in the full definable universe", Australasian Journal of Logic 9 (2010) p. 10]

Nikolai Nikolaevich Luzin I cannot consider the set of positive integers as given, for the concept of the actual infinite strikes me as insufficiently natural to consider it by itself. [N.N. Luzin, letter to K. Kuratowski, quoted in N.Y. Vilenkin: "In search of infinity", Birkhäuser, Boston (1995) p. 126]

Saunders Mac Lane The clear understanding of formalism in mathematics has led to a rather fixed dogmatic position which reads: Mathematics is what can be done within axiomatic set theory using classical predicate logic. I call this doctrine the *Grand Set Theoretic Foundation*. [...] It is my contention that this *Grand Set Theoretic Foundation* is a mistakenly one-sided view of mathematics and also that its precursor doctrine (Dedekind cuts) was also one-sided. [...] Second, set theory is largely irrelevant to the practice of most mathematics. Most professional mathematicians never have occasion to use the Zermelo-Fraenkel axioms, while others do not even know them. If they did know the axioms, they would rapidly observe that most of the mathematics they do could be satisfactorily based on a much weaker set of axioms – [S. Mac Lane: "Mathematical models: A sketch for the philosophy of mathematics", The American Mathematical Monthly, Vol. 88,7 (1981) p. 467f]

Menachem Magidor A typical reaction of a certain mathematical community when one of its central problems was shown to be independent is to try and ostracize the offending problem by labeling it as "ill defined", "vague" etc. [...] I find this kind of reaction somewhat intellectually dishonest. [M. Magidor: "Some set theories are more equal", ResearchGate (2015)]

Ron Maimon The notion of the continuum in physics is not as a collection of differentiated points gathered together into an abstract set, but as a limit of discrete structures with finite computations defined on them. The limiting process must be well defined, so that the answer to any experimental question to any accuracy can be answered by a finite computation. Any set-theoretic property of the collection of the real numbers which relies on separating out individual points from one another in a non-computable way and talking about them using logical properties of the individual points which involve undecidable questions is always going to be unphysical. [...] So I consider Banach Tarsky type results to be far worse than unphysical, they go so far as to be non-mathematical – they should be considered false even as pure mathematics. [...] There is absolutely no mathematical theorem which depends on uncountable choice which is of use to mathematicians, and it is well past the time to scrap this nonsense. [...] The finite computations are more fundamental than sets, in the sense that the properties of computation are absolute and independent of axiom systems, the axioms are only important to the degree they describe computation. [...] The demonstration that the uncountable ordinals are not absolute is simply from noting that all the models of an axiom system can be countable. It means that you don't need to take the uncountable ordinals seriously. [R. Maimon in "Does the Banach-Tarski paradox contradict our understanding of nature?", Physics.StackExchange (2 Feb 2012) & (25 Jul 2013) & (9 Aug 2013)]

Yuri I. Manin Already during Cantor's life time, the reception of his ideas was more like that of new trends in the art, such as impressionism or atonality, than that of new scientific theories. It was highly emotionally charged and ranged from total dismissal (Kronecker's "corrupter of youth") to highest praise (Hilbert's defense of "Cantor's Paradise"). (Notice however the commonly overlooked nuances of both statements which subtly undermine their ardor: Kronecker implicitly likens Cantor to Socrates, whereas Hilbert with faint mockery hints at Cantor's conviction that Set Theory is inspired by God.) [Y.I. Manin: "Georg Cantor and his heritage", arXiv (2002) p. 10]

Pedro Mascarós We say that in an interval, there exist numbers that we can't even construct ... Let's choose them or let's make a set with them because we have clear evidence that they are really in some place over there, ok, but ... can we really choose them if we don't know where they really are or their names? Is it not like a set of words that you cannot say or a set of invisible and untouchable ducks or something like that? [P. Mascarós in "Infinity: does it exist??", YouTube (28 Sep 2014)]

Robert Leon Massey And G. Cantor's 'diagonal' argument for proving higher order infinity can not be simulated by finite memory computers as R. Penrose claims. Real computers only generate a finite number of rational numbers or repeating integers if let run long enough because their determinism and only a finite number of different possible states makes them start repeating results when their count mechanisms overflow (possible after the universe collapses). [Bob Massey in "Cantor's transfinite numbers", sci.math (3 Nov 1996)]

Miles Mathis I have seen many proofs of Cantor's theorem that the irrationals (or reals) are uncountable, and none is at all convincing. [...] *The primary operational fact here is that no matter how many irrationals you have to count, you will always have an integer available to count it with. Always. Therefore, the claim that there are more irrationals or reals than integers or rationals is nonsense.* [...] All the math that takes place in the transinfinite is quite simply false. Notice that I do not say it is physically baseless, or mystical, or avant garde, or any other half-way adjective. It is false. It is wrong. It is a horrible, terrible mistake, one that is very difficult to understand. It is further proof that Modern math and physics have followed the same path as Modern art and music and architecture. It can only be explained as a cultural pathology, one where self-proclaimed intellectuals exhibit the most transparent symptoms of rational negligence. They are outlandishly irrational, and do not care that they are. They are proud to be irrational. They believe – due to a misreading of Nietzsche perhaps – that irrationality is a cohort of creativity. Or it is a stand-in, a substitute. A paradox therefore becomes a distinction. A badge of courage. [...] If we somehow survive this cultural pathology, the future will look upon our time in horror and wonderment. [M. Mathis: "Introductory remarks on Cantor"]

Rinette Mathlener Indeed, the interpretation of actual infinity leads to contradictions as seen in the paradoxes of Zeno. It is difficult for a human being to understand actual infinity. Our logical schemes are adapted to finite objects and events. Research shows that students focus primarily on infinity as a dynamic or neverending process. Individuals may have contradictory intuitive thoughts at different times without being aware of cognitive conflict. [R. Mathlener: "Die problematiek van die begrip oneindigheid in wiskundeonderrig en die manifestasie daarvan in irrasionale getalle, fraktale en die werk van Escher", Dissertation, University of South Africa, Pretoria (2008)]

Damien Mattei In Cantor theory infinite is considered as if you have an infinite process. Then it exists a step when this process is terminated. It seems to me that this the contrary of the notion of infinite. [D. Mattei: "Cantor's diagonal proof", sci.math (3 Apr 1998)]

Ardeshir Mehta All that Cantor *has* proved is that X is not on the list *above* n, no matter how large n may be. But he has *not* proved that X is not in the (not-yet-listed – and indeed *unlistable*, because infinitely large) part of the table *after* n ... and he *cannot* prove that! After all, no matter how large n gets, the part of the table *after* n still remains infinitely larger than the part of the table *before* n. And note that no matter how large n gets, the list after n can never be "brought up" to be included in the part of the list *before* n. This is because the part of the list *before* n is *necessarily* finite, while the part of the list *after* n must be infinitely long! [A. Mehta: "A simple argument against Cantor's diagonal procedure" (2001)]

Oualid Merzouga The situation is even worse when you realize that most of the numbers (in the current modern mathematic framework) are not numbers we even comprehend, we can't represent them in any way, we can't do arithmetic with them, we don't even know about them. [O. Merzouga in "Difficulties with real numbers as infinite decimals I", YouTube, link expired]

James R. Meyer According to Ross & Littlewood, every ball labeled n that is put into the vase is eventually removed in a subsequent step, the step n, and hence, by noon the vase must be empty. It's easy to see the fallacy. The process is an infinite unending process that is not completed. [J.R. Meyer: "The balls in the urn paradox (Ross-Littlewood)" (2018)]

Heinz Middelmann In this book it is shown that *all* infinite sets have same cardinality as the set of prime numbers und therefore as the set of natural numbers. This also holds for the power sets of the infinite sets. There is only one infinity in mathematics. [H. Middelmann: "Gezähmte Unendlichkeiten" (Sep 2018)]

John Stuart Mill Numbers are in the strictest sense names of objects. Two is certainly the name of things that are two, two spheres, two fingers and so on. [J.S. Mill in a remark to J. Mill: "An analysis of the phenomena of the human mind II", A.M. Kelley, New York (1967) p. 92]

Jon Miller There are even (intelligent) people who (claim that they) are unable to believe in infinite sets at all. To them, Cantor's argument makes no sense. There are people who deny the law of the excluded middle. To them, proving that not-A is false does not prove that A is true. [J. Miller in "Problem with Cantor's diagonal argument", sci.math (15 Feb 2002)]

S.S. Mirahmadi The set $\mathbb{N}$ of all natural numbers does not exist. [S.S. Mirahmadi, Qom seminary, Qom, Iran (Sep 2013)]

François Napoléon Marie Moigno Are actually infinite numbers possible? Can we by adding unit after unit or groups of units after groups of units arrive at an actually infinite number? To the question thus posed the simple sensible answer is no, without hesitation, obviously no. [...] there is no passage from the finite to the infinite possible, no connection between the numbers and the infinite can be assigned. *Galilée, Torricelli, Guldin, Cavalieri, Newton, Leibnitz, Rolle, Gerdil* and a lot of others also have clearly shown this. [M. Moigno: "Impossibilité du nombre actuellement infini", Appendix to Augustin Cauchy: "Sept lecons de physique générale", Gauthier-Villars, Paris (1868) pp. 77 & 81]

Philip Molyneux The use of the term "infinity" as a noun, and even the use of a symbol for it (whether ∞, $\aleph_0$, or whatever), seems to imply that it is a defined entity or quantity, and one to which a comparative or even possibly a superlative could possibly be applied. However, this is evidently a category error, for the use of the alternative ("AngloSaxon") form of the word as "endlessness" reveals that it refers to the *absence* of a feature to a process, that is, to the absence of any limitation to the length of this process. The parallels here are such terms as "blackness" (the absence of light in the environment, or the absence of reflectivity for a surface) or "vacuum" (the absence of material content) where again comparatives or superlatives evidently cannot be applied – one material object cannot be "blacker" than another, and one void cannot be more vacuous than another. Such considerations in viewing "infinity" otherwise then is what the science-popularising author Lancelot Hogben may have meant when referred to this area as a "semantic quagmire". [P. Molyneux: "Some critical notes on the Cantor diagonal argument", viXra (2017) p. 9]

Stephen Montgomery-Smith I can see Kronecker's point of view, which I guess is that Cantor's theories depend upon the existence of mathematical objects that don't seem to exist in real life (e.g. what is a real number, really?). If the anti-Cantorians argued at this level, I think that I would essentially be in agreement with them. [S. Montgomery-Smith in "Objections to Cantor's theory (Wikipedia article)", sci.math (19 Jul 2005)]

Adrian W. Moore The (truly) infinite, I claim, can never be subjugated. Indeed I would go further: the (truly) infinite, as a unitary object of thought, does not and cannot exist. This is not to say that the concept of the infinite has no legitimate use. One such use, if I am right, is precisely to claim that the infinite does not exist. [A.W. Moore: "The infinite", 3rd ed., Routledge, Oxon (2019) Preface]

When it is claimed that P(ω) is not unconditionally uncountable, we have no way of understanding this except as the demonstrably false claim that it is not uncountable at all. [A.W. Moore: "Set theory, Skolem's paradox and the Tractatus", Analysis 45 (1985) pp. 13-20]

Gregory H. Moore Despite his acceptance of Zermelo's Axiom, Poincaré rejected Zermelo's proof because of its use of an impredicative definition. For Poincaré, a definition of an object A was impredicative if A was defined in terms of a class B, of which A was a member. In Zermelo's proof, A was L_γ, and B was the family of all gamma-sets. Such impredicative definitions, claimed Poincaré, were born of the actual infinite and in turn sired the paradoxes. Zermelo would not agree. [...] Evidently, Zermelo did not know of the English debate in 1906 over his proof and Axiom by G.H. Hardy, E. Hobson, Jourdain, and Russell. Of these mathematicians, only Hardy and Jourdain accepted Zermelo's Axiom of Choice, and none was quite satisfied with Zermelo's proof. [G.H. Moore: "The origins of Zermelo's axiomatization of set theory", Journal of Philosophical Logic 7,1 (1978) pp. 316 & 320]

Andrzej Mostowski Most striking are results concerning the continuum hypothesis. Roughly speaking they show that practically every hypothesis concerning powers of regular cardinals is compatible with axioms of Zermelo-Fraenkel. [...] Such results show that axiomatic set-theory is hopelessly incomplete. Certainly nobody expected the axioms of set-theory to be complete, but it is also certain that nobody expected them to be incomplete to such a degree. [...] Of course if there are a multitude of set-theories then none of them can claim the central place in mathematics. [A. Mostowski: "Recent results in set theory" in I. Lakatos (ed.): "Problems in the philosophy of mathematics", North Holland, Amsterdam (1967) p. 93f]

Luboš Motl Nothing in physics depends on the validity of the axiom of choice because physics deals with the explanation of observable phenomena. [L. Motl in "Does the axiom of choice appear to be 'true' in the context of physics?" (edited), Physics.StackExchange (10 Nov 2012)]

Carl Mummert However, there is not even a formula that unequivocally defines a well ordering of the reals in ZFC. [...] Worse, it's even consistent with ZFC that *no* formula in the language of set theory defines a well ordering of the reals (even though one exists). That is, there is a model of ZFC in which no formula defines a well ordering of the reals. [C. Mummert in "Is there a known well ordering of the reals?", Math.StackExchange (11 Oct 2010)]

Jan Mycielski Imagine for a while that infinite objects of classical mathematics are as inconsistent as a triangular circle. Does this force us to do mathematics in the framework proposed by the finitists or constructivists? It is the purpose of this paper to show that the answer is no. [J. Mycielski: "Analysis without actual infinity", Journal of Symbolic Logic 46,3 (1981) p. 625]

David Hilbert in 1904 [...] wrote that sets are thought-objects which can be imagined prior to their elements. At the request of the referee who asked what is a thought-object let me add: I understand it to be a thought about an object which may exist or not. Thus it is an electrochemical event in the brain or/and its record in the memory. In particular it is a physical thing in space time. Of course it is difficult to characterise any physical phenomena. But we have the ability to recognize thoughts as identical or different, just as we have the ability to recognize a silent lightning from a thunderous one. Hence I understand Hilbert's words as follows: mathematicians imagine sets which do not exist, but their thoughts about sets do exist and they can arise prior to the thoughts of most elements in those sets. [J. Mycielski: "Russell's paradox and Hilbert's (much forgotten) view of set theory" in G. Link (ed.): "One hundred years of Russell's paradox: mathematics, logic, philosophy", De Gruyter, Berlin (2004) p. 533]

Edward Nelson But numbers are symbolic constructions; a construction does not exist until it is made; when something new is made, it is something new and not a selection from a pre-existing collection. [E. Nelson: "Predicative arithmetic", Princeton Univ. Press, Princeton (1986) p. 2]

If I give you an addition problem like

$$\begin{array}{r} 37460225182244100253734521345623457115604427833 \\ +\ 52328763514530238412154321543225430143254061105 \end{array}$$

and you are the first to solve it, you will have created a number that did not exist previously. [E. Nelson: "Confessions of an apostate mathematician", Princeton]

Let us distinguish between the *genetic*, in the dictionary sense of pertaining to origins, and the *formal*. Numerals (terms containing only the unary function symbol S and the constant 0) are genetic; they are formed by human activity. All of mathematical activity is genetic, though the subject matter is formal. Numerals constitute a *potential infinity*. Given any numeral, we can construct a new numeral by prefixing it with S. Now imagine this potential infinity to be completed. Imagine the inexhaustible process of constructing numerals somehow to have been finished, and call the result *the set of all numbers*, denoted by ℕ. Thus ℕ is thought to be an *actual infinity* or a *completed infinity*. This is curious terminology, since the etymology of "infinite" is "not finished".

We were warned. Aristotle: Infinity is always potential, never actual. Gauss: I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics.

We ignored the warnings. With the work of Dedekind, Peano, and Cantor above all, completed infinity was accepted into mainstream mathematics. Mathematics became a faith-based initiative.

Try to imagine ℕ *as if it were real.*

A friend of mine came across the following on the Web:

www.completedinfinity.com

Buy a copy of ℕ!

Contains zero – contains the successor of everything it contains – contains only these.

Just $100.

Do the math! What is the price per number?

Satisfaction guaranteed!

Use our secure form to enter your credit card number and its security number, zip code, social security number, bank's routing number, checking account number, date of birth, and mother's maiden name.

The product will be shipped to you within two business days in a plain wrapper.

My friend answered this ad and proudly showed his copy of ℕ to me. I noticed that zero was green, and that the successor of every green number was green, but that his model contained a red number. I told my friend that he had been cheated, and had bought a nonstandard model, but he is color blind and could not see my point.

I bought a model from another dealer and am quite pleased with it. My friend maintains that it contains an ineffable number, although zero is effable and the successor of every effable number is effable, but I don't know what he is talking about. I think he is just jealous. [E. Nelson: "Hilbert's mistake", Princeton (2007)]

John von Neumann At present we can do no more than note that we have one more reason here to entertain reservations about set theory and that for the time being no way of rehabilitating this theory is known. [J. v. Neumann: "Eine Axiomatisierung der Mengenlehre", Journal für die reine und angewandte Mathematik 154 (1925) p. 240]

Anne Newstead We argue [...] that the classical realist account of the continuum has explanatory power in mathematics and should be accepted, much in the same way that "dark matter" is posited by physicists to explain observations in cosmology. In effect, the indefinable real numbers are like the "dark matter" of real analysis. [A. Newstead: "On the reality of the continuum discussion note", Philosophy 83,1 (2008) Abstract]

Mike Oliver In fact it {{set theory}} makes further falsifiable predictions that you would not have been able to make by considering finite objects alone. [...] AC does not make new falsifiable predictions, but by facilitating the *conceptualization* of set theory, it makes new falsifiable predictions easier to make – in practice, for humans. [M. Oliver in "! Cantor", sci.math (26 Apr 1999)]

Tony Orlow I completely agree with your conclusion, Wolfgang. [...] There are exactly as many paths as nodes, as they are produced one at a time from the addition of new nodes. [T. Orlow in "A new view on the binary tree", sci.math (4 Jan 2015)]

Christopher Ormell Russell concludes that the main body of real numbers 'must be' of the 'lawless' variety. The author scrutinises these so-called 'lawless decimals' and concludes that they are mythical. [C. Ormell: "The continuum: Russell's moment of candour", Philosophy 81,4 (2006) Abstract]

Rob Osborn Actually, Cantor's arguments of infinity are disputed by many including the math itself. [...] to state that one could make a new number from a collection of unknowns is ludicrous as Cantor suggested. All I say is show me the math showing how one infinity couldn't include any number not already in that set. Cantor's diagonal argument only checks numbers in a finite set not an infinite set as he thinks. [R. Osborn in "The infinite. Part 6. Mathematics, physics and religion in the 19th century", By Common Consent (18 Jan 2012)]

Geng Ouyang [...] the potential infinite and actual infinite, number conception and numerical theory as well as limit theory in traditional infinite system are all with some fatal defects. [G. Ouyang in "Is there a cognitive breaking point in mathematics?", Quora (11 Dec 2016)]

Eray Ozkural Merely a finitist viewpoint is sufficient, and it resolves each and every antinomy in the set theory and logic with no extra effort. [E. Ozkural in "Objections to Cantor's theory (Wikipedia article)", sci.math (20 Jul 2005)]

Donald G. Palmer Cantor has not proven that the diagonal constructive process results in a unique infinite decimal. The missing part of the proof is that he assumes the limit of this process produces a unique decimal. He does not prove this point, but states it. [D.G. Palmer in "Cantor and infinite size", sci.math (28 May 1999)]

Juan-Esteban Palomar Tarancón [...] it is proved that Cantor's Theorem need not imply the existence of a tower of different-size infinities, because the impossibility of defining a bijection between any infinite countable set and its power can be a consequence of the existence of any intrinsic property which does not depend on size. [J.-E. Palomar Tarancón: "The existence of intrinsic set properties implies Cantor's theorem. The concept of cardinal revisited.", Int. J. Open Problems Compt. Math. 5,2 (2012) Abstract]

I also think that infinity denotes an endless process and it has a potential or virtual reality. Nevertheless, this claim requires a multivalued logic, or at least, a three valued one, in which the truth-values are True, False and Undefined. [J.-E. Palomar Tarancón: "Does actual infinity exist?", ResearchGate (26 Aug 2012)]

Dear Wolfgang, I appreciate your manuscript {{the present Source Book}}. It is very valuable for me. [J.-E. Palomar Tarancón: "Does actual infinity exist?", ResearchGate (17 Nov 2016)]

Rohit Parikh Does the Bernays' number 67^{257729} actually belong to every set which contains 0 and is closed under the successor function? The conventional answer is yes but we have seen that there is a very large element of fantasy in conventional mathematics which one may accept if one finds it pleasant, but which one could equally sensibly (perhaps more sensibly) reject. [R. Parikh: "Existence and feasibility in arithmetic", Journal of Symbolic Logic 36 (1971) p. 507]

Steve Patterson *There are no infinite sets.* [...] Infinite sets do not exist; Cantor was wrong; and it will take nothing less than an intellectual revolution to place mathematics back on firm foundations. [S. Patterson: "Cantor was wrong | There are no infinite sets" (20 Jul 2016)]

Joachim Pense Here we only talk about pathologies of modelling the continuum as a "point set". Of course all real numbers occuring "in the nature" are afflicted with imprecision. [...] For me they don't exist as individuals. [J. Pense in "Wenigstens hierüber sollte in einer sci-Newsgroup Einigkeit erzielt werden können", de.sci.mathematik (18 Jun 2006)]

Karma Peny But any set of unique (non-repeating) natural numbers, excluding zero, must contain at least one number that is equal-to or greater-than the size of the set. Why does (and how can) this rule suddenly not hold when the set contains 'infinitely many' elements? It seems we can pick and choose which fundamental rules of mathematics suddenly no longer apply where infinity is involved, as long as our choices support the idea that infinity is a valid concept. All this is, of course, complete nonsense. [K. Peny: "Let's visit infinity for a bit of fun", Extreme Finitism (14 Sep 2014)]

Mathematics would be a more staid and boring subject without all the weird and wondrous things that infinity brings with it. But do we really want mystic beliefs at the heart of a discipline that is traditionally associated with logic and rigour? [K. Peny: "Mainstream mathematics is based on a belief in the supernatural", Extreme Finitism (21 Oct 2014)]

Jaroslav Peregrin Our view of the world is so infiltrated by contemporary mathematics based on Cantor's set theory that we count infinite sets among the uttermost realities. However, on second thought it is quite clear that there is no infinite set we could really encounter within our 'real' world. Everyone of us can be confronted (at once, but also during the whole span of his life) with at most a finite number of objects. The only aspect of reality that may be felt as amounting to infinity is *unlimitedness*, the possibility to continue various processes over and over without any limit. In other words, there is no *actual* infinity, there is at most *potential* infinity. [J. Peregrin: "Structure and meaning", Semiotica 113 (1997) p. 83]

Benedictus Pererius But that there is no actual infinity is not only in accordance with the decrees of the Philosopher {{Aristotle}} but also with the holy scripture which reads God made all in weight, number, and measure. [B. Pererius: "De communibus omnium rerum naturalium principiis et affectionibus libri quindecim" Vol. X (1579) p. 568]

Juan A. Perez Cantor's proofs of nondenumerability are refuted by analyzing the logical inconsistencies in implementation of the *reductio* method of proof and by identifying errors. [J.A. Perez: "Addressing mathematical inconsistency: Cantor and Gödel refuted", arXiv (2010)]

Tilman Pesch The line all points of which have a finite distance is itself finite. [T. Pesch: "Institutiones philosophiae naturalis secundum principia S. Thomae Aquinatis", Freiburg, Herder (1880) § 425]

David Petry The "anti-Cantorian" view has been around ever since Cantor introduced his ideas. [...] It was the advent of the internet which revealed just how prevalent the anti-Cantorian view still is; [D. Petry: "Objections to Cantor's theory", sci.logic & sci.math (20 Jul 2005)]

So why are intellectuals attracted to such stupid ideas as Cantor's Theory? Due to the structure of our society, intellectuals tend to value cleverness, consistency, complexity, and even sensationalism, more highly than truth. We live in a sick society. Cantor's Theory sucks more than any theory has ever sucked before. [D. Petry: "Cantor's theory sucks", sci.math (18 May 2005)]

Here's a quote from Doron Zeilberger: [...] That sounds to me a great deal like what W. Mueckenheim is advocating. I wonder why WM wastes his time playing with losers on sci.math. [D. Petry: "Mueckenheim's views have some support from well respected mathematicians", sci.math (20 Jul 2014)]

In a recent article, WM wrote: "Existence in mathematics means existence in mathematical discourse. [...] ideas that nobody can have, do not exist. To assume the existence of ideas that nobody can have means to introduce most primitive religious ideas that destroy all sober thinking including mathematics." What WM is saying there is quite reasonable, though hardly original. I would guess that the whole point he has been trying to make in the 30,000 or so articles he has posted in the last two decades is that Cantor's diagonal argument does not compel us to believe that there exist things we cannot talk about, which leads to the conclusion that Cantor's talk about uncountable sets is silliness. I think he's right. [D. Petry: "WM starts to make sense", sci.math (4 Nov 2015)]

As I see it, the mathematics community has been taken over by a destructive quasi-religious cult. I feel I have a moral obligation to speak out. [D. Petry: "The subject formerly known as mathematics", sci.math (19 Dec 2015)]

Charles Petzold Turing's conclusions cast real doubts on a Platonistic interpretation of mathematics and imply instead an extreme Constructivist philosophy where mathematics is limited by time, resources, and energy. [C. Petzold: "Reading Brian Rotman's 'ad infinitum …'" (25 May 2008)]

Martin Pitt [...] there are only countably many irrational numbers which we can describe with an algorithm (in a sense, those numbers which only have a finite amount of intrinsical information in them). So you cannot represent almost all real numbers with algorithms anyway. [M. Pitt in "Difficulties with real numbers as infinite decimals II", YouTube (8 May 2012)]

Dima Podolsky Can't you just add the new number to the list? Yeah, it would create a new decimal, so add it to the list. Just like adding fractions onto the end of the fraction list. [D. Podolsky in "Infinity is bigger than you think – Numberphile", YouTube (6 Jul 2012)]

Henri Poincaré One of the characteristic features of Cantorism is that, instead of rising to the general by erecting more and more complicated constructions, and defining by construction, it starts with the *genus supremum* and only defines, as the scholastics would have said, *per genus proximum et differentiam specificam*. Hence the horror he has sometimes inspired in certain minds, such as Hermite's, whose favourite idea was to compare the mathematical with the natural sciences. For the greater number of us these prejudices had been dissipated, but it has come about that we have run against certain paradoxes and apparent contradictions, which would have rejoiced the heart of Zeno of Elea and the school of Megara. Then began the business of searching for a remedy, each man his own way. For my part I think, and I am not alone in so thinking, that the important thing is never to introduce any entities but such as can be completely defined in a finite number of words. Whatever be the remedy adopted, we can promise ourselves the joy of the doctor called in to follow a fine pathological case. [H. Poincaré: "Science and method: The future of mathematics", Nelson, London (1914) p. 44f]

There is no actual infinity. The Cantorians forgot this, and so fell into contradiction. [H. Poincaré: "Science and method: Last efforts of logisticians", Nelson, London (1914) p. 195]

Concerning the second transfinite cardinal number $\aleph_1$, I am not completely convinced of its existence. We reach it by considering the collection of ordinal numbers of cardinality $\aleph_0$; it is clear that this collection must have a higher cardinality. But the question is whether it is closed, that is whether we may talk about its cardinality without contradiction. In any case an actual infinite can be excluded. [H. Poincaré: "Über transfinite Zahlen", Teubner, Leipzig (1910) p. 48]

N.A. Popov, V.A. Levin, A.N. Popov We think we have succeeded in finding the mistake of Cantor which is pointed to by Bertrand Russell. Though he failed to solve the paradox of Cantor, he made the important step along the required investigation line. [N.A. Popov, V.A. Levin, A.N. Popov: "Resolution of Cantor's set theory contradictions based on analysis of unsatisfiable definitions", Proceedings of the Second International A.D. Sakharov Conference on Physics, I.M. Dremin, A.M. Semikhatov (eds.), World Scientific, Singapore (1997) p. 313f]

Gerhard Prandstätter I have needed some time to understand what you want to express with this argument {{In the binary tree there cannot be more separated paths than separators, namely nodes.}} and how simple it is. If I am not completely in error I would say it is irrefutable. [...] Since the number of paths (real numbers) always remains less than the number of nodes, it is guaranteed that the number of paths (real numbers) can reach *simple infinity* at most. [G. Prandstätter in "Wikipedia: Paradoxien der Mengenlehre", de.sci.mathematik (31 Mar 2007)]

Willard Van Orman Quine As a foundation for mathematics, then, set theory is far less firm than what is founded upon it; for common sense in set theory is discredited by the paradoxes. [W.V.O. Quine: "The ways of paradox and other essays", Harvard University Press (1966) p. 31f]

What is called *giving* the meaning of an utterance is simply the uttering of a synonym, couched, ordinarily, in clearer language than the original. If we are allergic to meanings as such, we can speak directly of utterances as significant or insignificant, and as synonymous or heteronymous one with another. [...] Consider, for example, the crisis which was precipitated in the foundations of mathematics, at the turn of the century, by the discovery of Russell's paradox and other antinomies of set theory. These contradictions had to be obviated by unintuitive, *ad hoc* devices; our mathematical myth-making became deliberate and evident to all. [W.V.O. Quine: "On what there is", in "From a logical point of view", 2nd ed., Harvard Univ. Press (1953) pp. 11f & 18]

Ken Quirici To show there is no bijection from a set *S* to its power set *T*: [...] The problem for me is: yes, there's a clear contradiction. But we're actually making *two* assumptions: the existence of the bijection, *and* the existence of the set *D*. Can't we simply say that the bijection could still exist, but there would be no set *D* defined as above, for it? [K. Quirici: "Cantor diagonal argument blues", sci.math (27 Jan 2005)]

Frank Ramsey Suppose a contradiction were to be found in the axioms of set theory. Do you seriously believe that a bridge would fall down? [F. Ramsey, quoted in D. MacHale: "Comic sections", Dublin (1993)]

Harold Ravitch Gödel, in the "Supplement to the Second Edition" of "What is Cantor's Continuum Problem?" remarked that a physical interpretation could not decide open questions of set theory, i.e. there was (at the time of his writing) no "physical set theory" although there is a physical geometry. [H. Ravitch: "On Gödel's philosophy of mathematics" Chapter I (1998)]

Charles Bernard Renouvier Renouvier, *in the interest of the PC* {{Principle of Contradiction}}, fought a life long battle against "completed" infinity [...] for him, the rejection of infinity was just a special expression of the PC. [E. Conze: "Der Satz vom Widerspruch" (1932), "The principle of contradiction", Lexington Books (2016) p. 304]

John Ringland Furthermore, in the context of computational metaphysics, representation is equivalent to existence. If something is represented and it takes part in the overall simulation of the universe then it exists in that universe but if it cannot be represented then it cannot exist. So if actual infinities exist then there cannot be any discrete computational foundation to reality but so far no actual infinites have ever been discovered. Even with the domain of pure mathematics, infinities can only exist because they are symbolically represented and never actually represented. No one has ever written out an infinite number of integers thereby actually representing the set of integers. It is only ever referred to but never fully represented. If one required sets to be fully represented then mathematics could not operate on actual infinite sets; it could only operate on potentially infinite sets which always have finite representations (e.g. {1, 2, 3}) but which are unlimited in their length. Such sets are arbitrarily large but always have a definite finite size. [J. Ringland: "Does infinity exist?"]

Peter Ripota The measure of all sets with countably many elements is 0; the measure of all sets with uncountably many elements is 1. In between and beyond there is nothing. Using the axiom of choice this immediately leads to a contradiction. For instance it is possible to exhaust the circumference of the circle by means of this controversial (but necessary) axiom in such a clever way that a set is created, the measure of which is either 0 or $1\cdot\infty = \infty$. However, that cannot be because the circumference of the circle is well known to have measure π. Speedily the mathematicians weaseled out in the usual manner by asserting: There are nonmeasurable sets. What those sets are they will decide if once again it does not work. [P. Ripota: "Von ρ zu Ω"]

Tom Robertson How can an infinite list, such as any which is generated by any function which maps natural numbers to real numbers, be completed? If it can't, and if it therefore can't be "Cantorized", how can it prove that there are real numbers not on the list? [T. Robertson: "A loophole in Cantor's argument?", sci.math (23 Apr 1999)]

Abraham Robinson (i) Infinite totalities do not exist in any sense of the word (i.e., either really or ideally). More precisely, any mention, or purported mention, of infinite totalities is, literally, *meaningless*. (ii) Nevertheless, we should continue the business of Mathematics "as usual", i.e., we should act as *if* infinite totalities really existed. [...]

I feel quite unable to grasp the idea of an actual infinite totality. To me there appears to exist an unbridgeable gulf between sets or structures of one, or two, or five elements, on one hand, and infinite structures on the other hand [...] I must regard a theory which refers to an infinite totality as *meaningless* in the sense that its terms and sentences cannot posses the direct interpretation in an actual structure that we should expect them to have by analogy with concrete (e.g., empirical) situations. [A. Robinson: "Formalism 64", North-Holland, Amsterdam, p. 230f]

The great fascination that contemporary mathematical logic has for its devotees is due, in large measure, to the ever increasing sophistication of its techniques rather than to any definitive contribution to our understanding of the foundations of mathematics. Nevertheless, the achievements of logic in recent years are relevant to foundational questions and it behooves the logician, at least once in a while, to reflect on the basic nature of his subject and perhaps even to report his conclusions. [A. Robinson: "From a formalist's point of view", Dialectica 23,1 (1969) p. 45]

Brouwer's intuitionism is closely related to his conception of mathematics as a dynamic activity of the human intellect rather than the discovery of an immutable abstract universe. This is a conception for which I have some sympathy and which, I believe, is acceptable to many mathematicians who are not intuitionists. [A. Robinson quoted in J. Dauben: "Abraham Robinson. The creation of nonstandard analysis", Princeton University Press, Princeton, NJ (1995) p. 461]

Marc Rochow {{From the argument of the binary tree}} it follows: Georg Cantor's set theory is a product of fantasy, not in touch with reality and without any theoretical foundation. [M. Rochow: "Die Geschichte des Unendlichen" (17 Jul 2010)]

Brian Rotman The author's entry point is an attack on the notion of the mathematical infinite in both its potential and actual forms, an attack organized around his claim that any interpretation of "endless" or "unlimited" iteration is ineradicably theological. [B. Rotman: "Ad infinitum ... The ghost in Turing's machine", Stanford University Press (1993) Advertisement]

Carlo Rovelli The idea that the mathematics that we find valuable forms a Platonic world *fully independent of us* is like the idea of an Entity that created the heavens and the earth, and happens to very much resemble my grandfather. [C. Rovelli: "Michelangelo's stone: An argument against Platonism in mathematics", arXiv (2015) p. 6]

Juha Ruokolainen At least from the computational point of view we can only possess and process a finite amount of finitely precise information in a finitely long period of time. If we do not take into account any physical or other limitations to available space and time resources, then we may allow us to possess and process indefinitely large yet finite amount of indefinitely yet finitely precise information in an indefinitely yet finitely long period of time. [J. Ruokolainen: "Constructive nonstandard analysis without actual infinity", Dissertation, Univ. Helsinki (2004)]

Bertrand Russell Russell had formulated his Multiplicative Axiom, later recognized as equivalent to the Axiom of Choice, independently of Zermelo but had serious doubts about its truth. [G.H. Moore: "The origins of Zermelo's axiomatization of set theory", Journal of Philosophical Logic 7 (1978) pp. 307-329]

At first it seems obvious, but the more you think about it, the stranger the deductions from this axiom {{of Choice}} seem to become; in the end you cease to understand what is meant by it. [B. Russell quoted in N.Y. Vilenkin: "In search of infinity", Birkhäuser, Boston (1995) p. 123]

The cardinal contradiction is simply this: Cantor has a proof that there is no greatest cardinal, and yet there are properties (such as "$x = x$") which belong to *all* entities. Hence the cardinal number of entities having a property must be the greatest of cardinal numbers. Hence a contradiction. [p. 31] The objections to the theory are [...] that a great part of Cantor's theory of the transfinite, including much that it is hard to doubt, is, so far as can be seen, invalid if there are no classes or relations; [p. 45] An *existent* class is a class having at least one member. [p. 47] Whether it is possible to rescue more of Cantor's work must probably remain doubtful until the fundamental logical notions employed are more thoroughly understood. And whether, in particular, Zermelo's axiom is true or false is a question which, while more fundamental matters are in doubt, is very likely to remain unanswered. The complete solution of our difficulties, we may surmise, is more likely to come from clearer notions in logic than from the technical advance of mathematics; but until the solution is found we cannot be sure how much of mathematics it will leave intact. [p. 53] [B. Russell: "On some difficulties in the theory of transfinite numbers and order types", Proc. London Math. Soc. (2) 4 (1907)]

Charles Sauerbier This paper offers a contrary conclusion to Cantor's argument, together with implications to the theory of computation. [...] The conclusions Cantor offered are a non sequitur. [C. Sauerbier: "Cantor's problem", arXiv (2009) pp. 1 & 5]

Erdinç Sayan I want to argue that the set of real numbers is, like the set of natural numbers (or the set of counting numbers), denumerably infinite, not nondenumerably infinite. [E. Sayan: "Contra Cantor: How to count the 'uncountably infinite'", Academia (2 Sep 2016) link expired]

Vladimir Y. Sazonov The "vague" set F of feasible numbers intuitively satisfies the axioms $0 \in F$, $F + 1 \subseteq F$ and $2^{1000} \notin F$, [V.Y. Sazonov: "On feasible numbers", Logic and Computational Complexity (31 May 2005) Abstract]

Arthur Schönflies It is well kown that set theory has lead to certain logical paradoxes which more and more are considered as an extremely unpleasant addition. A *simple* resolution of the appearing contradictions has not yet occured, as far as I know. [...] Hitherto logicians have only operated with logical chains consisting of a finite number of conclusions. But since *Cantor* mathematicians operate with infinite chains; [A. Schönflies: "Über die logischen Paradoxien der Mengenlehre", Jahresbericht der Deutschen Mathematiker-Vereinigung (1906) pp. 19 & 23]

Peter Schurr When addition, multiplication, and raising to a power (and their inverse functions) cannot change the quantity of the considered set, then mathematics is no longer concerned. What is tried here, namely to extend the principle of relativity by means of so-called "cardinalities" to different infinities, and hereby to fetch the actual infinite into the realm of mathematics, appears to me not only very dubious but fatally false. [P. Schurr: "Cantors Ende", de.sci.mathematik (3 Jan 2006)]

Kurt Schütte If we define the real numbers in a strictly formal system, where only finite derivations and fixed symbols are permitted, then these real numbers can certainly be enumerated because the formulas and derivations on the basis of their constructive definition are countable. [K. Schütte: "Beweistheorie", Springer (1960)]

Hermann A. Schwarz Today I received by mail an offprint of "Mittheilungen zur Lehre vom Transfiniten" {{Communications concerning the theory of the transfinite}} with a handwritten dedication: "H.A. Schwarz in memory of our old friendship dedicated by the author." After having had the opportunity to go through it leisurely, I cannot conceal that it appears to me as a pathological aberration. What on earth have the Fathers of the Church to do with the irrational numbers? I really hope my fear might not come true, that our patient has left the straight and narrow like the poor Zöllner {{Johann Karl Friedrich Zöllner (1834-1882) was a professor of astrophysics who later got involved in depth in philosophical studies and after all became an adherent of spiritism}} who never found the way back to scientific business. The more I think over these cases the more I am forced to get aware of the similarity of symptoms. Might we manage to lead the poor young man back to serious work! Otherwise it will come to a bad end with him. [H.A. Schwarz, letter to K. Weierstrass (17 Oct 1887)]

Martin Semerád And this is the point – the real numbers can be written down as some binary records – but also as decadic ($10^{\mathbb{N}}$) – and also unaric (or $\mathbb{N}$-aric) records – and what you get are still real numbers – but now they are countable. Because cardinality of $\mathbb{R}$ depends on the representation! [M. Semerád in "Cantor was right!", sci.math (19 Jun 2005)]

Doron Shadmi Strictly speaking, Actual infinity is too strong to be used as an input. Potential infinity (which never reaches Actual infinity, and therefore cannot be completed) is the name of the game. [...] Cantor uses simultaneously two different models [...] that are clearly contradicting each other. Therefore this proof does not hold. *In short, the transfinite system does not exist.* [D. Shadmi: "Transfinities?", HSN Forum Mathematics (12 Nov 2004)]

Andrew P. Shane *I propose an algorithm that (in infinite time) will enumerate all of the real numbers between zero and one.* [A.P. Shane: "An algorithm for the enumeration of all real numbers between 0 and 1", Rensselaer Polytechnic Inst., Troy, NY (10 Apr 1999) link expired]

Gary Shannon I define the following function to be recursive [...] This procedure will put the integers in one-to-one correspondence with the reals between 0 and 1 without omitting any reals. [G. Shannon in "Cantor and the mad man", sci.math (12 Jan 2002)]

Jim N. Sharon How large is an infinite set? The size of such a set is infinity, but infinity is a concept that means "keeps going". The concept of a biggest number is meaningless, and therefore so is the concept of the size of an infinite set. [J.N. Sharon: "Infinite nonsense 2: Cantor" (1 Sep 2016) link expired]

Saharon Shelah When modern set theory is applied to conventional mathematical problems, it has a disconcerting tendency to produce independence results rather than theorems in the usual sense. The resulting preoccupation with "consistency" rather than "truth" may be felt to give the subject an air of unreality. [S. Shelah: "Cardinal arithmetic for skeptics", arXiv (1992) p. 1]

Stephen G. Simpson *Ultrafinitism*: Infinities, both potential and actual, do not exist and are not acceptable in mathematics. *Finitism*: Potential infinities exist and are acceptable in mathematics. Actual infinities do not exist and we must limit or eliminate their role in mathematics. *Predicativism*: We may accept the natural numbers, but not the real numbers, as a completed infinite totality. Quantification over ℕ is acceptable, but quantification over ℝ, or over the set of all subsets of ℕ, is unacceptable. *Infinitism*: Actual infinities of all kinds are welcome in mathematics, so long as they are consistent and intuitively natural.

Of these four positions, the finitist one seems to be the most objective. [...]

The real world contains many potential infinities. Examples are: counting, the rotation of the earth about the sun, the human reproduction cycle, division of a piece of metal into smaller pieces, the accumulation of wealth, etc. However, the real world does not appear to contain any actual infinities. For this reason, actual infinities are suspect. In order to maintain objectivity in mathematics, it seems necessary to limit the use of actual infinities. We introduce them only as "*convenient fictions*". [...] There is no clear way to integrate ZFC-based mathematics with the rest of human knowledge. The unity of mathematics is good, but the unity of human knowledge would be even better. [...] both set theory and religious faith can claim to be in a "strong" position vis a vis skeptics, by avoiding reliance on facts which can be questioned. I reject such claims on grounds of lack of objectivity. [...]

The bulk of core mathematical theorems fall at the lowest levels {{in reverse mathematics}}. This suggests that higher set theory may be largely irrelevant. [...] A large portion of core mathematics, sufficient for applications, is validated in a finitistically provable way. [S.G. Simpson: "Potential versus actual infinity: insights from reverse mathematics" (2015)]

Thoralf Skolem There is no possibility of introducing something absolutely uncountable, but by a pure dogma. [T. Skolem: "Über einige Grundlagenfragen der Mathematik", Norske Videnskaps-Akademi, Mat.-naturv. Klasse No. 4, Oslo (1929)]

In order to obtain something absolutely nondenumerable, we would have to have either an absolutely nondenumerably infinite number of axioms or an axiom that could yield an absolutely nondenumerable number of first-order propositions. [...] It is easy to show that Zermelo's axiom system is not sufficient to provide a complete foundation for the usual theory of sets. [p. 296] I believed that it was so clear that axiomatization in terms of sets was not a satisfactory ultimate foundation of mathematics that mathematicians would, for the most part, not be very much concerned with it. But in recent times I have seen to my surprise that so many mathematicians think that these axioms of set theory provide the ideal foundation for mathematics; therefore it seemed to me that the time had come to publish a critique." [p. 300f] [T. Skolem: "Some remarks on axiomatized set theory" (1922) quoted in J. van Heijenoort: "From Frege to Gödel – A source book in mathematical logic, 1879-1931", Harvard University Press, Cambridge (1967)]

He {{Poincaré}} is reported to have said in a talk at the international congress of mathematicians at Rome in 1908 that in future set theory would be considered as a disease from which one has recovered. [...] It seems indeed that Hilbert wants to maintain Cantor's opinions in their old absolutistic sense. I find that rather strange. It is indicative that he never felt the necessity to consider the relativism that I have proven for every finitely formulated set axiomatics. He also said that he did not wish to be expelled from Cantor's paradise. It is curious to compare this statement with that one mentioned earlier, that set theory is a disease. [...] Firstly I have given a more precise grounds for the general set theoretic relativism, which has the particular consequence that the absolutely uncountable has no justification on an axiomatic basis. [T.

Skolem: "Über die Grundlagendiskussionen in der Mathematik", Den Syvende Skandinav. Matematikerkongr. Oslo (1929). "Selected Works in Logic", J.E. Fenstak (ed.), Scand. Univ. Books, Universitetsforlaget, Oslo (1970)]

Presumably, the reason for this division of the contents of "Einleitung in die Mengenlehre" into two different books is that the subject matter has grown too large. The reviewer, however, is not enthusiastic about this division since such a textbook as the present one will be read primarily by students and they might form the impression that classical set theory is securely founded just as other parts of mathematics, e.g. arithmetic. Such an impression would, however, be misleading. If it were not so, we could omit the entire modern foundational research without real loss to mathematics. To the reviewer it seems unfortunate that classical set theory is developed in a separate book so that all scruples – or almost all of them – are reserved for the second volume. This might have the effect that most readers of this present volume will probably not become acquainted with the criticisms at all. It is true that some hints to such scruples are given, but most students might not think that they are important. On the other hand, it must be conceded that the lack of knowledge of the results of foundational research will not mean much to mathematicians who are not especially interested in the logical development of mathematics. [T. Skolem: "Review of A.A. Fraenkel: Abstract set theory", Mathematica Skandinavica 1 (1953) p. 313]

Hartley Slater The key question therefore is: if there is a determinate number of natural numbers, then by what process is it determined? Replacing "the number of natural numbers" with "Aleph zero" does not make its reference any more determinate. The natural numbers can be put into one-one correspondence with the even numbers, it is well known, but does that settle that they have the same number? We have equal reason to say that they have a different number, since there are more of them. [H. Slater: "The uniform solution of the paradoxes" (2004) link expired]

Jim Slattery If, as I think most physicists tend to believe, space is impossible without matter then the states of the universe are limited to the possible permutations of the atoms in the universe and their internal structure. This appears to be finite and it is difficult to state that a number that could not be represented in the universe is real. [J. Slattery in "Does actual infinity exist?", ResearchGate (7 Jan 2013)]

Jerzy Slupecki I understand free variables, for me they are places for substituting individuals. But I do not understand quantifiers since they refer often to actually infinite universes of abstract objects and I do not believe in the existence of such universes. [J. Slupecki, private communication reported in J. Mycielski: "On the tension between Tarski's nominalism and his model theory (definitions for a mathematical model of knowledge)", Annals of Pure and Applied Logic 126 (2004) pp. 215-224]

Lawrence Spector For numbers to be useful to calculus and science, they must have names; the word number must have its customary meaning. [L. Spector: "Are the real numbers really numbers?", TheMathPage (2015) link expired]

Rudolf Sponsel Clearly that {{infinite set}} does not exist as a completed and finished entity, except as a self-contradictory delusion – and that has only a delusional reality content. [R. Sponsel in "Ist das schon 'mal jemandem aufgefallen?", de.sci.mathematik (23 Dec 2015)]

Radhakrishnan Srinivasan Physics today has been overrun by modern mathematicians, who have plugged in infinitary reasoning wholesale into theories of Physics. I am sure the old-timers, including mathematical physicists like Poincare, will be turning in their graves at this development. [R. Srinivasan in "Cantor's uncountability theory explains Casimer effect?", sci.math (8 Apr 2011)]

Radhakrishnan Srinivasan, H.P. Raghunandan Infinite sets are not permitted in NPAR and quantification over proper classes is banned; hence Cantor's diagonal argument cannot be legally formulated in NRA, and there is no 'cardinality' for any collection ('super-class') of real numbers. [R. Srinivasan, H.P. Raghunandan: "Foundations of real analysis and computability theory in non-Aristotelian finitary logic", arXiv (2005) Abstract]

Andrew Stanworth Cantor's assumption of the existence of a 'simply infinite sequence' leads to a clear contradiction and must, therefore, be false. [A. Stanworth: "The natural numbers are uncountable", sci.logic (5 Jul 2004)]

Christian Stapfer [...] considered from outside "existence" as deductibility in ZFC can always supply only countable objects. [C. Stapfer in "[WM] Der Cantorsche Satz", de.sci.mathematik (13 Jun 2005)]

Don Stockbauer The answer to Cantor is the cybernetic one, that there is just the potential infinity, which can exist in the physical world, e.g., when you program a spacecraft to report once a day "forever", knowing that physical processes will end it, and then there's the actualized infinity, a mathematical playtoy, which cannot fit into the real world. [D. Stockbauer in "Update: Objections to Cantor's theory", sci.math (18 Aug 2005)]

Gabriel Stolzenberg I am of the opinion – and I share this opinion with the other so-called "constructivists" among the mathematicians – that the science of pure mathematics in the last part of the 19th century has been caught in a certain intellectual trap during its efforts to give itself a strict form and to create appropriate foundations, and that the mathematicians since that time, supported by logicians, have gotten more and more tangled in that trap. I would like to show the constitution of this trap: how it is composed from certain structures of logic and language, why it is so easy to fall into the trap, and what happens when having been trapped. As a mathematician I would also like to do something to rescue my discipline from this trap. But that is another story. Further you should know that what I call "a trap" by most pure mathematicians, who look at it from inside, is considered as an intellectual paradise, and in fact this is not a contradiction. [G. Stolzenberg: "Kann die Untersuchung der Grundlagen der Mathematik uns etwas über das Denken sagen?" in P. Watzlawick (ed.): "Die erfundene Wirklichkeit. Wie wissen wir, was wir zu wissen glauben?", Piper, München (1985)]

Albrecht Storz After having in depth scrutinized Dr. Mückenheim's arguments and having seen also several interesting sites by other authors in the internet [...] I come to the conclusion that there are definitely properly thought-out and worked-out attacks on the concept of transfinite numbers. As far as I can judge, these studies in general come to the conclusion that the assumption of concepts of actual infinity raises contradictions which again and again have to be compensated by additional assumptions in order to obtain a consistent concept (the universal set is excluded by an axiom, the power set of infinite sets is accepted, but the sum of all natural numbers, for instance, is declared as non-existent, and so on). [A. Storz: "Hat Cantor doch geirrt?", de.sci.mathematik (7 Jan 2005)]

Peter Suber [...] the Skolem paradox would create a serious contradiction [...] Most mathematicians agree that the Skolem paradox creates no contradiction. But that does not mean they agree on how to resolve it. [P. Suber: "The Löwenheim-Skolem theorem" (1997)]

Alfred Tarski People have asked me, "How can you, a nominalist, do work in set theory and logic, which are theories about things you do not believe in?" ... I believe that there is value even in fairy tales and the study of fairy tales. [A. Burdman-Feferman, S. Feferman: "Alfred Tarski – life and logic", Cambridge Univ. Press (2004) p. 52]

Rudolf Taschner To talk about the infinite, even to talk about numbers is, according to Hilbert, a mere façon de parler – like the chess queen shorn of all female features is nothing but a worthless piece. And like pawns and Kings on the chess game neither cultivate fields nor rule countries but are silent stage extras, rigid pieces in the hands of the chess players, we find the situation of modern mathematics: Its calculations concern meaningless symbols, its theorems have no relation to reality, its problems apply to a thinking that autistically is revolving around itself. Only a marginal share of so-called constructive mathematicians overcome the meaningless language game in that they, following the eminent scholars Brouwer and Weyl, deliberately put mathematics into the dialectics between the naive ideal of the absolute claim to truth and the serene deconstruction. Most mathematicians however accept the message as the final say that modern mathematics consists of idle chatter. And it is miraculous to see how they feel happy in this remaining ruin. [R. Taschner: "Das Unendliche ist nur ein Wort – oder irren die Mathematiker womöglich?", Die Presse, Wien (18 Jan 2012)]

To appropriately encounter the infinite requires to refrain from the mania to subjugate the infinite like a given whole as an object to mathematics – like the number. The infinite is rather a limit notion that in a fundamental way evades human thirst of knowledge. [R. Taschner: "Der Zahlen gigantische Schatten", 4th ed., Springer, Wiesbaden (2017) p. 209]

Neil Tennant A constructivist version of a mathematical theory is adequate for all the applications to be made of the theory within natural science: [N. Tennant: "Logic, mathematics, and the natural sciences", ResearchGate (2007)]

Christian Thiel The constructivistic foundational critic has found the following faults: First it is not allowed to start with the assumption that the collection of real numbers is a set. [...] The definition of the binary sequence d^* relies on the totality of all binary sequences including the infinite binary sequence d^*, since the construction has to be done for all b_{kk}. So the given receipe of construction is even contradictory, because it requires to construct a binary sequence which differs from all binary sequences, in particular from itself. This contradiction sheds as much doubt on the assumption of the existence of the set of all binary sequences as on the assumption of their countability. [C. Thiel: "Philosophie und Mathematik", Wissenschaftliche Buchgesellschaft, Darmstadt (1995) p. 197f]

Chris M. Thomasson All I can say is that .999... will never equal 1 with respect to infinite precision. [...] 1 - 0.999... will always be greater than zero with respect to infinite precision. [C.M. Thomasson in "The common mistake", sci.math (10 Oct 2016)]

Wolfgang Thumser It is clear that the set of all numbers having a finite description is countable [...] Hence, forgetting all "moonshine" real numbers and restricting ourselves to the set *M* of all finitely describable numbers, we might possibly come to the conclusion that *M*, although being countable, is not finitely describably countable. For the friend of finite descriptions the classical "uncountability" is therefore not expressed, in case of *M*, by its giant magnitude but rather by its undescribable counting-sequence. [W. Thumser in "Das Kalenderblatt 100117", de.sci.mathematik (23 Jan 2010)]

I know that many – how do you say? – "Mengenlehrer" {{set-teachers}} imagine the natural numbers as kind of sack containing everything that never has been inserted. This, in my opinion, philosophical nonsense (with all its consequences recognized by you, including the potential and actual nonsense) is owed to an antiquated interpretation of the formal logical propositions, going back to Cantor's definition as a collection. [W. Thumser in "Anwort an Kluto", de.sci.mathematik (9 Nov 2008)]

William P. Thurston On the most fundamental level, the foundations of mathematics are much shakier than the mathematics that we do. Most mathematicians adhere to foundational principles that are known to be polite fictions. For example, it is a theorem that there does not exist any way to ever actually construct or even define a well-ordering of the real numbers. There is considerable evidence (but no proof) that we can get away with these polite fictions without being caught out, but that doesn't make them right. Set theorists construct many alternate and mutually contradictory "mathematical universes" such that if one is consistent, the others are too. This leaves very little confidence that one or the other is the right choice or the natural choice. [W.P. Thurston: "On proof and progress in mathematics", arXiv (1994) p. 10]

Ray Tomes [...] mathematicians tell me that there are an infinite number of integers which do not have an infinite number of digits in them. I say that is impossible because for any given maximum chosen integer there are exactly that many smaller integers and so only for an infinite maximum are there an infinite number of smaller ones. [R. Tomes: "I say Cantor was wrong!", sci.math (5 Jan 1999)]

Shack Toms It seems that the Cantor proof, at least in that form, might only be proving that the position on the list cannot be made definite. [S. Toms: "Cantor's diagonalization", sci.math (13 Apr 1997)]

Salvatoris Tongiorgi Let *X* be the origin of a line. Are there points in the line which have an infinite distance from *X* or not? If no such points are given the line is finite. [...] *Actually infinite multitudes are inconsistent*. [S. Tongiorgi: "Institutionis philosophicae", 4th ed., Goemaere, Brussels (1868) p. 171f]

Gert Treiber The discrepancy between cardinality and ordinality in the transfinite raised in me serious doubts about Cantor's theory. [G. Treiber: "Die Grundlagenkrise der Mathematik – Ein Wissenschaftsskandal", tredition, Hamburg (2014) p. 41]

Jim Trek Calculus does not need any concept of infinity in order to provide limits, derivatives, integrals, and differentials. [J. Trek: "Finitists", sci.math (5 Feb 1999)]

Toru Tsujishita Nonstandard mathematics shows new ways of making mathematical discourse more intuitive without losing logical rigor and giving more flexible ways of constructing mathematical objects. We may say that by discriminating between "actual finiteness" and "ideal finiteness", we obtain a better system of handling infinity than the "actual infinity" offers. [T. Tsujishita: "Alternative mathematics without actual infinity", arXiv (2012) p. 6]

Valentin F. Turchin For *actual infinity* we have no place in our system of concepts. On the intuitive level, we cannot imagine anything that would qualify as actual infinity, because neither we, nor our evolutionary predecessors ever had anything like that in experience. When we try to imagine something infinite, e.g., infinite space, we actually imagine a process of moving from point to point without any end in sight. This is potential, not actual, infinity. [...] Thus we cannot use the concept of actual infinity at all. [V.F. Turchin: "Infinity", Principia Cybernetica Web (Sep 1991)]

Bill Vallicella The numbers are not 'out there' waiting to be counted; they are created by the counting. In that sense, their infinity is merely potential. [B. Vallicella: "On potential and actual infinity", Maverick Philosopher (4 Aug 2010)]

Rod Vance I certainly like the idea that finite computations are more fundamental than sets and by "constructivist" I probably emphasise less the intuitionist logic and more the idea of things being defined by a finite "computation" (constructive proof). [R. Vance in "Does the Banach-Tarski paradox contradict our understanding of nature?", Physics.StackExchange (9 Aug 2013)]

K. Vela Velupillai Ostensibly, Cantor won the intellectual battle, but only 'temporarily'. [K.V. Velupillai: "Freedom, anarchy and conformism in academic research", ASSRU (Sep 2011) p. 7]

Peter Velzen Infinity is not a number, it is a process that has no end. So an infinit amount of anything can never exist. [P. Velzen in "Difficulties with real numbers as infinite decimals II", YouTube (8 May 2012)]

Naum Yakovlevich Vilenkin The most contentious aspect of set theory is the attempt to construct all mathematics on a set-theoretic base (the so-called "bourbakism"). Many scholars hold radically different views on this issue. [N.Y. Vilenkin: "In search of infinity", Birkhäuser, Boston (1995) p. 135]

Slavica Vlahovic, Branislav Vlahovic The main part of the paper is devoted to show that the real numbers are denumerable. [S. Vlahovic, B. Vlahovic: "Countability of the real numbers", arXiv (2004) p. 2]

Gilbert Voeten Why do the fans of Cantor always insult their antagonists to be crank? After all Cantor himself was crank. Who thinks to have overcome Infinity must be crank. That Cantor was crank is a historical fact. [G. Voeten in "! Cantor", sci.math (27 Apr 1999)]

Vladimir Voevodsky [...] mathematics is on the verge of a crisis, or rather, two crises. The first is connected with the separation of "pure" and applied mathematics. It is clear that sooner or later there will be a question about why society should pay money to people who are engaged in things that do not have any practical applications. [V. Voevodsky in "Интервью Владимира Воеводского" (1 Jul 2012), translated by J. Baez]

Petr Vopenka Probably the most convincing answer was given by P. Vopenka in his Alternative Set Theory – [...] According to him, actual infinity is only an illusion and the whole classical set theory can be embedded into AST where actual infinity does not exist. [V. Novak in "Does actual infinity exist?", ResearchGate (13 Oct 2012)]

Adriaan van der Walt This is motivated in part one by showing that Cantor's famous diagonal proof, which is an algebraic formulation of Euclidean Cosmology, rests on the fallacy that an infinite decimal fraction can be identified by specifying its finite digits. The argument of this part includes a proof that the set of equivalence classes of Cauchy sequences is countable. [A. van der Walt: "Cantor's fallacy and the Leibnizean cosmology" viXra (2015)]

Gertrud Walton In earlier versions of this webpage I had included a longer argument to show that Cantor's diagonal procedure fails. The fallaciousness of Cantor's abstract procedure, on careful inspection, is immediately evident; the simplest possible presentation should therefore suffice. [...] For a collection of classical mathematics (potential infinite only) we should say that the collection has n columns and 2^n rows. But even ignoring this classical restriction, it is clear that, for an "infinite" collection also, the diagonal traverses only a small part of the collection. [G. Walton: "Cantor's diagonal: an instance of the absurd fallaciousness of abstract procedure", sapere aude (19 Jul 2012)]

Nik Weaver ZFC is not a good choice to be the standard foundation for mathematics. It is unsuitable in two ways. Philosophically, it makes sense only in terms of a vague belief in some sort of mystical universe of sets which is supposed to exist aphysically and atemporally (yet, in order to avoid the classical paradoxes, is somehow "not there all at once"). Pragmatically, ZFC fits very badly with actual mathematical practice insofar as it postulates a vast realm of set-theoretic pathology which has no relevance to mainstream mathematics. We might say that it is both theoretically and practically unsuited to the foundational role in which it is currently cast. [N. Weaver: "Is set theory indispensable?" (2009) p. 17]

Curt Welch When I was shown Cantor's diagonal proof that the number of reals was not countable back in college, I thought it was a fascinating proof. It seemed to uncover some great mystery about the nature of numbers that was not at first obvious. It sounded very logical and I quickly embraced it as fact. Lately however, I've come to see things very differently. I now believe the proof is totally bogus. And the huge body of work built on top of the concept is likewise, totally bogus. [C. Welch: "Cantor's diagonal proof wrong?", sci.math (14 Nov 2004)]

Eduard Wette The author aims [...] to make cognizant numerically the exterior as well as the interior finite limits of controllably regulated mathematics [...] At a time where metamathematical considerations still are based on set theory it may appear bold or foolish if one tackles on such a low level to criticize not only the ideas of "actual" infinities in classical mathematics but also the attitude to the "potentially" infinite in constructivistic mathematics and to suspect them as self-deceptions by "verbally" suggested topics. [E. Wette: "Vom Unendlichen zum Endlichen", Dialectica 24,4 (1970) pp. 303f & 321]

Hermann Weyl Cantor's notion of *countability* depends on the sequence of natural numbers. This notion is known to have caused *Richard's antinomy*. Its common version reads: The possible combinations of finitely many letters form a countable set, and since every determined real number must be definable by a finite number of words, there can exist only countably many real numbers – in contradiction to Cantor's classical theorem and its proof. [H. Weyl: "Das Kontinuum", Veit, Leipzig (1918) p. 18]

Brouwer made it clear, as I think beyond any doubt, that there is no evidence supporting the belief in the existential character of the totality of all natural numbers, and hence the principle of excluded middle in the form "Either there is a number of the given property γ, or all numbers have the property ¬γ" is without foundation. [...] The sequence of numbers which grows beyond any stage already reached by passing to the next number, is a manifold of possibilities open towards infinity; it remains forever in the status of creation, but is not a closed realm of things existing in themselves. That we blindly converted one into the other is the true source of our difficulties, including the antinomies – a source of more fundamental nature than Russell's vicious circle principle indicated. Brouwer opened our eyes and made us see how far classical mathematics, nourished by a belief in the "absolute" that transcends all human possibilities of realization, goes beyond such statements as can claim real meaning and truth founded on evidence. According to this view and reading of history, classical logic was abstracted from the mathematics of finite sets and their subsets. (The word finite is here to be taken in the precise sense that the members of such set are explicitly exhibited one by one.) Forgetful of this limited origin, one afterwards mistook that logic for something above and prior to all mathematics, and finally applied it, without justification, to the mathematics of infinite sets. This is the Fall and original sin of set-theory, for which it is justly punished by the antinomies. Not that such contradictions showed up is surprising, but that they showed up at such a late stage of the game! Thanks to the notion of "*Wahlfolge*", that is a sequence in *statu nascendi* in which one number after the other is *freely chosen* rather than determined by law, Brouwer's treatment of real variables is in the closest harmony with the intuitive nature of the continuum; this is one of the most attractive features of his theory. But on the whole, Brouwer's mathematics is less simple and much more limited in power than our familiar "existential" mathematics. It is for this reason that the vast majority of mathematicians hesitate to go along with his radical reform. [H. Weyl: "Levels of infinity: Selected writings on mathematics and philosophy", Peter Pesic (ed.), Dover Publiations (2012) p. 140f]

The leap into the beyond occurs when the sequence of numbers that is never complete but remains open toward the infinite is made into a closed aggregate of objects existing in themselves. Giving the numbers the status of ideal objects becomes dangerous only when this is done. [p. 38] It cannot be denied, however, that in advancing to higher and more general theories the inapplicability of the simple laws of classical logic eventually results in an almost unbearable awkwardness. And the mathematician watches with pain the larger part of his towering edifice which he believed to be built of concrete blocks dissolve into mist before his eyes. [p. 54] In a recent appraisal of Russell's contribution to mathematical logic he {{Gödel}} says that the paradoxes reveal "the amazing fact that our logical intuitions are self-contradictory". I confess that in this respect I remain steadfastly on the side of Brouwer who blames the paradoxes not on some transcendental logical intuition which deceives us but on an error inadvertendly commited in the passage from finite to infinite sets. [p. 234] [H. Weyl: "Philosophy of mathematics and natural science", Princeton Univ. Press (2009)]

Thomas Whichello Cantor came to his beliefs about infinity owing to the fact that mathematics had not yet been soundly analyzed on a grammatical basis. Once the nouns are clearly distinguished from the adjectives, many of the inferences which he derived from his technically correct proofs become invalidated, and our traditional and intuitive conception of infinity is restored. [T. Whichello: "An attempted refutation of Georg Cantor's inferences" (2018)]

Bruno Whittle In §1 I will give an initial argument against the claim that Cantor established that there are infinite sets of different sizes. [B. Whittle: "On infinite size" in K. Bennet, D.W. Zimmerman (eds.): "Oxford studies in metaphysics, vol. 9", Oxford Univ. Press (2015) p. 3]

Norman J. Wildberger *Does mathematics require axioms?*

Occasionally logicians inquire as to whether the current 'Axioms' need to be changed further, or augmented. The more fundamental question – *whether mathematics requires any Axioms* – is not up for discussion. That would be like trying to get the high priests on the island of Okineyab to consider not whether the Divine Ompah's Holy Phoenix has twelve or thirteen colours in her tail (a fascinating question on which entire tomes have been written), but rather whether the Divine Ompah exists at all. Ask *that* question, and icy stares are what you have to expect, then it's off to the dungeons, mate, for a bit of retraining.

Mathematics does not require 'Axioms'. The job of a pure mathematician is not to build some elaborate castle in the sky, and to proclaim that it stands up on the strength of some arbitrarily chosen assumptions. The job is to *investigate the mathematical reality of the world in which we live*. For this, no assumptions are necessary. Careful observation is necessary, clear definitions are necessary, and correct use of language and logic are necessary. But at no point does one need to start invoking the existence of objects or procedures that we cannot see, specify, or implement.

The difficulty with the current reliance on 'Axioms' arises from a grammatical confusion, [...] People use the term 'Axiom' when often they really mean *definition*. Thus the 'axioms' of group theory are in fact just definitions. We say exactly what we mean by a group, that's all. [...]

And yes, all right, the Continuum hypothesis doesn't really need to be true or false, but is allowed to hover in some no-man's land, falling one way or the other depending on *what you believe*. Cohen's proof of the independence of the Continuum hypothesis from the 'Axioms' should have been the long overdue wake-up call. In ordinary mathematics, statements are either true, false, or they don't make sense. If you have an elaborate theory of 'hierarchies upon hierarchies of infinite sets', in which you cannot *even in principle* decide whether there is anything between the first and second 'infinity' on your list, *then it's time to admit that you are no longer doing mathematics*.

Whenever discussions about the foundations of mathematics arise, we pay lip service to the 'Axioms' of Zermelo-Fraenkel, but do we ever use them? Hardly ever. With the notable exception of the 'Axiom of Choice', I bet that fewer than 5% of mathematicians have ever employed even one of these 'Axioms' explicitly in their published work. The average mathematician probably can't even remember the 'Axioms'. [...]

Sequences generated by algorithms can be specified by those algorithms, but what possibly could it mean to discuss a 'sequence' which is not generated by such a finite rule? Such an object would contain an 'infinite amount' of information, and there are no concrete examples of such things in the known universe. This is metaphysics masquerading as mathematics. [N.J. Wildberger: "Set theory: Should you believe?" (2005)]

This is a valuable initiative! Thanks for putting up these interesting quotes. Many I have not seen before. Let's hope that we can start to generate some critical mass so that people start to have the courage to face the music: modern pure mathematics is in serious logical strife, and the faster we acknowledge it and move forward, the better. This is especially important for young people: if you are a student, make sure you start to give these issues some serious thought! You are the generation that will need to move mathematics to a better place. [N.J. Wildberger: "Comment on 'Kritik der transfiniten Mengenlehre'", Facebook (7 Jan 2015)]

Ludwig Wittgenstein If it were said: "Consideration of the diagonal procedure shews you that the concept 'real number' has much less analogy with the concept 'cardinal number' than we, being misled by certain analogies, inclined to believe", that would have a good and honest sense. But just the *opposite* happens: one pretends to compare the "set" of real numbers in magnitude with that of cardinal numbers. The difference in kind between the two conceptions is represented, by a skew form of expression, as difference of extension. I believe, and I hope, that a future generation will laugh at this hocus pocus. [II.22]

Imagine set theory's having been invented by a satirist as a kind of parody on mathematics. – Later a reasonable meaning was seen in it and it was incorporated into mathematics. (For if one person can see it as a paradise of mathematicians, why should not another see it as a joke?) [V.7]

The curse of the invasion of mathematics by mathematical logic is that now any proposition can be represented in a mathematical symbolism, and this makes us feel obliged to understand it. Although of course this method of writing is nothing but the translation of vague ordinary prose. [V.46]

"Mathematical logic" has completely deformed the thinking of mathematicians and of philosophers, by setting up a superficial interpretation of the forms of our everyday language as an analysis of the structures of facts. Of course in this it has only continued to build on the Aristotelian logic. [V.48] [L. Wittgenstein: "Remarks on the foundations of mathematics", Wiley-Blackwell (1991)]

The expression "and so on" is nothing but the *expression* "*and so on*". [p. 282]

There is no such thing as "the cardinal numbers", but only "cardinal numbers" and the concept, the form "cardinal number". Now we say "the number of the cardinal numbers is smaller than the number of the real numbers" and we imagine that we could perhaps write the two series side by side (if only we weren't weak humans) and then the one series would end in endlessness, whereas the other would go on beyond it into the actual infinite. But this is all nonsense. [p. 287]

In mathematics description and object are equivalent. "The fifth number of the number series has these properties" says *the same* as "5 has these properties". The properties of a house do not *follow* from its position in a row of houses; but the properties of a number are the properties of a position. [p. 457] [L. Wittgenstein: "Philosophical grammar", Basil Blackwell, Oxford (1969)]

[…] there is no path to infinity, *not even an endless one*. [...] All right, the path must be endless. But if it is endless, then that means precisely that you can't walk to the end of it. [§ 123]

It isn't just impossible "for us men" to run through the natural numbers one by one; it's *impossible*, it means nothing. [...] you can't talk about *all* numbers, because there's no such thing as *all* numbers. [§ 124]

There's no such thing as "all numbers" simply because there are infinitely many. [§ 126]

The infinite number series is only the infinite possibility of finite series of numbers. It is senseless to speak of the *whole* infinite number series, as if it, too, were an extension. [...] If I were to say "If we were acquainted with an infinite extension, then it would be all right to talk of an actual infinite", that would really be like saying, "If there were a sense of abracadabra then it would be all right to talk about abracadabraic sense perception". [§ 144]

Set theory is wrong because it apparently presupposes a symbolism which doesn't exist instead of one that does exist (is alone possible). It builds on a fictitious symbolism, therefore on nonsense. [§ 174] [L. Wittgenstein: "Philosophical remarks", Wiley-Blackwell (1978)]

Klaus D. Witzel Prof. Dr. Wolfgang Mückenheim has advanced the received mathematics of infinity from the greatest of their time like no one before him, and we all shall be deeply grateful that peer reviewed science has made this long awaited progress against superstition during our lifetime. [K.D. Witzel: "Review of W. Mückenheim: Mathematik für die ersten Semester", Amazon (12 May 2015)]

Dan Wood I have a method of mapping a subset of the integers, with a many to one relationship, to the reals thereby demonstrating that there are more integers than reals in that, with my mapping, there are an infinite number of integers which map to each real. [D. Wood: "Cantor and the mad man", sci.math (14 Jan 2002)]

Andy Wright If you allow leading zeros to your real numbers, you can have as many of them as you like, so 3 could be written as 03, 003, or even ...0003, i.e. every natural number can be written as a unique infinite string of digits, just like the real numbers. What's stopping anybody then from proving, using Cantor's diagonalisation argument, that the infinity of the naturals is as large as the infinity of the reals? [A. Wright in "Proof – There are more real numbers than natural numbers", YouTube (29 May 2009)]

Feng Ye This book [...] also shows that the applications of those classical theories to the finite physical world can be translated into the applications of strict finitism, which demonstrates the applicability of those classical theories without assuming the literal truth of those theories or the reality of infinity. [F. Ye: "Strict finitism and the logic of mathematical applications", Springer (2011)]

Alexander S. Yessenin-Volpin I [...] developed to a considerable extent the Anti-traditional, program of Foundations of Mathematics aimed at the banishing of beliefs from Foundations. [...] I have, starting in 1973-75, changed the original approach [...] and replaced it by a new approach which can be considered as "finitistic" in a sense close to that once specified by J. Herbrand but now additionally specified in accordance with the prototheories. [A.S. Yessenin-Volpin: "About infinity, finiteness and finitization (in connection with the foundations of mathematics)", Springer, Lecture notes in mathematics, Vol. 873 (2006) p. 274f]

I have seen some ultrafinitists go so far as to challenge the existence of 2^{100} as a natural number, in the sense of there being a series of "points" of that length. There is the obvious "draw the line" objection, asking where in $2^1, 2^2, 2^3, ..., 2^{100}$ do we stop having "Platonistic reality"? Here this ... is totally innocent, in that it can easily be replaced by 100 items (names) separated by commas. I raised just this objection with the (extreme) ultrafinitist Yessenin-Volpin during a lecture of his. He asked me to be more specific. I then proceeded to start with 2^1 and asked him whether this is "real" or something to that effect. He virtually immediately said yes. Then I asked about 2^2, and he again said yes, but with a perceptible delay. Then 2^3, and yes, but with more delay. This continued for a couple of more times, till it was obvious how he was handling this objection. Sure, he was prepared to always answer yes, but he was going to take 2^{100} times as long to answer yes to 2^{100} then he would to answering 2^1. There is no way that I could get very far with this. [H.M. Friedman: "Philosophical Problems in Logic", Seminar Notes (2002)]

Eliezer S. Yudkowsky The Banach-Tarski Gyroscope is an intricate mechanism believed to have been constructed using the Axiom of Choice. On each complete rotation counterclockwise, the Banach-Tarski Gyroscope doubles in volume while maintaining its shape and density; on rotating clockwise, the volume is halved. When first discovered, fortunately in the midst of interstellar space, the Banach-Tarski Gyroscope was tragically mistaken for an ordinary desk ornament. Subsequently it required a significant portion of the available energy of the contemporary galactic civilization to reverse the rotation before nearby star systems were endangered; fortunately, the Banach-Tarski Gyroscope still obeys lightspeed limitations on rotation rates, and cannot grow rapidly once expanding past planetary size. After the subsequent investigation, the Banach-Tarski Gyroscope was spun clockwise and left spinning. [E. Yudkowsky in "The Banach–Tarski paradox", Hacker News (28 Dec 2008)]

Doron Zeilberger By hindsight, it is not surprising that there exist undecidable propositions, as meta-proved by Kurt Gödel. Why should they be decidable, being meaningless to begin with! The tiny fraction of first order statements that are decidable are exactly those for which either the statement itself, or its negation, happen to be true for *symbolic* integers. A priori, every statement that starts "for every integer *n*" is completely meaningless. [D. Zeilberger: "'Real' analysis is a degenerate case of discrete analysis", International Conference on Difference Equations and Applications, Augsburg, Germany (1 Aug 2001) p. 8]

We have to kick the misleading word "undecidable" from the mathematical lingo, since it *tacitly* assumes that infinity is real. We should rather replace it by the phrase "not even wrong" (in other words utter nonsense), that cannot *even* be resurrected by talking about *symbolic* variables. Likewise, Cohen's celebrated meta-theorem that the continuum hypothesis is "independent" of ZFC is a great *proof* that none of Cantor's $\aleph$s make any (ontological) sense. [D. Zeilberger: "Opinion 108" (2010)]

Read Wolfgang Mückenheim's fascinating book! {{"Die Geschichte des Unendlichen", 7th ed., Maro, Ausgsburg (2011)}} I especially like the bottom of page 112 and the top of page 113 {{cp. section "Scrooge McDuck" in this source book}}, that prove, *once and for all*, that (at least) the *actual* infinity is pure nonsense. [D. Zeilberger: "Addendum to Opinion 68" (2011)]

Alexander A. Zenkin *First hidden necessary condition of Cantor's proof.* – In the middle of the XX c., meta-mathematics announced Cantor's set theory "naive" and soon the very mention of the term "actual infinity" was banished from all meta-mathematical and set theoretical tractates. The ancient logical, philosophical, and mathematical problem, which during millenniums troubled outstanding minds of humankind, was "solved" according to the principle: "there is no term – there is no problem". So, today we have a situation when Cantor's theorem and its famous diagonal proof are described in every manual of axiomatic set theory, but with no word as to the "actual infinity". However, it is obvious that if the infinite sequence of Cantor's proof is *potential* then no diagonal method will allow to construct an *individual* mathematical object, i.e., to *complete* the *infinite* binary sequence. Thus, just the *actuality* of the infinite sequence is a *necessary* condition (a Trojan Horse) of Cantor's proof, and therefore the traditional, set-theoretical formulation of Cantor's theorem {{$\mathbb{X}$ is *uncountable*}} is, from the standpoint of classical mathematics, simply wrong and must be re-written as follows without any contradiction with any logic. [...] If $\mathbb{X}$ is a*ctual*, then $\mathbb{X}$ is *uncountable*. [A.A. Zenkin: "Scientific intuition of genii against mytho-'logic' of Cantor's transfinite 'paradise'", Proceedings of the International Symposium "Philosophical insights into logic and mathematics", Nancy, France (2002) p. 3]

The fact to be demonstrated is that ultimately Cantor's diagonal proof engages us in an endless, *potentially* infinite and quite senseless paradoxical "game of two honest tricksters" (a new set-theoretical paradox) which, as Wittgenstein alleged, "has no relation to what is called a deduction in logic and mathematics". [A.A. Zenkin: "Logic of actual infinity and G. Cantor's diagonal proof of the uncountablility of the continuum", Review of Modern Logic 9 (2004) p. 28]

Cantor's 'paradise' as well as all modern axiomatic set theory {{AST}} is based on the (self-contradictory) concept of actual infinity. Cantor emphasized plainly and constantly that all transfinite objects of his set theory are based on the actual infinity. Modern AST-people try to persuade us to believe that the AST does not use actual infinity. It is an intentional and blatant lie, since if infinite sets, $\mathbb{X}$ {{the real numbers of the unit interval}} and $\mathbb{N}$, are potential, then the uncountability of the continuum becomes unprovable, but without the notorious uncountablity of continuum the modern AST as a whole transforms into a long twaddle about nothing and really is a pathological incident in history of mathematics from which future generations will be horrified. [A. Zenkin, letter to D. Zeilberger (20 Dec 2005)]

Ernst Zermelo The requirement that every element of a set shall be a set itself seems questionable. Formally that may work and simplifies the formalism. But what about the application of set theory on geometry and physics? [E. Zermelo, letter to A. Fraenkel (20 Jan 1924)]

Could not just this seemingly so fruitful hypothesis of the infinite have introduced straight contradictions into mathematics, thereby destroying the basic nature of this science that is so proud upon its consistency? ["On the hypothesis of the infinite", Ernst Zermelo's Warsaw notes W4 (p. 171), reported in H.-D. Ebbinghaus, V. Peckhaus: "Ernst Zermelo: An approach to his life and work", Springer (2007) p. 292]

By its nature, traditional "Aristotelian" logic is finitary and, hence, not suited for a foundation of the mathematical sciences. Therefore, there is a necessity for an extended "infinitary" or "Platonic" logic which rests on some kind of infinitary "intuition" [E. Zermelo: "Thesen über das Unendliche in der Mathematik", unpublished (17 Jul 1921)]

By "relativising" the notion of set, I believe to be able to contradict Skolem's "relativism" that would like to represent the whole of set theory in a *countable* model. It is simply *impossible* to give *all* sets in a constructive way [...] and any theory, founded on this assumption, would be no set theory at all. [E. Zermelo, letter to E. Artin (?) (25 May 1930)]

Chaohui Zhuang In Cantor's diagonal argument, the contradiction originates from the hidden presumption that the definition of Cantor's number is complete. The contradiction shows that the definition of Cantor's number is incomplete. Thus Cantor's diagonal argument is invalid. [C. Zhuang: "Wittgenstein's analysis on Cantor's diagonal argument", philpapers (2010)]

VI Contradictions of transfinite set theory

In this chapter we will disprove the three basic features of transfinite set theory by means of a diversity of arguments:

(1) There is no actual infinity. For instance there is no complete infinite set $\mathbb{N}$ with $\aleph_0$ identifiable natural numbers.

(2) Assuming that $\aleph_0$ exists and is consistent, countability turns out to be a self-contradictory notion. For instance, there is no bijection between $\mathbb{N}$ and $\mathbb{Q}$.

(3) Even if $\aleph_0$ and countability are assumed to exist free of self-contradictions and to make sense, there are no uncountable sets.

It is not an easy task to disprove transfinite set theory. Always when in history a contradiction has raised its ugly head its derivation has been denounced as invalid. The set of all sets, the most natural "universe" of set theory, has been excluded by hastily added axioms. The Löwenheim-Skolem paradox has been explained away by different notions of countability in "inner" and "outer" models. The antinomies raised by Vitali, Hausdorff, and Banach and Tarski have been "resolved" by declaring the sets used for the contradictions as not measurable. Of course every such result must be rejected in a theory of **Z**ero **F**indable **C**ontradictions. Only then the boasting can be maintained that hitherto no-one has found a contradiction, where the "hitherto" is always proudly emphasized, to the end of showing incorruptible objectivity, connected with unshakeable trust in the immaculate continuance of this state though.

In this chapter we argue and extrapolate by applying simple finite logic. The usual counter argument that infinite sets behave differently from or even contrary to finite sets would destroy the basis of set theory too, namely the conclusion from finite to infinite bijections.

Besides of irrefutable arguments and stringent proofs there are other aspects presented in the following which are not refutations in the proper sense of the word but only show that set theory requires the belief in useless, discontinuous, and unmathematical properties of "the infinite". We will see many infinite sequences[1] where with every step the target moves further and further away: the counting of all positive even numbers, the enumeration of all positive fractions, the bankruptcy of Scrooge McDuck, and many similar sequences will show this. But by simple, naive, and completely irrational belief, it is asserted that "a limit" exists which can be "reached", and that "in this limit" all elements of the set of positive even numbers are smaller than its cardinal number, all positive fractions will have been enumerated, and Scrooge McDuck will have gone bankrupt, because the set-limit is considered somehow superior to the mathematical (improper) limit of the sequence of cardinalities.

In cases where for a sequence of sets the limit of the cardinal numbers differs from the cardinal number of the limit set, a mathematician has to trust in mathematics only, which yields the former, but not in set theory, which yields the latter.

[1] In the following the terms of a sequence are sometimes written below each other for greater clarity and to spare separators between them. – There is enough inexpensive space.

$\mathbb{N}$ does not exist – the self-contradictoriness of the notion $\aleph_0$

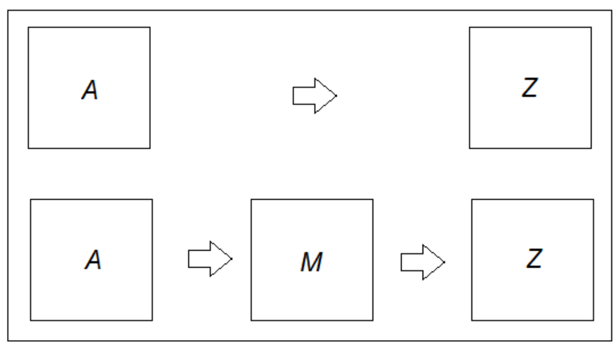
The pigeonhole principle

If each of the first n positive integers has a *unary representation* in form of a string that is shorter than n then, by the pigeonhole principle, there must be two different positive integers defined by the same unary representation. Clearly this is absurd.

Same holds in case of $\aleph_0$ finite strings. $\aleph_0$ is a fixed quantity such that $\forall n \in \mathbb{N}$: $n < \aleph_0$. If each one of all $\aleph_0$ positive integers has a *unary representation* in form of a string that is shorter than $\aleph_0$ then, by the pigeonhole principle, there must be two different positive integers defined by the same unary representation. Clearly this is absurd too.

Three reservoirs

Consider a reservoir A containing all natural numbers and another reservoir Z being empty. According to set theory it is possible, by addition and subtraction of sets, to transfer all natural numbers from A to Z.



Now add an intermediate reservoir M. By addition and subtraction of sets transfer the natural numbers, one after the other, from A to M and from M to Z but obeying the condition[1] that number n is not transferred from M to Z before number $n + 1$ has arrived in M from A.

According to this procedure always one number remains in M so that M is never empty and $\mathbb{N}$ is never complete in Z.

Set theory however replaces "never" by "in the infinite" – and then "in the infinite" $\mathbb{N}$ is in Z. In that case we get a result for the sets S_n residing in M that is typical for limits of sequences of sets:

$$1 = \text{Lim}_{n\to\infty} |S_n| \neq |\text{Lim}_{n\to\infty} S_n| = |\{ \}| = 0 .$$

[W. Mückenheim: "History of the infinite", current lecture, chapter XII. Illegal border: "Minimal super task", MathOverflow (10 Jul 2014) deleted]

[1] This condition is in no way artificial because for every natural number taken, there is another natural number not yet taken.

Discontinuity of transfinity (I)

Every finite set of positive even integers $2k$ contains at least one number that is larger than the cardinal number of the set. For instance every term $F_n = \{2, 4, 6, ..., 2n\}$ of the sequence of Finite Initial Segments (FIS) of positive even integers

$$\{2\}, \{2, 4), \{2, 4, 6\}, \{2, 4, 6, 8\}, \{2, 4, 6, 8, 10\}, \{2, 4, 6, 8, 10, 12\}, ... \qquad (*)$$

contains greater numbers than its cardinal number $|F_n| = |\{2, 4, 6, ..., 2n\}| = n$. Replacing one or more of the positive even integers of a FIS by larger positive even integers (smaller are not available) cannot remedy this result. So we can even state the

<u>Theorem</u> Every finite set S of positive even integers $2k$ contains at least one integer $2k > |S|$.

The surplus of greater integers, marked red in (*), grows without bound. The quotient of cardinal numbers of "upper set" (containing the integers $> n$) and "lower set" (containing the integers $\leq n$), namely $|\{2k \mid 2k > n\}| / |\{2k \mid 2k \leq n\}|$ is never less than 1. The first FISs yield

$$\frac{|\{2\}|}{|\{\ \}|} > 1,\ \frac{|\{4\}|}{|\{2\}|} = 1,\ \frac{|\{4,6\}|}{|\{2\}|} = 2,\ \frac{|\{6,8\}|}{|\{2,4\}|} = 1,\ \frac{|\{6,8,10\}|}{|\{2,4\}|} = \frac{3}{2},\ \frac{|\{8,10,12\}|}{|\{2,4,6\}|} = 1,\ \frac{|\{8,10,12,14\}|}{|\{2,4,6\}|} = \frac{4}{3}\ .$$

The minimum is reached for FISs of the form $\{2, 4, 6, ..., 4n\}$

$$\frac{|\{2(n+1),\ 2(n+2),\ ...,\ 4n\}|}{|\{2,\ 4,\ 6,\ ...,\ 2n\}|} = 1\ .$$

For all other finite sets of positive even integers the quotient is larger.

Let (G_n) be the sequence of cardinal numbers of the "upper sets", then the reciprocal sequence

$$(1/G_n) = 1, 1, 1/2, 1/2, 1/3, 1/3, 1/4, 1/4, 1/5, 1/5, ... \to 0$$

converges to zero. Therefore the sequence (G_n) has the improper limit ∞. This is in contradiction with the set theoretic postulate that in the limit there are more even numbers than any even number indicates, i.e, the upper set is empty.

Indexing the green and the red numbers of the above sequence (*) separately, we get

$$\{\ \} \cup \{2_1\}, \{2_1\} \cup \{4_1\}, \{2_1\} \cup \{4_1, 6_2\}, \{2_1, 4_2\} \cup \{6_1, 8_2\}, \{2_1, 4_2\} \cup \{6_1, 8_2, 10_3\}, ...$$

In the limit the index sets $\{1, 2, 3, ...\}$ and $\{1, 2, 3, ...\}$ would be expected. But the discontinuity of set theory results in the index sets $\mathbb{N}$ and $\emptyset$. It is amazing that the limits would be $\{1, 2, 3, ...\}$ and $\{1, 2, 3, ...\}$, i.e., $\mathbb{N}$ and $\mathbb{N}$, if the indices were accumulated as before but logically disconnected from the red surplus numbers.

[W. Mückenheim: "Die Geschichte des Unendlichen", 7th ed., Maro, Augsburg (2011) p. 111f]

Discontinuity of transfinity (II)

Discontinuity is obvious from some bijections with ℕ having gaps. Simple bijections like

n	$2n$	n^2	$1/n$
1	2	1	1
2	4	4	1/2
3	6	9	1/3
...	...	...	...
$\aleph_0$	$\aleph_0$	$\aleph_0$	0

do not reveal the effect but it becomes clearly visible in more involved examples. $n \leftrightarrow 2^n$ and similar bijections given in the table below are discontinuous since

$$2^n < \aleph_0 \quad \text{for} \quad n < \aleph_0 \quad \text{and} \quad 2^n = 2^{\aleph_0} > \aleph_0 \quad \text{for} \quad n = \aleph_0 .$$

This bijection is undefined between $\aleph_0$ and $2^{\aleph_0}$. There is a gap where 2^n is neither finite nor infinite. But if $\aleph_0$ is considered a *number* which can be in trichotomy with natural numbers, then this discontinuity is highly suspect. $n \leftrightarrow 2^{2^{n!}}$ exhibits indefiniteness over even a larger gap. All the rows with question marks in the following table show gaps of indefiniteness. They are unanswered by set theory, which, on the other hand, is ready to calculate hyper-inaccessible cardinals minutely. Common calculus simply would replace the question marks by 0 or ∞.

$1/2^{n!}$	$1/n!$	$\log_2 n$	n	2^n	$n!$	$2^{n!}$	$2^{2^{n!}}$
1/2	1	0	1	2	1	2	4
1/4	1/2	1	2	4	2	4	16
1/64	1/6	$\log_2 3$	3	8	6	64	2^{64}
...	...	...	...	...	...	...	...
?	?	?	?	?	?	?	$\geq \aleph_0$
...	...	...	...	...	...	...	...
0	?	?	?	?	?	$\geq \aleph_0$	$\geq 2^{\aleph_0}$
...	...	...	...	...	...	...	...
0	0	?	?	?	$\geq \aleph_0$	$\geq 2^{\aleph_0}$	$\geq 2^{2^{\aleph_0}}$
...	...	...	...	...	...	...	...
0	0	?	?	$\geq \aleph_0$	$\geq \aleph_0$	$\geq 2^{\aleph_0}$	$\geq 2^{2^{\aleph_0}}$
...	...	...	...		...	...	...
0	0	?	$\geq \aleph_0$	$\geq 2^{\aleph_0}$	$\geq \aleph_0!$	$\geq 2^{\aleph_0!}$	$\geq 2^{2^{\aleph_0!}}$
...	...	...	...	...	...	...	...
0	0	$\geq \aleph_0$	$\geq 2^{\aleph_0}$	$\geq 2^{2^{\aleph_0}}$	$\geq 2^{\aleph_0}!$	$\geq 2^{(2^{\aleph_0})!}$	$\geq 2^{2^{(2^{\aleph_0})!}}$

Of course this table could easily be extended by loglog...log$n$ as well as by higher powers of 2 or other numbers.

[W. Mückenheim: "The meaning of infinity", arXiv (2004)]

Failure of universal quantification

By means of universal quantification, $\forall n \in \mathbb{N}$: $P(n)$, allegedly all $\aleph_0$ natural numbers can be addressed individually and provided with a property P. But every natural number n that we can address individually belongs to a Finite Initial Segment Of Naturals (FISON) $F_n = (1, 2, 3, ..., n)$ that is followed by an actual infinity of $\aleph_0$ natural numbers – if set theory is right.

So what means quantifying over all natural numbers n? Does it mean to take only those natural numbers which have the characteristic property to be followed by infinitely many natural numbers? Then we don't get all of them because always infinitely many are missing. Or do we take all n with no exception? Then we include some which are not followed by infinitely many others and hence are not natural numbers because they are lacking this characteristic property.

As the infinite set following every F_n is much larger than this F_n, almost all (i.e., all except finitely many) natural numbers cannot be addressed individually. In fact every putative user of the universal quantifier fails infinitely often to select a number of the final sequence of $\aleph_0$ natural numbers and never succeeds. The claim that all natural numbers can be identified and referred to individually, for instance in a bijection of $\mathbb{N}$ with a countable set, is false. Who would have ever identified or referred to a natural number that has more predecessors than successors?

To illustrate this, we can divide the complete sequence of natural numbers into say 100 consecutive finite intervals of equal size and an infinite rest. If we refer to any natural number n, then $100 \cdot n$ is a natural number too and belongs to a FISON. (Of course we can subjugate $100 \cdot n$ to the same multiplication.) So every natural number belongs to the first of 100 similar finite intervals, i.e., to less than 1 % of $\mathbb{N}$. A counter argument stating that this 1% of $\mathbb{N}$ is all of $\mathbb{N}$ does not impair the argument: We cannot refer to more than the first percent of $\mathbb{N}$.

Instead of 100 every factor K could be used. Thus all natural numbers belong to a tiny, nearly vanishing initial segment of the complete set $\mathbb{N}$ – if such a set exists somewhere. But whether or not it exists, it would not have any effect since almost all of its elements are inaccessible.

Same holds for the rational numbers. Every decimal period can be extended to a period of K-fold length – it remains a rational number with a *small* period. Also for every fraction the numerator or the denominator can be multiplied by any K without leaving the domain of *small* fractions.

Modern set theorists try to save $\mathbb{N}$ by storing all natural numbers, with the omission of their natural order, in a big bag. The natural order of the natural numbers however is their most important feature. It is already embossed to the sequence S0, SS0, SSS0, ... by the Peano axioms. So set theorists have to forget this order. But then $\mathbb{N}$ is unsuitable for enumerating purposese or mathematical induction.

To make a long story short: A complete set $\mathbb{N}$ of identifiable numbers does not exist in potential infinity and is not accessible in actual infinity.

[W. Mückenheim: "Matheology § 295", sci.math (22 Jun 2013). W. Mückenheim: "What means quantifying over all natural numbers?", sci. math (5 Oct 2016)]

$\aleph_0$ destroys the translation invariance of finite strings.

A unary representation of a natural number n is a finite string or word of n similar symbols. If the sequence (n) in unary representation is written into rows, one number beneath the other,

o
oo
ooo
...

then no row covers $\aleph_0$ symbols because $\forall n \in \mathbb{N}$: $n < \aleph_0$ is a theorem of set theory.

If the same sequence (n) is written into a single row or if all symbols, maintaining their column, are shifted or projected into the first row (or a new row number zero), then this row covers $\aleph_0$ symbols (because there is an infinitely increasing sequence of words). This destroys translation invariance of words, namely the fact (among others necessary for mathematical discourse) that changing the place of writing must not change the meaning of the written.

Every set with N different natural numbers (here and in the following always positive integers are meant) in unary representation contains at least one representation with at least N symbols. In unary representation, there is no "other digit" but always only "one more symbol". Since there is no natural number with more than every finite set of symbols, there cannot be a set with more than all finite numbers of symbols either. The "actually infinite" set of $\aleph_0 > n$ symbols is a contradictio in adiecto. This can also be seen by the symmetry between rows and columns: The figure covers only those columns which are covered by at least one row – and no row covers $\aleph_0$ columns, i.e., more than all rows cover.

Like the sequence (s_n) of lines of length $s_n = 1 - 1/n$ covers all points of the unit interval except its supremum 1, the sequence (n) contains all numbers less than its supremum $\aleph_0$. But if $\aleph_0$ is not the number of elements of an uncountable set but merely the supremum which is never reached, this not only stands in opposition to Cantor's idea of the reality of sets and their elements, but it is devastating for the definition of real numbers by infinite sequences: A potentially infinite digit sequence with less than $\aleph_0$ digits does not define, till each digit, anything but a rational interval.

Same holds for the ordinal number ω. If it exists, then it is only the supremum and not the maximum of $\mathbb{N}$. Its column in the figure

1
1, 2
1, 2, 3
...

is far beyond all columns occupied by natural numbers because $\forall n \in \mathbb{N}, \forall X < \omega$: $n + X < \omega$.

[W. Mückenheim: "Das Zauberdreieck", de.sci.mathematik (22 Oct 2011)]

Discussion: When all finite rows o, oo, ooo, ... are projected into a single row number zero, L_0, then no state emerges where L_0 is longer than all. "No, exactly that is the case." [Franz Fritsche alias "Me" in "Grundpfeiler der Matheologie", de.sci.mathematik (1 Aug 2016)] My reply: Let us project, in a supertask, every row into L_0 but so that L_0 is always cleared between two steps (that is obviously without any effect, if the rows are processed in their natural order). Never $\aleph_0$ symbols o will show up in L_0. But when we project all rows into L_0 simultaneously, then $\aleph_0$ symbols o will show up there?

Discussion: L_0 is longer than all finite rows. "Replace *all* by *every*, then the statement obviously gets correct." [A. Leitgeb in "Grundpfeiler der Matheologie", de.sci.mathematik (16 Aug 2016)] My reply: When shifting *every* row into the first one, we do not get a row L_0 longer than all?

Duality in set theory

Theorem Every arbitrarily small ε-neighbourhood of 0 with $0 < \varepsilon < 1/n$ contains ω terms of the sequence $(1/n)_{n\in\mathbb{N}}$.

Proof is trivial. The dual statement is constructed by replacing 0 by ω, $1/n$ by n, and "<" by ">".

Dual Theorem Every arbitrarily large X-neighbourhood of ω with $\omega > X > n$ contains 0 terms of the sequence $(n)_{n\in\mathbb{N}}$.

Proof: $\neg\exists\, n$ with $n + X = \omega$ for $X < \omega$
$\neg\exists\, n$ with $n = \omega$
$\neg\exists\, n$ with $n > \omega$.

If ω exists, then we can talk about "all natural numbers which are followed by something". It would be wrong however to say that *all natural numbers* ℕ are followed by infinitely many natural numbers. Since all natural numbers are less than ω, we can say that *all natural numbers* are followed by the ordinal numbers ω, ω + 1, ω + 2, ... Then, according to the above dual theorem, *all natural numbers* must be followed *first* by $\aleph_0$ ordinals X which are smaller than ω.

Then, however, it is wrong to consider ω the limit of all natural numbers[1] let alone to identify ω with ℕ. Between a set and its limit there must not exist anything foreign to that set. Therefore we cannot accept Russell's statement: "Thus, for instance, the smallest of the infinite integers is the limit of the finite integers, though all finite integers are at an infinite distance from it." [Bertrand Russell: "Mathematics and the metaphysicians" from "Mysticism and logic and other essays", George Allen & Unwin, London (1917) p. 92]

[1] "Es ist sogar erlaubt, sich die neugeschaffene Zahl ω als Grenze zu denken, welcher die Zahlen ν zustreben, wenn darunter nichts anderes verstanden wird, als daß ω die erste ganze Zahl sein soll, welche auf alle Zahlen ν folgt, d. h. größer zu nennen ist als jede der Zahlen ν" [Cantor, p. 195].

Inclusion monotony

The sequence of all FISONs is strictly inclusion-monotonic $\{1\} \subset \{1, 2\} \subset \{1, 2, 3\} \subset \ldots$.

The limit $\mathbb{N}$ is not contained, since every FISON has a last number. On the other hand no natural number can be missing in their union. But for an inclusion monotonic sequence, the union is a term of the sequence. Every union of FISONs is contained in at least one of the unioned FISONs. This holds without any bound for arbitrarily large sets of FISONs including the infinite set too, because all its FISONs are finite and placed at finite positions.

Every natural number n is in some FISON F_i

$$\forall n\ \exists i: n \in F_i$$
$$\forall n\ \forall i: (n \leq i \Leftrightarrow n \in F_i) \wedge (n > i \Leftrightarrow n \notin F_i)\ .$$

If the union of FISONs nevertheless is larger than all FISONs, then there must exist at least two FISONs, F_j and F_k, such that

$$\exists j, k, m, n: m \in F_j \wedge m \notin F_k \wedge n \notin F_j \wedge n \in F_k\ .$$

Such FISONs cannot be found because FISONs obey inclusion monotony.

In words: If $\aleph_0$ is a *number* in trichotomy with all cardinal numbers including all natural numbers, and if there are $\aleph_0$ natural numbers, i.e., more than are in any FISON $\{1, 2, 3, ..., n\}$, then, as a consequence, no FISON contains all natural numbers. So the natural numbers must be dispersed over two or more FISONs, each of which is containing less than $\aleph_0$ elements. That is mathematically contradicted by the inclusion monotony of the FISONs.

By inclusion monotony every FISON contains all numbers of its predecessors. A counterargument claims that inclusion monotony does no longer hold "in the infinite". But every FISON has only a finite number of predecessors. Since there is no FISON with infinitely many predecessors, all FISONs come before "the infinite". Inclusion monotony holds for all FISONs.

[W. Mückenheim in "Diagonal wanderings (incongruent by construction)", sci.math (17 May 2009)]

Discussion: "For inclusion-monotonic sequences like the initial segments of $\mathbb{N}$ the union of all terms is the limit of the sequence." [Andreas Leitgeb in "Grundpfeiler der Matheologie", de.sci.mathematik (31 Jul 2016)] My reply: This statement is obviously wrong because every union of FISONs is finite. Every set of FISONs contains only finite FISONs (by definition) which come at finite positions. See section "All gapless unions are not enough to produce $\mathbb{N}$".

Discussion: This argument forces set theorists to confess that part of mathematics cannot be accepted in set theory: "Inclusion monotony is false, so I don't care what or what does not play a role for it." [Martin Shobe in "It is easy to understand that transfinite set theory is wrong", sci.math (15 Jan 2016)].

$\omega + 1$ unions

Every FISON $F_n = \{1, 2, 3, ..., n\}$ is the union of its predecessor $F_{n-1} = \{1, 2, 3, ..., n\text{-}1\}$ and $\{n\}$. In this way we accumulate ω or $\aleph_0$ finite unions of the form

$$(... ((\{1\} \cup \{2\}) \cup \{3\}) \cup ... \cup \{n\}) = \{1, 2, 3, ..., n\}$$

none of which yields the set $\mathbb{N}$ of all natural numbers although all natural numbers are present as elements in the set of all FISONs created in this way. But if we union all these FISONs for another time, i.e., all the unsuccessful attempts to establish $\mathbb{N}$ (which is also the set of all last elements of the FISONs), then we get

$$\{1\} \cup \{1, 2\} \cup \{1, 2, 3\} \cup ... \cup \{1, 2, 3, ..., n\} \cup ... = \{1, 2, 3, ...\} = \mathbb{N}\ .$$

We execute union number $\omega + 1$ over what already had been unioned before (each FISON is in infinitely many unions), and without adding anything further we get a larger set than has been existing before, namely the set of all last elements of the FISONs. The infinite union

$$\{1\} \cup \{1, 2\} \cup \{1, 2, 3\} \cup ... = \mathbb{N} \qquad \text{(A)}$$

is in the *limit* of the sequence of all finite unions

$$\{1\}, \{1, 2\}, \{1, 2, 3\}, ... \qquad \text{(B)}$$

since in both cases all natural numbers are applied. But $\mathbb{N}$ is not contained in (B) or (C):

$$\{1\}, \{1\} \cup \{1, 2\}, \{1\} \cup \{1, 2\} \cup \{1, 2, 3\}, ... \ . \qquad \text{(C)}$$

A matter of notation

$$\{1, 2, 3, ...\} = \mathbb{N}$$

is the set of all natural numbers. The order does not matter. It can be chosen arbitrarily but has to remain the same in the following. If every number is prepended by all its predecessors

$$\{1, 1, 2, 1, 2, 3, ...\} = \mathbb{N}$$

nothing is changed (since multiple appearance of an element in curly brackets does not matter). But if some curly brackets (or parentheses) are inserted,

$$\{\{1\}, \{1, 2\}, \{1, 2, 3\}, ...\} = \mathbb{F} \quad \text{or} \quad \{(1), (1, 2), (1, 2, 3), ...\} = \mathbb{F}$$

then, while no natural number is removed, $\mathbb{N}$ is lost. Now there is the set $\mathbb{F}$ of all FISONs without the set $\mathbb{N}$ which is no longer present, neither as a subset nor as an element. So the mere insertion of braces destroys $\mathbb{N}$ – although never two numbers, which are neighbours in the original sequence, have been separated by braces.

All gapless unions of FISONs are not enough to produce ℕ.

The following sequence contains all initial gapless unions of FISONs; all are finite, none is ℕ.

$\{1\} = \{1\}$
$\{1\} \cup \{1, 2\} = \{1, 2\}$
$\{1\} \cup \{1, 2\} \cup \{1, 2, 3\} = \{1, 2, 3\}$
$\{1\} \cup \{1, 2\} \cup \{1, 2, 3\} \cup \{1, 2, 3, 4\} = \{1, 2, 3, 4\}$
$\{1\} \cup \{1, 2\} \cup \{1, 2, 3\} \cup \{1, 2, 3, 4\} \cup \{1, 2, 3, 4, 5\} = \{1, 2, 3, 4, 5\}$
...

But the terms of the sequence contain, as elements of FISONs, all natural numbers and all FISONs (on the right-hand side), and also all initial gapless unions of FISONs (on the left-hand side). An infinite union ℕ that is larger than all FISONs is impossible because there are no further FISONs that could be unioned in order to surpass themselves.

The intersection of all endsegments is not empty.

Definition: $E_n = (n, n+1, n+2, n+3, ...) = \{k \in \mathbb{N} \mid k \geq n\}$ is called the endsegment of n.

According to set theory the union of all FISONs is ℕ, and the intersection of all endsegments is empty. For every element n there is a first endsegment E_{n+1} not containing it. This result however will be refuted in the following.

Every endsegment E_n is the finite intersection of itself and all its predecessors and contains $\aleph_0$ natural numbers in common with all other endsegments. For every endsegment E_m with $m < n$ this is clear. For every endsegment E_{n+k} it follows by contradiction: Assume that E_n has less than $\aleph_0$ elements in common with E_{n+k}, then there is an endsegment, namely E_{n+k} having less than $\aleph_0$ elements in common with its predecessor E_n. Contradiction.

Discussion: "The intersection of any finite set of endsegments contains $\aleph_0$ numbers. The intersection of any infinite set of endsegments is empty." [Jim Burns in "Why is it relevant to know potential infinity?", sci.logic (17 Jul 2019)] My reply: The largest infinite intersection of endsegments is the intersection of all endsegments. – But not more. Every endsegment however *is a finite* intersection of itself and its predecessors. We agree that every finite intersection has $\aleph_0$ elements in common with $E_1 = \mathbb{N}$ and all its other predecessors and successors at finite places. There are no further endsegments by which the intersection could be extended in order to diminish this result. Therefore the infinite intersection of all these endsegments is not empty.

Sequence of exponents

The sequence (a_k) of terms $a_k = 10^{-1} + 10^{-2} + ... + 10^{-k}$ has, as absolute values of its exponents, the set of all FISONs (1), (1, 2), (1, 2, 3), ... ℕ does not belong to this set because the sequence does not contain its limit 0.111... . But all natural numbers are present. That means we have all natural numbers in the set but we don't have ℕ. This proves that by "all natural numbers" we have to understand a potentially infinite collection, but not an actually infinite, completed set.

(1) There is no natural number that is missing in all exponents of the sequence.
(2) There is no infinite set of natural numbers in any term of the sequence (because there is no infinite set of powers in any term).
(3) There is no natural number that, once it has appeared in the exponent of a term a_k, will be missing in the exponent of a later term a_{k+j}, with $j > 0$.

From (1) we see that every $n \in \mathbb{N}$ is in the sequence (a_k) as the absolute value of an exponent. If the complete infinite set ℕ is actually existing, then it is in the union of all those FISONs.
From (3) we see that all n that are in the union of all exponents also are in a single exponent.
From (2) we see that no actually infinite set of natural numbers is in any exponent of a term a_k.

Arbitrariness in set theory: "Every" is not "all".

Every initial sequence of digits $D_n = d_1d_2...d_n$ of the antidiagonal D differs from the first n entries L_n of the Cantor-list L. From this it is concluded that D differs from *all* entries of the Cantor-list and therefore D is not in the list. $\forall n \in \mathbb{N}: D_n \notin L_n \Rightarrow D \notin L$.

Every initial sequence of digits $D_n = d_1d_2...d_n$ of the antidiagonal D differs from every element of the set $\mathbb{X}$ of all irrational numbers because it is rational. This is *not* used to argue that D is not in the set $\mathbb{X}$ of irrational numbers. $\forall n \in \mathbb{N}: D_n \notin \mathbb{X} \Rightarrow D \notin \mathbb{X}$.

Every rational number can be indexed by a natural number. From this it is concluded that *all* rational numbers can be indexed by natural numbers.

Every natural number leaves the overwhelming majority of rationals without index. From this it is not concluded that *all* natural numbers leave the overwhelming majority of rationals without index.

It is possible for every $n \in \mathbb{N}$ to enumerate the first n rational numbers $q_1, q_2, q_3, ..., q_n$ – for instance like Cantor did it. Set theorists claim that this proves the possibility of enumerating all rational numbers, i.e., listing them as a sequence.

It is possible for every $n \in \mathbb{N}$ to well-order these first n rational numbers $q_1, q_2, q_3, ..., q_n$ by size. Set theorists do not claim that this proves the possibility of well-ordering all rational numbers by size.

Cantor's Theorem B

Here are some excerpts from Cantor's papers concerning the fact that every set of ordinal numbers has a smallest element:

• "Among the numbers of the set (α') there is always a *smallest* one."[1] A simple proof has been shown by Zermelo, the editor of Cantor's collected works.[2]

• "Theorem B: *Every embodiment of different numbers of the first and the second number class has a smallest number, a minimum.*"[3]

• "Would the index α' not run through all numbers of the second number class then there had to be a smallest number α that it does not reach."[4]

• "But from the proven theorems about well-ordered sets in § 13 it also follows easily that every multitude of numbers, i.e., every part of Ω contains a *smallest* number."[5]

These theses however have to be rejected by present day set theorists (to protect Cantor's works from destruction). Among the FISONs of $\mathbb{N}$ there is not, in any enumeration, a first one that is required to yield the union $\mathbb{N}$. (See section: "What FISONs are necessary to house all natural numbers?") Usually this is apologized by the fact, that even in

$$\{1, 2\} \cup \{2, 3\} \cup \{3, 1\} = \{1, 2, 3\} \qquad (*)$$

it is impossible to find a first set which cannot be omitted from the union to yield {1, 2, 3}. But this argument fails. It is not *a set of sets* which is subject to Cantor's theorem B but only every set of *ordinal numbers*. Therefore we always have to enumerate the sets. In case of FISONs this is simple. We apply the natural order: $\{1, 2, 3, ..., n\} \rightarrow n$. Of course every other enumeration would also do. In case of the sets (*) we can use the written order from left to right. Then the first set not to be omitted is {2, 3} because after having omitted {1, 2} already, 2 would then be missing in the union. When choosing the order

$$\{3, 1\} \cup \{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$$

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[1] "Unter den Zahlen der Menge (α') gibt es immer eine *kleinste*." [Cantor, p. 200]

[2] "Daß es in jeder Menge (α') transfiniter Zahlen immer eine kleinste gibt, läßt sich folgendermaßen einsehen. Es sei (β) die Gesamtheit aller (endlichen und unendlichen) Zahlen β, welche kleiner sind als *alle* Zahlen α'; solche Zahlen muß es geben, z. B. die Zahl 1, sofern diese nicht selbst zu α' gehört und dann natürlich die kleinste der Menge ist. Unter den Zahlen β gibt es nun entweder eine größte β_1, so daß die unmittelbar folgende β'_1 nicht zu (β) gehört, aber $\leq \alpha$' ist für jedes α', dann gehört β'_1 selbst zu (α') und ist ihre kleinste. Oder aber die Zahlen β enthalten keine größte, dann besitzen sie (nach dem zweiten Erzeugungsprinzip) eine 'Grenze' β', welche auf alle β zunächst folgt, also wieder $\leq$ jedem α' ist, und diese Zahl β' muß dann wieder notwendig zu (α') gehören und die kleinste aller α' darstellen." [E. Zermelo in Cantor, p. 208f]

[3] Satz B. "*Jeder Inbegriff von verschiedenen Zahlen der ersten und zweiten Zahlenklasse hat eine kleinste Zahl, ein Minimum.*" [Cantor, p. 332]

[4] "Würde nun der Index α' nicht alle Zahlen der zweiten Zahlenklasse durchlaufen, so müßte es eine kleinste Zahl α geben, die er nicht erreicht." [Cantor, p. 349]

[5] "Aber aus den in § 13 über wohlgeordnete Mengen bewiesenen Sätzen folgt auch leicht, daß jede Vielheit von Zahlen, d. h. jeder Teil von Ω eine *kleinste* Zahl enthält." [Cantor, p. 444]

we could drop {3, 1} but then {1, 2} is the first set necessary, because otherwise 1 would be missing in the union. Another example[1] is this one:

$$\{1, 2\} \cup \{2, 3\} \cup \{3, 4\} \cup ... = \mathbb{N} . \qquad (**)$$

Here obviously {1, 2} is the first set that cannot be omitted (it is required in every order) because 1 is not contained in any further set, whereas {3, 4} is the second set which cannot be omitted (in the order suggested in (**)) because otherwise 3 would be missing in the union. So it is not only inclusion monotony which proves that no FISON is required or necessary to yield the union $\mathbb{N}$ but also the theorem that every well-defined set of ordinal numbers has a smallest element.

Since from every FISON we know by definition that it is neither sufficient nor necessary to yield the union of FISONs $\mathbb{N}$, we find

$$\bigcup\{F_1, F_2, F_3, ...\} = \mathbb{N} \;\;\Rightarrow\;\; \bigcup\{\,\} = \mathbb{N} .$$

This is a highly counterintuitive result[2], at least for those who believe in the union resulting in $\mathbb{N}$. But it becomes easily understandable and clearly obvious without any induction in the form

$$\{F_1, F_2, F_3, ..., F_n\} = \mathbb{N} \;\;\Rightarrow\;\; \{\,\} = \mathbb{N}$$

or

$$\text{Butter \& Bread \& Beef} = \mathbb{N} \;\;\Rightarrow\;\; \{\,\} = \mathbb{N} .$$

[W. Mückenheim: "Sometimes the conclusion from 'every' on 'all' is desired but sometimes it is forbidden." sci.math (26 Apr 2016)]

Why not apply Theorem B to prove the fact that no natural number is sufficient to make the set $\mathbb{N}$ actually infinite?

Theorem   The sequence of natural numbers is not actually infinite.

Proof: The natural numbers 1, 2, ..., n do not produce an actually infinite set. If there were natural numbers capable of producing an actually infinite set, then one of them was the smallest, call it a, such that the theorem was valid for all $x < a$ but not for $x \le a$. Contradiction.

Of course the natural numbers are potentially infinite. This cannot be disproved. With respect to *this* fact, proofs like the present one would be hilarious – and frequently they have been called so. But when the critics were silenced, then the meaning of infinity would be changed back on the quiet and unnoticed from potential to actual. – A really perfidious procedure.

That's why set theorists refuse to understand the difference between potential and actual infinity. Their standard procedure would become obvious.

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[1] I am indebted to Mr. J. Rennenkampff for the instructive, although not inclusion monotonic, examples (*) and (**) [J. Rennenkampff in "Von seinen Jüngern verleugnet", de.sci.mathematik (20 Apr 2016)].
[2] Usually the handwaving claim is: "But infinitely many FISONs have to remain!"

$\aleph_0$ and induction

By induction we can prove for all natural numbers $n \in \mathbb{N}$:

$$1 + 2 + 3 + ... + n = n(n+1)/2 \ . \qquad (*)$$

All positive numbers obeying this rule are natural numbers. If the natural numbers obeying eq. (*) are removed from the set $\mathbb{N}$ of all natural numbers, what remains? Nothing. Further positive numbers satisfying (*) are not existing. Therefore $\mathbb{N}$ can be defined by this property:

$$\mathbb{N} = \{x \mid \sum_{k=1}^{x} k = \frac{x(x+1)}{2}\} \ .$$

Zermelo's infinite set S is defined by induction too: $\{\ \} \in S \wedge (X \in S \Rightarrow \{X\} \in S)$.

By the same induction we can prove $\{\ \} = \mathbb{N}$ as follows:

From the set $\mathbb{F}$ of all finite initial segments of natural numbers (FISONs):

$$\mathbb{F} = \{(1), (1, 2), (1, 2, 3), ...\} = \{F_1, F_2, F_3, ...\}$$

we can remove, without changing the union $\mathbb{N}$ of the remainder, the set F defined by induction:

$$(1) \in F$$
$$(1, 2, 3, ..., n) \in F \quad \Rightarrow \quad (1, 2, 3, ..., n+1) \in F \ .$$

Obviously

$$\forall n \in \mathbb{N}: \bigcup(\mathbb{F} \setminus \{F_1, F_2, F_3, ..., F_n\}) = \bigcup(\{F_{n+1}, F_{n+2}, F_{n+3}, ...\}) = \mathbb{N} \ .$$

By induction we see that none of the sets $\{F_1\}$, $\{F_1, F_2\}$, $\{F_1, F_2, F_3\}$ ..., when removed from $\mathbb{F}$, changes the union of the remainder.

Now remember that complete sets can be defined by induction:

If $\forall n \in \mathbb{N}$: n is given, then $\{1, 2, 3, ...\} = \mathbb{N}$ is given.
If $\forall n \in \mathbb{N}$: F_n is given, then $\{F_1, F_2, F_3, ...\} = F = \mathbb{F}$ is given.
If $\forall n \in \mathbb{N}$: $\mathbb{F} \setminus \{F_1, F_2, ..., F_n\}$ is given, then $\mathbb{F} \setminus \{F_1, F_2, F_3, ...\} = \mathbb{F} \setminus F = \{\ \}$ is given.
If $\forall n \in \mathbb{N}$: $\bigcup(\mathbb{F} \setminus \{F_1, F_2, ..., F_n\}) = \mathbb{N}$ is given, then $\bigcup(\mathbb{F} \setminus F) = \bigcup(\mathbb{F} \setminus \mathbb{F}) = \mathbb{N}$ is given.

Conclusion: If $\mathbb{N}$ and $\mathbb{F}$ are complete and exhaustible, i.e., if $\aleph_0$ exists, then $\bigcup\{\ \} = \mathbb{N}$.

What FISONs are necessary to house all natural numbers?

Consider the sequence of finite initial segments of natural numbers (FISONs) $F_k = (1, 2, 3, ..., k)$.

Definition: F_k is necessary $\Leftrightarrow$ F_k contains a number n that is not in any F_{k+j} with $j > 0$.

Alternatively: F_k is necessary $\Leftrightarrow F_k \cup F_{k+1} \cup F_{k+2} \cup ... = \mathbb{N} \wedge F_{k+1} \cup F_{k+2} \cup F_{k+3} \cup ... \neq \mathbb{N}$.

Finite example: $\{1\} \cup \{1, 2\} \cup \{1, 2, 3\} = \{1, 2, 3\}$ is true, and $\{1\} \cup \{1, 2\} = \{1, 2, 3\}$ is false. In both cases the FISONs $\{1\}$ and $\{1, 2\}$ are not necessary – neither to make the true statement true nor to make the false statement false.

Theorem *If it is possible* to have all natural numbers in the union of a set of FISONs then we need no FISON at all for this sake, i.e., then $\bigcup\{\ \} = \mathbb{N}$.

Proof:

F_1 is not necessary.

If F_n is not necessary, then also F_{n+1} is not necessary.

The implication does not rely on the existence of an always larger Fison $F_{n+2} \supset F_{n+1}$, although this is existing, but above all on the fact that neither endsegment contains more than a finite set of natural numbers and therefore is useless – whether or not a successor is existing.

This holds for every $n \in \mathbb{N}$. We can remove all FISONs without removing $\mathbb{N}$ completely.

In case of potential infinity every FISON can be removed too. But there is no "all FISONs". "Every" never concerns "all". Therefore never all are removed.

Discussion: "By complete induction it can be proved only that finitely many can be removed, infinitely many remain, therefore OK." [A. Leitgeb in "Grundpfeiler der Matheologie", de.sci.mathematik (28 Jul 2016)] This opinion is contradicted by the fact that complete induction proves $1 + 2 + 3 + ... + n = n(n+1)/2$ to be true for *infinitely* many natural numbers. None remains.

Three pillars of mathematics

In the present section we have seen that transfinite set theory is incompatible with these three pillars of mathematics:

- Complete induction is valid for infinitely many natural numbers.
- All infinitely many FISONs obey inclusion monotony.
- Every well-defined set of natural numbers has a minimum.

At least one of them must be violated and overthrown if transfinite set theory is to be accepted.

Dark numbers in set theory

If, contrary to what has been proved in the preceding section, actual infinity is to be maintained, then there must exist a set Y of undefinable "natural" numbers. That will be discussed here.

A natural number is defined as a positive (sometimes also as a non-negative) integer that can be identified, by its digit-sequence or by its name or by a less concise description like "the number of days of a week", so that it can be communicated. Only when a number n is identified we can use it in mathematical discourse and determine the trichotomy properties of n and every multiple of n with respect to every other identifiable natural number. We can count, in principle, from 1 to n and to every multiple of n. The set $\mathbb{N}$ of natural numbers n is defined as the set that contains all these natural numbers as elements and nothing else.

But according to set theory $\mathbb{N}$ is also actually infinite. While potential infinity only would require that for every Finite Initial Segment Of Naturals (FISON) $F_n = (1, 2, 3, ..., n)$ there exists a larger one

$$\forall F_n \, \exists F_m \colon |F_n| < |F_m| \land F_n \subset F_m$$

actual infinity exchanges quantifiers and states

$$\exists \mathbb{N} \, \forall F_n \colon |F_n| < |\mathbb{N}| \land F_n \subset \mathbb{N} \, .$$

$\mathbb{N}$ differs from all F_n because it has a greater cardinal number, i.e., $\mathbb{N}$ has more elements than all F_n. This is only possible if it has at least one element that is not in any F_n. But it turns out that no such element can be found. Therefore the actually infinite set $\mathbb{N}$ must embrace elements that cannot be identified. We call them dark numbers.

Theorem   Almost all natural numbers are not identifiable.

Proof: Consider the set $\mathbb{F} = \{F_n \mid n \in \mathbb{N}\}$ of all identifiable FISONs F_n and observe that its union $\mathbb{N}$ does not change when one of the FISONs is omitted:

$$F_1 \cup F_2 \cup F_3 \cup ... \cup F_{n-1} \cup F_{n+1} \cup F_{n+2} \cup F_{n+3} \cup ... = \mathbb{N} \, .$$

There is no identifiable FISON whose omission would change the result of the union. So every FISON can be omitted; the set of FISONs that can be omitted is the set $\mathbb{F}$ of all FISONs existing.

Further we observe that those FISONs which are predecessors, i.e., subsets, of the omitted F_n can be omitted too with no effect so that we get

$$F_{n+1} \cup F_{n+2} \cup F_{n+3} \cup ... = \mathbb{N}$$

for all n. Since there is no first FISON that cannot be omitted without effect, we get

$$\bigcup\{ \} = \{ \} = \mathbb{N} \, .$$

Of course this result is false because $\mathbb{N}$ is not empty. But the usual explanation, that every FISON can be omitted only as long as there remain larger FISONs, does not hold, firstly because a FISON without successor does not exist at all and therefore cannot be omitted, and secondly because we can see that as long as F_n is a proper subset of $\mathbb{N}$, it is neither necessary nor sufficient in the union to give $\mathbb{N}$. This holds for all FISONs how large they ever may be. Therefore we must accept the implication: *if* $\mathbb{N}$ is the union of all identifiable FISONs, *then* $\mathbb{N}$ is empty:

$$F_1 \cup F_2 \cup F_3 \cup ... = \mathbb{N} \;\Rightarrow\; \{\,\} = \mathbb{N}\,. \qquad (*)$$

By contraposition, it follows that $\mathbb{N}$ is not the union of only all FISONs of identifiable natural numbers. What remains? Numbers that cannot be identified: Dark numbers. ■

Conclusion: If $\mathbb{N}$ is larger than all unions of FISONs (see section "All gapless unions of FISONs are not enough to produce $\mathbb{N}$"), then it must contain something larger than all FISONs. Since $\mathbb{N}$ is infinite but no FISON is, almost all elements of $\mathbb{N}$ are dark numbers.

Every identifiable natural number is followed by infinitely many natural numbers including identifiable numbers and a set Y, with $|Y| = \aleph_0$, of dark numbers. Always many further numbers of the infinite endsegment can be identified and added to the first segment. But beyond those there will always remain an endsegment embracing $\aleph_0$ not identifiable natural numbers, the set Y of dark numbers. This is *the* set Y that remains absolutely inaccessible. Set theorists will dispute the existence of this set Y but will not be able to refute it.

Since the sequence of natural numbers cannot have two consecutive actually infinite segments, the first segment of the sequence, consisting of FISONs up to every identifiable natural number, is not actually infinite but always finite, that is: potentially infinite.

The implication (*) only holds if $\mathbb{N}$ is larger than every FISON, because only then every FISON is too small to be relevant for the union. In case of a potentially infinite view $\mathbb{N}$ is not a fixed set but the temporary maximum F_{max} of an ever increasing sequence of FISONs:

$$F_1 \cup F_2 \cup F_3 \cup ... \cup F_{max}\,. \qquad (**)$$

In this view not all FISONs exist simultaneously, there is always a next one and therefore not all FISONs of the union can be omitted without changing the result of (**). Every identifiable natural number is finite, has a finite decimal representation, and belongs to a finite initial segment.

Alternative proof using endsegments.

Definition: $E_n = (n, n+1, n+2, n+3, ...)$ is called the endsegment of n.
Definition: A natural number n is identifiable if it can occupy the first position of an endsegment.

Only this one number can vanish and vanishes between E_n and E_{n+1}:

$$E_n \setminus E_{n+1} = \{n\}. \qquad (***)$$

For every identifiable natural number n there is a first endsegment, E_{n+1}, where it vanishes. However the sequence (E_n) of endsegments

$$E_1, E_2, E_3, ... = E_1, E_1 \cap E_2, E_1 \cap E_2 \cap E_3, ...$$

contains only members of cardinality $\aleph_0$:

$$|E_1| = \aleph_0, |E_1 \cap E_2| = \aleph_0, |E_1 \cap E_2 \cap E_3| = \aleph_0, ...$$

This can be shown by induction: $E_1 = \mathbb{N}$ has cardinality $\aleph_0$. If $|E_n| = \aleph_0$ then $|E_{n+1}| = \aleph_0 - 1 = \aleph_0$. Further, inclusion monotony shows that all natural numbers of E_n are contained in all preceding endsegments, and almost all are contained in all succeeding endsegments. No natural number can be found[1] that is in any E_n but not in $E_1 = \mathbb{N}$. For any two endsegments we have $|E_m \cap E_n| = \aleph_0$. Since an endsegment E_n cannot have cardinality $\aleph_0$ without containing $\aleph_0$ elements, there is an infinite set Y subset of all endsegments. This is shown by contradiction: If there is an endsegment E_n that has not $\aleph_0$ elements in common with any succeeding endsegment E_{n+k}, then there is an endsegment, namely E_{n+k}, that has $\aleph_0$ elements but not $\aleph_0$ elements in common with a preceding endsegment, namely E_n, contradicting inclusion monotony and commutativity of $E_n \cap E_m$.

The infinite intersection $E_1 \cap E_2 \cap E_3 \cap ...$ is said to be empty in set theory although there is no additional endsegment available that could reduce the infinite sets even by one natural number. This is in contradiction with (***). The only way to save $E_1 \cap E_2 \cap E_3 \cap ... = \emptyset$ in spite of (***) is to assume a subset Y in all E_n that can remain without exhibiting itself. No element of Y can be identified and can occupy the first position of any endsegment and get lost. ■

Another indication of dark numbers is the fact that no smallest number can be identified that is required to make the endsegment (n, n+1, n+2, n+3, ...) actually infinite. It is not 1, it is not 2, it is not any *identifiable* number.

Remark: We do not use $\lim_{n\to\infty} |E_1 \cap E_2 \cap ... \cap E_n| = |\lim_{n\to\infty} E_1 \cap E_2 \cap ... \cap E_n|$ here but only the simple argument: The infinite intersection of endsegments contains only endsegments.
Remark: $\lim_{n\to\infty} E_n = \emptyset$ for only identifiable n would be tantamount to $\omega, \omega, \omega, ... \to 0$.
Remark: $\forall n \in \mathbb{N}$: $|\omega - n| = |\omega| = \aleph_0$. So there is no definable natural number in the set with $\aleph_0$ elements. What are these elements if not dark numbers?
Remark: For every accessible $n \in \mathbb{N}$ there is a first endsegment where it disappears. But *every* identifiable $n \in \mathbb{N}$ that disappeares leaves infinitely many successors. Without the existence of the set Y or an empty endsegment the intersection of endsegments cannot be empty.
Remark: The dark numbers solve the puzzle why $\lim_{n\to\infty} |E_n| \neq |\lim_{n\to\infty} E_n|$. The right-hand side concerns only identifiable numbers, the left-hand side concerns all numbers.

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[1] An empty intersection would require an empty endsegment, which has been excluded, or two numbers j and k and two endsegments E_m and E_n such that $j \in E_m \land j \notin E_n \land k \notin E_m \land k \in E_n$, which can be excluded by inclusion monotony, $E_n \supseteq E_{n+1}$, of the sequence (E_n).

Discussion: "The intersection of sets is defined by reference to elements not cardinalities, and this exercise shows that they are not interchangeable – even if any finite subcollection of sets has a non-empty (even infinite) intersection, it does not follow that the intersection of all the sets is non-empty." [Mark Bennet in "Can the intersection over all sets of a particular infinite sequence be empty?", Math.StackExchange (7 Oct 2018)] My reply: If all natural numbers are accessible, then there is no cardinality C of a set without C elements in that set that can be referred to.

Discussion: "To show that no number is element of *every* endsegment it is sufficient to show that every number is not element of at least one endsegment. The latter however is trivial: $n \notin E_{n+1}$." [Diedrich Ehlerding in "Dark matter in der Zahlentheorie", de.sci.mathematik (23 Oct 2018)] My reply: That concerns only identifiable numbers. To show that the intersection of endsegments is not empty is trivial: Every intersection is that endsegment with fewest elements. None is empty.

Discussion: "The mistake in WM's fallacy is that the intersection only then would be not empty if every 'endsegment' had numbers in common with all its predecessors *and simultaneously* with all its successors. The first condition is trivial [...] the second one is simply false." [Detlef Müller in "Dark matter in der Zahlentheorie", de.sci.mathematik (24 Oct 2018)] My reply: The second condition follows from the first condition. If there is an endsegment having less than $\aleph_0$ numbers in common with all its successors, then there is a first successor violating the first condition.

Discussion: At *every* finite number of steps there are almost all elements not at the first position. "And almost all steps left to execute." [Martin Shobe in "Can universal quantification circumvent this fact? If so, how? If not, why is it called universal?", sci.math (24 Oct 2018)] My reply: Correct, but this situation never changes. They are left to execute forever. This proves that almost all natural numbers cannot be selected or identified.

Discussion: "ZFC has no notion of accessible / non-accessible, it doesn't say that the elements of ω are accessible." [Jan Burse in "The nature of truth", sci.logic (9 Mar 2019)] My reply: The basis of human intellectual activity, which is much deeper than any artificial theory and is prerequisite also to set theory, is endowed with the notion of accessibility and distinguishes it from inaccessibility in practice.

Discussion: "What kind of 'proof' are you talking about?" [Jan Burse, loc cit] My reply: The best possible one, not refutable by axioms and modern logic: Predicting failure to access a natural number having less than infinitely many successors with 100 % certainty.

Discussion: "The problem is that you use a sequence instead of a series. [Jan Burse in "Is there a suitable English word to describe this situation?", sci.math (30 Jul 2019)] My reply: The infinite intersection $E_1 \cap E_2 \cap E_3 \cap ... = \emptyset$ is empty according to set theory. But there is no endsegment making $E_1 \cap E_2 \cap E_3 \cap ... \neq \emptyset$ wrong. Therefore the non-empty intersection embraces "all endsegments", i.e., it is infinite according to set theory.

Discussion: "The infinite product $1/1 \cdot 1/2 \cdot 1/3 \cdot ... = 0$ according to analysis. But there is no first term $1/n$ making $1/1 \cdot 1/2 \cdot 1/3 \cdot ... \cdot 1/n \neq 0$ wrong." [Franz Fitsche in "Natural logic vs Foundations of Mathematics", sci.math (31 Jul 2019)] My reply: There is no infinite product in analysis. Every product is finite and positive. Only the *limit* is 0. Set theory however has also the infinite product.

Limits of sequences of sets

The real sequence $(1/n)_{n\in\mathbb{N}}$, although being a set too, has limit 0 because the *numerical values of the elements* are concerned. If however *numbers of elements* are measured, necessarily with quantized result, then a proper limit can and does only exist when the number gets constant.

Limit of sequences of cardinal numbers

Sometimes one hears the argument that the limit of the cardinal numbers of the sets of a sequence need not be equal to the cardinal number of the limit of the sequence of sets:

$$\lim_{n\to\infty} |S_n| \neq |\lim_{n\to\infty} S_n| \ .$$

This view neglects that cardinality $|S_n| > 0$ is inseparably connected with elements populating S_n. They cannot wander around in the limit "in the infinite" after all finite steps. Therefore we obtain from a sequence with $\forall n \in \mathbb{N}$: $S_n \neq \emptyset$ a non-empty limit.

Moreover that possibility would destroy analysis, since, if the limit cardinal number could not be calculated, using analysis, from the cardinal numbers of the sets of the sequence, i.e., if an arbitrary jump of the limit was possible, then a sequence like $(1/n)$ could acquire the limit 1. Further it would destroy set theory because then the first infinite cardinality $\aleph_0$, by definition the cardinality of the limit $\mathbb{N}$ of the sequence of all its finite initial segments $F_n = (1, 2, 3, ..., n)$, could, in principle, jump to any other value like, for instance, $|\mathbb{R}|$.

Every set of n positive integers contains at least one integer $m \geq n$. The sequence of least maximum numbers $m_{\min}(n)$ is dominating the sequence n. According to analysis we have then

$$\lim_{n\to\infty} m(n) \geq \lim_{n\to\infty} n \ .$$

As long as there are no unnatural integers m, there cannot be an unnatural number n of elements.

Undefined limit

According to set theory the following sequence of sets of pairs, where the first number is consecutively increasing and the second number marks the position of the pair, can be understood as a supertask:

$S_1 = ((1, 1))$
$S_2 = ((2, 1), (3, 2))$
$S_3 = ((4, 1), (5, 2), (6, 3))$
$S_4 = ((7, 1), (8, 2), (9, 3), (10, 4))$
... .

It has the limit { } in the first numbers (because ℕ becomes exhausted) and ℕ in the second numbers (because finally all indices are issued). What would be the result if both numbers simultaneously were written on physical objects?

The limit ∅ of the sequence of ordered sets

$$(1, 1), (1, 2), (1, 3), ...$$

is not existing *in the classical domain* because

$$(1, 0), (1, 0), (1, 0), ... \to (1, 0)$$

and

$$(1, 1), (2, 2), (3, 3), ... \to (\infty, \infty) .$$

For autonomously existing variables, separated by a comma, we have in the classical domain

$$\lim (a_n, b_n) = (\lim a_n, \lim b_n) .$$

Set theory appears to show quantum entanglement, such that

$$\lim (a_n, b_n) \neq (\lim a_n, \lim b_n) .$$

Only then the sequence $((1, n))_{n\in\mathbb{N}}$ could "converge" towards ∅. But if parentheses make any effect

$$\begin{array}{ccc} \{1\} & , & \{1\} \\ \{1\} & , & \{2\} \\ \{1\} & , & \{3\} \\ \vdots & , & \vdots \\ \downarrow & , & \downarrow \\ \{1\} & , & \emptyset \end{array} \qquad\qquad \begin{array}{c} (\ \{1\}\ ,\ \{1\}\ ) \\ (\ \{1\}\ ,\ \{2\}\ ) \\ (\ \{1\}\ ,\ \{3\}\ ) \\ \vdots\ \vdots\ \ ,\ \ \vdots\ \vdots \\ \downarrow \\ \emptyset \end{array}$$

then not even constants *a* and *b* will remain the same when being included in parentheses (*a*, *b*).

Discussion: "No, the sequence of ordered pairs is not just 'considering the two sequences separately'. It's a third sequence." [Martin Shobe in "Entangled states in set theory?", sci.logic (27 Feb 2016)] "The parentheses change the meaning. Once again, you are surprised that when you change something, something changes." [Martin Shobe in "Entangled states in set theory?", sci.logic (28 Feb 2016)] My reply: Note that *in the finite domain* nothing changes at all.

Discussion: "If $\lim (a_n, b_n) = (\lim a_n, \lim b_n)$ does it follow that $\lim (b_n, a_n) = (\lim b_n, \lim a_n)$ regardless of the meaning of (..., ...) ?" [Jürgen Rennenkampff in "Entangled states in set theory?", sci.logic (29 Feb 2016)] My reply: It is not independent of the meaning of the parentheses. For a_n/b_n for instance it would be relevant. But obviously an ordering does not play a role for two sequences simply existing side by side. This is always true when the parentheses include two sequences of sets which according to set theory "converge" to limits.

Set theory depends on notation.

The sequence

$$\{1/1\}, \{1/2\}, \{2/1\}, \{1/3\}, \{2/2\}, \{3/1\}, \{1/4\}, \{2/3\}, \{3/2\}, \{4/1\}, \{1/5\}, \ldots \to \{\ \}$$

is an enumeration of the (not cancelled) positive fractions. Therefore this sequence including all fractions of the form n/n must have an empty limit if no fraction remains uncounted (all uncancelled fractions appear as finite terms of the sequence), cp. section 2.1 "Countable sets". So, the sequence of only all fractions of the form n/n

$$\{1/1\}, \{2/2\}, \{3/3\}, \ldots \to \{\ \}$$

a fortiori has an empty limit. This subsequence, when evaluated however,

$$\{1\}, \{1\}, \{1\}, \ldots \to \{1\}$$

is mathematically the same and has the limit $\{1\}$, i.e., it is not empty. This proves that set theory makes its results depending on whether or not fractions have been cancelled. That is incompatible with mathematics.

If in the sequence (S_n) with $S_n = \{n \mid n\}$ the separator symbol is interpreted as division mark, then $\lim_{n\to\infty} \{n \mid n\} = \lim_{n\to\infty} \{1\} = \{1\}$. If the separator symbol is interpreted as a comma, then $\lim_{n\to\infty} \{n \mid n\} = \{\ \}$. Can the fractions n/n representing 1 be exhausted like the natural numbers?

The sequence of ordered sets

$$(1, 1), (1, 2), (1, 3), \ldots$$

is a template for sequences like

$$\{1/1\}, \{1/2\}, \{1/3\}, \ldots \to \{\ \}$$

or

$$\{1^1\}, \{1^2\}, \{1^3\}, \ldots \to \{1\}\ .$$

Limits of some sequences depending on notation:

$\{1_1\}, \{1_2\}, \{1_3\}, \ldots \to \{1\}$ whether or not the symbols are evaluated.
$\{1^1\}, \{2^1\}, \{3^1\}, \ldots \to \{\ \}$ when the symbols are evaluated.
$\{1^1\}, \{2^1\}, \{3^1\}, \ldots \to \{\ ^1\}$ when the symbols are not evaluated.
$\{0^1\}, \{0^2\}, \{0^3\}, \ldots \to \{0\}$ whether or not the symbols are evaluated.
$\{1^0\}, \{2^0\}, \{3^0\}, \ldots \to \{1\}$ when the symbols are evaluated.
$\{1^0\}, \{2^0\}, \{3^0\}, \ldots \to \{\ ^0\}$ when the symbols are not evaluated.

[W. Mückenheim: "Who know's?", sci.math (15 & 20 Feb 2016)]

Problems of defining a bijection

The proof of equinumerosity by bijection between infinite sets is facilitated by mathematical induction, cp. for instance section 2.1 "Countable sets" where rules for enumerating the sets of rational numbers and algebraic numbers are given. It is clear that these sets have to be well-ordered in order to apply the given rules and that always only subsets[1], in particular finite initial segments, are considered.

The question is how to pass from the rule for every finite initial segment of the ordered set to the complete, actually infinite set. Modern set-theorists do not like this question and usually refuse to answer it (because there is no answer). They argue that all elements are mapped simultaneously. But of course every sequence can be analyzed in mathematics step by step. That is the core of "counting" and "being countable".

Cantor has coined and applied various terms for the process of mapping in his collected works like mapping- or assignment law, mapping- or assignment modus, mapping- or assignment procedure, mapping- or assignment process, and mapping- or assignment relation.[2]

The problem is that a rule applied by mathematical induction (or any other means involving finite initial segments) will never yield the set theoretic limit "empty set" for a sequence (S_n) with

$$S_n = (n, n+1, n+2, ..., 2n)$$

(cp. section 2.16 "Set-theoretical limits of sequences of sets"). Further it is impossible to argue that and how the limit set, if existing as a real set, can combine the two features, namely being empty and having cardinal number $\aleph_0$, i.e., resulting from a sequence of sets with infinitely increasing cardinal numbers. According to mathematical analysis the limit ∞ is following from the sequence of cardinal numbers

$|(1, 2)| = 2$
$|(2, 3, 4)| = 3$
$|(3, 4, 5, 6)| = 4$
$|(4, 5, 6, 7, 8)| = 5$
... .

Cantor called it ω or later, when he recognized the problems of transfinity and tried to overcome them by splitting the infinite in ordinal numbers and cardinal numbers, $\aleph_0$.

---

[1] Zwei geordnete Mengen M und N nennen wir "*ähnlich*", wenn sie sich gegenseitig eindeutig einander so zuordnen lassen, daß wenn m_1 und m_2 irgend zwei Elemente von M, n_1 und n_2 die entsprechenden Elemente von N sind, alsdann immer die Rangbeziehung von m_1 zu m_2 innerhalb M dieselbe ist wie die von n_1 zu n_2 innerhalb N. Eine solche Zuordnung ähnlicher Mengen nennen wir eine "Abbildung" derselben aufeinander. Dabei entspricht jeder Teilmenge M_1 von M (die offenbar auch als geordnete Menge erscheint) eine ihr ähnliche Teilmenge N_1 von N. [Cantor, p. 297]

[2] We find the terms Zuordnungsgesetz, Zuordnungsmodus, Zuordnungsprozeß, Zuordnungsverfahren, Zuordnungsverhältnis. [Cantor, pp. 239, 283f, 286f, 291, 293, 305f, 413]

Improper limits

A proper limit is a state that is approached better and better by the terms of a sequence. The meaning of an improper limit is only that the sequence like (n) or (2^n) increases beyond any given bound. It does not exist as or represent a quantity. Is

$$2^{\lim_{n\to\infty} n} = \lim_{n\to\infty} 2^n \text{ ?}$$

In calculus we cannot decide what ∞/∞ is. But often the unbounded increase on both sides is accepted as the improper limit ∞. Many write 2·∞ = ∞, for instance. In this sense the above equality is obvious. And when we refrain from using exponential notation, then both sides simply read 2·2·2·... = 2·2·2·... so that there cannot be any difference.

Now let $S_n = \{n, n+1, n+2, ...\}$. Why is $0 = |\lim_{n\to\infty} S_n| \neq \lim_{n\to\infty} |S_n| = \infty$?

Also in this case we have improper limits only, showing a never ending process:
- $\lim_{n\to\infty} S_n = \{ \}$ expresses the fact that n will not be in sets following upon S_n.
- $\lim_{n\to\infty} |S_n| = \infty$ expresses the fact that infinitely many naturals will follow upon every n.

It is very simple. No contradiction. No exhaustion. And therefore no proof of complete bijection or countability of infinite sets. But many will refuse to understand this because it is so tempting and easy to confuse infinite sets with finite sets and to think that infinite sets could be finished and enumerated too. Usually the strict application of analysis is disparaged as "intuition", sometimes even denounced as "moral":

"The limit of the cardinalities {{of the above S_n}} is $\aleph_0$. Why do you neglect that fact and choose arbitrarily the contrary?" [Heinrich in "Discussion between Steven Gubkin and Heinrich", MathEducators.StackExchange (8 Nov 2017)] "You are unwilling to engage with the precise definitions of things, preferring some sort of moral arguments. Since these are delicate issues, they require rigor, not simply intuition." [Steven Gubkin, loc cit]

Remarkable sequences of sets and their different limits

The following sequences are constructed by always removing the terms with n and inserting the terms with n+1. Since *no term stays forever* the set limit is empty and the cardinality of the set limit is 0. Applying actual infinity we "get ready". Then all natural numbers have been exhausted.

(A_n) with $A_n = \{n\}$ has $\lim_{n\to\infty} A_n = \{ \}$, $|\lim_{n\to\infty} A_n| = 0$, $\lim_{n\to\infty} |A_n| = 1$.
(B_n) with $B_n = \{n^n\}$ has $\lim_{n\to\infty} B_n = \{ \}$, $|\lim_{n\to\infty} B_n| = 0$, $\lim_{n\to\infty} |B_n| = 1$.
(C_n) with $C_n = \{a + b\cdot n\}$ has $\lim_{n\to\infty} C_n = \{a\}$, $|\lim_{n\to\infty} C_n| = 1$, $\lim_{n\to\infty} |C_n| = 1$.
(D_n) with $D_n = \{-1/n, 1/n\}$ has $\lim_{n\to\infty} D_n = \{ \}$, $|\lim_{n\to\infty} D_n| = 0$, $\lim_{n\to\infty} |D_n| = 2$.

If the set limits, i.e., the sets "at ω", are empty, the limits of the sequences of cardinalities differ from the cardinalities of the sets "at ω".

Difference between union and limit of an infinite sequence

Mark with a pencil, without lifting it in between, all points 1, 1/2, 1/4, 1/8, ... of the sequence $(1/2^n)_{n\in\mathbb{N}_0}$. Get every point of the interval (0, 1] marked. The limit point 0 remains unmarked.

Mark with a pencil, without lifting it in between, all points 1, 2, 4, 8, ... of the sequence $(2^n)_{n\in\mathbb{N}_0}$. Get every point of the interval [1, ω) marked. The limit point $\omega = \aleph_0$ remains unmarked.

Two complementary results

The union of all closed real intervals

$$A(n) = [1/n, 1] = \{x \in \mathbb{R} \mid 1/n \le x \le 1\}$$

is the half-open interval $(0, 1] = \{x \in \mathbb{R} \mid 0 < x \le 1\}$ because for every $n \in \mathbb{N}$ the interval $A(n)$ is contained therein. The smallest closed interval containing this union is [0, 1]. It contains an additional element, not present in any of the unioned intervals $A(n)$, namely 0. This is also the analytical limit because the analytical limit of the sequence $(1/n)$ is 0, and no change can be caused by connecting, on paper or only in mind, the points $1/n$ to 1.

Natural numbers do not form open intervals. There are only closed intervals, e.g., FISONs

$$F(n) = \{1, 2, 3, ..., n\} .$$

Therefore the union of all FISONs cannot be embraced by an open interval but only by the closed interval {1, 2, 3, ..., ω} embracing all FISONs and, analogously to the additional 0 above, the additional element ω which is not contributed by any $F(n)$ to the union.

Conclusion: The union of all FISONs, not containing the limit ω of (n), is less than its limit $\mathbb{N}$.

A yawning chasm

The sequence $\left(\frac{n-1}{n}\right)_{n\in\mathbb{N}}$ of points $\frac{n-1}{n}$ is a strictly monotonically increasing sequence, i.e., according to analysis it does not attain its limit 1 as a term. The points with coordinates $\frac{n-1}{n}$ lie on the real axis in the interval [0, 1). When connecting every point geometrically by a line to the origin 0 we get the sequence of intervals $\left(\left[0, \frac{n-1}{n}\right]\right)_{n\in\mathbb{N}}$. Obviously these connections have no influence on the limit, such that $\lim\limits_{n\to\infty}\left[0, \frac{n-1}{n}\right] = \left[0, \lim\limits_{n\to\infty}\frac{n-1}{n}\right] = [0,1]$.

The "set-theoretical limit" is the union of all intervals. Since 1 is not included, it is only the half-open interval [0, 1). This however differs from analysis because *every* $x < 1$ is contained in *infinitely many* closed intervals $\left[0, \frac{n-1}{n}\right]$ of the sequence together with *infinitely many* y such that $x < y < 1$. The least upper bound of analysis is 1 and therefore the analytical limit of the sequence of intervals is [0, 1].

We see here a tiny but insurmountable difference between the analytical limit and the set-theoretical union, usually denoted as limit. This is usually acknowledged. "In the infinite" however, where this gap grows to a yawning chasm, it is neglected. If the sequence of finite initial segments of natural numbers (FISONs)

$$\{1\}, \{1, 2\}, \{1, 2, 3\}, ..., \{1, 2, 3, ..., m\}, ... \to \mathbb{N}$$

is considered, then their *limit* $\mathbb{N} = \{1, 2, 3, ...\}$ is identified with the union of FISONs which, according to the above, is not correct. The limit $\mathbb{N}$ with $|\mathbb{N}| = \aleph_0$ is not reached by the union as the limit ω is not reached by the sequence of ntural numbers

$$1, 2, 3, ... \to \omega .$$

Usually it is understood that not all $\aleph_0$ natural numbers are in any FISON

$$\neg\exists m \in \mathbb{N}\ \forall n \in \mathbb{N}: n \in \{1, 2, 3, ..., m\}$$

but it is claimed that the union of all FISONs contains all $\aleph_0$ natural numbers

$$\forall n \in \mathbb{N}\ \exists m \in \mathbb{N}: n \in \{1, 2, 3, ..., m\} .$$

These statements however are simply expressing the properties of the potentially infinite set of FISONs. In fact they do not concern $\aleph_0$ naturals. This can be seen when the latter statement is given more precisely, replacing "n" by "n and all its predecessors", i.e., by "$\{1, 2, 3, ..., n\}$":

$$\forall n \in \mathbb{N}\ \exists m \in \mathbb{N}: \{1, 2, 3, ..., n\} \subset \{1, 2, 3, ..., m\}$$

making clear that by "$\forall n$" never $\aleph_0$ natural numbers are addressed, because always only FISONs are accepted, limiting the quantification "$\forall n \in \mathbb{N}$" to a never actually infinite set, the elements of which can always be accommodated in other FISONs.

From the limits considered above we have to draw the conclusion that the union of all FISONs is not the analytical limit ω and does not contain $|\mathbb{N}| = \aleph_0$ natural numbers.

[W. Mückenheim: "Fundamental theorem", sci.math (9 & 11 Nov 2016)]

The limit depends on direction.

In the sequence

01
0011
000111
00001111
...

little by little every index is covered by zero. The limit is an infinite sequence of zeros 000..., *if* we start to enumerate from the left-hand side as is usual in European literature and in the notation of digits behind the decimal point. Starting from the right-hand side, as is usual in Arabic literature and in denoting the digits of integers, the limit is an infinite sequence of ones 111... . When indexing the digits alternatingly like

..., -5, -4, -3, -2, -1, 1, 2, 3, 4, 5, ...

then the limit of the sequence has infinitely many zeros at the left-hand side and infinitely many ones at the right-hand side. What may happen when writing from top to bottom?

The limit depends on indexing.

Always add two digits 1 to the right-hand side and shift that one with the smallest index remaining behind the decimal point to the left-hand side immediately in front of the decimal point:

$1_1.1_2$
$1_11_2.1_31_4$
$1_11_21_3.1_41_51_6$
$1_11_21_31_4.1_51_61_71_8$
... .

According to set theory this sequence has limit ω because all natural indices are accumulated on the left-hand side.

Always add two digits 1 to the left-hand side and shift that one with the smallest index remaining in front of the decimal point to the right-hand side immediately behind the decimal point:

$1_2.1_1$
$1_41_3.1_21_1$
$1_61_51_4.1_31_21_1$
$1_81_71_61_5.1_41_31_21_1$
... .

According to set theory this sequence has limit 1/9 (at most) because all natural indices are accumulated on the right-hand side.

Always add two digits 1, one to the left-hand side and one to the right-hand side:

$1_1.1_2$
$1_3 1_1.1_2 1_4$
$1_5 1_3 1_1.1_2 1_4 1_6$
$1_7 1_5 1_3 1_1.1_2 1_4 1_6 1_8$
... .

According to set theory this sequence has limit ω because all odd natural indices are accumulated on the left-hand side.

When not indexing, it is impossible to distinguish these three cases. The limit is always ω because infinitely many digits are gathered in front of the decimal point.

The limit depends on representation.

When numbers are understood as sets (since in ZFC everything is a set) we can write

$$1, 2, 3, \ldots \to \{\ \} \,. \qquad (*)$$

For the sequence of the finite initial segments of $\mathbb{N}$ we obtain

$$\{1\}, \{1, 2\}, \{1, 2, 3\}, \ldots \to \mathbb{N} \,.$$

But when representing the natural numbers according to von Neumann (cp. section 2.12.6), then we get the sequence of natural numbers with a limit different from that of (*):

$$\{0\}, \{0, 1\}, \{0, 1, 2\}, \ldots \to \mathbb{N} \,.$$

Discussion: "The only convergent sequences in the discrete topology are eventually constant sequences, and so the limit formulas in your question are simply false if you use the discrete topology on the power set." [Lee Mosher in "Is there a reference for this paradox?", Math.StackExchange (30 Sep 2016), meanwhile deleted] That is correct in principle but: "The discrete topology is the only one accessible here since sets consist of discrete elements. They are quantized. The limit formulas have been developed for sets with discrete elements. That they are useless is just shown again by my observation." [Hans, loc cit]

Discussion: "Well, you are arguing that the two limits should be the same because the individual terms represent the same natural numbers. But this is nonsense! The limit is defined via the terms themselves – that is, what specific sets they are. Conflating two different definitions of 'natural number' is of course going to lead to contradictions." [Noah Schweber, loc cit] My reply: Here we use *two different expressions* for *one and the same notion* of natural number. If both sets express the same natural numbers, like II = 2 = {0, 1}, then the limits must be identical, unlike $\emptyset$ and $\mathbb{N}$.

Discussion: "These are simply different sets, and it's no paradox that you use different codewords in different codes. This is no more a paradox than observing that the word for 2 starts with 't' in English, but it starts with 'd' in French. How can that be? It's the same number after all! But the codes are different." [Mitchell Spector, loc cit] "The codes may be different: 2 or II or two, but if $\mathbb{N}$ is coded as { } then there went something wrong." [Hans, loc cit]

Disappearing sequences (I)

Consider a sequence of sets where we, starting from $\{2^0\}$, replace $\{2^n\}$ by $\{2^{n+1}\}$. We get the sequence of singletons with empty limit:

$$\{1\}, \{2\}, \{4\}, \{8\}, ... \to \{ \} .$$

In unary representation this sequence

{I}, {II}, {IIII}, {IIIIIIII}, ... → { }

has an empty limit too, although the continuously doubling strokes, like slipper animalcule (paramecium), cannot know that they are interpreted as natural numbers and eventually will have to disappear and, if indexed, yield the unearthly picture $\{ {}_1, {}_2, {}_3, {}_4, {}_5, ...\}$.

Note that the different representations of natural numbers by Zermelo and von Neumann (cp. section 2.12.6) cause the same problem.

[W. Mückenheim: "Das Kalenderblatt 120413", de.sci.mathematik (12 Apr 2012). W. Mückenheim: "Slipper animalcule or natural numbers", sci.math (1 Jul 2016)]

Many mathematicians don't know at all about this problem (see 2.16 "Set-theoretical limits of sequences of sets"): "Where are you getting this 'empty limit' bullshit?" [Dan Christensen in "Slipper animalcule or natural numbers", sci.math (1 Jul 2016)] "By what mythical rule does WM claim that the limit of a sequence of non-empty sets must be empty?" [Virgil, loc cit (1 Jul 2016)]

Others have a very healthy intuition: "It seems to me that this set is just getting fuller, not empty." [Konyberg, loc cit (1 Jul 2016)]

Only very few can recognize the far-reaching consequences: "Ingenious. An empty set with $\aleph_0$ strokes! How much more screwed up can set theory get ..." [John Gabriel, loc cit (7 Jul 2016)]

Disappearing sequences (II)

Consider the $\omega + 1$ terms

$$(1), (2), (3), ..., (\omega), (\omega + 1) .$$

There is no finite ordinal number missing, there is no gap. This fact remains unchanged, when we double each number x to get $2x$ and add the predecessor of each one:

$$(1, 2), (3, 4), (5, 6), ..., (2\omega - 1, 2\omega), (2\omega + 1, 2\omega + 2) .$$

Again double each last number $2x$ to get $4x$ and add the three predecessors of $4x$

$$(1, 2, 3, 4), (5, 6, 7, 8), (9, 10, 11, 12), ..., (4\omega - 3, 4\omega - 2, 4\omega - 1, 4\omega),$$
$$(4\omega + 1, 4\omega + 2, 4\omega + 3, 4\omega + 4)$$

and so on. At the nth doubling-step we get

$$(1, ..., 2^n), (2^n + 1, ..., 2\cdot 2^n), (2\cdot 2^n + 1, ..., 3\cdot 2^n), ..., (2^n\omega - 2^n + 1, 2^n\omega - 2^n + 2, ..., 2^n\omega),$$
$$(2^n\omega + 1, 2^n\omega + 2, ..., 2^n\omega + 2^n) .$$

Obviously all ordinal numbers from 1 to $\omega + 1$ are in the first and every further row. But in the limit ($n \to \omega$) only the set of all finite ordinal numbers (1, 2, 3, ...) from the first sequence (within the first parentheses) remains. All other parentheses have disappeared (or are empty). Since n *approaches* but does not *reach* ω, no multiple of 2^n is defined there.

[W. Mückenheim: "Das Kalenderblatt 100729", de.sci.mathematik (5 Aug 2010)]

Disappearing sequences (III)

Consider the $\omega + 1$ terms and manipulate them similar to section "Disappearing sequences (II)":

$$(1), (2), (3), ..., (\omega), (\omega + 1)$$
$$(1, 2), (3, 4), (5, 6), ..., (2\omega - 1, 2\omega), (2\omega + 1, 2\omega + 2)$$
$$(1, 2, 3), (4, 5, 6), (7, 8, 9), ..., (3\omega - 2, 3\omega - 1, 3\omega), (3\omega + 1, 3\omega + 2, 3\omega + 3)$$

and so on. At the nth step we get

$$(1, ..., n), (n + 1, ..., 2n), (2n + 1, ..., 3n), ..., (n\omega - n + 1, n\omega - n + 2, ..., n\omega),$$
$$(n\omega + 1, n\omega + 2, ..., n\omega + n) .$$

Obviously all ordinal numbers from 1 to $\omega + 1$ are in the first and every further row. But in the limit only the set of all finite ordinal numbers (1, 2, 3, ...) from the first sequence (within the first parentheses) remains. All other parentheses have disappeared (or are empty).

The union of the first row is $(1, 2, 3, 4, 5, 6, 7, 8, 9, ..., \omega + 1)$.
The union of the second row is $(1, 2, 3, 4, 5, 6, 7, 8, 9, ..., 2\omega + 2)$.
The union of the third row is $(1, 2, 3, 4, 5, 6, 7, 8, 9, ..., 3\omega + 3)$.
...
The union of the *n*th row is $(1, 2, 3, 4, 5, 6, 7, 8, 9, ..., n\omega + n)$.
...
The limit of the unions is $(1, 2, 3, 4, 5, 6, 7, 8, 9, ..., \omega\omega + \omega)$.

A not disappearing set

The set $\{0\} \cup \{1\} \cup \{2\} \cup ...$ is in general recognized as $\mathbb{N}$.

The set $... ((... ((((\{0\} \cup \{1\}) \setminus \{1\}) \cup \{2\}) \setminus \{2\}) ... \cup \{n\}) \setminus \{n\}) ...$ might be recognized as $\{0\}$.

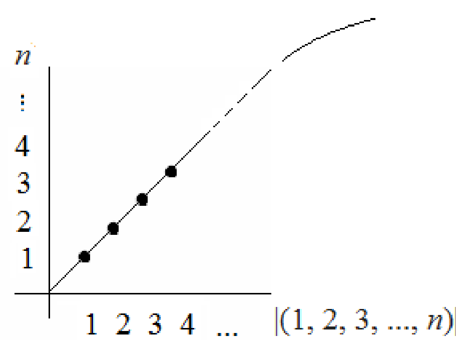
The set $... ((... ((((\{0\} \cup \{1\}) \setminus \{0\}) \cup \{2\}) \setminus \{1\}) ... \cup \{n\}) \setminus \{n - 1\}) ...$ is empty because all singletons will have been removed "in the limit".

But according to Cantor the cardinal number of a set remains constant, if instead of its elements m, m', m'', ... other things are substituted [Cantor, p. 413], i.e., if only elements are exchanged. Therefore the set has the cardinal number 1 after every removal and that will never change.

The bent graph

A basic theorem of set theory is

$$\forall n \in \mathbb{N}: n < \aleph_0 .$$



This strict inequality causes a remarkable feature of set theory. While for every finite initial segment of natural numbers $(1, 2, 3, ..., n)$ the last ordinal number and the cardinal number $|(1, 2, 3, ..., n)|$ are identical, this rule is violated "in the limit", i.e., for the whole set $\mathbb{N}$. There are $\aleph_0$ numbers each of which is less than $\aleph_0$. The straight graph of the function

$$|(1, 2, 3, ..., n)| = n$$

somehow has to change its slope "in the infinite".

[W. Mückenheim: "Das Unendliche", de.sci.mathematik (5 Jan 2005)]

Separating sequences

The sequence S of sets

$$S(n) = (n, n\text{-}1, ..., 3, 2, 1)$$

has a limit because both LimSup and LimInf exist and are equal, namely

$$\omega^* = (..., 3, 2, 1) .$$

If $S(n)$ is divided into two sets $S_1(n)$ containing all elements larger than $|S(n)|/2$ and $S_2(n)$ containing all elements not larger than $|S(n)|/2$, then the sequence S_2 has limit ω^*, and the limit of the sequence S_1 is empty. Here are the first terms of the sequences S_1 and S_2:

n	S_1	S_2
1	(1)	( )
2	(2)	(1)
3	(3, 2)	(1)
4	(4, 3)	(2, 1)
...	...	... .

Now take all elements of the set $S_2(n)$ which are not larger than $|S_2(n)|/2$ and put them into a new set $S_3(n)$. Here are the first terms of the three sequences S_1, S_2, S_3:

n	S_1	S_2	S_3
1	(1)	( )	( )
2	(2)	(1)	( )
3	(3, 2)	(1)	( )
4	(4, 3)	(2)	(1)
...	...	...	... .

This procedure can be continued. If there are m sets, then form a new set S_{m+1} containing all elements which are not larger than $|S_m(n)|/2$.

For $m \to \infty$ we have infinitely many sequences S_k, $1 \le k \le m$, the limit of each sequence being empty. Nevertheless the union of the terms $S_k(n)$ of these sequences (i.e., the series), taken for fixed n, contains n natural numbers. And the limit of this union is ω^* with cardinal number $\aleph_0$.

[W. Mückenheim: "Die Geschichte des Unendlichen", 7th ed., Maro, Augsburg (2011) p. 115]

Fibonacci-sequences with fatalities

The Fibonacci-sequence

$$f(n) = f(n\text{-}1) + f(n\text{-}2) \qquad \text{for } n > 2 \qquad \text{with } f(1) = f(2) = 1 \,,$$

the first recursively defined sequence in human history (Leonardo of Pisa, 1170-1240), should be well known. A pair of rabbits that reproduces itself monthly as from the completed second month on will yield a stock of 144 pairs after the first 12 months

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 .

If we assume that each pair reproduces itself after two months for the first and last time and dies afterwards, we get a much more trivial sequence:

1, 1, 1, ...

However the rabbits behind these numbers change. If we name them in the somewhat unimaginative but effective manner of the Old Romans, we get Prima, Secunda, Tertia, Quarta, Quinta, Sexta, Septima, Octavia, Nona, Decima and so on.

A more interesting question is brought up, if the parent pair dies immediately after the birth of its second child pair. Then the births $g(n)$ in month n can be traced back to pairs who have been born in months n-2 and n-3

$$g(n) = g(n\text{-}2) + g(n\text{-}3) \,.$$

The number $f(n)$ of pairs in month n is given by those born in month n, i.e. $g(n)$ and those already present in month n-1, i.e., $f(n$-1), minus those who died in month n (i.e. those who were born in month n-3):

$$\begin{aligned}
f(n) &= g(n) + f(n\text{-}1) - g(n\text{-}3) = g(n\text{-}2) + f(n\text{-}1) \\
g(n\text{-}2) &= f(n) - f(n\text{-}1) \\
g(n\text{-}2) &= g(n\text{-}4) + g(n\text{-}5) \\
&= f(n\text{-}2) - f(n\text{-}3) + f(n\text{-}3) - f(n\text{-}4) \\
&= f(n\text{-}2) - f(n\text{-}4) \,.
\end{aligned}$$

For $n > 4$ we have with $f(1) = 1, f(2) = 1, f(3) = 2, f(4) = 2$.

$$f(n) = f(n\text{-}1) + f(n\text{-}2) - f(n\text{-}4) \,.$$

The number of pairs during the first 12 months is

1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21 .

The sequence grows slower than the original one, but without enemies or other restrictions it will grow beyond every threshold. If we wait $\aleph_0$ days (or use the trick that the duration of pregnancy is halved in each step, facilitated by genetic evolution), we will get infinitely many pairs (although the set-theoretic limit of living pairs is empty because for *every* pair the date of death can be determined) – a nameless number, alas of nameless rabbits, because they cannot be distinguished. The set of all Old-Roman names has been exhausted already, and even all of Peano's New-Roman names S0, SS0, SSS0, ... have been passed over to pairs which already have passed away. (Since infinitely many have passed away during $\aleph_0$ days, the set of names has been exhausted.) That is amazing, since none of the pairs of the original and much more abundant Fibonacci sequence has to miss a name.

So we obtain from set theory: The cultural assets of distinguishability of distinct objects by symbols, names, or thoughts do not belong to the properties of Cantor's paradise. Like in the book of genesis, before Adam began to name the animals, we have a nameless paradise – not mathematics though.

But this sequence with fatalities can also be obtained without fatalities (killings), namely if each pair has to pause for two months after each birth in order to breed again in the following month. Mathematically, there is no difference. (Pair *P*, that originally dies away after breeding its second child pair *S*, takes the position of *S* and pauses for two months like the fresh pair *S* would have done.) Set theory, however, yields a completely different limit in this case. The limit set of living rabbits is no longer empty, but it is infinite – and every rabbit has a name.

[W. Mückenheim: "Das Kalenderblatt 120412", de.sci.mathematik (11 Apr 2012)]

The six sisters

Once upon a time there were six sisters who got lost in the dark forest. There an evil sorcerer found them and decided, first to fatten them and then to eat them.

When the sisters recognized their fate they craved mercy. The evil sorcerer, after short reflection, set them a task. Every sister should name a sequence of sets. The evil sorcerer promised to release those girls who, at the end, could show him a set with $\aleph_0$ many elements. (Of course he knew that none of them had ever heard of $\aleph_0$ – as I said he was an evil sorcerer.)

The two eldest, Anna and Bertha, began. They knew the even numbers already. So they simply used the sequence of finite initial segments of even numbers {2, 4, 6, ..., 2*n*}. Since each one should have her own sequence, Bertha took all numbers that were larger than *n* and Anna took the rest. So the sisters believed to share just and fair.

n	*A*(*n*)	*B*(*n*)
1	{ }	{2}
2	{2}	{4}
3	{2}	{4, 6}
4	{2, 4}	{6, 8}
5	{2, 4}	{6, 8, 10}
6	{2, 4, 6}	{8, 10, 12}
...	...	... .

Alas, this was fateful for Bertha. The evil sorcerer decided that Bertha's sequence had the empty set as its limit set, because for *every* even number there is a natural number, a step *n*, where it enters Anna's set.

Christa and Dora, who had observed Bertha's misfortune, were cleverer. They chose the sequences

n	$C(n)$	$D(n)$
1	{ }	{1}
2	{2}	{1}
3	{2}	{1, 3}
4	{2, 4}	{1, 3}
5	{2, 4}	{1, 3, 5}
6	{2, 4, 6}	{1, 3, 5}
...	...	... .

And the evil sorcerer had to confess that both had solved their task.

Now Edda and Frida remained. Frida had already recognized the whole lot, but she looked for a way to break the spell of poor Bertha. And when Edda started, Frida took a really sophisticated choice:

n	$E(n)$	$F(n)$
1	{ }	$\{\square_1\}$
2	{2}	$\{\square_1\}$
3	{2}	$\{\square_1, \square_2\}$
4	{2, 4}	$\{\square_1, \square_2\}$
5	{2, 4}	$\{\square_1, \square_2, \square_3\}$
6	{2, 4, 6}	$\{\square_1, \square_2, \square_3\}$
...	...	... .

Here identical indices should not imply equal elements but only distinguish free places. And although the evil sorcerer threatened her harshly, she did not reveal whether her squares should be filled with Bertha's or Dora's elements. This was too much for the evil sorcerer. His brain overheated and finally burst. And all six sisters were free and could run home.

Frida shows that the results of set theory depend on arbitrarily selected notation. That must never be accepted in science and not even in fairy tales.

[W. Mückenheim: "Die Geschichte des Unendlichen", 7th ed., Maro, Augsburg (2011) p. 113f]

There is no countable set

Diagonalization applied to the set of natural numbers

The set of natural numbers can be written in binary notation as a sequence

...00000
...00001
...00010
...00011
...00100
... .

This sequence can be diagonalized

...0000**1**
...000**1**1
...00**1**10
...0**1**011
...**1**0100
... .

The antidiagonal number $d_1d_2d_3... = 111...$ consists only of bits 1. For its initial segment of n bits we find

$$D(n) = d_1d_2d_3...d_n < 2^{n+1} .$$

As there is no infinite index n, we have

$$\forall n \in \mathbb{N}: D(n) \in \mathbb{N} .$$

Further for every finite initial segment of the sequence, there is a $D(n)$ that is not an element of that finite initial segment.

The set of natural numbers is uncountable.

For comparison: Cantor's original diagonal argument also holds only for every finite initial segment of any given sequence of real numbers. (Infinite sequences of digits without a structural plan do not define real numbers. See section "Sequences and limits".)

[W. Mückenheim: "Infinite sets are non-denumerable", arXiv (2003)]

Can the manner of marking influence the result?

Let (S_n) be the sequence of sets $S_n = \{n\}$ with $n \in \mathbb{N}$. This sequence has an empty limit set.

Let (T_n) be the sequence of sets $T_n = \{I_1, I_2, I_3, ..., I_n\}$ where we have indexed notches I in order to distinguish them. S_{n+1} emerges from s_n by adding notch number $n+1$. (A unary system is the historically first kind of representing natural numbers.) This sequence has not an empty limit set. The sequence of notches diverges towards ω, the seqence of sets of indices diverges towards |$\mathbb{N}$.

[W. Mückenheim: "Can the manner of marking influence the result?", sci.math (7 Jul 2014)]

A minimum

Let $F_n = \{1, 2, 3, ..., n\}$ be the nth finite initial segment of the set of natural numbers (FISON). Then the sequence (a_n) defined by $a_n = \min(10^{100}, |\mathbb{N}\backslash F_n|) = 10^{100}$ has limit 10^{100} in analysis but limit 0 in set theory because $\lim_{n\to\infty} F_n = \mathbb{N}$.

[W. Mückenheim: "Matheology § 295", sci.math (26 Jun 2013)]

Counting like an old-fashioned station clock

Hilbert proudly boasted that it is possible to simply count beyond the infinite: "These are Cantor's first transfinite numbers, the numbers of the second number class as Cantor calls them. We get to them simply by counting beyond the normal countable infinite, i.e., in a very natural and uniquely defined consistent continuation of the normal counting in the finite." [D. Hilbert: "Über das Unendliche", Math. Annalen 95 (1925) p. 169]

But sometimes this counting comes to a hold like an old-fashioned station clock that pauses at every full minute. Let us use commercial calculation for doing the counting. The results have been double underlined.

$$\begin{array}{lll} \{1\} & \{1\} & \{1\} \\ \{1, 2\} & \{1, 2\} & \{1, 2\} \\ \underline{\{1, 2, 3\}} & \{1, 2, 3\} & \{1, 2, 3\} \\ \underline{\underline{\{1, 2, 3\}}} & \underline{\ldots\qquad\qquad} & \ldots \\ & \underline{\underline{\{1, 2, 3, \ldots\}}} & \underline{\{1, 2, 3, \ldots\}} \\ & & \underline{\underline{\{1, 2, 3, \ldots\}}} \end{array}$$

The last step has not increased the sum set although more than before has been added, namely the infinite set $\mathbb{N}$ which is larger than all FISONs. The reason is that the preceding step yields a sum set that is larger than all summed sets. For inclusion-monotonic sequences of sets this should be impossible. But let us simply continue the counting "beyond the normal countable infinite, i.e., in a very natural and uniquely defined consistent continuation of the normal counting in the finite".

{1}	{1}	{1}	{1}
{1, 2}	{1, 2}	{1, 2}	{1, 2}
{1, 2, 3}	{1, 2, 3}	{1, 2, 3}	{1, 2, 3}
...	...	...	...
{1, 2, 3, ...}	{1, 2, 3, ...}	{1, 2, 3, ...}	{1, 2, 3, ...}
{1, 2, 3, ..., *a*}	{1, 2, 3, ..., *a*}	{1, 2, 3, ..., *a*}	{1, 2, 3, ..., *a*}
{1, 2, 3, ..., *a*}	{1, 2, 3, ..., *a*, *aa*}	{1, 2, 3, ..., *a*, *aa*}	{1, 2, 3, ..., *a*, *aa*}
	{1, 2, 3, ..., *a*, *aa*}	...	...
		{1, 2, 3, ..., *a*, *aa*, ...}	{1, 2, 3, ..., *a*, *aa*, ...}
			{1, 2, 3, ..., *a*, *aa*, ...}

The last step has not increased the sum set although more than in the preceding step has been added. The reason is again that the preceding step yields a sum set that is larger than all summed sets. The *sums* resulting from these summations are as follows

{1}
{1, 2}
{1, 2, 3}
...
{1, 2, 3, ...}
{1, 2, 3, ...}
{1, 2, 3, ..., *a*}
{1, 2, 3, ..., *a*, *aa*}
{1, 2, 3, ..., *a*, *aa*, *aaa*}
...
{1, 2, 3, ..., *a*, *aa*, *aaa*, ...}
{1, 2, 3, ..., *a*, *aa*, *aaa*, ...}
{1, 2, 3, ..., *a*, *aa*, *aaa*, ..., *b*}
{1, 2, 3, ..., *a*, *aa*, *aaa*, ..., *b*, *bb*}
{1, 2, 3, ..., *a*, *aa*, *aaa*, ..., *b*, *bb*, *bbb*}
...
{1, 2, 3, ..., *a*, *aa*, *aaa*, ..., *b*, *bb*, *bbb*, ...}
{1, 2, 3, ..., *a*, *aa*, *aaa*, ..., *b*, *bb*, *bbb*, ...}
...
{1, 2, 3, ..., *a*, *aa*, *aaa*, ..., *b*, *bb*, *bbb*, ..., *c*}
... .

and so on. Hilbert's counting is stuck on. From time to time it comes to a hold. The reason is that the union of all natural numbers shall be a transfinite number. That is impossible because the natural numbers count themselves, and there is nothing in the natural numbers larger than all natural numbers. This is disputed by set theorists, because they count in the following way:

$$1, 2, 3, ..., \omega, \omega+1, \omega+2, \omega+3, ..., \omega+\omega, \omega+\omega+1, ...$$

[W.Mückenheim: "Das Kalenderblatt 091206", de.sci.mathematik (11 Dec 2009)]

Two similar properties with different results

Every element q of the set $\mathbb{Q}$ of all rational numbers can be enumerated. In every step n we get another number q_n which has the properties (1) to have an index n and (2) to have $\aleph_0$ successors which are not enumerated in or before step n.

Now we can conclude that the property (1) is inherited by all elements of the complete set $\mathbb{Q}$ which therefore may be called a countable set, but although property (2) is also inherited by all elements of the complete set $\mathbb{Q}$ it must not be called a discountable set, that is a set all elements of which can be subtracted without changing the cardinality of the remainder.

[W. Mückenheim in "Two properties", sci.math. (25 Jan 2016)]

The ketchup effect

If you have ever used a new bottle of tomato ketchup, you will have experienced the ketchup effect: First barely some drops leave the bottle, but when you shake harder the whole contents splashes on your plate.

Same happens in transfinite set theory. When indexing the rational numbers, there will remain infinitely many real intervals, each one populated by infinitely many not indexed rational numbers. This remains so for all finite indices. But as soon as you have used infinitely many indices (i.e., never) all intervals are populated exclusively by indexed rationals.

[W. Mückenheim: "The ketchup effect", sci.math (10 Jun 2015)]

Failure of Cantor's first diagonal method

The diagram in section "2.1 Countable sets" allegedly shows a way to enumerate all positive fractions. It is called Cantor's first diagonal method because the enumeration proceeds parallel to the diagonal (from the lower left to the upper right).

1/1, 1/2, 1/3, 1/4, ...
2/1, 2/2, 2/3, 2/4, ...
3/1, 3/2, 3/3, 3/4, ...
4/1, 4/2, 4/3, 4/4, ...
... .

In fact the diagram shows quite the contrary. Every square of fractions that you are going to enumerate has a diagonal that cannot be traversed without enlarging the square and getting a new diagonal which cannot be traversed without enlarging the square and getting a new diagonal ... . Further you will never reach the diagonal of the "whole square". Therefore it is clear that *at least* half of all positive fractions will not become enumerated.

A counter argument says "the diagonal does not exist". This is astounding because the complete first row of $\aleph_0$ unit fractions "exists" as well as the complete first column of $\aleph_0$ natural numbers. Further in Cantor's second diagonal argument (cp. section "2.2.3 Cantor's second uncountability proof") the complete diagonal of $\aleph_0$ digits "exists".

[W. Mückenheim in "Ueberdeckung der rationalen Zahlen", de.sci.mathematik (8 Jun 2011)]

Uncountability of the rational numbers shown without diagonalizing (I)

The incompleteness of any enumeration of the reals and even of the rationals of the real interval (0, 1) can be shown without diagonalization, i.e., without constructing an antidiagonal number.

Assume an enumeration (r_k) as complete as possible of the real numbers

$$r_k = 0.r_{k1}r_{k2}r_{k3}...$$

of the unit interval (i.e., a Cantor-list). In every row replace every digit behind the diagonal digit r_{kk} by zero.[1] Get a rational number. Then in every row replace every digit from the first digit behind the decimal point to the diagonal digit by 1

$$q_k = 0.q_{k1}q_{k2}q_{k3} \;...\; q_{kk} = 0.111...1 \;.$$

Now the list contains only $\aleph_0$ different rational numbers

0.1
0.11
0.111
0.1111
... .

This procedure shows that there is no complete enumeration of the rational numbers (and hence of the real numbers) of the real interval (0, 1). After having created the list 0.1, 0.11, 0.111, ... without changing the number of entries, it is obvious that all possible places are occupied by different rational numbers. But no irrational numbers and no rational numbers with digits other than 0 and 1 are present. So they are missing from the whole list.

Note: It is allowed to insert any desired missing number between two rows or to prepend it to the list. But then the whole procedure has to be performed again. (When inserting a constructed antidiagonal number into the list in Cantor's original version, you would also be asked to repeat the diagonalization.)

[1] This would not change the resulting antidiagonal number. The digits beyond the diagonal digit are irrelevant for the construction of the antidiagonal number in Cantor's original diagonal procedure.

Uncountability of the rational numbers shown without diagonalizing (II)

Assume an enumeration of all positive rationals. Write them into a Cantor-list. Replace every entry by its natural index. Get the list

1
2
3
...

All proper fractions are missing. In order to re-introduce them, you have to remove naturals.

Of course you can also place the proper fractions before the first row or between two rows. But don't forget then, what you always recommend in case a layman wants to put the "antidiagonal number" into the list: Repeat the whole procedure with the new list!

Mirroring digits (I)

$\forall n \in \mathbb{N}$: If 10^{-n} is defined, then 10^{n} is defined.

If 10^{-1}, 10^{-2}, 10^{-3}, ..., 10^{-n} is defined, then 10^{1}, 10^{2}, 10^{3}, ..., 10^{n} is defined.

If $\sum_{k=1}^{n} 10^{-k}$ is defined, then $\sum_{k=1}^{n} 10^{k}$ is defined.

Obviously these transformations do not depend on the number of terms or the number of exponents, but solely on the condition that all exponents are natural numbers.

If, as is usually claimed, the infinite digit sequence 0.111... contains only the complete set of all negative integers as exponents, why is there no complete decimal representation with only the complete set of all positive integer exponents? The answer is that there is no complete set of integers. Only the limit of an infinite sequence or sum is a fixed quantity.

10^{-1}, 10^{-2}, 10^{-3}, ... cannot be mirrored to 10^{1}, 10^{2}, 10^{3}, ...

$\sum_{k=1}^{\infty} 10^{-k}$ cannot be mirrored to $\sum_{k=1}^{\infty} 10^{k}$.

[W. Mückenheim in "Abzählbare Liste aller Irrationalzahlen eines Intervalls – hier bitte!", de.sci.mathematik (24 Feb 2006)]

Mirroring digits (II)

The nth term of the following sequence consists of a real number which has n digits 1 before and n digits 1 behind the decimal point

1.1
11.11
111.111
... .

There is every natural number n of digits 1 before the decimal point possible because for every n there is a larger $n + 1$. Behind the decimal point however, there can be more than every natural number of digits 1, namely completed infinity. Note: Completed infinity is impossible before and behind the decimal point. But behind the decimal point the error is too small to be observable for most mathematicians.

Well-ordering the rational numbers by magnitude (I)

If all (positive) rational numbers exist, then all permutations should exist, because each number has a finite index, i.e., it is in finite distance from the first number 0 which is enumerated by 1. But then also the permutation with all rational numbers enumerated and sorted according to their magnitude should be obtainable within $\aleph_0$ steps and, therefore, should exist. Let

$$q_1, q_2, q_3, q_4, q_5, ...$$

be an enumeration of the positive rational numbers. Consider the first two elements

$$(q_1, q_2), q_3, q_4, q_5, ...$$

and order them by magnitude:

$$(q_1', q_2'), q_3, q_4, q_5, ...$$

Consider the first three elements

$$(q_1, q_2, q_3), q_4, q_5, ...$$

and order them by magnitude:

$$(q_1'', q_2'', q_3''), q_4, q_5, ...$$

Continue such that in the nth step the first n elements are ordered by magnitude. In the limit all enumerated rational numbers have been well-ordered by magnitude – if limits of non-converging sequences have any meaning. Otherwise, the enumeration of any non-converging sequence is meaningless too.

Well-ordering the rational numbers by magnitude (II)

It is well known that Cantor gave a method of enumerating all positive rational numbers (see section "2.1 Countable sets"). He constructed the sequence, the finite initial segments of which are repeated here – not because they are unknown, but because I wish to apply a new method to them:

1/1
1/1, 1/2
1/1, 1/2, 2/1
1/1, 1/2, 2/1, 1/3
1/1, 1/2, 2/1, 1/3, 3/1
1/1, 1/2, 2/1, 1/3, 3/1, 1/4
... .

This sequence

$$\frac{1}{1};\frac{1}{2},\frac{2}{1};\frac{1}{3},\frac{3}{1};\frac{1}{4},\frac{2}{3},\frac{3}{2},\frac{4}{1};\frac{1}{5},\frac{5}{1};\frac{1}{6},\frac{2}{5},\frac{3}{4},\frac{4}{3},\frac{5}{2},\frac{6}{1};\frac{1}{7},\frac{3}{5},\frac{5}{3},\frac{7}{1};\frac{1}{8},\frac{2}{7},\frac{4}{5},\frac{5}{4},\frac{7}{2},\frac{8}{1};\frac{1}{9},\frac{3}{7},\frac{7}{3},\frac{9}{1};...$$

will never end.

For every natural number we will have achieved less than $10^{-1000000000000000000000000000000000000}$ of the complete task, but, since every step is well-defined and absolutely fixed, we conclude from the fact that the enumeration holds up to *every* rational number that the enumeration holds for *all* rational numbers.

Now apply the same method, but, in addition, always put the finite initial segments in proper order by size (which is no problem as long as they are finite, i.e., as long as we are enumerating with finite natural numbers only – and other naturals are not known). This new method will change the sequence as follows

1/1
1/2, 1/1
1/2, 1/1, 2/1
1/3, 1/2, 1/1, 2/1
1/3, 1/2, 1/1, 2/1, 3/1
1/4, 1/3, 1/2, 1/1, 2/1, 3/1
1/4, 1/3, 1/2, 2/3, 1/1, 2/1, 3/1
... .

This sequence will never end, but, since also here every step is well-defined and absolutely fixed, we can conclude from the fact that enumeration and ordering hold up to *every* rational number that they hold for *all* rational numbers too.

Why don't we accept the second method, or, alternatively, why do we accept the first one?

Why zero has been included in the set of natural numbers.

For every natural number we have

$$|\{1, 2, 3, ..., n\}| = n$$

and, if a proper limit X exists,

$$\lim_{n\to X}|\{1, 2, 3, ..., n\}| = \lim_{n\to X} n \ .$$

Hence we have always

$$|\{1, 2, 3, ..., X\}| = X \ .$$

The following equations suggest themselves

$$|\{1, 2, 3, ..., \aleph_0\}| = \aleph_0$$
$$(1, 2, 3, ..., \omega) = \omega$$

but are unpopular in set theory. Correct is there

$$(1, 2, 3, ..., \omega) = \omega + 1 \ .$$

This is much easier to learn (and to teach) if the pupil starts with

$$|\{0, 1, 2, 3, ..., n\text{-}1\}| = n \ .$$

Tertium non datur

It was Brouwer who vehemently opposed the application of tertium non datur in the infinite. Let us take a simple finite example: The function $f(x) = 1/x$ is undefined at $x = 0$. Therefore it is not continuous there. According to tertium non datur it is discontinuous there. But the function is not defined in x. Therefore the function, not being there, cannot be discontinuous there either.

Same holds for the actual infinite.

For every natural number $n \in \mathbb{N}$ there exists a FISON $F_n = (1, 2, 3, ..., n) \subset \mathbb{N}$ with $n \in F_n$. Therefore all natural numbers belong to finite initial segments. Actual infinity never enters the stage.

But for every FISON F_n there exists a natural number $m \in \mathbb{N}$ with $m \notin F_n$. Therefore $\mathbb{N}$ is larger than every finite initial segment, i.e., $\mathbb{N}$ is actually infinite.

Set theory accepts the latter but is forgetful of the former. In fact tertium non datur fails.

Debunkers

Debunking doubts on actual infinity is usually executed by means of potential infinity or by the "typical representant".

- All positive numbers are greater than 0.
- All negative numbers are less than 0.
- For any set of positive integers, there exists one integer which is the smallest.
- Infinite parallel lines never intersect.

[Larry Freeman: "False proofs" (2006)]

Nothing wrong, because here we can always replace "all" by "every". But the properties or *qualities* of the typical representant do not distinguish between *quantities* within sets.

Intercession

Instead of the self-contradictory concept of bijection, the concept of intercession, denoted by $\succ$ (symbolizing a zipper), may serve as a measure for infinite sets. Two infinite sets, A and B, *intercede* (each other) if they *can be* put in an intercession, i.e., if they can be ordered such that between two elements of A there is at least one element of B and, vice versa, between two elements of B there is at least one element of A.

The intercession includes Cantor's definition of equivalent (or equipotent) sets. Two equivalent sets always intercede each other, i.e., they can always be put in an intercession.

The intercession of sets with nonempty intersection, e.g., the intercession of a set with itself, requires the distinction of identical elements. As an example an intercession of the set of positive integers and the set of even positive integers is given by

$$(1, 2, 3, 4, ...) \ \& \ (2, 4, 6, 8, ...)' \succ (1, 2', 2, 4', 3, 6', 4, 8', ...) .$$

The intercession is an equivalence relation, alas it is not as exciting as the bijection. *All* infinite sets (like the integers, the rationals, and the reals) belong, under this relation, to *one and the same equivalence class*. The sets of rational numbers and irrational numbers, for instance, intercede already in their natural order. There is no playing ground for building hierarchies upon hierarchies of infinities, for accessing inaccessible numbers, and for finishing the infinite. Every set which is not finite, simply is infinite, namely potentially infinite.

[W. Mückenheim: "Inconsistent sets", sci.math (13 Jul 2005). W. Mückenheim: "Intercession", PlanetMath.Org (3 May 2007). W. Mückenheim: "The infinite in sciences and arts", Proceedings of the 2nd International Symposium of Mathematics and its Connections to the Arts and Sciences (MACAS 2), B. Sriraman, C. Michelsen, A. Beckmann, V. Freiman (eds.), Centre for Science and Mathematics Education, University of Southern Denmark, Odense (2008) pp. 265-272]

ArithmoGeometry

In ArithmoGeometry (not to be confused with an arithmetico-geometric sequence) digits, letters, or other symbols are used to construct geometric figures. ArithmoGeometry serves mainly to prove that there does not exist $\aleph_0$ as a "constant quantum, *fixed in itself*, but beyond all finite magnitudes" [Cantor, p. 374] but only the improper limit ∞ which is not a fixed quantum but merely indicates infinite growth.

Asymmetry in the limit

The arithmogeometric figure formed by the sequence

0.1
0.11
0.111
0.1111
...

has $\aleph_0$ rows because, according to set theory, the union of rows has $\aleph_0$ elements. But the figure has definitely not $\aleph_0$ columns, since the limit 0.111... with $\aleph_0$ digits is not contained in any of the $\aleph_0$ rows (indexed by the number of digits 1). So the *set theoretical limit* of the sequence of triangles with equal height and width (which is explicitly defined as nothing else but the *union* of the finite triangles, i.e., of ordered sets (r_1, r_2, r_3, ..., r_n) of finite rows r_n)

$$\begin{array}{lllllllllllllllllll}
0.1 & , & 0.1 & , & 0.1 & , & 0.1 & , & \ldots & = & r_1 & , & r_1 & , & r_1 & , & r_1 & , & \ldots \\
 & & 0.11 & & 0.11 & & 0.11 & & & & & & r_2 & & r_2 & & r_2 & & \\
 & & & & 0.111 & & 0.111 & & & & & & & & r_3 & & r_3 & & \\
 & & & & & & 0.1111 & & & & & & & & & & r_4 & &
\end{array}$$

has not this symmetry in number of rows and number of columns. In this limit the height is greater than the width. (Analysis would preserve symmetry, namely ∞ rows and ∞ columns.)

According to set theory all infinite arithmogeometric figures like

1		1		*a*		•
12	or	1, 2	or	*bb*	or	••
123		1, 2, 3		*ccc*		•••
1234		1, 2, 3, 4		*dddd*		••••
...		...		...		...

contain $\aleph_0$ *finite* rows (like $\mathbb{N}$ has $\aleph_0$ *finite* elements) since all rows form a completed set without a largest element that is bounded by the smallest transfinite cardinal number $\aleph_0$. By the same argument however these figures should also contain $\aleph_0$ columns. This is a contradiction since the length of each row is independent of the total number of rows. The rows do not support each other in accomplishing $\aleph_0$ columns together. And $\mathbb{N}$ has no infinite number.

Two different limits of one and the same sequence

The following sequence of arithmogeometric triangles has been constructed by *appending* finite rows below the foregoing rows:

$$\begin{array}{llllllll}
1 & , & 1 & , & 1 & , & 1 & , \; \ldots \; . \\
 & & 2, 1 & & 2, 1 & & 2, 1 & \\
 & & & & 3, 2, 1 & & 3, 2, 1 & \\
 & & & & & & 4, 3, 2, 1 &
\end{array}$$

It has, according to set theory, a limit which is defined as the union, namely $\aleph_0$ natural numbers in every column but, by definition, not $\aleph_0$ natural numbers in any row.

The following sequence of arithmogeometric triangles has been constructed by *prepending* finite columns to the left-hand side of the foregoing columns:

$$\begin{array}{llllllll}
1 & , & 1 & , & 1 & , & 1 & , \; \ldots \; . \\
 & & 2, 1 & & 2, 1 & & 2, 1 & \\
 & & & & 3, 2, 1 & & 3, 2, 1 & \\
 & & & & & & 4, 3, 2, 1 &
\end{array}$$

It has, according to set theory, a limit which is defined as the union, namely $\aleph_0$ natural numbers in every row but, by definition, not $\aleph_0$ natural numbers in any column.

Unless we say how the construction has to be performed, both sequences are identical up to every finite step. It is impossible to find the limit. This proves that the results of set theory depend on what we *intend* to do but not on what we write. – This is another proof that set theory cannot be in accordance with mathematics which is independent of what we intend to do but depends only on what is written.

[W. Mückenheim: "Set theory depends on what we intend to do and not on what we do.", sci.math (28 Mar 2015)]

The rotating triangle

This asymmetry does not only disqualify set theory from every scientific application, but offers itself to being contradicted by an arithmogeometric method: It would be completely unclear, what side of the infinite triangle is the first one to finish infinity $\aleph_0$, when adding the terms of the sequence *a*, *bb*, *ccc*, *dddd*, *eeeee*, ... in a rotating manner:

$$\begin{array}{llllllllll}
 & & & & & & d & & d & \\
 & & & & c & & dc & & dc & \\
a & , & a & , & ac & , & dac & , & dac & , \quad \ldots \\
 & & bb & & bbc & & dbbc & & dbbc & \\
 & & & & & & & & eeeee &
\end{array}$$

for example

7
76
746
7436
74136
742236
7555556
8 ... .

1 is added in step 1, 22 is added in step 2, 333 is added in step 3, ..., *nnn...n* is added in step *n* ... . There is no edge added with $\aleph_0$ symbols. Alas height and width are claimed to have $\aleph_0$ symbols.

[W. Mückenheim: "Das Kalenderblatt 120411", de.sci.mathematik (10 Apr 2012)]

The alternating triangle

It is not even necessary to add strings on all three sides of the triangle. The sequence of numbers (arrays of digits) 1, 22, 333, 4444, 55555, ... is used alternatingly in horizontal and vertical order to construct an infinite sequence of arithmogeometric triangles. The first number, 1, remains in the origin of the coordinate system:

								5		
				3		3		53		
1	,	1	,	31	,	31	,	531	,	... .
		22		322		322		5322		
						4444		54444		

Since there are infinitely many rows, there must be infinitely many digits in every column. But that is excluded by the always finite arrays of digits added in vertical direction. Since there are infinitely many columns, there must be infinitely many digits in every row. But that is excluded by the always finite arrays of digits added in horizontal direction. Height and width of the triangle remain always finite by definition.

The terms consist of two components, each of which does not raise a contradiction when observed as a single triangle. The fifth term, for instance, is decomposed into the two separate triangles

5		
53		
531	and	
53		22
5		4444 .

The square

The sequence (D_n) of arithmogeometric triangles defined on $\mathbb{N}\times\mathbb{N}$ with D_n defined by

$$D_n(x, y) = y \quad \text{for} \quad 0 < x \le n \quad \text{and} \quad x \le y \le n$$

...
4444
333
22
1

contains in its rows the arrays of digits 1, 22, 333, ... that can be interpreted as natural numbers. According to set theory, every infinite subset of natural numbers has cardinality $\aleph_0$. Therefore the limit of this sequence has height $\aleph_0$ rows whereas no row has $\aleph_0$ digits.

The sequence (E_n) of arithmogeometric triangles defined on $\mathbb{N}\times\mathbb{N}$ with E_n defined by

$$E_n(x, y) = x \quad \text{for} \quad 0 < y \le n \quad \text{and} \quad y \le x \le n$$

```
   4
  34 ...
 234 ...
1234 ...
```

contains in its columns the arrays 1, 22, 333, ... that can be interpreted as natural numbers too. In set theory the limit of this sequence has width $\aleph_0$ columns whereas no column has $\aleph_0$ digits.

Both sequences of triangles pasted together give a sequence of squares (the diagonal is covered twice which does not matter).

...
4444 ...
3334 ...
2234 ...
1234 ... .

What about $\aleph_0$ in the limit figure?

Remark: After digit 9 always continue with digit 1 or use only the digits 1 and 2:

...
2222 ...
1112 ...
2212 ...
1212 ... .

Another model of the same principle is this

1
12
123
1234
...

which together with the complementary triangle yields the square

1111 ...
1222 ...
1233 ...
1234 ...
... .

A trijection

Definition: A ternary relation on three sets A, B, and C which is pairwise surjective and injective is called a trijection. The trijection is a set of triples (a_i, b_i, c_i) where $a_i \in A$, $b_i \in B$, and $c_i \in C$.

Consider the following infinite matrix $M = (m_{ij})$

1 1 1 1 1 1 1 1 ...
1 1 0 0 0 0 0 0 ...
1 1 1 0 0 0 0 0 ...
1 1 1 1 0 0 0 0 ...
... .

For all $i \in \omega$ including $i = \omega$ there is a trijection between the initial segments of the first column, the diagonal, and the first row

$$(m_{11}, ..., m_{i1}) \leftrightarrow (m_{11}, ..., m_{ii}) \leftrightarrow (m_{11}, ..., m_{1i})$$

such that all elements belonging to an initial segment of column, diagonal, and row are 1's. But there is no such trijection for all $1 < i \in \omega$ including $i = \omega$ between the first column, the diagonal and the ith row

$$(m_{11}, ..., m_{i1}) \leftrightarrow (m_{11}, ..., m_{ii}) \leftrightarrow (m_{n1}, ..., m_{n\mathrm{i}})$$

because there is no row (except the first one) with ω 1's. The diagonal gets its ω digits from ω rows none of which has ω digits. The question is how this can be understood geometrically?

[W. Mückenheim: "A problem of set geometry", sci.math.research (25 Jul 2007)]

Gravity effects detected in transfinite set theory.

The difference between potential and actual infinity can even be photographed: Infinity, to find use in set theory, must split off. The following pictures of a movie of an ever-expanding square were taken to show this for the first time (alas the camera got defective in the decisive phase):

o	,	oo	,	ooo	,	oooo	,	... .
		oo		ooo		oooo		
				ooo		oooo		
						oooo		

For each finite square we find height = width. For the infinite figure however, height > width, namely a complete or finished infinite sequence of $\aleph_0$ *finite* strings.

[W. Mückenheim: "Das Kalenderblatt 120406", de.sci.mathematik (5 Apr 2012)]

Supertasks

The noblest property of the sequence of natural numbers is its suceptibility to being interrupted at every desired position. Therefore every enumeration of an infinite set is a supertask already, yet this notion is usually reserved for processes covered by the keyword Tristram Shandy (cp. section 3.3). We will stick to this habit. (See also section "Different limits in set theory and analysis".)

In the present section we will learn about some nonsensical results of supertasks. But with respect to relativity theory we can transfer the events from the time axis to the real axis and vice versa. This emphasizes that bijections between infinite sets are supertasks too. [W. Mückenheim: "The impossibility of a supertask on the time axis shows ...", sci.math (24 Feb 2017)]

Scrooge McDuck

Every day Scrooge McDuck earns 10 enumerated dollars and returns 1 enumerated dollar. If he happens to return the right numbers, he will get unmeasurably rich. If he happens to return the wrong numbers, he will go bankrupt. If he always returns the smallest number the set theoretic limit of the sequence of his possessions s_n at step n is the empty set and $\text{CardLim}_{n\to\infty}\, s_n = 0$. On the other hand his richness grows by \$9 per step, such that $\text{LimCard}_{n\to\infty}\, s_n = \aleph_0$. What is the correct result? None – because actual infinity is not available. But the limit is never the empty set. If McDuck gives away always the largest number then his wealth would grow constantly. And if he gives away randomly chosen numbers, then the answer is unknown. *Anyhow, a result depending on the labels cannot have any scientific relevance.* In mathematics, there is only the improper limit: Scrooge McDuck's richness grows without bound but is never infinite. [W. Mückenheim: "Die Geschichte des Unendlichen", 7th ed., Maro, Augsburg (2011) p. 112 f]

The solution of McDuck's paradox

This is a step-by-step procedure, a construction, and every received dollar is returned.

Considered in potential infinity there is no "all", neither of dollars nor of steps.
For *every* n in ℕ: The dollars 1 to n are returned. *Every* n belongs to a finite initial segment which is followed by a potential infinity of others. No contradiction appears. But potential infinity disallows to prove equinumerosity of received and returned dollars.

Considered in actual infinity, there are all dollars but also all steps.
- *All* dollars are received and returned.
- *All* steps are not sufficient for that task, because for *all* n in ℕ: after step n there are $9n$ dollars not returned. Therefore the set of not returned dollars is not empty. Transactions are only possible at finite steps n (there is no action possible "between all n and ω"). So, even if "the cardinality function is not continuous", there is a contradiction with an empty set of not returned dollars.

In most concentrated form:

• McDuck's wealth $W_n = |S_n|$ can only change with n.
• For every n, W_n is positive and increasing.
• In the limit after all n, $W = |\{ \}|$. – Contradiction.

An attempt to save transfinite set theory is to claim that the empty limit set does not mean a state after all natural numbers but only indicates that all received dollars are returned somewhere. This explanation fails already because unions and intersections to calculate the set limit range over k to ∞ (cp. 2.16 Set-theoretical limits of sequences of sets). Further, if interpreted in actual infinity, the failure of completing the return at any finite step disproves the complete return.

Another attempt, to argue by the limits of analysis, is besides the point: Let q_n = 0.0...01...10... be the sequence of rational numbers having digit 1 in kth position if Scrooge McDuck possesses dollar note number k at day n. So q_n has $9n$ digits 1, but in the limit of that sequence all these digits 1 are gone. Is this is a contradiction in the notion of limit?

No. The analytical limit 0 of the sequence (q_n) is nothing that the terms q_n would "evolve to". It is simply a real number that is approximated better and better by the terms of the potentially infinite sequence. The limit of the *number of digits* 1 of the q_n in mathematics is simply the *improper* limit ∞, i.e., for every number there is a larger one (but never ω or $\aleph_0$ is reached).

For the sequence of sets things are quite different. There one set S_n is transformed into its successor by adding and removing dollars. If their infinity is actual such that it is possible to complete the set, but if no dollar remains forever, then the complete loss of all dollars leaves the empty set. That is not mathematically possible. There is an infinite supply.

A "limit" with quantized numbers, integers or cardinalities, different from all terms of the sequence, is impossible per se. Simple to see in the present case: For every step n there are elements. In the "limit" there are none. Contradiction. Mathematically reasonable is only the limit $\lim_{n\to\infty} 1/W_n = 0$. It does not require an actually infinite or complete sequence (W_n).

Donald Duck

Donald Duck will never become as rich as his uncle Scrooge McDuck. If he gets some money, he soon spends it, except that he always keeps one dollar as an emergency ration.

He starts with two dollars, spends one, gets another one, spends one, and so on, forever, since he is a cartoon character. He marks his dollars with felt-tip pen in order to spend always the oldest one: $\{1, 2\}, \{2\}, \{2, 3\}, \{3\}, \{3, 4\}, \{4\}, ...$

The set-theoretical limit shows which numbers Donald will possess forever. It is the empty set. The set-theoretical *interpretation* of this limit however says that Donald will *spend all* natural numbers. (This interpretation is required to prove that all rational numbers or all entries of a Cantor-list carry indices. The set of not enumerated rational numbers or entries must be empty.) Fact is however, that Donald always keeps a dollar. The set of not spent dollars is not empty by definition. Even the limit cannot be less than 1 $. (Never all rational numbers or all entries of a Cantor-list are enumerated (where "never" means never and not "at ω") because there is no "all".)

This solves the apparent contradiction: Set theory proves for every natural number *n* that the set of not spent numbers *up to n* is empty. But it is also true that *every* number *n* belongs to a finite initial segment upon which infinitely many further numbers follow. Donald returns every number and nevertheless always keeps one, because every number is followed by infinitely many.

[W. Mückenheim: "Die Geschichte des Unendlichen, ch. XII", current lecture]

Daisy Duck

Daisy Duck daily receives and spends her dollar pocket money. If she gets banknotes enumerated by natural numbers she will be bankrupt in the limit: $\{1\}, \{2\}, \{3\}, ... \to \{\ \}$. If she gets always the same banknote, say number 1, she will own this banknote in the limit $\{1\}, \{1\}, \{1\}, ... \to \{1\}$. But what is the limit result when she gets coins $\{a\}, \{b\}, \{c\}, ...$ where it is unknown whether or not $a = b, b = c, ...$?

A merry-go-round

A merry-go-round is a very apt picture of the infinite sequence of natural numbers. If you look at it from the side you see always carousel horses appearing and disappearing – and now and then a snow-white elephant[1].

If set theory is true, then the sequence of natural numbers $\{n\}$ has as its limit the empty set. That means in the limit all carousel horses including the snow-white elephant have disappeared.

This limit is needed to "explain" why Scrooge McDuck (see section "Scrooge McDuck") has gone bankrupt (if he has issued the wrong dollar notes, but not if he has issued other dollar notes or if he had decided to run his transactions by coins). Further this limit is needed to "prove" that all fractions can be counted and that a complete Cantor-list can be prepared.

Of course never all carousel horses will have escaped your horizons – and also the snow-white elephant will persist in returning from time to time. McDuck will never go bankrupt, not even in the limit – if a limit exists. Do you really like to trust in a theory that teaches the contrary?

[W. Mückenheim: "Das Karussell", de.sci.mathematik (7 Mar 2016)]

[1] Mit einem Dach und seinem Schatten dreht / sich eine kleine Weile der Bestand / von bunten Pferden, alle aus dem Land, / das lange zögert, eh es untergeht. / Zwar manche sind an Wagen angespannt, / doch alle haben Mut in ihren Mienen; / ein böser roter Löwe geht mit ihnen / und dann und wann ein weißer Elefant. [Rainer Maria Rilke: "Das Karussell", Neue Gedichte (1907)]

Enumerating all positive rational numbers as a supertask

Bijections between different infinite sets can lead to paradoxes like that of Tristram Shandy that Adolf Fraenkel tells us in order to explain this fundamental feature of set theory (cp. section 3.3). This method lies at the basis of the countability-notion. (That's why Fraenkel reports the story.) For instance, the enumeration of all positive rational numbers runs as follows: Insert into an urn all rationals between 0 and 1 and, if not among them, the rational q_1 that we want to count as the first one. Then remove q_1. Next insert all remaining rationals between 1 and 2 and, if not yet inserted, the rational q_2 that we want to count as the second one. Then remove q_2. Next insert all remaining rationals between 2 and 3 and, if not yet inserted, the rational q_3 that we want to count as the third one. Then remove q_3. And so on: Insert all remaining rationals between n and $n + 1$ and, if not yet inserted, the rational q_n that we want to count as the nth one. (Up to every step n only a finite number n of rationals has been counted. But we insert in every step an infinite number of rationals.) According to set theory, all rationals have been counted "in the limit". According to mathematics this method fails, as has been shown by the 10-to-1 example in the section "Scrooge McDuck". Obviously Fraenkel's 365-to-1 example and Cantor's ∞-to-1 example cannot do better.

The error of set theory is the assumption that indexing *every* positive rational number amounts to indexing *all* positive rational numbers. For infinite sets this assumption fails. There are always only finitely many rational numbers indexed whereas infinitely many remain without index.

Not enumerating all positive rational numbers (I)

Let (s_n) be a sequence of sets s_n of positive rational numbers q such that for $n = 1, 2, 3, ...$

$$s_{n+1} = (s_n \cup \{q \mid n < q \leq n+1\}) \setminus \{q_{n+1}\}$$

with

$$s_1 = \{q \mid 0 < q \leq 1\} \setminus \{q_1\}$$

and

$$q_1 = 1/1,\ q_2 = 1/2,\ q_3 = 2/1,\ q_4 = 3/1,\ q_5 = 1/3,\ ...$$

the positive rational numbers, indexed by zigzag Cauchy-diagonalization of the matrix of positive fractions with repetitions cancelled (indexing all fractions would not change the overall result)

1/1, 1/2, 1/3, 1/4, ...
2/1, ~~2/2~~, 2/3, ~~2/4~~, ...
3/1, 3/2, ~~3/3~~, 3/4, ...
4/1, ~~4/2~~, 4/3, ~~4/4~~, ...
... .

The set s_n contains the rationals of the interval $(0, n]$ which have not got an index $k \leq$ n. When investigating this case for all natural numbers, we get two limits, one for the sequence of sets and one for the sequence of cardinal numbers:

$$\lim_{n\to\infty} s_n = \{ \}$$

is indicating that no rational remains without natural index.

$$\lim_{n\to\infty} |s_n| = \infty$$

is indicating that the set of rational numbers without natural index has infinitely many elements, not only for every s_n but also in the limit.

My questions: Why is the first limit considered more reliable than the second one? Has the second limit a mathematical meaning? If so what is it?

My answers: $\lim_{n\to\infty} s_n$ is meaningless since it is impossible to exhaust an infinite set. There is an infinite supply; this is indicated by $\lim_{n\to\infty} |s_n| = \infty$.

[Bacarra: "Why do you trust the limit of a sequence of sets more than the limit of cardinal numbers?", MathOverflow (7 Jul 2014)]

Some answers in MathOverflow agreed with the following one: "So, the cardinal number of a set is not a continuous function in the natural topology on the set of subsets in which the first limit is taken. It is surprising, perhaps, but by no means undermining anything." [fedja]
My reply: It is undermining the (completely unjustified) assumption that the infinite set could be "finished" such that an empty limit set could result.

Not enumerating all positive rational numbers (II)

Cantor's enumeration of the positive rationals $\mathbb{Q}_+$ (mentioned in a letter to R. Lipschitz on 19 Nov 1883) is ordered by the ascending sum (a+b) of numerator a and denominator b of $q = a/b$, and in case of equal sum, by ascending numerator a. Since all *fractions* will repeat themselves infinitely often, repetitions will be dropped in enumerating the *rational numbers*. This yields the sequence

$$\begin{array}{l} 1/1, \\ 1/2,\ 2/1, \\ 1/3,\ 3/1, \\ 1/4,\ 2/3,\ 3/2,\ 4/1, \\ 1/5,\ 5/1, \\ 1/6,\ 2/5,\ 3/4,\ 4/3,\ 5/2,\ 6/1, \\ \ldots\ . \end{array} \qquad (*)$$

It is easy to see that at least half of all fractions of this sequence belong to the first unit interval (0, 1]. Therefore, while every positive rational number q gets a natural index n in a finite step of this sequence, there remains always a set s_n of positive rational numbers less than n which have not got an index less than n

$$s_{n+1} = (s_n \cup \{q \mid n < q \leq n+1\}) \setminus \{q_{n+1}\} \quad \text{with} \quad s_1 = \{q \mid 0 < q \leq 1\} \setminus \{q_1\} \,.$$

Since all terms of the sequence (s_n) are infinite, $|s_n| = \infty$ for every $n \in \mathbb{N}$ and in the limit. But also the geometric measure of connected unit intervals without any indexed rational number below n, so-called *undefiled* intervals, is increasing beyond every bound. This is shown by the following

Theorem For every $k \in \mathbb{N}$ there is $n_0 \in \mathbb{N}$ such that for $n \geq n_0$: $(n-k, n] \subset s_n$.

Proof: Let $a(n)$ be the largest fraction indexed up to n. Up to every n at least half of the natural numbers are mapped on fractions of the first unit interval. $a(n)$ is increasing in steps without missing any natural number, i.e., without gaps. Therefore n must be about twice $a(n)$, precisely:

$$n \geq 2a(n) - 1 \,.$$

Examples of largest fraction $a(n)$ up to step n (compare the steps of sequence (*) given above):

$a(n) = 1/1$ for $n = 1, 2$
$a(n) = 2/1$ for $n = 3, 4$
$a(n) = 3/1$ for $n = 5, 6, 7, 8$
$a(n) = 4/1$ for $n = 9, 10$
$a(n) = 5/1$ for $n = 11, 12, 13, 14, 15, 16$
$a(n) = 6/1$ for $n = 17, 18, 19, 20$
... .

Therefore for any $n_0 \geq 6$ we can take $k = n_0/2$. Then the interval $(n_0/2, n_0] \subset s_{n_0}$.

This means, there are arbitrarily large sequences of undefiled unit intervals (containing no rational number with an index n or less) in the sets s_n.

Remark: It is easy to find an undefiled interval of length $10^{1000^{100000000000}}$ or any desired multiple in some set s_n and all following sets. Everybody may impartially examine himself whether he is willing to believe that nevertheless all rational numbers can become enumerated "in the limit".

Remark: Cantor does neither assume nor prove that the whole set $\mathbb{N}$ is used for his enumeration (in fact it cannot be proved). Cantor's argument is this: Every natural is used, so none is missing. He and most set theorists interpret this without further ado as using all elements of $\mathbb{N}$. But *every* natural belongs to a finite initial segment which is followed by infinitely many others.

Remark: Although more than half of all naturals are mapped on fractions of the first unit interval, never (for no n) more than $1/10^{1000^{100000000000}}$ % of all fractions of this interval will become enumerated. In fact it can be proved for every natural number n, that not the least positive interval $(x, y]$ of rational numbers is ever completely or at least for the most part enumerated.

[W. Mückenheim: "Consider a tree-structure", sci.math (3 Apr 2015)]

Not enumerating all positive rational numbers (III)

1) Every fraction enumerated by zigzag diagonalization belongs to a small finite triangle. The remainder of the infinite matrix, i.e., at least half of all fractions, will never be touched.
2) The set of intervals with almost all their rationals not enumerated is, according to analysis, *all* intervals. Therefore the claim of *complete* (vollständige) enumeration cannot be taken seriously.
3) The bijection requires *clever* pairing, i.e., it depends on the choice of labels. This is a strictly unscientific procedure because scientific results do never depend on the labeling.

Not enumerating all positive rational numbers (formal proof)

Let j, k, n denote natural numbers and let I be the set of positive real unit intervals $(k - 1, k]$. Further let $q_1, q_2, q_3, \ldots$ be any enumeration of all (cancelled) positive fractions.

Consider the sequence (S_n) of sets S_n of such unit intervals $s_k = (k - 1, k]$ which contain one and therefore also infinitely many rational numbers not enumerated by $j \leq n$:

$$S_n = \{s_k \in I \mid k \leq n \wedge \exists q(q \in \mathbb{Q} \cap (k - 1, k] \wedge \neg\exists j \leq n\colon q = q_j)\} \ .$$

This sequence of sets of unit intervals with the specific property of containing not enumerated fractions has the limit

$$\lim_{n\to\infty} S_n = \{s_k \mid k \in \mathbb{N}\} = I$$

i.e., in the limit, after having used up all natural numbers available, there are all unit intervals containing infinitely many not enumerated fractions.

Remark: Sometimes it is claimed that, by some unknown power, "in the limit" all rationals are enumerated. But this formal approach has been applied specifically only to intervals having at least one not enumerated rational number. Therefore the limit concerns only such intervals.

[Claus: "Is this limit of a sequence of sets correct?", MathOverflow (2 Oct 2015)]

A ZF-compatible conception of infinity

In Cantor's or any other's enumeration of the rational numbers every enumerated rational number belongs to a finite initial segment of the sequence. However each finite initial segment is followed by infinitely many rational numbers. (Same is true for the natural numbers, but that does not prove equinumerosity.) So Cantor's "bijection" is lacking surjectivity in $\mathbb{Q}$ (and $\mathbb{N}$).

Proof: The enumeration occurs, or at least can be analyzed, in steps because the sequence of natural numbers is well-ordered and the result of the enumeration can be pursued up to each step. For every step n, every unit interval $(k, k+1]$ with $k < n$ is populated by infinitely many rational numbers without index and with comparatively few indexed rational numbers only. Since this does *never* change, we can calculate, using analysis, the limit of such unit intervals where nearly all rational numbers are missing an index. Assuming we could do all steps n, the result is "all unit intervals". This implies the existence of infinitely many rational numbers without index.

Usually set theorists are not accepting this proof since we cannot determine a rational number which never gets an index. This argument is based on an erroneous conception of infinity.

Since every indexed rational number is defined (by its index) each one belongs to a finite initial segment of the sequence. The set of defined rational numbers is never infinite. Infinity can only be spread out by those rational numbers which are following upon the defined ones. Of course, each of these can be defined too. Nevertheless all defined rational numbers belong to a finite set which is followed by an infinity of undefined rational numbers. How many of them ever may become defined, the situation does *never* change. It remains so – in infinity.

This conception of infinity satisfies the ZF-axiom of infinity (not the axiom of extensionality though) as well as the analytical limit of unit intervals with almost only not enumerated rational numbers, avoiding Tristram-Shandy-like paradoxes.

The only possibility to get rid of undefined and not enumerated rational numbers is to switch to potential infinity. There the undefined numbers simply do not (yet) exist.

[W. Mückenheim: "Perception of infinity avoiding paradoxes", sci.math (18 Jan 2017)]

Different limits in set theory and analysis

Consider an urn and infinitely many actions performed within one hour (the first one needs 1/2 hour, the second one 1/4 hour, the third one 1/8 hour, and so on). We fill successively pairs of consecutively enumerated marbles into the urn. Between every two steps of filling we remove the marble with the lowest number from the urn.

Fill in 1, 2, remove 1.
Fill in 3, 4, remove 2.
Fill in 5, 6, remove 3.
Fill in 7, 8, remove 4.
Continue over the full hour.

According to set theory in the limit the urn is empty, because for every number the time of removal can be determined (see section "Set-theoretical limits of sequences of sets"). It is simply neglected that *always* another number is added. So for the set S of numbers residing in the urn we obtain

$$\aleph_0 = \lim_{\text{time}\to 1\text{h}} |S| \neq |\lim_{\text{time}\to 1\text{h}} S| = |\{\ \}| = 0 .$$

This prevents any application of set theory to reality although it was Cantor's definite aim to apply set theory to reality (cp. the section "Cantor on sciences"). And it prevents any application to mathematics too, because in mathematics the limit of the sequence of real numbers (constructed from the natural numbers in the urn and the natural numbers that have left the urn such that the natural numbers contained in the urn are separated by a decimal point from those removed already) yields the following sequence (given in vertical order)

```
  21.
   2.1
 432.1
  43.21
6543.21
 654.321
87654.321
 8765.4321
   ... .
```

For infinitely many exchanges we have a contradiction between set theory and mathematics: Set theory yields as the limit the empty urn because for every marble the step can be determined, when it is removed. The (improper) analytical limit is ∞.

The above sequence however gets confusing when multi-digit natural numbers are involved. In a simpler representation, we abbreviate an odd natural number by 1 and an even natural number by 2 or, for the sake of simplicity of the continued fraction discussed below, an odd natural number by 1 and an even natural number by 0:

```
  21.            01.
   2.1            0.1
 212.1          010.1
  21.21          01.01
2121.21        0101.01
 212.121        010.101
21212.121      01010.101
 2121.2121      0101.0101
   ...            ... .
```

The analytical limit of these sequences is ∞ but set theory predicts a limit < 1 because there remain no digits in front of the decimal point.

The latter sequence can easily be represented as a continued fraction

$$\cfrac{\cfrac{\cfrac{\cfrac{10^0}{10}+10^1}{10}+10^2}{10}+10^3}{10}+\ldots\ .$$

Since the logarithm is a strictly increasing function of its argument and gives the number of digits of r_n by $[\lg r_n] + 1$, we have a contradiction in that, according to mathematics, there will be a set of infinitely many digits on the left-hand side of the decimal point in the limit. The (improper) limit is infinite, proved by the proper limit 0 of the sequence of the reciprocals $(1/r_n)$:

Cauchy:

$$\frac{1}{\cfrac{\cfrac{\cfrac{\cfrac{10^0}{10}+10^1}{10}+10^2}{10}+10^3}{10}+\ldots} = 0$$

Cantor:

$$\frac{1}{\cfrac{\cfrac{\cfrac{\cfrac{10^0}{10}+10^1}{10}+10^2}{10}+10^3}{10}+\ldots\ .} > 1$$

Indexing the bits yields

$0_2 1_1.$
$0_2.1_1$
$0_4 1_3 0_2.1_1$
$0_4 1_3.0_2 1_1$
$0_6 1_5 0_4 1_3.0_2 1_1$
$0_6 1_5 0_4.1_3 0_2 1_1$
$0_8 1_7 0_6 1_5 0_4.1_3 0_2 1_1$
$0_8 1_7 0_6 1_5.0_4 1_3 0_2 1_1$
... .

What is the limit of the sequence of the sets of indexes on the left-hand side? What is the limit of the decimal numbers?

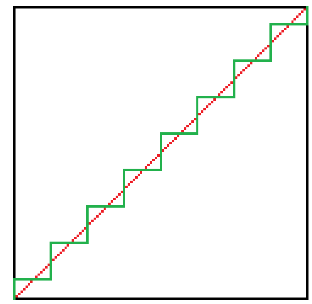
[W. Mückenheim: "Das Paradoxon des Tristram Shandy", HS Augsburg, Forschungsbericht 2012, wmm wirtschaftsverlag, Augsburg (2012) pp. 242-244. W. Mückenheim: "Das Kalenderblatt 120410 ", de.sci.mathematik (9 Apr 2010). W. Mückenheim: "History of the infinite, XII", current lecture]

Consider the simple function

$$f(n) = |\{1, 2, 3, ..., n\}|/|\{n+1, n+2, n+3, ...\}| = 0, 0, 0, ...\ .$$

By analysis the limit is 0. Set theory gives a limit > 1 (since the denominator gets exhausted).

The Manhattan paradox



Approximate the diagonal of the unit square by rectangular stairs of equal height and width. Double the number of stairs, i.e., halve their size. Repeat this procedure again and again. Then the limit of the length of the curve approximating the diagonal is the limit of the sequence 2, 2, 2, ..., namely 2.

If however a *last term* could be obtained such that for every stair lenght and height became zero, represented by a single point, then the curve would be the diagonal with total length $\sqrt{2}$. In that case never two or more points would lie side by side in horizontal or the vertical direction.

Then the limit 2 would be false. But of course that case cannot be realized because every bisection is followed by another one and the single points are never arrived at in a last step.

Same is true for McDuck (see section "Scrooge McDuck"). If all dollars could be spent, then McDuck became bankrupt. Then the improper mathematical limit ∞ was wrong.

Conclusion: It is not possible, as set theorists try to suggest, that there are two different limits peacefully coexisting. There can be only one limit. And since there is no last step but every bisection has a successor, this limit is of course the improper mathematical limit.

Some notes:

- If the limit of the staircase curve was the diagonal, then every fractal had a smooth limit.
- If the limit $\sqrt{2}$ of the length of the diagonal was reached by bisecting the stairs, then the diagonal consisted of $\lim_{n\to\infty} 2^n = \aleph_0$ points – not an uncountable number.
- The staircase curve is similar to the Cantor set (see 3.11) if the removal of thirds is replaced by removing and re-inserting halves. If the Manhattan-limit was the diagonal, then the Cantor set was countable.

The real numbers

What is a real number?

A *real number* is an algorithm that supplies an infinite sequence of digits (or bits etc.). This satisfies the axiom of trichotomy with respect to every rational number.

A *general pointer* to a real number is a finite expression (like "2" or "my net income in 2017" or "the square root of two" or "√2" or "π") which has been related to the real number in at least one physical system, usually a dictionary, or a text book, or a letter, or a brain.

A *special pointer* to a real number contains the space-time coordinates or other identification properties of actual constructions of infinite sequences of digits (or bits etc.).

All algorithms and all pointers belong as elements to the countable set of all finite expressions. Therefore any uncountability of the set of real numbers can be excluded.

Existence of a real number

There are three alternatives:

1) A number exists if it can be individualized in mathematical discourse such that its numerical value can be calculated in principle by every mathematician without any error.

2) A number exists if it can be individualized in mathematical discourse such that its numerical value can be calculated in principle by every mathematician with error less than any given epsilon.

3) A number exists if it cannot be individualized but if there is some proof showing that this number exists.

Grades of definition

Some real numbers are *extremely well defined*: The small natural numbers like 3 or 5 can be grasped at first glance, even in unary representation.

A real number is *very well defined*, if its value (compared with the unit) can be determined without any error, like all rational numbers the representations of which have complexity that can be handled by humans or computers.

A real number is *well defined*, if its value can be determined in principle with an error as small as desired, i.e., the number can in principle be put in trichotomy with every very well defined rational number. The irrational numbers with definitions that can be handled by humans or computers belong to this class.

Finite formulas versus listings of strings (I)

If you get a string of information but can't wait until the end-signal is given, how sure can you be to know the correct meaning? Not at all! Therefore the following equivalence, usually applied in set theory, is *wrong* because it is obtained by reversion of implication.

A finite formula defines an infinite string. ⇔ An infinite string defines a finite formula.

We can never obtain a finite formula like 1/9 or √2 from an infinite string of digits unless we know the last term (which is impossible by definition). Every form of information transfer (and what else are strings of digits?) requires an end-of-file signal. Infinite strings of digits without a defining formula are therefore unsuitable for mathematical purposes.

It is impossible to define a real number by an infinite string of digits. A real number can only be defined by a finite definition. But it is well known that the set of all finite definitions is countable.

Finite formulas versus listings of strings (II)

A finite definition D can define an infinite sequence or string S.

$$D: f(n) = 1/n$$

defines the infinite sequence $S = 1, 1/2, 1/3, \ldots$ with limit $\lim_{n\to\infty} f(n) = 0$. Sometimes simply a series is used as an abbreviated definition of its limit. So the string 0.111..., abbreviating the infinite series 1/10 + 1/100 + 1/1000 + ..., is also used to denote its limit 1/9.

But it is impossible to obtain a limit from the terms of an infinite sequence or string. For instance we could write "0." and let follow 10^{1000} lines filled with solely the digit 0. (It would obviously not be possible to write infinitely many such lines.) The reader could not be sure that zero is meant because after the last digit that he sees there could follow an unexpected change.

Usually in set theory the sequence or string S and its finite definition D are understood to be equivalent. But in fact we have only

$$D \Rightarrow S\,.$$

The reversal

$$S \Rightarrow D$$

is simply wrong – a popular logical fallacy.

[W. Mückenheim in "Why can no one in sci.math understand my simple point?", sci.math (18 Jun 2010)]

Decimal representation of real numbers

For computing purposes usually non-negative decimal fractions abbreviated by digit sequences are applied. The non-integer part is written as

$$\sum_{n\in\mathbb{N}} \frac{z_n}{10^n} = \frac{z_1}{10^1} + \frac{z_2}{10^2} + \frac{z_3}{10^3} + \dots := 0.z_1 z_2 z_3 \dots . \qquad (*)$$

This representation remains an approximation if the sequence of partial sums does not become constant, i.e., unless there exists an index n_0 such that $z_n = 0$ for all $n \geq n_0$. A quasi-strictly[1] monotonically increasing sequence of partial sums cannot contain its limit. This fact concerns decimal representations of fractions the denominators of which contain prime factors different from 2 or 5 and all irrational numbers.

By changing the base, fractions can be represented by eventually constant digit sequences. In the ternary system 1/3 has the representation 0.1. A representation of an irrational number by a digit sequence however is not feasible – not even by an infinite sequence. *Every* decimal fraction simultaneously defines the finial summand and the common denominator of a rational partial sum and therefore not the irrational limit of the sequence which differs from every rational partial sum.

Since the digits, even if "all" could be noted and compared, do not define the limit but only the infinite sequence of all partial sums, there must exist a unique formula supplying the limit. This formula simultaneously supplies every sequence of digits as far as required to determine the trichotomy relation with other numbers. Such a formula can be very simple. Already the fraction "1/9" or the finite expression "0.111..." (here written with eight symbols) are sufficient to determine the infinite string of ones. In other cases more complicated formulas are required, like Newton's series for e or Wallis' product for $\pi/2$.

Expressions like $a_1 + a_2 + a_3 + \dots$ or 0.111... in general are not understood as series but tacitly are interpreted as their limits. Therefore a correct notation like $0.111\dots \to 1/9$ is written without further ado as $0.111\dots = 1/9$. This simplifying convention does no harm, neither in computing nor in the mathematics based on potential infinity because nobody would expect completeness of the terms of a series. If however this completeness is demanded by axioms or assumed because of other reasons, then we have to distinguish between the series, i.e., the sequence of partial sums, and its limit. Otherwise an irrational number would be identified with an infinite series of decimal fractions.

[W. Mückenheim: "Mathematik für die ersten Semester", 4th ed., De Gruyter, Berlin (2015) p. 194f]

If there are $\aleph_0$ digits, then $\aleph_0$ digits fail to define a real number. If the digit sequence is only potentially infinite, then it cannot define more than a converging sequence of rational intervals.

---

[1] A sequence like the series (*) increases strictly monotonically, if there is no digit 0. In case the sequence contains finite subsequences of digits 0 but never becomes constant, I call the sequence quasi-strictly monotonically increasing.

Sequences and limits

Abstract Irrational numbers cannot be represented by infinite digit-sequences. A digit sequence is only an abbreviated notation for an infinite sequence of rational partial sums. Irrational numbers are limits of sequences, incommensurable with any grid of decimal fractions.

Introduction Strictly monotonic sequences do not assume their limit. Rarely the terms of the sequence and its limit are confused. But this situation changes dramatically when sequences of partial sums of series are involved. It is customary in textbooks to identify the infinite sum over all terms of a series and the limit of this series, often called its "sum".

In the following we will see that this is imprecise and point out an important consequence. *A limit is not defined by the infinite sequence of partial sums because the sequence cannot be given in the necessary completeness. Only a finite formula can determine both the terms of the sequence of partial sums and the limit as well.*

Theorem A non-terminating series of decimal fractions does not determine a real number.
Corollary A non-terminating digit sequence does not determine a real number.

Proof: The limit of a strictly monotonic sequence is not among its terms. Strictly monotonic sequences like $(10^{-n})_{n\in\mathbb{N}}$ or $\left([1+1/n]^n\right)_{n\in\mathbb{N}}$ or $(\sum_{k=1}^{n}10^{-k!})_{n\in\mathbb{N}}$ sufficiently show this. None of the $\aleph_0$ indexed terms is equal to the limit 0, e, and Liouville's number L, respectively.

The same distinction has to be observed with series. There must not be a difference in the mathematical contents whether the partial sums are written separately like

$$3,\ 3.1,\ 3.14,\ 3.141,\ 3.1415,\ \ldots \tag{1}$$

or are written in one line with interruptions

$$(((((3.)1)4)1)5)\ldots \tag{2}$$

or without interruptions

$$3.1415\ldots\ . \tag{3}$$

The infinite sequence of digits d_n is completely exhausted by all terms of the Cauchy-sequence of rational partial sums of decimal fractions. The intended meaning as a sequence of rational partial sums according to (1) can be expressed also by $\sum_{n=0}^{0,1,2,\ldots}\frac{d_n}{10^n}$. The infinite sum $\sum_{n\in\mathbb{N}_0}\frac{d_n}{10^n}$ given by (3) is merely an abbreviation: All partial sums are written in one and the same line *without adding the limit*. The same is expressed by (2) because writing or not writing parentheses here must not change the result. In all cases *none of the $\aleph_0$ decimal fractions is left out*. The "sum" of the series, i.e., the limit of the Cauchy-sequence of partial sums, is not established by any term with natural index $n \in \mathbb{N}$. But only all these $\aleph_0$ terms are given in equations (1) to (3) as well as on the left-hand sides of the following examples whereas the limits are given on the right-hand sides:

$$\sum_{n\in\mathbb{N}} \frac{1}{10^{n!}} \neq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{10^{k!}} = \sum_{k=1}^{\infty} \frac{1}{10^{k!}} = L$$

$$\sum_{n\in\mathbb{N}_0} \frac{1}{n!} \neq \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} = e$$

$$\sum_{n\in\mathbb{N}} \frac{1}{2^n} \neq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2^k} = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1\ .$$

Digits are simply too coarse-grained to represent irrational limits of Cauchy-sequences. To represent $\lim_{n\to\infty} 1/10^n$ by an infinite digit sequence, we would need infinitely many digits 0 preceding the digit 1. Whereas it is obvious that this is impossible, the infinitely many digits 1 required for the expansion of $\lim_{n\to\infty}\sum_{k=1}^{n} 1/10^k = \sum_{k=1}^{\infty} 1/10^k = 1/9$ are usually swallowed without scruples. But it is as obvious that digits 0 and digits 1 do not allow for a different treatment with respect to the fact that never infinitely many can precede one of them. This leads us to the often asserted double-representation of periodic rationals. $\forall n \in \mathbb{N}$ the sum of the nth terms of the two complementary sequences

$$\begin{array}{lcl} (1/10^n) & = & 0.1,\ 0.01,\ 0.001,\ \ldots \ \to\ 0 \\ \underline{(1 - 1/10^n)} & \underline{=} & \underline{0.9,\ 0.99,\ 0.999,\ \ldots \ \to\ 1} \\ \quad 1 & = & \ \ 1\ ,\quad 1\ ,\quad 1\ ,\ \ldots \ \to\ 1 \end{array}$$

is 1. Since all $\aleph_0$ digits are not sufficient to realize the limit 0 of the first sequence, all $\aleph_0$ digits of 0.999... are not sufficient to realize the limit 1 of the second sequence. Only when explicitly taking the *limits* of the sequences, we get 0 and 1, respectively. For *series*, taking the limit is usually assumed without saying and does not cause mistakes in numerical calculations, but if we look at the matter with advisable mathematical precision, we see

$$0.999... = \sum_{n\in\mathbb{N}} \frac{9}{10^n} \neq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{9}{10^k} = \sum_{k=1}^{\infty} \frac{9}{10^k} = 1.$$

The usual proof for 0.999... = 1, namely $10\cdot 0.999... = 9.999... = 9 + 0.999... \Rightarrow 9\cdot 0.999... = 9$ holds in the limit only. The series 0.999... *is not a number* but a sequence of partial sums. Like a vector it can be multiplied such that $10\cdot(0.9, 0.99, 0.999, ...) = (9, 9.9, 9.99, ...)$ but it is impossible to isolate one 9 from infinitely many terms. ■

<u>Conclusion</u>: As a result we can state that an infinite digit sequence $0.d_1d_2d_3...$, abbreviating an infinite sequence of partial sums of decimal fractions, also called an infinite series $(\sum_{k=1}^{n} d_k/10^k)_{n\in\mathbb{N}} \equiv \sum_{n\in\mathbb{N}} d_n/10^n$, is not a number (unless eventually becoming constant). 3.1415... for example is an abbreviation of a sequence of rational partial sums converging to π. This sequence is purely rational although we cannot find a fraction m/n = 3.1415... with a common denominator

covering all terms of the sequence. This disadvantage however is shared by sequences like $(10^{-n})_{n \in \mathbb{N}}$ too. We cannot find a fraction with a common denominator covering all terms of the sequence all of which are rational with no doubt.

A periodic decimal fraction has *as its limit* a rational number. A non-periodic decimal fraction has *as its limit* an irrational number. *But it is not this number.* In case of periodic decimal fractions it is possible, by changing the base, to obtain a terminating digit sequence. Irrational numbers have no decimal expansion, no representation by digits or bits, not even by infinitely many. They are *incommensurable* with *every* rational measure expanded by digits or bits. An irrational number requires a generating formula F in order to calculate every digit of the infinite digit sequence S and *in addition* to calculate the limit. This formula F may be interpreted as the number as well as the limit. It may be involved or as simple as "0.111..." which is a finite formula (consisting of eight symbols) allowing to obtain every digit of the sequence converging to 1/9.

The implication $F \Rightarrow S$ cannot be reversed because without F the sequence S cannot be obtained in the completeness required, i.e., including *all* its terms such that *none* is missing and undefined.

Consequence: The mathematical facts discussed above also apply to all infinite sequences of digits or bits appearing in the folklore version of Cantor's diagonal argument or in the Binary Tree argument. Sequences of digits or bits are never representing irrational numbers let alone transcendental numbers. Therefore Cantor's diagonal argument as well as the Binary Tree argument do not concern the cardinality of the set of irrational numbers.

[W. Mückenheim: "Sequences and limits", Advances in Pure Mathematics 5 (2015) pp. 59-61]

Sequences and limits (Discussion)

"Once real numbers are introduced (usual means are Dedekind cuts, equivalence classes of Cauchy rational sequences, axiom system), the infinite non-recurring decimal fraction notation 3.14159... is not precise, what are the dots? What is the *n*th decimal digit?" [János Kurdics in Discussion of "Sequences and limits", Advances in Pure Mathematics 5 (Jan 2015)]

"The author is mistaken. A digit sequence such as 3.14159... denotes the limit of a sequence of partial sums, e.g. the limit of the sequence 3, 3.1, 3.14, 3.141, ... It does not denote the sequence of partial sums itself." [David Radcliffe in Discussion of "Sequences and limits", Advances in Pure Mathematics 5 (Jan 2015)] "The author is correct. You are mistaken." [John Gabriel, loc cit] "There are infinitely many partial sums. Every digit finishes one partial sum. More digits are not available." [W. Mückenheim, loc cit]

"0.111... (as a number in decimal system) is not equal to any of its partial sums but, on the contrary, is larger than any partial sum, and 1/9 is the only number which is both larger than every partial sum and having in every open interval around it contained nearly all partial sums. That means that 0.111... is exactly equal to 1/9." [Andreas Leitgeb in "Grundpfeiler der Matheologie", de.sci.mathematik (1 Aug 2016)] My reply: 0.111... *is* all partial sums!

"The statement $\sum_{i\in\mathbb{N}} 9\cdot 10^{-i} = 1$ is not the same as the statement $\forall n \in \mathbb{N}$: $\sum_{i=1}^{n} 9\cdot 10^{-i} \neq 1$." [Andreas Leitgeb in "Grundpfeiler der Matheologie", de.sci.mathematik (6 Aug 2016)] My reply: The sum is over all those n only, for none of which identity is reached. The statement $\sum_{i\in\mathbb{N}} 9\cdot 10^{-i} = 1$ is wrong. Sequences have no numerical values.

Résumé

Every digit of 0.111... is defining a partial sum differing from 1/9. To claim that the sequence of all digits together produces exactly 1/9 means to have two different meanings for the complete sequence of $\aleph_0$ digits, namely simultaneously denoting the two different objects of mathematics

- the sequence of all partial sums ≠ 1/9
- the numerical value 1/9.

Nobody would trust in 3.25 = 1 + 1 + 1 + 1 - 1 + 1 - 1 +-... Why should we trust in π = 3.1415...? To write π as a decimal fraction is as impossible as to write 0 as a positive fraction.

A finite or infinite sequence has no numerical value. Every real number is the *limit* of an infinite sequence of rational numbers. (More technically a real number is an equivalence class of limits of converging sequences of rational numbers, so-called Cauchy sequences. But that is irrelevant in this context.) Some sequences get eventually constant and contain their limit as an element. 1.000... contains its limit 1 whereas 0.999... does not contain its limit 1.

A real number is *more or less defined*, if it can be communicated such that a receiver with *more or less* mathematical knowledge understands *more or less* the same as the sender. Results of calculations that only in principle can be finished belong to this class.

Even *Non-numbers* can be *defined* like the greatest prime number or the smallest positive rational or the reversal of the digit sequence of π or the lifetime of the universe measured in seconds.

Mathematical objects without definitions, however, cannot exist since all *mathematical objects* by definition have no other form of existence than *existence by definition*.

Brouwer's ω-sequences

According to Brouwer we can create infinite sequences of type ω by laws or algorithms, for instance given as finite expressions like 0.101010... in binary, 0.444... in decimal, or 0.AAA... in hexadecimal notation. In that way also every other rational number can be written.

A sequence without a generating formula can only be represented intensionally, that is by a catalogue or list of terms. But lawless choice sequences will always remain in the state of being finite. Irrational numbers in fact are not available as sequences of digits.

Undefinable objects of mathematics?

Modern set theorists often react surprised when the distinguishability of all objects of mathematics, in particular of all numbers, is required as a basic feature of mathematics. But can mathematics tolerate objects that *in principle*, i.e., even in an infinite and eternal universe, could not be defined such that they are distinct from all other objects?

The answer of Wipedia is yes: "A real number is called computable if there exists an algorithm that yields its digits. Because there are only countably many algorithms, but an uncountable number of reals, almost all real numbers fail to be computable. Moreover, the equality of two computable numbers is an undecidable problem. Some constructivists accept the existence of only those reals that are computable. The set of definable numbers is broader, but still only countable." ["Real number", Wikipedia] This text has obviously been writen by mathematicians of a generation that has been trained to be attracted by counter-intuitive opinions.

Cantor originally defined: "By a 'set' we understand every collection *M* of definite well-distinguished objects *m* of our visualization or our thinking (which are called the 'elements' of *M*) into a whole." [Cantor, p. 282] Forced by antinomies his fully extensional definition has become obsolete. However even the modern axiom of restricted extensionality persists in stating: "Sets are completely defined by their elements." But how could sets be defined and distinguished by elements that could not?

Note Hilbert's statement: "Working with the infinite can only be secured by the finite." [D. Hilbert: "Über das Unendliche", Math. Annalen 95 (1925) 190] Well, in the finite every mathematical object is definable. What does that mean? Every object that can be applied in mathematics must be applied by its finite name (because an object of mathematics has no other form of existence). This name, wherever defined, is one of perhaps many definitions of the object. A number or finite string of symbols is definable if mathematicians can talk about it.

Cantor defended his invention of transfinite numbers, with respect to the fact that the distinction between numbers is their most notable property, by explaining that cardinal numbers satisfy just this requirement "because the completed infinite comes in different modifications which are *distinguishable* with utmost sharpness by the *so-called* '*finite* human mind'." [G. Cantor, letter to R. Lipschitz (19 Nov 1883)]

"Beyond the finite there exists [...] an infinite stepladder of certain modis the nature of which is not finite but infinite, which however like the finite can be determined by definite, well-defined *numbers* distinguishable from each other." [Cantor, p. 176]

The *numerical character* of all cardinal numbers is not only implied by the generic term "number" but also by their trichotomy properties: "*Let a and b be any two cardinal numbers, then we have either a = b or a < b or a > b*." [Cantor, p. 285]

To apply the notion of "cardinality" or "number" requires *different* elements, i.e., elements that are *distinct* and *can be distinguished* and well-ordered. Only *definable* elements can be uniquely related to each other. One-to-one mappings require to *distinguish* each "one". This is stressed by Cantor's frequent use of the phrase "element by element", the basic principle of bijection.

"If two well-defined manifolds, *M* and *N*, can be related completely, element by element, to each other [...], then for the following the expression may be permitted that these manifolds have the same cardinality or that they are equivalent." [Cantor, p. 119]

"Every well-defined set has a defined cardinality; two sets are ascribed the same cardinality if they mutually uniquely, element by element, can be mapped onto each other." [Cantor, p. 167]

"Two sets are called 'equivalent' if they mutually uniquely, element by element, can be mapped onto each other." [Cantor, p. 380 & 441]

Note "well-defined" and "element by element".

"If every element of set *M* can be related to one and only one corresponding element of set *N* and vice versa, and if there is never an obstacle or halt in this process of assignment, then both infinite sets are in bijection." [Cantor, p. 239]

Note "process of assignment" which excludes undefined steps. Under axiom of choice this extrapolation can even be extended to uncountable sets: All well-ordered sets can be compared. They have the same cardinal number if they, by preserving their order, can be uniquely mapped or counted onto each other. "Therefore all sets are 'countable' in an extended sense, in particular all 'continua'." [G. Cantor, letter to R. Dedekind (3 Aug 1899)]

Of course the elements of the sets have to be distinguishable. Richard and Poincaré insisted in the countability of all definitions. But: "It is assumed that the system $\{B\}$ of notions B, which possibly have to be used to define real number-individuals, is finite or at most countably infinite. This assumption must be mistaken because otherwise this would imply the wrong theorem that the continuum of numbers has cardinality $\aleph_0$." [G. Cantor, letter to D. Hilbert (8 Aug 1906)]

Meanwhile nobody doubts that this theorem is true. And of course its implication is true too.

Defined list of all definable real numbers

Assume that $\mathbb{R}$ and *all* its subsets are actually existing. Then also the set of all *definable* real numbers is existing, although remaining unknown. It is countable and has a well-order as a sequence or list. It does not matter whether the order of the list is definable. We define lots of real numbers like the first prime number to be found in year 2222 or the diagonal number I produced this morning. Only *that* kind of definitions counts because they are finite and thus countable. In this very same way the antidiagonal of our example is defined by the following

Definition: Take all subsets of $\mathbb{R}$. One of them contains all definable real numbers. There exists a sequential well-ordering of this set. Produce the antidiagonal with the usual provisions.

It does not matter whether or not the list has a *definable* well-ordering. Every countable set has a sequential well-ordering. The above definition yields another defined real number. This is a contradiction. Obviously the initial assumption is false.

There is no uncountable set

On Cantor's first uncountability proof

Implicitly, there is a serious restriction of the proof (cp. section 2.2.2) with respect to the set of numbers that it is to be applied to: the *complete* set of real numbers is required as the manifold investigated. If only one of them is removed, the proof fails because just this one could be the common limit $\alpha^\infty = \beta^\infty$.

Cantor took his result as evidence in favour of the existence and uncountability of the set $\mathbb{T}$ of all transcendental numbers which were shortly before discovered by Liouville. Nevertheless his proof fails, if applied to the set $\mathbb{T}$ alone. The reason is again, that "any infinite sequence" like $\alpha, \alpha', \alpha'', ...$ or $\beta, \beta', \beta'', ...$ need not converge to a transcendental limit. Already the absence of a single number, zero for instance, cannot be tolerated, because it is the limit of several sequences.

This situation, however, is the same if only the set $\mathbb{Q}$ of all rational numbers is considered. Therefore both sets, $\mathbb{Q}$ and $\mathbb{T}$, have the same status with respect to *this* uncountability proof. And we are not able, *based on this very proof*, to distinguish between them.

The proof can feign the uncountability of a countable set. If, for instance, the alternating harmonic sequence

$$\omega_\nu = (-1)^\nu/\nu \rightarrow 0$$

is taken as the sequence, yielding the intervals $(\alpha, \beta) = (-1, 1/2)$, $(\alpha', \beta') = (-1/3, 1/4)$, ..., we find that its limit 0 does not belong to the sequence, although everything here is countable.

The alternating harmonic sequence does not, of course, contain all real numbers, but this simple example demonstrates that Cantor's first proof is not conclusive. *Based upon this proof alone*, the uncountability of this and every other alternating convergent sequence must be claimed. Only from *some other information* we know their countability (as well as that of $\mathbb{Q}$), but how can we exclude that *some other information*, not yet available, in the future will show the countability of $\mathbb{T}$ or $\mathbb{R}$?

Anyhow, the countability properties of an infinite set will not be altered by adding or removing one single element. Cantor's first uncountability proof does not apply to the set $\mathbb{R} \setminus \{r\}$, with r being any real number. This shows its insufficiency.

[W. Mückenheim: "On Cantor's important proofs", arXiv (2003)]

A severe inconsistency of set theory

Following basics of set theory are applied:

- Mappings between infinite sets can always be completed, such that at least one of the sets is exhausted.
- The real numbers can be well ordered.
- The relative positions (Lagenbeziehungen) of real numbers enumerated by a finite set of natural numbers can always be determined, in particular the maximum real number below a given limit.
- Between any two real numbers, there exists always a rational number.

For the sake of simplicity this proof is restricted to positive numbers. The extension to all numbers is obvious.

<u>Theorem</u> The set of all positive irrational numbers $\mathbb{X}_+$ can be mapped into the set of all positive rational numbers $\mathbb{Q}_+$ leading to $|\mathbb{X}_+| \leq |\mathbb{Q}_+|$.

Proof: Let the sets $\mathbb{Q}_+$ and $\mathbb{X}_+$ be well-ordered. Define two sets, one of them containing only the number zero, $Q = \{0\}$, and the other one being empty, $X = \{ \}$. Take the first element $\xi_1 \in \mathbb{X}_+$. Select the largest rational number $q \in Q$ with $q < \xi_1$ (in the first step, this is obviously $q = 0$). Between two different real numbers like q and ξ_1 there is always a rational number, $q_1 \in \mathbb{Q}_+$, with $q < q_1 < \xi_1$. Transfer ξ_1 to the set X and transfer q_1 to the set Q. Then choose the next positive irrational number, $\xi_2 \in \mathbb{X}_+$, select the largest number $q \in Q = \{0, q_1\}$ with $q < \xi_2$. There is a rational number, $q_2 \in \mathbb{Q}_+$, with $q < q_2 < \xi_2$. Transfer ξ_2 to the set X and transfer q_2 to the set Q. Continue until one of the sets $\mathbb{Q}_+$ or $\mathbb{X}_+$ is exhausted which, according to the axiom of choice, will unavoidably occur.

If the set $\mathbb{Q}_+$ were exhausted prematurely and no q_n remained available to map ξ_n on it, this proof would fail. We would leave the countable domain and could not make use of Cantor's "Lagenbeziehung" to select the largest rational number $q \in Q$ with $q < \xi_n$. But that cannot occur because there is *always* a rational number between two real numbers. As long as rational numbers $q_n \in \mathbb{Q}_+$ are available, the set of pairs (q_n, ξ_n) remains countable and there are also natural numbers n available as indices, because all our positive rational numbers have been enumerated by natural numbers. Therefore, we do not leave the countable domain and do not need transfinite induction. But our mapping process runs until one of the sets is exhausted. By tertium non datur this set, if any, can only be $\mathbb{X}_+$.

The mapping supplies $|Q| = |X|$ at every stage while finally $X_{\text{fin}} = \mathbb{X}_+$ and $Q_{\text{fin}} \subseteq \mathbb{Q}_+$. ($Q_{\text{fin}}$ even must be a proper subset of $\mathbb{Q}_+$ because there are further rational numbers between every two elements of Q_{fin}.) The result $|\mathbb{X}_+| \leq |\mathbb{Q}_+|$ completes the proof. ■

[W. Mückenheim: "On the abundance of the irrational numbers", arXiv (2004). W. Mückenheim: "A severe inconsistency of transfinite set theory", arXiv (2004)]

A severe inconsistency of set theory (Discussion)

A talk with the above contents and headline has been delivered at the annual meeting of the German Mathematical Society (DMV), section logic, at Heidelberg (14 Sep 2004). During the discussion the question was raised whether the proof requires transfinite induction.
My answer: Transfinite induction is not required as long as the process occurs in the countable, i.e., in the finite domain. This is true as long as rational numbers are available because their set is countable. Cantor's enumeration of the rational numbers does not apply transfinite induction.

Another discussion of my proof appeared on Math.StackExchange where probabilityislogic asked: "... I read this article which seems to provide a rather simple proof showing that the rationals have a cardinality at least as big as the irrationals, which would contradict the above paragraph. So at least one of the results should be wrong? However I cannot find the fallacy in the reasoning in the article ..." [probabilityislogic: "Axiom of choice: Can someone explain the fallacy in this reasoning?", Math.StackExchange (26 Apr 2011)]

"There is also an error in the proof on page 5, in that when one does an argument like that by transfinite induction one has to show that a choice is possible even at limit stages." [Carl Mummert, loc cit]
My answer: As above. There is no transfinity in the domain of natural indices k of q_k.

"By assuming that the rationals are not exhausted before the irrationals, the author is already assuming the result that is being proved, that the cardinality of the irrationals is no bigger than the cardinality of the rationals." [Carl Mummert, loc cit]
My answer: I do not *assume* that the rationals are not exhausted, but I use the *fact* that they are dense even after removing any finite set. Since all rationals are said to be countable, I never reach "the infinite".

"The author tries to address this lower on page 5, but that argument is only valid if only finitely many rationals have been chosen so far in the construction." [Carl Mummert, loc cit]
My answer: Correct! And this situation does never change.

"Well-order the positive reals. Consider building a set of rationals. For each real in turn, add a rational less than (conventional order) the real that has not been added so far. As the rationals are dense, this can always be done. This makes a bijection between the reals and the rationals." [Ross Millikan, loc cit]
My answer: Correct.

"He claims that q' will always exist 'because there is always a rational between any two real numbers'. But this assertion is empty: the existence of such a rational does not imply the existence of a rational *that has not yet been chosen*. The author never actually establishes the existence of such a q'." [Arturo Magidin, loc cit]
My answer: This argument is very unapt because all used rationals have been removed from the set $\mathbb{Q}_+$ and have been transferred to the set Q. Since in every step only a finite set of rationals has been used, there is always a further rational as required in the set $\mathbb{Q}_+$.

"The idea of exhausting infinite sets is that the process is not finitary, but we describe 'all' the steps needed for it to finish, even if by one-step-at-a-time we can never even reach the first limit point." [Asaf Karagila, loc cit]
My answer: This opinion is sensible but is contrary to set-theory. Cantor claims exhausting of an infinite set in a similar procedure: "It is clear that in this manner *definite* points of the sequence (5) can be assigned to *all* intervals of the sequence (3). Because of their being dense [...] there are infinitely many points of the *required* relative position, and the mapping process resulting from our rule will *never come to a halt*." [Cantor, p. 239]

"The bijection described in the article will exhaust exactly after ω steps ..." [Asaf Karagila, loc cit]
My answer: This is amazing, because in the last paragraph above the same author denied that the bijection could be completed by one step at a time, contrary to Cantor's claim.

"...while you still have at least $2^{\aleph_0}$ many steps to go with the irrationals." [Asaf Karagila, loc cit]
My answer: This is simply a claim taken from elsewhere. But an inconsistency is not removed by showing that one of the contradictory results is supported by firm belief.

"This is similar to proving that every ordinal number is finite: Start with 0, finite. Then assume n is finite therefore $n + 1$ is finite. Continue until you exhaust the class of ordinal numbers. Therefore all ordinals are finite." [Asaf Karagila, loc cit]
My answer: True. However that does not contradict my proof, but actual infinity. In potential infinity always everything is finite.

"This is wrong because eventually we exhaust natural numbers and we find ourselves at the realm of infinite ordinals, as we did not specify what is going to happen at the limit stages, this induction can (and will) fail at the first limit point – ω." [Asaf Karagila, loc cit]
My answer: Again, the author contradicts his own argument that infinite sets cannot be exhausted. We will never "find ourselves in the realm of infinite ordinals" – other than by delusions.

"There is no need to look at the article to which you refer. It must be wrong, and it is the author's job, not yours or mine, to find out where the mistake is." [Gerry Myerson, loc cit]
My answer: That's why ZFC is also called the theory of Zero Findable Contradictions!

"There are simple proofs of the correct result all over the place. You can read them, and understand them, and then you will know which to believe. [...] I do have a problem with 'is the author's argument sound?' when you can *know* that the author's argument can't *possibly* be sound." [Gerry Myerson, loc cit]
My answer: Again, that's why ZFC is free of contradictions!

"The problem, as Asaf points out, is the limit points. Think about this: given any natural number n, there is always an *infinite* quantity of natural numbers strictly larger than n; but this does not mean that we can 'keep picking' natural numbers, one for every real: once we get 'to the limit' (ω), we'll be out of natural numbers." [Arturo Magidin, loc cit]
My answer: Never anybody got to the limit. In case of this proof this would mean to run out of rationals.

"We can certainly keep picking them 'one-at-a-time', but this induction only gets us to the elements of ω, not even to ω itself." [Arturo Magidin, loc cit]
My answer: If it gets us to *all* elements of ℕ while ω is not reached, then ω will never be reached.

"In transfinite recursion, you *also* need to say what to do at a step which is *neither* the first step, nor is the 'immediately next step' to any step. ω (the first ordinal that follows all natural numbers) is an example of such a step. They are the 'limit ordinals' others have mentioned." [Arturo Magidin, loc cit]
My answer: Since in my proof a limit is not reached, no transfinite induction is required.

"I think that the main mistake he makes is that he doesn't understand the difference between standard induction and transfinite induction ..." [N. S., loc cit]
My answer: I think that the main mistake N. S. makes is that he does not understand that transfinite induction is not required as long as there are enumerated rationals.

"He seems to think that well ordered means the set is like the natural numbers..." [N. S., loc cit]
My answer: As long as only enumerated rationals are used, the set *is* like the natural numbers.

Recently this proof has been discussed for another time. "'We can say that there are no different infinities. If the axiom of choice is abolished, then well-ordering of the continuum and of larger sets is impossible, and there is no chance of attributing a cardinal number to those sets. If the axiom of choice is maintained then the continuum can be proved countable, also contradicting transfinite set theory.' I was just getting comfortable with ω, $\omega+1$, $\omega 2$, ω^2, ω^ω [...] Is there really only one ∞?" [Michael Tiemann: "W. Mückenheim claims a severe inconsistency of transfinite set theory; true?", Math.StackExchange (9 Jul 2015)]

"What Mückenheim claims is wrong. [...] There may be contradictions in set theory, but what Mückenheim writes does not expose any." [Daniel Fischer, loc cit]
My answer: A remarkably void argument.

"You need injectivity to conclude something like that." [Cameron Williams, loc cit]
My answer: Injectivity is proved in my case by the fact that every irrational number is mapped on its own rational number.

"The glaring flaw I found was this [...] In fact, $\mathbb{Q}_+$ *is* exhausted first, and so the proof *does* fail." [Tanner Swett, loc cit]
My answer: Unfortunately the claimed "fact" about the "glaring flaw" has not been supported by any argument other than italicizing the "is". Infinite sets *never* get exhausted step-by-step.

"Perhaps he doesn't understand that an enumeration of the rationals as $q(0)$, $q(1)$, $q(2)$, ... etc cannot preserve the arithmetic order." [user254665, loc cit]
My answer: Since I never gave the least indication to assume a preservation of the arithmetic order, this statement shows the lack of the critic's comprehension.

The fact that between two irrational numbers always a rational number is tracked down reminds of a trident or three-pronged fork. So I call this proof procedure *forking*.

The list of everything

The set of all finite expressions can be written in every finite alphabet, for instance in binaries:

0	or	*a*
1		*b*
00		*aa*
01		*ab*
10		*ba*
11		*bb*
000		*aaa*
...		... .

This is the list of everything containing $\aleph_0$ finite expressions. Of course some of its words may have different meanings, according to the language applied. (Definitions of languages can be found in some later parts of the list.) The only item that is missing is a diagonal word. Therefore *it is easy to see that an antidiagonal word cannot be constructed. And even if it could, it would not mean anything because all meaningful words have to be finite.*

The list of everything contains all finite words. Infinite sequences of symbols without a finite formula creating them cannot be transmitted and cannot be used in mathematical monologue, dialogue, or discourse. The meaning of each word depends on the used language. But since every language has to be devised and stored in at least one memory, there are only finitely many languages. Therefore the list of all possible meanings is countable.

The list of everything, by enumerating all finite expressions u, maps $\mathbb{N}$ to the set O of all objects o of discourse such that every object o is in the image of infinitely many natural numbers. (Every object o can be addressed by infinitely many words u.) This mapping is a surjection $\mathbb{N}$ to O including a surjection from a subset of $\mathbb{N}$ to the subset $\mathbb{R}$ of O, but it is tacitly assumed that a bijection $\mathbb{N} \leftrightarrow \mathbb{R}$ can be obtained from it because every infinite subset of $\mathbb{N}$ can be put in bijection with $\mathbb{N}$. So this mapping can be called a bijection from $\mathbb{N}$ to $\mathbb{R}$.

[W. Mückenheim in "Why worry about the axiom of choice?", MathOverflow (4 Jul 2010)]

Sometimes advocates of set theory claim that subsets of countable sets can be uncountable. They call them "subcountable". This is in contraction with set theory: "Every infinite subset of a countable set is an infinite countable set again." [Cantor, p. 152]

Same holds for distinguishing "countable" and "listable" as opposed to each other because "all 'definable' (computable) reals cannot be explicitly listed. This is *not* the same as being uncountable." [Peter Webb in "Why can no one in sci.math understand my simple point?", sci.math (16 Jun 2010)] "The constructable reals are countable but an enumeration can not be constructed (otherwise the diagonal argument would lead to a real that has been constructed)." [Dik T. Winter in "Cantor's diagonalization", sci.math (7 Apr 1997)]
My answer: Of course this is a contradiction. But set theorists appear to be unable even to remotely consider the possibility that set theory could be self-contradictory. "Countable" and "listable" or "ordered as a sequence" *is* the same: "Consider any point set M which [...] has the

property of being countable such that the points of M can be imagined in the form of a sequence". [Cantor, p. 154] "ordering of all algebraic numbers in a sequence, their countability". [G. Cantor, letter to R. Dedekind (10 Jan 1882)]

Other advocates of set theory claim that elements can be distinguished and put into an order without a possibility to distinguish them: "Labels aren't relevant to distinguishing anything. [...] Anonymous elements (even in countable models, where they *could* be labeled in principle, but just don't happen to be in practice) are absolutely key to all kinds of results in this field." [George Greene in "Listing rationals", sci.logic (17 Jan 2016)]
My answer: One of the results in this field is the self-contradictory belief that although most elements of an uncountable set cannot be identified in any possible language, the position of every element can be defined and therefore identified in a well-ordering – called "extended sequence" by Cantor: "All sets are therefore '*countable*' in an *extended sense*, in particular all '*continua*'". [Cantor, p. 447]

"Do you tell your poor students that $\mathbb{R}$ is not like $\mathbb{Q}_+$ in this regard – that no one has found a successor function for $\mathbb{R}$ like my S for $\mathbb{Q}_+$? If you do admit that no such function is known, do your dare to tell them why?" [Ben Bacarisse in "How to apply uncountable alphabets?", sci.logic (17 Jan 2019)]
My answer: Of course, it is not a deep mystery. There is your successor function S for the sequence of pairs of natural numbers, and there is my successor function for the sequence of finite expressions of letters. There is no successor function for only real numbers because many real numbers cannot be expressed by digits. We need words of letters as they are used in other circumstances too.

Countability of the real numbers

Consider an inertial system[1] in an infinite and eternal universe. All rational spatio-temporal coordinate quadruples (x, y, z, t) belong to a countable set. Every real number that is thought, written, mentioned, or in any other way used as an individual exists in a domain of its own in this inertial system. Every domain consists of infinitely many rational coordinate quadruples. Take one of those that are in contact with the physical definition of the real number and map it onto that real number. Then a countable subset of rational coordinate quadruples surjects all instances of real numbers (some of them even more than once) including all antidiagonal numbers. This is a "Cantor-list" where the enumerating set $\mathbb{N}$ is replaced by a countable subset of rational coordinate quadruples. Therefore the cardinality of all real numbers cannot surpass $\aleph_0$.

Of course this argument does not only concern the real numbers but all material and immaterial elements of the universe.

[W. Mückenheim in "Who's up for a friendly round of debating Cantor's proof?", sci.math (23 Aug 2011)]

[1] Even a countably infinite set of inertial systems would satisfy the conditions of the argument.

Enumerating all real numbers

The assumption that all natural numbers could be exhausted in connection with the wrong idea that a real number could be defined by an infinite digit sequence has lead to the diagonal argument. But even under these assumptions it is possible to enumerate all real numbers. To enumerate sets like all rational numbers or all algebraic numbers requires an intelligent way. To show that the real numbers are countable we need an even more sophisticated method. It does not lead to a sequence but to a sequence of sequences and therefore to a countable result like ω^2.

The virtue of this method is that it avoids to squander the natural numbers: The entries of the Cantor-list are not enumerated by all natural numbers but by powers of 2. The antidiagonals of this list (never more than a countable set) are enumerated by the powers of 3. If a new list is created then its entries are enumerated by the powers of 5 and its antidiagonals (never more than a countable set) are enumerated by the powers of 7. This is going on, in every step using a new prime number. The set of prime numbers will never get exhausted.

Enumerating all real numbers (Discussion)

"Uncountable sets result only from the very clumsy way in that set theorists count and enumerate. Every antidiagonal and every real number that results from any 'uncountability-proof' belongs to a countable set because all proofs can be enumerated. Therefore uncountable sets, if existing anywhere, are inaccessible and not provable.

In fact it is the asssumption that all natural numbers could be exhausted, which leads astray. But even under this asssumption it would be necessary to enumerate in an intelligent way in order to prove countability. An example are the algebraic numbers. Why do set theorists resist to enumerate all antidiagonals in an intelligent way?

Example: All Cantor-lists can be enumerated. The digonal of Cantor-list number *n* is enumerated by the *n*th prime number and the entries are enumerated by powers of the *n*th prime number. Every antidiagonal and every real number that results from any 'uncountability-proof' belong to one and the same countable set." [W. Mückenheim: "Enumerating of infinite sets requires intelligence", sci.math (17 Jul 2016), sci.logic (17 Jul 2016)]

"The definition of such a set is obviously impredicative: You have defined a set that contains an element that differs from each of its elements." [J. Rennenkampff in "Enumerating of infinite sets requires intelligence", sci.logic (17 Jul 2016)]

This is an unjustified counter argument as has been noted by the following author too:

"That doesn't make the definition impredicative. Impredicativity is more about quantifying over a domain that *does* contain the thing defined. And impredicativity is in any case far too complex and irrelevant a notion to be invoking here." [G. Greene in "Enumerating of infinite sets requires intelligence", sci.logic (18 Jul 2016)]

"The fact that some *one* thing belongs to a countable set *does not prevent* that one thing *from proving that some 'particular' set is uncountable!*" [G. Greene in "Enumerating of infinite sets requires intelligence", sci.logic (18 Jul 2016)]

"None-the-less WM has presented a contradiction." [graham in "Enumerating of infinite sets requires intelligence", sci.logic (19 Jul 2016)]

On Hessenberg's proof (I)

Hessenberg's proof (see section 2.4) fails in infinite infinity. This statement sounds rather strange, but it is required to distinguish infinities since Cantor and his disciples have finished infinity.

Hessenberg derives the uncountability of the power set of $\mathbb{N}$ from the limit-set M of *all* natural numbers which are not in their image-sets. M cannot be enumerated by a natural number n. If M is enumerated by n, and if n is not in M, then n belongs to M and must be in M, but then n does not belong to M, and so on.

If "*all*" is replaced by "*every*" and if we keep in mind that every natural number is succeeded by infinitely many natural numbers (and preceded by only finitely many), we get the following sequential explanation of the "paradox":

Every set $M_k = \{n_1, n_2, ..., n_k\}$ containing all natural numbers up to n_k, which are not mapped on image-sets containing them, can be mapped by any number m not yet used in the (always incomplete) mapping. This number m is not in M_k and therefore has to be included as $m = n_{k+1}$ into the set M_k. Doing so we get the set $M_{k+1} = M_k \cup \{m\}$. There remain infinitely many further natural numbers available to be mapped on M_{k+1}. Choose one of them, say m'. Of course, m' is not in M_{k+1} and therefore has to be included as $m' = n_{k+2}$ into M_{k+1}, such that we get $M_{k+2} = M_{k+1} \cup \{m'\}$. This goes on and on without an end. The mapping is infinite. As long as there is no limit-set M, there cannot be a contradiction obtained from not finding a natural number to be mapped on M.

On Hessenberg's proof (II)

We will show that the impossible set does not exist and that the paradox-generating requirement cannot be satisfied, even if the mapping is defined between equivalent sets.

Define a bijective mapping from $\{1, a\}$ on $\mathcal{P}(\{1\}) = \{\{\ \}, \{1\}\}$, where a is a symbol but not a number. There are merely two bijections possible. The set M of all *numbers* which are not mapped on a set containing them cannot be mapped by a *number m* although M is in the image of both the possible mappings:

f: $1 \to \{1\}$ and $a \to \{\ \}$ with $M_f = \{\ \}$.
g: $1 \to \{\ \}$ and $a \to \{1\}$ with $M_g = \{1\}$.

Here we have certainly no problem with lacking elements in the domain. Nevertheless Hessenberg's condition cannot be satisfied. Both the sets, $\{M_f, m_f, f\}$ and $\{M_g, m_g, g\}$ with $m = 1$ the only available *number*, are impossible sets:

If $m = 1$ is mapped on $\{1\}$, then the set of all *numbers* which are not mapped on a set containing them is the empty set. It cannot be mapped by a number because the only number has already been mapped otherwise.

If $m = 1$ is mapped on { }, then the set of all *numbers* which are not mapped on a set containing them is {1}. It cannot be mapped by a number because the only number has already been mapped otherwise.

Hessenberg's proof does not concern the question whether or not $\aleph_0 < 2^{\aleph_0}$.

[W. Mückenheim: "On Cantor's Theorem", arXiv (2005). W. Mückenheim: "Die Geschichte des Unendlichen", 7th ed., Maro, Augsburg (2011) p. 119]

On Hessenberg's proof (III)

If the set of all $2^{\aleph_0}$ subsets of the natural numbers exist, i.e., if the precondition of Hessenberg's proof is satisfied, then one should expect that also all permutations of the natural numbers exist and (by the bijection of $\mathbb{N}$ and $\mathbb{Q}$) one should further expect that also all permutations of the rational numbers, each rational number indexed by a natural number, exist. Each permutation is a well-ordering, and one of them would be the well-ordering of $\mathbb{Q}$ that is simultaneously the ordering by size. This is a contradiction. Like Hessenberg's assumption.

The halting problem

The construction given in section 2.3.2 does not yield a number. We should put 3 if a program never outputs a *k*th digit. That may happen, if it never halts. But in order to find that out, we would have to wait infinitely long. This is not a practicable advice. It can be demonstrated by paraphrasing "if the *k*th program never outputs a *k*th digit" by "as soon as it happens that the output does not happen".

Chaitin himself explains the infeasibility of this procedure:

"It must be uncomputable, by construction. Nevertheless, as was the case in the Richard paradox, it would seem that we gave a procedure for calculating Turing's diagonal real *r* digit by digit. How can this procedure fail? What could possibly go wrong?

The answer is this: The only noncomputable step has got to be determining if the *k*th computer program will *ever* output a *k*th digit. If we could do that, then we could certainly compute the uncomputable real *r* of {{Sec. 2.3.2}}.

In other words, {{Sec. 2.3.2}} actually *proves* that there can be no algorithm for deciding if the *k*th computer program will ever output a *k*th digit.

And this is a special case of what's called Turing's halting problem. In this particular case, the question is whether or not the wait for a *k*th digit will ever terminate. In the general case, the question is whether or not a computer program will ever halt.

The algorithmic unsolvability of Turing's halting problem is an extremely fundamental meta-theorem. It's a much stronger result than Gödel's famous 1931 incompleteness theorem. Why? Because in Turing's original 1936 paper he immediately points out how to derive incompleteness from the halting problem.

A formal axiomatic math theory (FAMT) consists of a finite set of axioms and of a finite set of rules of inference for deducing the consequences of those axioms. Viewed from a great distance, all that counts is that there is an algorithm for enumerating (or generating) all the possible theorems, all the possible consequences of the axioms, one by one, by systematically applying the rules of inference in every possible way. This is in fact what's called a breadth-first (rather than a depth-first) tree walk, the tree being the tree of all possible deductions.

So, argued Turing in 1936, if there were a FAMT that always enabled you to decide whether or not a program eventually halts, there would in fact be an algorithm for doing so. You'd just run through all possible proofs until you find a proof that the program halts or you find a proof that it never halts.

So uncomputability is much more fundamental than incompleteness. Incompleteness is an immediate corollary of uncomputability. But uncomputability is *not* a corollary of incompleteness. The concept of incompleteness does not contain the concept of uncomputability." [Gregory Chaitin: "How real are real numbers?", arXiv (2004)]

The key to the halting problem as well as to Hessenberg's argument is this: There is no complete enumeration of all computable sequences and there is no complete set of natural numbers which are mapped on subsets not containing them *because there is no completeness at all in infinity*. Neither Turing's complete sequence of computable sequences nor Hessenberg's complete mapping from $\mathbb{N}$ to $\mathcal{P}(\mathbb{N})$ are admissible notions in mathematics. Therefore the premises of these arguments remain undefined.

The divergence proof of the harmonic series by Nicole d'Oresme

Nicole d'Oresme (1323-1382) proved the harmonic series to be divergent. Alas his proof needs $\aleph_0$ sums of the form (1/2) + (1/3 + 1/4) + (1/5 + 1/6 + 1/7 + 1/8) + (1/9 + ... + 1/16) + ... requiring in total $2^{\aleph_0+1} - 1 = 2^{\aleph_0}$ unit fractions. If there were less than $2^{\aleph_0}$ natural numbers (or if $2^{\aleph_0}$ was larger than $\aleph_0$), then there were also less than $2^{\aleph_0}$ unit fractions and d'Oresme's proof would fail. The harmonic series could not diverge and mathematics would supply wrong results. [W. Mückenheim: "The meaning of infinity", arXiv (2004). W. Mückenheim: "Die Geschichte des Unendlichen", 7th ed., Maro, Augsburg (2011) p. 118]

This is the same procedure with the terminating binary representations of the rational numbers of the unit interval. Each terminating binary representation $q = 0.abc...z$ (including 0) is an element out of $2^{\aleph_0+1} - 1 = 2^{\aleph_0}$ elements.

An argument of set theory says that the function 2^n "is not continuous at infinity"

$$\lim_{n\to\infty} 2^n = \aleph_0 \neq 2^{\aleph_0} .$$

Therefore $\aleph_0 = \lim_{n\to\infty} 2^n \neq 2^{\lim_{n\to\infty} n} = 2^{\aleph_0}$. (*)

But then, why is $|\lim_{n\to\infty} \{1, 2, ..., n\}| = |\mathbb{N}| = \aleph_0 = \lim_{n\to\infty} |\{1, 2, ..., n\}|$ taken for granted? Why is continuity assumed in this case? Couldn't as well $|\lim_{n\to\infty} \{1, 2, ..., n\}| = 0$? And what remains if in (*) exponentiation is replaced by what it originally abbreviates, namely by repeated multiplication: 2·2·2·... ≠ 2·2·2·... ???

The Binary Tree

The complete infinite Binary Tree contains all infinite bit sequences. The *limits* of these bit sequences, prepended by "0.", represent the real numbers of the unit interval [0, 1]. When the following arguments had been devised, it was generally accepted however that the *paths* of the Binary Tree represent all *real numbers* of the unit interval [W. Mückenheim: "Has this paradox been known in literature?", MathOverflow (29 Jun 2010)] Meanwhile this mistake has been clarified (see section "Sequences and limits"). But also the set of infinite bit sequences is uncountable according to Cantor's second uncountability proof (cp. section 2.2.3). This will be contradicted in the following.

The complete infinite Binary Tree consists of nodes (bits or binary digits 0 and 1) which are indexed by natural numbers and connected by edges such that every node has two and only two child nodes. Node number $2n + 1$ is called the left child of node number n, node number $2n + 2$ is called the right child of node number n.

The set $\{a_k \mid k \in \mathbb{N}\}$ of nodes (bits) a_k is countable as shown by the indices of the nodes.

```
Level          Bits                   Nodes

  0             0.                     a0.
               /  \                   /  \
  1           0    1                a1    a2
             / \   / \             / \    / \
  2         0   1  0   1          a3  a4 a5  a6
           /\  /\  /\  /\         /\  /\  /\  /\
  3       0 ...                  a7 ... .
```

A path p is a subset of nodes having the indices

$$0 \in p$$

and

$$n \in p \;\Rightarrow\; (2n + 1 \in p \;\text{ or }\; 2n + 2 \in p \;\text{ but not both})\,.$$

The sets of indices are infinite subsets of $\mathbb{N}_0$ e. g. $\{0, 1, 3, 7, ...\}$ or $\{0, 2, 6, 14, ...\}$. Every path denotes an infinite sequence of bits, e. g., 0.000... or 0.111..., the limit of which is a real number of the unit interval [0, 1]. The examples given above are paths with only left children and only right children, respectively.

Irrational numbers cannot be represented by paths in the complete infinite Binary Tree. The Binary Tree of only terminating rational numbers is, as far as nodes and edges are concerned, identical with the complete infinite Binary Tree. But how can the periodic and irrational paths be inserted in the Binary Tree of terminating rational numbers to get the complete infinite Binary Tree? Not at all! 1/3 for instance has no binary representation (see section "Sequences and limits").

The extended Binary Tree is obtained by extending the ordinary Binary Tree to the upper side:

```
   ...
0 1 0 1
 \/  \/
  0  1
   \ /
    .
   / \
  0  1
 /\  /\
0 1 0 1
   ... .
```

The upper side can be understood as indicating the integer part, i.e., the bits in front of the radix point.

Equal number of distinguishable paths and nodes

The basic structure is the branching at a node o

```
  |
  o
 / \
```

where the number 2 of edges leaving a node is equal to the number of 1 incoming edge plus 1, represented by the node:

$1 + 1 = 2$.

All paths that can be distinguished on a certain level are distinguished by nodes (or edges). Therefore the number of distinguishable paths grows with the number of nodes. Every node increases the number of distinguishable paths by 1. The number of distinguishable paths is identical to the number of nodes (+1). It can be made equal to it by an additional pre-rootnode o:

```
Level          o
               |
  0            0.
              / \
  1          0   1
            / \ / \
  2        0  1 0  1
              ... .
```

The number of incoming distinguishable paths on a level plus the number of nodes at this level is the number of distinguishable paths leaving this level.

Even "in the infinite", should it exist, a path cannot branch into two paths without creating a node. Because a node is defined as a branching point, no increase in distinguishable paths is possible without the same increase in nodes.

Not necessary to mention, at every level the cross-section of the Binary Tree, i.e., the number of nodes at that level, is finite. And, as an upper estimate, even lining up all $\aleph_0$ nodes on a single level would limit the set of paths to $2\aleph_0$.

Construction of the Binary Tree

A countable set can be constructed by using always half of the remaining time for the next step. An uncountable set cannot be constructed such that uncountably many elements can be distinguished. So it is possible to construct $\mathbb{N}$ and with it all its subsets. But these subsets cannot be distinguished unless it is indicated which elements are to combine. Therefore we find:

- The Binary Tree can be constructed because it consists of countably many nodes and edges.
- The Binary Tree cannot be constructed because it consists of uncountably many *distinct* paths.

Construction by finite (initial segments of) paths

The Binary Tree can be constructed by $\aleph_0$ finite initial segments of paths, briefly called finite paths. The *n*th finite path ends at node *n* (in the following figure indicated by its index):

```
Level   0            0
                    /  \
        1          1    2
                  / \   / \
        2        3   4  5   6
                /\  /\  /\  /\
        3      7 ... .
```

There is no node and no edge missing. There is no path missing, that can be defined by its nodes.

However, the path 0.000... is constructed as soon as all its subpaths (or at least infinitely many) have been constructed, for instance using the paths 0.111..., 0.0111..., 0.00111..., ..., since then no node of 0.000... is missing.

At each level the number of nodes doubles. We start with the (empty) finite path at level 0 and get

$2^{n+1} - 1$ finite paths within the first n levels.

The number of all levels of the Binary Tree is called $\aleph_0$ although there is no level number $\aleph_0$. But mathematics uses only the number of terms of the geometric sequence. That results in

$2^{\aleph_0+1} - 1 = 2^{\aleph_0}$ finite paths within the whole infinite Binary Tree.

The bijection of finite paths with their last nodes proves $2^{\aleph_0} = \aleph_0$.

[W. Mückenheim: "Physical constraints of numbers", Proceedings of the First International Symposium of Mathematics and its Connections to the Arts and Sciences, A. Beckmann, C. Michelsen, B. Sriraman (eds.), Franzbecker, Berlin (2005) pp. 134-141]

This construction is often presented in form of a (binary or decimal) list, cp. Table 3 in [W. Mückenheim: "The meaning of infinity", arXiv (2004)] (here given in section "Discontinuity of transfinity (II)"). Compare also some of the works quoted in chapter V. Nearly all proposed enumerations of the set of real numbers are based on potential infinity, i.e., on the enumeration of an infinite set of finite strings of bits or digits or letters.

Construction by finite initial segments

The complete infinite Binary Tree is the limit of the sequence of its initial segments B_k:

$$B_0 = a_0\,,\quad B_1 = \begin{array}{c} a_0 \\ / \\ a_1 \end{array},\quad B_2 = \begin{array}{ccc} & a_0 & \\ / & & \backslash \\ a_1 & & a_2 \end{array},\quad B_3 = \begin{array}{ccccc} & & a_0 & & \\ & / & & \backslash & \\ & a_1 & & & a_2 \\ / & & & & \\ a_3 & & & & \end{array},\ \ldots,\quad B_k = \begin{array}{ccccccc} & & & a_0 & & & \\ & & / & & \backslash & & \\ & a_1 & & & & a_2 & \\ / & & \backslash & & / & & \backslash \\ a_3 & & a_4 & & a_5 & & a_6 \\ \ldots & & & & & & \\ \ldots\, a_k & & & & & & \end{array},\ \ldots\,.$$

The structure of the Binary Tree excludes that there are any two initial segments, B_k and B_{k+1}, such that B_{k+1} contains two complete infinite paths both of which are not contained in B_k. Nevertheless the limit of all B_k is the complete Binary Tree including all (allegedly uncountably many) infinite paths. Contradiction. There cannot exist more than countably many infinite paths.

Alternative consideration: Obviously every B_k is finite. None does contain any infinite path. The infinite paths come into the play only after all B_k with $k \in \mathbb{N}$ (by some unknown mechanism). Therefore they cannot be identified by finite initial segments. If that is possible, however, this mechanism can also act in Cantor's diagonal proof such that the diagonal number enters the list only after all rows at finite positions.

[W. Mückenheim in "How many orders of infinity are there?", MathOverflow (27 Jun 2010)]

This is a splendid example of the ketchup effect (see section "The ketchup effect").

Construction by levels

Every *finite* Binary Tree T_n with n levels contains less paths than nodes. Down to level n there are $2^{n+1} - 1$ nodes but only 2^n path crossing all n levels. The union $\cup T_n = T_\infty$ of *all finite* Binary Trees T_n *covers all levels* enumerated by natural numbers. With respect to nodes and edges it is identical with the complete infinite Binary Tree T. T_∞ contains only a countable set of finite paths. But does T_∞ contain only finite paths?

The union of all indices of nodes of finite paths is the union of all finite initial segments of natural numbers $\{1\} \cup \{1, 2\} \cup \{1, 2, 3\} \cup ... \cup \{1, 2, 3, ..., n\} \cup ... = \{1, 2, 3, ...\}$.

This is also the set of all last elements of the finite segments, i.e., it is the set of all natural numbers, the set of *all* indices – there is no one left out. We find for instance that the union of all finite paths of T_∞ which always turn right, 0.1, 0.11, 0.111, ..., is the infinite path 0.111... . From this we can conclude that *every* infinite path belongs to the union T_∞ of all finite Binary Trees.

The union of all finite Binary Trees T_∞ and the complete infinite Binary Tree T are identical with respect to all nodes, all edges, and all paths (which would already have been implied by the identity of nodes and edges). But the set of all paths is countable in the Binary Tree T_∞ and allegedly uncountable in the same Binary Tree T.

[W. Mückenheim: "The infinite in sciences and arts", Proceedings of MACAS 2, B. Sriraman, C. Michelsen, A. Beckmann, V. Freiman (eds.), Centre for Science and Mathematics Education, University of Southern Denmark, Odense (2008) pp. 265-272]

Construction by infinite paths

The set of all finite paths (from the root-node to any other node) in the complete infinite Binary Tree is countable. Therefore the complete infinite Binary Tree has countably many *paths that can be identified by nodes*.

Infinite paths can only be defined by finite descriptions or expression like "always turn left", or "0.111...", or "the path which converges to 1/3" or "the path which converges to $1/\pi$". Note however that the set of all finite expressions, a superset of the set of all finite descriptions, has countable cardinality. Therefore the set of all paths in the complete infinite Binary Tree, as far as they can be identified, has countable cardinality too.

Construct the Binary Tree by all finite paths and append an arbitrary infinite tail to each path:

- Construct the Binary Tree by all finite paths that have the sequence 000... appended.
- Construct the Binary Tree by all finite paths that have the sequence 111... appended.
- Construct the Binary Tree by all finite paths that have the sequence 010101... appended.
- Construct the Binary Tree by all finite paths that have all bits converging to $1/\sqrt{2}$ appended.
- Construct the Binary Tree by all finite paths that have all bits converging to $1/\pi$ appended.

It is impossible to distinguish the constructed Binary Trees by their nodes and to determine what tails have been used and what infinite paths are missing.

Alternatively: Enumerate all nodes a_i of a Binary Tree and map them on infinite paths p_i such that $a_i \in p_i$. There is no further restriction. The mapping need not be injective. Assume there exists a path p_0 that is not the image of a node. Then p_0 would consist of nodes a_i only which have been mapped on other paths. (Otherwise a node remains that can be mapped on p_0.) So we get a contradiction:
• p_0 is distinct from all path on which its nodes are mapped.
• For all nodes a_i of p_0: p_0 is identical from the root node a_0 to node a_i with the path that a_i is mapped on. Therefore no node with finite index serves to identify p_0.

If p_0 consists only of nodes a_i with finite index i, this distinctness assumption is a contradiction.

In order to demonstrate the result vividly, remove the (remaining) nodes of every path that has been mapped by a node and construct from them another Binary Tree. Mathematical analysis is not able to discern which paths were used to construct the new Binary Tree. Further, since the original Binary Tree is void of nodes (cp. section "The power set of ℕ is not uncountable"), how should further paths in it be defined there?

Continuity of paths

At every finite level n the number of paths distinguishable there, the so-called cross-section of the Binary Tree at that height, is less than $\aleph_0$. In the whole tree the number of distinct paths is uncountable. How can that fact be reconciled with the continuity of paths? [W. Mückenheim: "Warnung vom Meister persönlich", de.sci.mathematik (24 May 2016]

Conquer the Binary Tree

Here is a variant of the construction by infinite paths, a game that only can be lost if set theory is true: You start with one cent. For a cent you can buy an infinite path of your choice in the Binary Tree. For every node covered by this path you will get a cent. For every cent you can buy another path of your choice. For every node covered by this path (and not yet covered by previously chosen paths) you will get a cent. For every cent you can buy another path. And so on. Since there are only countably many nodes yielding as many cents but uncountably many paths requiring as many cents, the player will get bankrupt before all paths are conquered. If no player gets bankrupt, the number of paths cannot surpass the number of nodes. [W. Mückenheim: "What can we learn from the new game CTBT that I devised for my students?", MathOverflow (2 Jul 2010). W. Mückenheim: "Die Geschichte des Unendlichen, ch. XII", current lecture]

Colour the Binary Tree

Assume that every path in a complete infinite Binary Tree represents a real number of the unit interval between 0 and 1. Assume that you have a can of red paint and that you can colour an infinite path of the tree with one can. Assume further that you get another can of red paint for every node that you are colouring for the first time. Then you will first accumulate an infinity of cans of red paint, but you will nevertheless not be able to colour all paths, since there are uncountably many paths in the tree (and you can win only countably many cans of red paint). So there remain uncountably many paths uncoloured in the first run. Start with another colour, say green. Also with green paint you will not finish. How many different colours will be required? [rainbow: "How many colours are required to colour the tree of real numbers of the unit interval?", MathOverflow (3 Jul 2013)]

"It sounds like you're painting nodes, in that case you can color all of them red on the first try." [François G. Dorais, loc cit]

"You seem to be ignoring the fact that, after you have colored a countable family of pathes, say P_0, P_1, ..., P_n, ..., there may be other paths Q that are not on this countable list but have, nevertheless, had all their nodes and edges colored. Perhaps the first node and edge of Q were also in P_1, the second node and edge of Q were in P_2, etc. [...] by choosing the sequence of P_n's intelligently, you can, in fact, ensure that this sort of thing happens for *every* path Q." [Andreas Blass, loc cit]

"After ω steps, all edges have been colored red. But, of course, only countably many paths have been painted. Are you finished or not?" [Gerald Edgar, loc cit]

My reply to all: However if you colour only all finite paths, then you will succeed with the countable set of cans that you are winning during the process. What part of the Binary Tree will then remain uncoloured? Nothing that contains or could be defined by nodes.

The power set of ℕ is not uncountable.

Every path in the Binary Tree represents a subset of ℕ: When the node at level n is 1, then the number n is contained in that subset. Otherwise, if the node at level n is 0, the number n it is not contained in that subset. Therefore there are precisely as many subsets of ℕ as infinite paths in the complete infinite Binary Tree.

0	*	*
	/ \	/ \
1	0 1	a_1 a_2
	/ \ / \	/ \ / \
2	0 1 0 1	a_3 a_4 a_5 a_6
	/\ /\ /\ /\	/\ /\ /\ /\
...	0 ...	a_7 ... .

Examples: The path $*111...$ represents $\mathbb{N}$, the path $*000...$ represents $\emptyset$, the path $*010101...$ represents all even numbers, the path $*11000...$ represents the finite initial segment $\{1, 2\}$.

In order to carry out the proof in full generality, we enumerate the paths according to the scheme on the right-hand side. Take an arbitrary path through node a_1, enumerate it by 1, and remove all its nodes. Then take another arbitrary path through node a_2, enumerate it by 2, and remove all its nodes which have not yet been removed. Repeat this action with all remaining nodes a_k until all nodes have been removed. This happens after at most countably many steps. Therefore at most countably many paths can have been enumerated.

If there are uncountably many paths in the Binary Tree, then most of them must stay in a Binary Tree which is void of nodes.

As a counter argument it has been asserted that when removing the sequence of all finite initial segments from $\mathbb{N}$, i.e., the sequence of paths $*1000...$ $*11000...$ $*111000...$ from the path $*111...$, the latter path is not removed. This however would imply, that the sequence of all finite initial segments, when put together after removing it, would represent $\mathbb{N}$ whereas $\mathbb{N}$ remains represented by the path $*111...$ in the Binary Tree too.

[W.Mückenheim: "The power set of |N is not uncountable", sci.logic (5 Jun 2016)]

This proof cannot be met other than by simply ignoring its consequences: "If you replace (as he should have done) 'delete' with 'count' then, obviously, if the rules let you count only one path for each node, you are doing nothing of any interest at all." [Ben Bacarisse, loc cit (10 Jun 2016)] or by stating counterfactually: "If all nodes of a path have been deleted, that does *not* mean you *ever* deleted the *path*!!! You could delete all and only *the finite* paths from the tree, and *that* would delete all the nodes, *yet you would never* have deleted *any infinite* path, and they would all still *exist* in any case, regardless of what had been deleted! [George Greene, loc cit (22 Jul 2016)]

My reply: And Elvis is alive too. But what kind of existence does he enjoy? To be serious: Even when accepting paths without nodes, there cannot be more than paths with nodes. At least one of the latter is needed to define one of the former.

The wondrous increase of paths

Construct a star such that every rational number of the unit interval has its own infinite bit sequence, starting from a common point "$*$" called root node. There are $\aleph_0$ such *not overlapping* sequences or individual paths.

Now put some paths together to build the infinite Binary Tree. Although the number of paths is not changed (only some paths are now partially overlapping each other) there are uncountably many paths in the end.

Here is a finite visualization. Start with the four paths 0.00, 0.01, 0.10, 0.11. When "$*$" stands in for "0.", we get the star

```
            1
            |
            0
            |
0 – 1  – * – 0 – 0      from which we construct the Binary Tree        *
            |                                                         / \
            1                                                        0   1
            |                                                       / \ / \
            1                                                      0  1 0  1 .
```

[W. Mückenheim: "The wondrous increase of paths", sci.math (4 Jul 2016)]

Distinguishing paths in the Binary Tree

Definition: Two infinite paths A and B can be distinguished at level n. $\Leftrightarrow$ There are two different nodes, a and b, at level n, such that a is in A and not in B, and b is in B and not in A.

Definition: Two infinite paths A and B are different. $\Leftrightarrow$ There exists a level n such that A and B can be distinguished at level n.

<u>Theorem</u> There is no level n where uncountably many paths can be distinguished.

<u>Corollary</u> In the complete infinite Binary Tree at most countably many infinite paths can be distinguished.

This does not imply that there exists a level n such that all paths A and B differ at that level. But for every pair of paths A and B, there is a level such that A and B have different nodes at that level. The simple result is that there cannot be more different paths than different nodes in the Binary Tree.

In order to distinguish a path A from another path P, you need a node of P that is not in A. The level, where this node can be found, does not matter. But of course, the level must have a finite index n, because there are no levels with infinite index ω or larger.

In order to distinguish A from n paths P_1, P_2, ..., P_n we need n nodes. It cannot be excluded that one node a of A is sufficient to distinguish A from all paths P_1, P_2, ..., P_n, but these are not n different paths unless there are nodes that distinguish P_1 from P_2 to P_n, and P_2 from P_3 to P_n and so on. In total n - 1 nodes are required to distinguish n paths P_1, P_2, ..., P_n.

In order to distinguish a further path from the former, there is at least one other node necessary. However, at most a countable set of nodes is available.

Fractional mapping

Map half of the root node 0 on a path $0.a_1$... that goes left and the other half on a path $0.a_2$... that goes right. It does not play a role how the chosen paths continue. Further map node a_1 on $0.a_1$... and map node a_2 on $0.a_2$... . So each of these paths carries 3/2 nodes.

```
    0.                 0.
  /    \             /    \
a1      a2          a1      a2
                   / \     / \
                  a3 a4   a5 a6
```

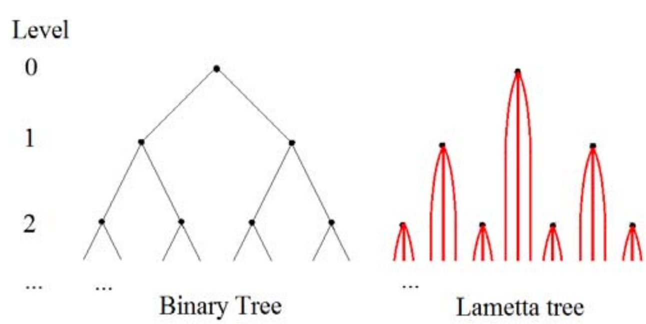
Choose arbitrarily four path starting with $0.a_1a_3$..., $0.a_1a_4$..., $0.a_2a_5$..., $0.a_2a_6$..., respectively, crossing the second level. Again it is irrelevant how they continue. Map a quarter of the root node and half of the node passed at level 1 and the full nodes at level 2 on these paths. So every path carries 7/4 nodes.

Continue such that 2^n of the paths crossing level n carry $2 - 2^{-n}$ nodes each. All paths defined by nodes will be included in the limit. Every path will carry 2 nodes if there are terminating nodes. Every path will carry one node if there are no terminating nodes.

[W. Mückenheim: "Cantor and the binary tree", sci.math (8 Jun 2005)]

The Lametta Tree

Define a tree L that has the same nodes as the ordinary complete infinite Binary Tree T but instead of paths running through the nodes let three paths or strings begin at every node of L without crossing any further nodes, like lametta, here marked in red.



For every $n \in \mathbb{N}$ the set of lametta strings in L crossing level n is larger than the set of distinct paths crossing level n in T. The set of lametta strings of L has cardinal number $3 \cdot \aleph_0 = \aleph_0$.

In classical mathematics this means that the lametta strings in L are not less numerous than the paths in T – even in the limit. How can set theorists argue that the paths in the ordinary complete infinite Binary Tree T become more numerous "in the infinite"?

[W. Mückenheim: "Consider a tree-structure", sci.math (2 Apr 2015). W. Mückenheim: "A direct comparison test for the binary tree", sci.logic (15 Oct 2017)]

The Lametta Tree (Discussion)

"Yesterday Otto asked this question in MathOverflow {{Otto: 'Where does the direct comparison theorem fail?', MathOverflow (20 Oct 2017)}}. [...] No 'professional mathematician' of MathOverflow could answer it. After a while it was migrated to Math.StackExchange and there it was immediately deleted. Otto has not acted in any way there, but now he has been suspended in Math.StackExchange. [...] So he has been punished, in Math.StackExchange, for applying, in MathOverflow, the simple mathematical rule: If the terms of the sequence (a_n) are always less than the terms of the sequence (b_n) then the limit of (a_n) cannot be larger than the limit of (b_n). Could it be that something is rotten in Math.StackExchange?" [Heinrich: "Why has Otto been suspended?", Math.StackExchange meta (21 Oct 2017)]

This question was deleted after half an hour without any explanation and Heinrich was suspended too. So it is definitively something rotten in Math.StackExchange and in the community dominated by set theorists.

"The number of paths leading from the root to some level is equal to the number of nodes at that level, and we already know that the set of nodes is countable. [...] That you draw lines all the way to 'infinity' does not help them to have any more to do with counting non-terminating paths than nodes do." [Conifold in "What are the 'undefinable numbers' in real analysis and philosophy?", Philosophy.StackExchange (20 Oct 2017)]
My reply: It is not necessary to have "any more to do with counting non-terminating paths than nodes do", and it is not a counter argument that "we already know that the set of nodes is countable". The number of paths that can be distinguished at some level is equal to the number of nodes at this level. Since the number of nodes at any level, including the limit, is not greater than $\aleph_0$, the number of distinct paths cannot be greater than $\aleph_0$ either. Of course this number cannot "explode" in the limit. If there were uncountably many *distinct* paths, then there should be a level, where we could see and *distinguish* all *distinct* paths. The lametta strings show that this is not the case, not even in the limit or, as set theorists prefer to say, "at infinity". But they seem to have a blind spot in this matter.

"But non-terminating paths are not accounted for in this way whether one interprets it in terms of cardinalities or not, so it is beside the point." [Conifold, loc cit]
My reply: The lametta strings are not terminating and they are more numerous than the paths of the tree at every finite level. This means, according to mathematics, even in the limit the set of paths cannot become larger than the set of lametta strings. This result cannot be circumvented.

"There is no need to circumvent your result since it is irrelevant." [Conifold, loc cit]
My reply: My result is a result of classical mathematics. You see: If set theory is preferred, then classical mathematics must be disregarded as irrelevant. That's what I wanted to show.

"You need an injection of such paths into red lines to say there are 'less' of them, all you do is provide injection of the terminating ones 'at each level'." [Conifold, loc cit]
My reply: If I were to prove that not all devils are red, I would have to show a blue or green or yellow devil according to you. *In mathematics* it would be sufficient to show that no devils can exist at all. I need no injection since it is presupposing finished infinity – a nonsense concept.

Cantor's "proofs" show merely that the presumed finished or actual infinity, required for his complete mappings and non-mappings, is an invalid premise.

"Yes, it is an irrelevant result of classical mathematics. This is not about classical mathematics vs set theory, your 'argument' is trivially erroneous. You need to decide if you argue from set theorists' premises (for reductio) or your own." [Conifold, loc cit]
My reply: I argue from mathematical premises – as Cantor originally also did (otherwise no-one would have been interested in his results) and as set theorists agreed upon until they faced contradictions that could be circumvented only by violating mathematics. Classical mathematics can be applied to check every field of science and of course also set theory. Everything contradicted by mathematical arguments like the majorant criterion above is scientifically rubbish. But nobody is forced by law to adhere to mathematics.

"Limits of continuous functions are not always continuous." [Conifold, loc cit]
My reply: First, the paths in the Binary Tree and in the Lametta Tree are as continuous as paths use to be. And second, a discontinuity could also happen in the enumeration of the rational numbers with the result that the set is uncountable. Without any "continuity" all transfinity fails.

"A path in the binary tree consists of one choice on each level, which can be coded by an infinite 0-1-string. Intuitively one would expect that there are more infinite strings than finite strings, and Cantor showed that this is indeed true." [Jan-Christoph Schlage-Puchta in "Where does the direct comparison theorem fail?", MathOverflow (19 Oct 2017)]
My reply: Here intuition fails. There are more lametta paths starting at every level than results of choices at every level. Cantor's "proof" is contradicted by the majorant criterion.

"Actually, the 'nature' of these 'lametta strings' is *completely immaterial* for your 'argument'." [Franz Fritsche alias "Me" in "The Lametta Tree", sci.math (19 Oct 2017)]
My reply: For every level of the tree L the lametta strings are more numerous than the *distinct* paths of T. In mathematics this proves that in the limit there cannot be more paths than lamettas.

"You know, there are *infinitely* many paths starting at the root of the tree. And there are *infinitely* many paths 'running through' each and every node in the tree." [Franz Fritsche, loc cit]
My reply: If there were *uncountably* many *distinct* paths, then there should be a level, where we could *distinguish* them. The lametta strings show that this is not the case, not even in the limit or "at infinity". (Cp. section "Distinguishing paths in the Binary Tree".)

Finally a general argument, repeated again and again: In order to prove an inconsistency of ZFC you must only use what ZFC uses.
My reply: Is it impossible to define the complete infinite Binary Tree in ZFC and to show that there are never more than $\aleph_0$ paths distinguishable *at any level*?

The commonly accepted answer is this: ZFC claims the existence of $2^{\aleph_0}$ paths in the Binary Tree but at any finite level the number of nodes is finite!
My reply: There are only finite levels in the complete infinite Binary Tree. Set theory claims $2^{\aleph_0}$ paths within this Binary Tree. Contradiction.

$2^{\aleph_0}$ paths separated from each other and plugged into the Binary Tree

Two paths, A and B, differ if A contains at least one node that is not in B. Vice versa this necessarily implies that B contains at least one node that is not in A.

A bunch of X paths is split into two bunches at their last common node N. What is X at most? How many paths can be in a bunch? Not more than there are nodes below N crossed by paths of the bunch which can further separate the paths of the bunch. Therefore every bunch can contain at most countably many paths which later can be separated.

The sequence $N(n) = 2^n$ of nodes N at level n gives us the number of path bunches that can be distinguished at level n. The preceding levels are irrelevant for this sake. This sequence has, assuming actual infinity of set theory, the limit $\aleph_0$.

Two paths are distinguishable if and only if they belong to different path bunches. This limits the number of distinguishable paths to $\aleph_0$.

Note that Cantor's theorem concerns only *distinguishable* paths. Paths which always inhabit the same path bunch are not distinguishable from each other.

A path bunch splits at a node in two bunches: 1 bunch + 1 node → 2 bunches. Therefore the number of path bunches cannot surpass the number of nodes.

Discussion: "This does not seem to be research level mathematics to me. However, if I understand the terminology 'bunch' and 'split', I think the error is that a single node can separate uncountably many different paths, and so even though there are only countably many nodes it is possible to have uncountably many paths so that any two of the paths are observably different. For example the node (0, 1, 0) separates any path starting in (0, 1, 0, 1) from any path starting in (0, 1, 0, 0)." [Carl Mummert in "Is even this clear contradiction incomprehensible to set theorists?", MathOverflow (23 Jun 2018)]

My reply: The expression "observably different" is clearly wrong.

Discussion: "Even if 'in the limit' there was only one bunch {{of paths distinguishable}} there would still be uncountably many paths."[Ben Bacarisse in "How many different paths can exist in the complete infinite Binary Tree?", sci.logic (10 Jul 2018)]

My reply: This is wrong by definition. Two paths can be distinguished if and only if they cross different nodes, i.e., if they are contained in different path bunches.

Discussion: "You are apparently trying to appeal to a theorem that says that if $a_n < A$ for all n then $\lim a_n \leq A$. This statement is false in the context;" [Jürgen Rennenkampff in "How many different paths can exist in the complete infinite Binary Tree?", sci.logic (15 Jul 2018)]

My reply: This statement is *never* false – wherever mathematics applies.

Distinguishing three kinds of Binary Trees

• *The complete infinite Binary Tree of actual infinity* is said to contain all real numbers of the interval [0, 1) as infinite paths, but this is an error (cp. section "Sequences and limits"). It contains all paths that always "do something", for instance always go right or alternate, but real numbers like 1/3 or 1/π are only approximated. A simple proof consists in the fact that the Binary Tree contains only levels with finite indices, where the paths provably differ from their limits.

Unfortunately the complete infinite Binary Tree cannot be used in full completeness in *mathematics*, because every level-index *n* belongs to the first percent of all natural numbers. (Proof: every *n* multiplied by 100 is a natural number which follows the same rule; cp. section "You can use only natural numbers of the first percent of $\aleph_0$".)

• *The potentially infinite Binary Tree* contains all levels that can be indexed by 1, 2, 3, and so on up to every level *n*. Every index belongs to a finite initial segment of ℕ. Beyond every index infinitely many further indices are following. (As far as the *numerical accessibility* is concerned, the complete infinite Binary Tree has the same extension as the potentially infinite Binary Tree.)

• *The Binary Tree of MatheRealism* contains not more levels than can be indexed by natural numbers which can be adressed by the computing system. As soon as the Kolmogorov-complexity of an index exceeds the computing capacities (memory space) of the used system, there is no chance to address that index. So there is a first index which cannot be addressed. Subsequent indices, which may have smaller Kolmogorov-complexity, are useless.

One could think that "the first index which cannot be addressed" is addressed by this very sentence, but that is an error, because the bits needed for this sentence are required to address the preceding indices (and the sentence would not define which number *n* it refers to).

Every system including the accessible universe has finite computing capacity.

Limits and the Binary Tree

The Binary Tree contains paths like 0.010101... converging to 1/3 and paths converging to irrational numbers but not these limits themselves. However, there is an exception with limits that are fractions with denominator divisible by 2 like 0.1000... or 0.000... .

Such numbers possess several representations. Consider 0 for instance. Besides being represented by the path 0.000... it is the limit of the sequence of paths 0.1t, 0.01t, 0.001t, ... → 0,000... where t is an arbitrary tail, for instance t = 000..., or t = 111..., or t = 010101..., etc., or a mixture of these.

When we assume that every path can be distinguished from all other paths, then the path 0.000... differs from all paths of a sequence like

0.111..., 0.0111..., 0.00111..., ... (S)

That means, when each path of this sequence is completely coloured, then the path 0.000... is not yet completely coloured. In other words, it is not possible to colour (or to cover) the Binary Tree by *different* sets of infinite paths. Each and every path is required.

On the one hand, this is clear, because every path of the sequence (S) has a tail of nodes consisting of bits 1 only, while 0.000... does not. Let's call this position A.

On the other hand, we cannot find a node 0 of the path 0.000... which is *not* covered by the sequence (S). That means, we *cannot* distinguish the path 0.000... from *all* other paths of the Binary Tree. We can colour or cover the whole Binary Tree by a set U of paths not containing the path 0.000... or by a set V containing it. Let's call this position B.

If (A) is correct, then there must be nodes in 0.000... that cannot be found and defined. That implies that actual infinity, the complete infinite Binary Tree, and its infinite paths do not exist. Because nodes that cannot be defined cannot yet exist. They only can "come into being".

If (B) is correct, then 0.000... cannot be distinguished from all paths of the sequence. That implies that in a Cantor-list like the following

0.1
0.01
0.001
0.0001
...

when replacing the diagonal digit 1 by 0, the resulting antidiagonal 0.000... cannot be distinguished from all entries. Therefore Cantor's diagonal argument fails in this special case and hence always, because it is based on a proof by contradiction which never must fail.

Result: There is no uncountable set of paths in the Binary Tree: In case (A) there is no actual infinity and therefore it cannot be surpassed. In case (B) there is actual infinity but it does not supply uncountable sets.

[W. Mückenheim: "Limits and the infinite complete Binary Tree and the contradiction of uncountability", sci.math (20 Jan 2017)]

The same result is obtained from the proof that every path has to be connected to the root node: There are only $\aleph_0$ supply points with two connections each, hence $2\aleph_0$ in total.

Similar techniques in Cantor's list and the Binary Tree

Cantor divides the set of all infinite digit sequences of the interval [0, 1] into a countable set A and the remainder R. The set A is written in form of a sequence, in plain language called a list. An antidiagonal digit sequence d is constructed, such that every digit d_n of d differs from the corresponding digit of the nth digit sequence of A. So Cantor shows for every $n \in \mathbb{N}$ that the finite initial segment $0.d_1d_2...d_n$ of d is not identical with the finite initial segment of one of the first n digit sequences of A. This is tantamount to showing that every finite initial segment of d is equal to one of the digit sequences of R (or one of the following digit sequences of A).

From the proof for *all* n it is concluded, that all digit sequences following in A can be excluded too, and the whole antidiagonal digit sequence d is in R.

If the Binary Tree contains the set of all bit sequences of the interval [0, 1] as its paths, then I divide them in a countable set A and the remainder R.

The set of paths A is utilized to cover all nodes. With every node unavoidably every finite initial segment of every path is covered too. That means, it is proved that every finite initial segment of every path is identical with the finite initial segment of a path of A.

From the proof for all n it is concluded, that every whole path is in A and therefore the set R is empty.

Should further paths be assumed to exist, they cannot be distinguished by nodes from the paths of A. So the uncountability of the bit sequences represented by those paths could not be proved by changing bits with finite indices.

Example: All finite initial segments of the path p = 0.111... are contained in paths of the set

0.1000...
0.11000...
0.111000...
... .

There are never two paths of the covering set A existing such that their union contains a larger finite initial segment of p than at least one of them. This means that only one path of A covers all finite initial segments of p.

Hilbert's hotel as a parable of Cantor's list

Hilbert's infinite hotel is completely filled with guests. But when another guest arrives he is accommodated too. Every resident guest is asked to move from room number n to n + 1. Even infinitely many guests will get rooms when every resident guest doubles his room number.

Between Cantor's list (see section "On the diagonal argument" below) and Hilbert's hotel there is only the arbitrary difference that Hilberts hotel is really infinite, unfinished, extendable whereas Cantor's list is not. Two different interpretations of one and the same infinity.

Only that allows to conclude that the antidiagonal, as a new guest differing from all resident guests or entries of the list, cannot be inserted, for instance into the first position when every other entry moves on by one room number. Without Cantor's arbitrary constraint even all infinitely many antidiagonals that ever could be constructed could be accommodated. Cantor's theorem would go up in smoke.

On the diagonal argument

The diagonal argument is one of the most famous proof-techniques. Here we will show how and why it fails on several occasions. The most fundamental reason is the lack of actual infinity. Further we have seen already that irrational numbers have no decimal representation, so that the diagonal argument does not concern irrational numbers at all and therefore does not prove anything about irrational numbers.

Infinite digit sequences without the attached powers of 10 (or another usable base) do not converge. With only few exceptions they accidentally jump to and fro and do not assume a limit because the trembling does never calm down. But just these sequences are produced by the diagonal argument. Cantor, in his original version [G. Cantor: "Über eine elementare Frage der Mannigfaltigkeitslehre", Jahresbericht der DMV I (1890-91) pp. 75-78], did not define limits at all; see section 2.2.3. Therefore the invention of the rule to avoid antidiagonal numbers with periods of nines does merely show a big misunderstanding of facts (cp. section "The nine-problem"). Of course the string 1,0,0,0,... differs from the string 0,9,9,9,...

Finally: The antidiagonal does not *exist* but is only *constructed*. If it had existed during the creation of the list, it would have been included, wouldn't it?

Controversy over Cantor's theory

The diagonal argument is applied to sequences of digits and produces a sequence of digits. But digits abbreviate fractions. Fractions are never irrational. The limit of a rational sequence can be irrational. But, as already mentioned, the diagonal argument does not concern limits, only fractions or digits, each of which belongs to a finite initial segment and is followed by infinitely many digits.

From an infinite sequence or listing of digits a limit can never be derived, because always infinitely many digits remain unknown. Only from a generating formula the limit can be obtained. But such formulas are not subject to the diagonal argument.

[W. Mückenheim: "Objection to the representations of irrational numbers by digits", Wikipedia (1 Oct 2016)]

A simple list

In the following list, every entry contains more digits 1 than its predecessors.

0.0
0.1
0.11
0.111
... .

The limit 0.111... is not in the list. Does it contain more digits than all entries of the list? If yes, where are they? If no, then the antidiagonal number (constructed when 0 is replaced by 1) is yet contained in the list.

After removing all entries with less digits than 0.111... nothing remains.

Of course d = 0.111... is not completed in any enumerated line but only in the infinite – alas there it is already welcomed by itself.

The projection of d on the horizontal axis is never complete (that would require a completed line). But its projection on the vertical axis is assumed to be complete. And from that part it is concluded in reverse that d is complete. Only by this incoherent arguing it is possible for d to differ from every line.

Not necessary to mention that in analysis this limit is not created by digits. We have to use finite definitions for what we never get by digits. The above list does never reach, create, or complete a string of digits without a tail of infinitely many zeros. – And in analysis "never" means *never* and not *in the infinite*!

What digit of 0.111... is not contained in its approximations?

Every number of the following sequence contains more digits 1 than all its predecessors:

0.000...
0.1000...
0.11000...
0.111000...
…

Does the limiting number 0.111…, not contained in the sequence, contain more digits 1 than all numbers of the sequence? If yes, which digits are that? If not, then 0.111… (which is also the antidiagonal number of the sequence) cannot be distinguished by means of digits from all numbers of the list.

It can be distinguished from *every* finite initial segment, but that does not mean anything with respect to the infinite sequence.

All digits 1 of 0.111... are in no single line but are in the complete list? Remove every line with less digits 1. Nothing remains.

Similar problem: Colour all paths of the Binary Tree of the form 0.111..., 0.0111..., 0.00111..., ..., then the path 0.000... has not been coloured but has no uncoloured node.

W.Mückenheim: "Das Kalenderblatt 091206", de.sci.mathematik (5 Dec 2009)]

The squareness of the list

The Cantor-list is a square, by the bijection between rows and columns in the diagonal. If it contains $\aleph_0$ rows then each entry contains also $\aleph_0$ digits. Then, by replacing 0 by 1, the antidiagonal number 0.111... of the list

0.000...
0.1000...
0.11000...
0.111000...
...

either is already in the list because all columns are filled with the digit 1 or there are less rows than are required to house the antidiagonal number.

But every row of the list contains *too many zeros*, infinitely many, i.e., more than ones (because finite is always less than infinite). That makes a representation 0.1, 0.11, 0.111, ... of all approximations to 1/9 impossible. Not even half of them can be represented.

In every Cantor-list there are only finitely many pre-diagonal digits and infinitely many post-diagonal digits. Therefore the "workable" list is always less than half of the complete list. If the matrix would be a square then not all diagonal digits would have infinitely many successors.

By symmetry same holds for the columns. The "workable" list must be a very small patch in the upper left of the infinite list. (See also section "Dark numbers in set theory".)

Cantor-list of all rational numbers (I)

Consider a Cantor-list that contains a complete sequence (q_k) of all rational numbers q_k of [0, 1). The first n digits of the antidiagonal number d are $0.d_1d_2d_3...d_n$. It can be shown *for every n* that the Cantor-list beyond line n contains infinitely many rational numbers q_k that have the same sequence of first n digits as the antidiagonal d.

Proof: There are infinitely many rationals q_k with this property. All are in the list by definition. At most n of them are in the first n lines of the list. Infinitely many must exist in the remaining part of the list. So we have obtained:

$$\forall n\ \exists k\colon 0.d_1d_2d_3...d_n = 0.q_{k1}q_{k2}q_{k3}...q_{kn}\,. \qquad (*)$$

This theorem is not less important than Cantor's theorem:

$$\forall k\colon d \neq q_k\,.$$

Both theorems contradict each other with the result that finished infinity as presumed for transfinite set theory is not a valid mathematical notion.

In order to remove the contradiction write Cantor's theorem in the correct form of potential infinity

$$\forall n\ \forall k \le n: 0.d_1d_2d_3...d_n \neq 0.q_{k1}q_{k2}q_{k3}...q_{kn}$$

as the complement of (*)

$$\forall n\ \exists k > n: 0.d_1d_2d_3...d_n = 0.q_{k1}q_{k2}q_{k3}...q_{kn} .$$

Cantor-list of all rational numbers (II)

Enumerate all rational numbers to construct a Cantor-list. Replace the diagonal digits a_{nn} by d_n in the usual way to obtain the antidiagonal number d. Beyond the nth row there are $f(n)$ rational numbers the first n digits of which are same as the first n digits $0.d_1d_2d_3...d_n$ of the antidiagonal number. And we can prove

$$\forall n \in \mathbb{N}\ \forall k \in \mathbb{N}: f(n) > k .$$

Define for every $n \in \mathbb{N}$ the function $g(n) = 1/f(n) = 0$. In analysis the limit of this function is $\lim_{n\to\infty} g(n) = 0$. So set theory with its limit $\lim_{n\to\infty} f(n) = 0$ is incompatible with analysis. Since analysis is a branch of mathematics, set theory is incompatible with mathematics.

Cantor-list of all rational numbers (III)

Cantor's argument constructs from a list (a_n) of real numbers another real number, the antidiagonal number d, that is not contained in the list. The argument is based on the completion of the antidiagonal number. But this assumption is wrong. The list contains only all finite initial segments

$$0.d_1, 0.d_1d_2, 0.d_1d_2d_3, ... \qquad (*)$$

of d. d itself is not constructed (and cannot be constructed).

In a list of all rational numbers, the antidiagonal number should be an irrational number. But it is not (cp. section "Sequences and limits"). It is only the infinite sequence (*) of all rational approximations. What differs from a listed number is always merely a rational approximation. All these, however, are already listed, by definition, in a list of all rational numbers.

Cantor-list of all rational numbers (IV)

The digits d_n of the antidiagonal number d are in bijection with $\mathbb{N}$.

Not all entries of a rationals-complete list differ at a finite index n from the antidiagonal number d. On the contrary, for *every* finite index n there are infinitely many duplicates of $0.d_1d_2d_3...d_n$. So if d differs by its digits from all rationals of the rationals-complete list, then countably many digits are not enough. The digits of d must be uncountable. There must be at least one more digit than those which are in bijection with $\mathbb{N}$ (similar to the antidiagonal, proving uncountability of the entries of a Cantor-list). This is obviously in contradiction with the notion of sequence.

Only terminating entries are applied in the diagonal argument.

Consider a Cantor-list with infinite digit sequences as entries a_n and antidiagonal number d:

$\forall n$ (for every n) $\in \mathbb{N}$: $(a_{n1}a_{n2}...a_{nn}) \neq (d_1d_2...d_n)$.
$\forall n$ (for every n) $\in \mathbb{N}$: $(a_{n1}a_{n2}...a_{nn})$ is terminating.
$\forall n$ (for every n) $\in \mathbb{N}$: $(d_1d_2...d_n)$ is terminating.

$\forall n$ (for all n) $\in \mathbb{N}$: $(a_{n1}a_{n2}...a_{nn}) \neq (d_1d_2...d_n)$.
$\forall n$ (for all n) $\in \mathbb{N}$: $(a_{n1}a_{n2}...a_{nn})$ is terminating.
$\forall n$ (for all n) $\in \mathbb{N}$: $(d_1d_2...d_n)$ is *not* terminating.

From every entry of the list only a finite initial segment is exploited. Therefore it is irrelevant whether or not infinite digit sequences are applied. The relevant entries are rational numbers. Hence the diagonal argument shows at best that there are irrational numbers! What a magnificent discovery!

[Hans: "Can the diagonal argument be extended to irrational numbers?", MathOverflow (4 Oct 2015)]

The diagonal argument depends on representation.

Consider a civilization that has not developed decimal or comparable representations of numbers. Irrational numbers are obtained from geometric problems or algebraic equations. They are defined by the problems where they appear and abbreviated by finite names – just as in human mathematics. If all rational numbers in an infinite list are represented only by their fractions and all irrational numbers by their finite names, it is impossible to apply Cantor's diagonalization with a resulting "antidiagonal number". Such a culture would not fall into the trap of uncountability. It is leading astray human mathematics too, because the infinite decimal representation does never allow us to identify an irrational number. Note the name decimal-*fractions*.

The nine-problem

A Cauchy-sequence has infinitely many ($\aleph_0$) *rational* terms. Since all terms of all Cauchy-sequences are rational, they belong to the countable set of rational numbers. The limit, if a non-terminating rational or an irrational number, differs from the terms of the sequence. Ignorance of these differences has lead to the "9-problem" in Cantor-lists: Provision has been made that the antidiagonal number cannot have the form 0.999... . However, this provision is not necessary. Cantor's diagonal-argument requires more precision than unwritten limits. Every digit appearing in a Cantor-list belongs to a Cauchy-sequence – not to its limit! The Cauchy-sequences abbreviated by 1.000... and 0.999... are quite different. The provision shows, however, that set theorists have been confusing sequences and their limits for about one hundred years. (See section 2.2.3. – Cantor himself did not make this provision.)

What about writing $\lim_{n\to\infty}$ in front of every line of a Cantor-list? Or what about writing every line of a Cantor-list twice, the second one always equipped with a $\lim_{n\to\infty}$? Subject and result of diagonalization are always digits, i.e., rational terms of Cauchy-sequences – whether or not these sequences stand for themselves or are used as names of irrational numbers. In a rationals-complete list, this always raises a contradiction.

Limits in Cantor-lists

Taking the limit could spoil the diagonal argument because the argument distinguishes only the finite terms of sequences the limits of which can agree. Here is a simple example. Consider the list of power sequences of reciprocals of prime numbers:

$$\begin{array}{llll}
1/2, & 1/4, & 1/8, & \dots \\
1/3, & 1/9, & 1/27, & \dots \\
1/5, & 1/25, & 1/125, & \dots \\
1/7, & 1/49, & 1/343, & \dots \\
1/11, & 1/121, & 1/1331, & \dots \\
\dots & & & \\
1/p_n, & 1/(p_n)^2, & 1/(p_n)^3, & \dots \\
\dots & & & \\
\to 0, & 0, & 0, & \dots .
\end{array}$$

It can be diagonalized by replacing $1/(p_n)^n$ by 0, resulting in the antidiagonal sequence 0, 0, 0, ... This is identical with the limit of the list, i.e., of the sequence of entries. In Cantor's original version, there is no problem. The limit is not enumerated and therefore does not appear in the argument. But if "automatically" the limit is assumed, then this has also to be done with the limit of the list. Otherwise the antidiagonal would have one more term than the list has lines – and the diagonal argument is superfluous, because the antidiagonal number cannot be in the list, caused already by this difference.

Note that even $\forall n \in \mathbb{N}$: $|d_n - a_{nn}| > 0$ does not exclude the limit $\lim_{n\to\infty} |d_n - a_{nn}| = 0$.

Different results of universal quantification?

(1) $\forall n \in \mathbb{N}$: $\sum_{k=1}^{n} \frac{9}{10^k} < 1$.

The sum over *all* these very terms however is said to be

$$\sum_{n\in\mathbb{N}} \frac{9}{10^n} = 1 .$$

Here it does not matter in the sum of *all* finite terms that *all* finite terms fail.

(2) $\forall n \in \mathbb{N}$: The digit sequence $0.d_1d_2d_3...d_n$ of the antidiagonal in Cantor's list differs from the first n entries a_k of the list

$$\forall n \in \mathbb{N}\ \forall k \le n: 0.d_1d_2d_3...d_n \neq 0.a_{k1}a_{k2}a_{k3}...a_{kn} .$$

And in the limit the antidiagonal is said to differ from all entries of the Cantor-list too.

Here we can conclude, from "each d_n fails" to the "failure of all". Unlike in case of the digits 9, it does matter in the limit that all finite terms fail. (Since every diagonal digit d_n belongs to a finite initial sequence of digits, like every 9 in (1), the conclusion to the infinite case requires the same logic as the conclusion in the first case.)

(3) $\forall n \in \mathbb{N}$: The nth level in the Binary Tree has

$$N(n) = 2^n$$

nodes. The limit is said to be actually infinite: $\aleph_0$.

Here again, like in (1) it does not matter in the limit that all finite terms fail to be infinite.

(4) $\forall n \in \mathbb{N}$: The number of paths in the Binary Tree that can be distinguished at level n is finite, namely

$$P(n) = 2^n .$$

In the limit however the number of paths that can be distinguished is said to be actually infinite.

Also here it does not matter that all finite levels fail to distinguish infinitely many paths. Moreover, it does not only not matter that $\forall n \in \mathbb{N}$: $P(n)$ is finite and equal to $N(n)$ but the actual infinity of $P(n)$ is much larger than that of $N(n)$, namely uncountable, $2^{\aleph_0}$.

Different results of universal quantification? (Answer)

There are no different results of quantification. To see this we have to be clear about real numbers first. Either a real number can be represented by an actually infinite sequence of digits (i.e., of partial sums of fractions) or this is not the case. If so, then all real numbers of the unit interval [0, 1) can also be represented as actually infinite and *continuous* paths in the Binary Tree. If not, then no real number can be represented in a Cantor-list either.

Now we consider the four questions.

(1) $\forall n \in \mathbb{N}$: $\sum_{k=1}^{n} \frac{9}{10^k} < 1$ implies $\sum_{n \in \mathbb{N}} \frac{9}{10^n} < 1$.

How else could logic work? If we gather a set of green balls, we will not get a red cube. If we take a set of failures, we will not get a success. We can even define: Sum all fractions $9/10^n$ that fail to yield 1. Like in every infinite sequence that means we have to take $\aleph_0$ fractions, namely one for every natural index $n \in \mathbb{N}$ that fails. Correct is

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{9}{10^k} = \sum_{n=1}^{\infty} \frac{9}{10^n} = 1 \ .$$

(2) With this interpretation the diagonal argument holds. The infinite sequence of failures, i.e., of diagonal digits d_n represents the real number $d = 0.d_1d_2d_3...$ that is not in the Cantor-list.

(3) With respect to the number of nodes $N(n)$ as function of the level number n in the Binary Tree we apply the same logic again. Like in (1) the terms of an increasing sequence do not include the limit. Since every $N(n) < \aleph_0$ there is no level with $\aleph_0$ nodes. Only the limit of $N(n)$ has $\aleph_0$ nodes. And since there are infinitely many levels with finite number of nodes, the complete, actually infinite Binary Tree has $\aleph_0$ nodes too.

(4) It is easy to see that at level n we can distinguish precisely $P(n) = 2^n$ paths. (It is irrelevant how many may be there "in principle". We consider in mathematics what we can distinguish.) This implies that we cannot distinguish more than countably many paths in the whole Binary Tree. An upper estimate is obtained by putting all nodes of all levels of the Binary Tree on one common level, say at ω. More is not possible to distinguish. But remember that the interpretation of infinite representations by digit sequences or paths has been introduced to circumvent the lack of uncountably many finite definitions. This approach has been proved to fail.

What is the answer to this seeming contradiction of uncountably many real numbers according to the diagonal argument and only countably many according to the Binary Tree? *There is no actual infinity at all*. There is no complete Cantor-list, and hence there cannot be a complete diagonal number. And of course there is no complete Binary Tree either. It has only served to prove its nonexistence like Cantor's diagonal argument only has served to prove the nonenxistence of the completed infinity assumed by him.

Approximatable numbers contradict Cantor's diagonal argument

Definition: A real number r is approximatable if for every natural number n the nth digit r_n can be found by a finite method. This does not require a finite formula for all its digits but only a method to find digit r_n by a finite calculation or decision.

Approximatable numbers are rational numbers, irrational numbers defined by a finite formula, and arbitrary digit sequences which are either appended by a finitely defined string or ending in the finite (since no infinite string can be obtained by arbitrary decision, i.e, without a finite formula). The ending digit sequences are rational numbers. The set of all approximatable numbers is clearly countable because every approximatable number is finitely defined.

If there is a list of n approximatable numbers, then their nth digits can be calculated and linked together. Therefore the diagonal d of every finite part of a list of approximatable numbers is an approximatable number too.

Theorem. There is no set of all approximatable numbers.

Proof by contradiction: Assume that the set of all approximatable numbers exists. List the set and consider the diagonal number d. For every n its digit d_n results from a finite computation, namely that supplying this digit $d_n = r_{nn}$. Note that a finite series of finite computations is a finite computation too. Construct the antidiagonal number according to a fixed algorithm. It is an approximatable number too but not in the list at any finite place. Contradiction.

Choosing and well-ordering undefinable elements?

The axiom of choice

The axiom of choice (AC) implies the proof that every set can be well-ordered, implying that it is always possible *to choose* an element from every non-empty set and to union the chosen elements into a set. "Choosing something" means pointing to or showing this something, or, if this something has no material existence, defining or labelling it by an index, i.e., by a finite word or name. For uncountable sets this is known to be impossible; a countable alphabet is not sufficient to label uncountably many elements (cp. section "The list of everything") there are only countably many indices.

Is there any remedy possible by using an uncountable alphabet? No, an alphabet is a simply ordered finite set. Otherwise you would never find most letters of the alphabet – cp. the telephone book. See also the sections "Comments on undefinable mathematics and unusable languages" and "On uncountable alphabets".

Therefore the proof implies that *two elements which cannot be identified, distinguished, and put in an order can be identified, distinguished, and put in an order.* This is a contradiction in ZFC, since in every model including the whole universe of set theory it is necessary to identify and to distinguish elements that shall be chosen. Since this is impossible for uncountably many elements, there is no well-ordering in any uncountable model. And the proof for a model without well-ordering (carried out by forcing, cp. section 3.2.3) is superfluous.

Some axioms like the axiom of choice

Consider a geometry that contains the axiom: "For every triple of points there exists a straight line containing them." When you ask for the straight line that contains the points (0|1), (0|2), (1|0) the masters of that theory reply that some straight lines cannot be constructed but that they certainly "exist". If you ask what in this case existence would mean, you are called a crank.

Here are some Axioms of similar scientific value.

- Axiom of two even primes: There are two even prime numbers. (But provably it is impossible to determine whether the second one is less or larger than 2.)
- Axiom of prime number triples: There is a second triple of prime numbers, besides (3, 5, 7). (But provably this second triple is not arithmetically definable.)
- Axiom of meagre sum: There is a set of n different positive natural numbers with sum $n^2/2$. (This axiom is not constructive. Provably no such set can be constructed.)
- Axiom of ultimate mathematical simplification: All mathematical problems are solved by whatever is declared as their solutions. (This axiom is guaranteed not less useful than the axiom of choice in considerably shortening proofs about uncountable sets.)
- Axiom of well-ordering: Every set can be well-ordered. (This axiom is not constructive. In most cases provably no set theoretic definition of a well-order can be found.)

How obvious a contradiction has to result from an additional axiom in order to reject it?

Zermelo's defence of the axiom of choice

The following defence of the unprovable but fundamental principle of choice is given by Zermelo: "Now, *unprovability* in mathematics is not at all tantamount to *invalidity*, because it is clear that not everything can be proved since every proof presupposes unproven principles. In order to discard such a fundamental principle one had to establish contradictory consequences in special cases; but none of my opponents has made an attempt." [E. Zermelo: "Neuer Beweis für die Möglichkeit einer Wohlordnung", Math. Ann. 65 (1908) p. 112]

With respect to uncountable sets this requirement has certainly been met by listing all countably many objects that can be identified and, therefore, can be ordered (cp. sections "The list of everything" and "The axiom of choice"). For countable sets choice is a true and basic principle.

An invalid step in Zermelo's proof of well-ordering

In his proof (cp. section 2.13) Zermelo considers any two different γ-sets M'_γ and M''_γ and concludes that always one of them is identical with a segment of the other.

"Because the first element of *every* γ-set is m_1 since the corresponding segment A does not contain an element, i.e., $M - A = M$. If now m' were the *first* element of M'_γ which differed from the corresponding element m'' then the corresponding segments A' and A'' must be equal and hence also the complementary sets $M - A'$ and $M - A''$ and as their distinguished elements m' and m'' themselves, contrary to the assumption." [E. Zermelo: "Beweis, daß jede Menge wohlgeordnet werden kann", Math. Ann. 59 (1904) p. 515]

Here Zermelo uses a step-by-step argument which is not allowed, because it would presuppose the countability and even finity of the real numbers:

Theorem All initial segments C of the well-ordered set $\mathbb{R}$ are countable.

Proof: Let $x \in \mathbb{R}\backslash C$ be the first real number of $\mathbb{R}\backslash C$, i.e., the first real number that follows on C in the well-ordering of $\mathbb{R}$, such that $C \cup \{x\}$ is the first uncountable initial segment of $\mathbb{R}$. This is a contradiction, because if C is countable, then $C \cup \{x\}$ is countable too. ■

Theorem All initial segments F of the well-ordered set $\mathbb{R}$ are finite.

Proof: Let $x \in \mathbb{R}\backslash F$ be the first real number of $\mathbb{R}\backslash F$, i.e., the first real number that follows on F in the well-ordering of $\mathbb{R}$, such that $F \cup \{x\}$ is the first infinite initial segment of $\mathbb{R}$. This is a contradiction, because if F is finite, then $F \cup \{x\}$ is finite too. ■

An infinite union of finite sets need no longer be finite. An infinite union of countable sets need no longer be countable. But an infinite union of segments of γ-sets must remain a γ-set, i.e., well-ordered? Why should it? Zermelo's step-by-step argument here collapses to unjustified belief.

[W. Mückenheim: "Das Kalenderblatt 091123", de.sci.mathematik (22 Nov 2009)]

What is the advantage of the expenses to accept AC?

Alice accepts the axiom of choice (AC). So there exists a well-order of ℝ in some platonic shelf. Alice cannot find it, but it is relieving to know that it exists because a good logician then can apply it nevertheless.

Carl denies AC. But he knows of Alice and therefore he knows of the existence of the well-order of ℝ. He cannot change it, in particular because he cannot find it. But he lives in the same world as Alice. So without the disadvantageous confession to believe in the nonsensical idea that undefinable and undistinguishable elements can be well-ordered, i.e., defined, distinguished, and chosen, he can nevertheless apply it too – since it is there, guaranteed by Alice.

[W.Mückenheim: "What is the advantage of the expenses to accept AC?", sci.math (3 Feb 2017)]

Well-ordering and Zorn's lemma

The axiom of choice implies that every set can be well-ordered (cp. section 2.13). But ordering of elements requires that these elements can be distinguished by labels. Cantor stressed that arguments against actual infinity become invalid "as soon as a principle of individuation, intention, and ordination of actually infinite numbers and sets has been found." [G. Cantor, letter to A. Schmid (26 Mar 1887)]. Since there are only countably many labels for individuation, the axiom of choice is a counterfactual axiom with respect to uncountable sets.

The axiom of choice is equivalent to Zorn's lemma (cp. section 2.12.9): If a partially ordered set *S* has the property that every chain has an upper bound in *S*, then the set *S* contains at least one maximal element. Without this feature, every element would have another next element. This successorship would not stop at natural indices but would run through all ordinal numbers. This argument is obviously as invalid here as in the original "proof" that every set can be well-ordered (cp. section 2.13). We will never reach any non-natural number when running through the natural numbers.

Accessing inaccessible numbers

An accessible number, according to Borel, is a number which can be described as a mathematical object. The problem is that we can only use some finite process to describe a real number, so only such numbers are accessible. We can describe rationals easily enough, for example either as, say, one seventh or by specifying the repeating decimal expansion 142857. Hence rationals are accessible. We can specify Liouville's transcendental number easily enough as having digit 1 in place n! and 0 elsewhere. Provided that we have some finite way of specifying the nth term in a Cauchy-sequence of rationals we have a finite description of the real number resulting as its limit. However, as Borel pointed out, there are a countable number of such descriptions. Hence, as Chaitin writes: "Pick a real at random, and the probability is zero that it's accessible – the probability is zero that it will ever be accessible to us as an individual mathematical object." [J.J. O'Connor, E.F. Robertson: "The real numbers: Attempts to understand", St. Andrews (2005)]

But how to pick this dark matter of numbers? Only accessible numbers can get picked. Unpickable numbers cannot appear anywhere, neither in mathematics nor in Cantor's lists. Therefore Cantor "proves" that the pickable numbers, for instance numbers that can appear as an antidiagonal number of a defined Cantor-list, i.e., the countable numbers, are uncountable.

Also "Chaitin's constant" (cp. 2.3.2 "An uncomputable real number") is not more nor less uncomputable than the smallest positive fraction. "If there is a Turing Halting problem that is undecidable within our own system of reasoning? Is it then still a proper defined number?" [Lucas B. Kruijswijk in "Shannon defeats Cantor = single infinity type", sci.math (13 Dec 2003)] "We could construct a similar number. $x_1 = 1$ iff God is real, zero otherwise (non computable!), $x_2 = 1$ iff there are 2 gods of equal power, $x_3 = 1$ iff there are 3 gods of equal power, ... [...] Now we can construct the non computable irrational!" [Herc, loc cit]

Comments on undefinable mathematics and unusable languages

A question similar to that dicussed in section "The axiom of choice (III)" was answered by Noah Schweber. [User: "Why is ZFC called free of contradictions?", Math.StackExchange (12 Apr 2017)] I will cite some parts (printed in blue) and give some comments.

NS: First of all, we can distinguish objects from each other, even if they're undefinable individually! Think about the real numbers. These are uncountable, so lots of real numbers are undefinable; however, any two real numbers can be distinguished from each other by saying which one is bigger.

WM: Every real number that you can compare with anything is defined (by your choice) and is thus definable. Undefinable real numbers cannot be put into any processing unit. You cannot compare undefinable real numbers because all decimal representations as far as you can use them are finite strings. You cannot even think of undefinable reals as individuals.

NS: The more fundamental problem, however, essentially boils down to the difference between definability without parameters and definability with parameters. This isn't an outright definition [...] in the usual sense, but it is a definition with parameters. [...] A well-ordering of an uncountable set (say, ℝ) is itself not necessarily definable in any good sense [...] most elements of an uncountable collection are undefinable – so most ordinals are undefinable! So there's no reason to believe that we can get a genuine definition [...] from this definition-with-parameters.

WM: No genuine definition, not in any good sense, no outright definition. That is true.

NS: How many definitions-with-parameters are there?

WM: Why is this admittedly insufficient topic elaborated further?

NS: Well, there's as many of these as there are parameters – that is, there are exactly as many definitions-with-parameters as there are objects in our universe, so we're not going to "run out".

And indeed every object a is definable by the formula-with-parameters $x = a$. This is of course a very odd notion of definability, but it's the one corresponding to the kind of definability you get from a well-ordering of a set.

WM: "Very odd" is an euphemism. It is not a definition.

NS: So the apparent contradiction is only coming from the conflation of the notion of definability without parameters and definability with parameters.

WM: "Apparent"?

NS: [...] there is no way to write the formula "x is definable" in the language of set theory.

WM: Therefore we use mathematics and find that all words of all languages are elements of one and the same countable set. What words are real definitions and what words are not is irrelevant since every subset of a countable set is a countable set.

NS: This is essentially due to Tarski's undefinability theorem.

WM: This theorem is nothing but an attempt to veil the contradiction.

NS: And in fact – and very surprisingly – even though intuitively there must be lots of undefinable elements in any model of ZFC, this turns out to not necessarily be the case!

WM: Of course you cannot take an undefinable element and use it as parameter. First you would have to define it.

NS: That's simply false. You really should read a text on basic model theory before you claim to have found a contradiction in ZFC.

WM: Every language is countable. Otherwise you could not use it.

NS: You're conflating a language with a usable language – or, more precisely, you're making an ontological assumption that every mathematical object is "knowable" in some sense.

WM: Using usable language is the basic principle of mathematics and even of set theory. Every axiom and theorem of ZFC is expressed in usable language. Try to express some mathematics in a not usable language. Try to send something via internet in a not usable language. Fail.

NS: There's no justification for this, though. If you take this as your starting point, then of course there are only countably many mathematical objects; but that's not a background assumption of mathematical practice. And in particular, it can't be turned into an actual contradiction from the ZFC axioms – all you can do is show that the ZFC axioms contradict your view of what the mathematical universe is.

WM: In mathematics we prove assertions. Therefore, take an undefinable element as a parameter. Show how you do it. Fail

NS: If you truly think you can produce a contradiction from the ZFC axioms, I encourage you to try to produce a computer-verified proof of 0 = 1 directly from the ZFC axioms.

WM: A proof will never convince hard-boilded set theorists believing in undefinable entities of anything. Consider geometry. Of course there can be not finitely definable straight lines in geometry so that always three points lie on one of them. No problem – since they are undefinable. So you can believe in their existence since no computer can falsify this existence if you accept it.

NS: [...] the OP advocates a philosophical position which might be called physicalism (I've not heard a specific name for this, although it's not too rare):

WM: Hopefully this position is upheld by all true mathematicians.

NS: namely, that there is a connection between mathematical existence and physical reality in that mathematical objects can only be said to exist if they are "representable" in the physical universe.

WM: It is not "physicalism" but simply reasonable to require that every object of mathematics is definable. Everything else is theology. Computer-verified proofs do never establish undefinable results. Computers don't believe in theology. You can see your error when you try to take two undefinable real numbers. How would you "take" them and put them into a computer?

NS: The existence of uncountable sets is incompatible with the physicalist philosophy [...] But this is completely uncontroversial.

WM: No it isn't at all. Even in the first part of this answer it was incidentally assumed that every real number can be *taken*, i.e., can be defined.

NS: So the OP has only argued for "ZFC is false" insofar as they have argued for "physicalism is true" – and they haven't done that at all.

WM: Remember what you have said about definitions with parameters. They are no definitions. Now it seems that these non-things are the cornerstones of mathematics.

Noah Schweber's comment on the comments reproduced above: If your manuscript mentioned that the thing I am arguing for is the plausible *formal consistency* of ZFC rather than its truth (let alone *truth free of intuitive assumption* – I've certainly never argued that ZFC should be seen as true *regardless of your philosophy of mathematics*), then you would be representing my position more accurately; and if you disclosed the fact that your own intuition about what mathematics means is not the only perspective on the subject, you would be representing your own position more honestly.

On uncountable alphabets

"By an *alphabet A* we mean a nonempty set of symbols." [H.-D. Ebbinghaus, J. Flum, W. Thomas: "Mathematical logic", 2nd ed., Springer, New York (1994) p. 11]

That is true but insufficient to describe an alphabet that can be applied to form words and expressions. An alphabet is an alphabetically or lexically ordered string or list of symbols, where "lexically ordered" means that an algorithm exists to find a desired word within finite time.

"The set $\{c_r \mid r \in \mathbb{R}\}$, which contains a symbol c_r for every real number r, is an example for an uncountable alphabet." [loc cit p. 13]

That is impossible because the set of symbols is countable – at least if transmitted through the internet, printed, or carved of matter within the universe.

"Finally, as Godel observes, his argument is restricted to countable vocabularies; Henkin proves the results for uncountable languages. [...] Henkin (Corollary 2) uses the uncountable vocabulary to deduce the full force of the Löwenheim-Skolem-Tarski theorem: a consistent first order theory has models in every infinite cardinality. [...] Henkin already points out that his proof (unlike Godel's) generalizes easily to uncountable vocabularies. [...] McKinsey (also noting the uncountable application) and Heyting give straightforward accounts in Mathematical Reviews of the result of Henkin's papers on first order and theory of types respectively with no comments on the significance of the result. Still more striking, Ackermann's review of Henkin's proof gives a routine summary of the new argument and concludes with 'The reviewer can not follow the author when he speaks of an extension to an uncountable set of relation symbols, since such a system of notations can not exist'." [John Baldwin: "The explanatory power of a new proof: Henkin's completeness proof" (25 Feb 2017)]

"Ref. kann dem Verf. aber nicht folgen, wenn er von der Möglichkeit einer mehr als abzählbaren Menge von primitiven Symbolen spricht, da es ein derartiges Bezeichnungssystem doch nicht geben kann." [W. Ackerman: Review of "L. Henkin: 'The completeness of the first-order functional calculus'", Journal of Symbolic Logic 15 (1950) p. 68]

Ackermann's clear and correct position seems "striking". "Logic" has really become a perverted subject – useful and interesting like chess competitions of mentally disabled persons!

"Roughly, a formal language could be completely mastered by a suitable machine, without any understanding. (This needs qualification where the formal language has an uncountable alphabet: In such case it is not clear that the formal language could be completely mastered by anything.)" [Geoffrey Hunter: "Metalogic", 6th ed., Univ. California (1996) p. 4f]

"However, in building up formulas, conjunctions of sets of formulas of power less than κ and quantifications over sets of variables of power less than λ are allowed. If κ is bigger than ω, this, of course, allows formulas which are not 'writable' in the sense of being finite strings of symbols. Also, in the book, no notions of recursiveness, admissibility, or even definability are applied to the formulas or sets of formulas considered." [W.P. Hanf: "Review of M.A. Dickmann: 'Large infinitary languages, model theory'", Bull. Am. Math. Soc. 83,2 (1977) p. 184]

"A well-known theorem of Mills asserts that there is a model of Peano Arithmetic *M* in an uncountable language such that *M* has no elementary end extension (e.e.e.). I ask whether every complete extension of *PA* in an uncountable language can have models of every cardinality with arbitrary large e.e.e.'s." [sharam: "Elementary end extensions of models of Peano arithmetic in uncountable languages", MathOverflow (11 Sep 2013)]

"You simply name all the submanifolds of your manifold and work with an uncountable language." [user 1686 in "Algebraic description of compact smooth manifolds?", MathOverflow (19 Nov 2009)]

"For uncountable languages, the logic is not complete with respect to the standard algebra $[0,1]_G$ and completeness fails already for propositional logic." [Emil Jeřábek in "Compactness and completeness in Gödel logic", MathOverflow (12 Mar 2014)]

"Extending Shelah's main gap to non first-order or even to first-order theories in an uncountable language is a major hard open problem". [Rami Grossberg in "Main gap phenomenon", MathOverflow (19 Mar 2015)]

"Anyway, it's impossible to countably axiomatize a (consistent) theory in an uncountable language, unless all but countably many of the symbols are left completely arbitrary by the axioms." [Emil Jeřábek in "Is the following theory countably axiomatizable?", MathOverflow (13 Dec 2016)]

"T_Ω is not countably axiomatizable for trivial reasons: it has an uncountable language, and says non-trivial things about each symbol in that language." [Noah Schweber, loc cit]

"You may also find it helpful to think of a computer whose keyboard has, not the usual 101 keys, but infinitely many keys; [...] In this book we'll study the 'sentences' that can be typed on such a keyboard [...] Also for simplicity of notation, we have chosen an alphabet that is only countably infinite. That alphabet is adequate for most applications of logic, but some logicians prefer to allow uncountable alphabets as well. (Imagine an even larger infinite computer keyboard, with *real numbers* written on the key caps!)" [Eric Schechter: "Classical and nonclassical logics", Princeton Univ. Press (2005) p. 207f]

"How could the manufacturer write a real number except the few which have their own names like 2 or √3 or 1/4 or π? The real numbers written on the key caps have to be individuals, i.e., it is not sufficient to distinguish each one from some 'given' real numbers but each one must differ from all other real numbers. How can that be possible by finite strings of symbols on the key caps? I assume consent that infinite strings of symbols don't carry any information that could be conveyed to the typist with sufficient completeness." [Wilhelm: "How to write an individual real number?", Math.Educators.StackExchange (27 Nov 2017)]

An infinite sequence of digits would not be sufficient since it never determines the number.

"It seems like it needs comment that this is only a hypothetical/theoretical thought experiment. No such keyboard is actually constructable. Not even the simpler countably infinite one." [Daniel R. Collins, loc cit]

"This is not a hypothetical question since a rather big group of logicians claims that uncountable alphabets can be used in formal languages. If so, we should know a means how to use them." [Wilhelm, loc cit]

"So, here are two assumptions that I think are built into your question. The manufacturer is writing on the keys with only a finite, or perhaps countable alphabet of symbols. The manufacturer can only fit a finite number (maybe unboundedly large) of symbols on each key. If assumption one fails, then you could just give each real number a symbol, and be done with it. If assumption two fails, you could name each real number by its perhaps infinitely long decimal expansion. If both assumptions hold, then the answer is 'They can't'." [Graham Leach-Krouse, loc cit]

"That's my opinion too. But before I accept your answer I would wait some time such that logicians claiming the contrary can support their position." [Wilhelm, loc cit] Of course such an explanation never came up. – The only "counter argument" was a lot of downvoting.

"Formal languages don't have 'meaning'; they only have syntax and grammar. Assigning meaning to the language is usually done by an interpretation." [Hurkyl, loc cit]

"It is kind of perversion to call meaningless strings a language. However the language is applied to achieve something not meaningless. Therefore an interpretation is required. There are however only countably many interpretations. Therefore almost all symbols of an uncountable set cannot be used purposefully. So what should they be good for?" [Wilhelm, loc cit]

"And yet, that is what people do." [Hurkyl, loc cit]

"When people pursue an insane occupation that does not make it sensible." [Wilhelm, loc cit]

"You want to use a finite/countable alphabet of symbols to construct the language of real numbers. That is simply not possible, so why do you keep on trying. You need an uncountably big alphabet to have any chance at all, and then you could simply make $\mathbb{R}$ itself your alphabet." [Dirk Liebhold, loc cit]

"If the accepted answer {{see Graham Leach-Krouse on p. 313}} is true, then there is no uncountable alphabet." [Wilhelm, loc cit]

"The point you're missing is that, if you take the real numbers to be your alphabet, they are the very symbols you use to write with." [Hurkyl, loc cit]

"How do you know what symbol corresponds to what number? How would you distinguish a completely unrelated set of symbols from symbols corresponding to real numbers? An alphabet is what is alphabetically ordered. Otherwise you cannot know the place where to hit the keys and to find the letters. Further you are missing the fact that there are only countably many finite symbols (whether defined by pixels or painted by hand)." [Wilhelm, loc cit]

"You know what symbol corresponds to what number because the symbol is the number." [Hurkyl, loc cit]

"A symbol is not a number. An alphabet is an alphabetically or lexically ordered set, namely a list or sequence of symbols that serve to form words while almost all real numbers cannot be defined let alone be chosen. The real numbers are definitely not an alphabet." [Wilhelm, loc cit]

"An alphabet is any set we want, as the term is used by logicians formalizing logic via sets." [Hurkyl, loc cit]

"If you want to describe a car you will not use the word 'butter'. An alphabet is an ordered set or list. That's the general meaning. Alphabetically or lexically ordered list is often enough used in mathematics. If you want to denote only a set why don't you call it set?" [Wilhelm, loc cit]

"If you taught your child some permutation of the order of the alphabet, would it hinder their ability to read? It seems that the alphabet is an unordered set, and the ordering is only used as a memorization aid." [Steven Gubkin, loc cit]

"It would hinder them to find words in a dictionary. The order is important. Therefore it is one of the first things to learn in school. And it is common among all users of the alphabet. In the same way the sequence of natural numbers is taught and not an unordered set." [Wilhelm, loc cit]

In mathematics we often highlight some essential feature of an object, and leave out others. Certainly alphabets with order are interesting, but alphabets without order are also interesting. [Steven Gubkin, loc cit]

"But alphabets with almost all letters undefinable are certainly not interesting in a mathematical or scientific context." [Wilhelm, loc cit]

"'There are only countably many different symbols existing' is patently false in the situation of an uncountable alphabet." [Hurkyl, loc cit]

"No it is obvious. Every symbol has to be encoded by sequences of bits (when provided in the internet) or by a grid (when being printed) or by atoms and molecules with reproducable distances (when a three years old child paints it). Note that every symbol has to be reproducable if two logicians want to exchange their results. Bare claim-and-belief is not a mathematical feature and was not a logical feature as long as logic was logic." [Wilhelm, loc cit]

"Well if you are imagining a keyboard with an infinite amount of keys, you would want to imagine an infinite amount of space to write the number on each key. So the unnamed real numbers would contain infinite space to write. But how would the typist distinguish them? Well how does the mathematician distinguish them? By their position. We don't read off the digits but locate them in the place where they belong." [Jeffery Thompson, loc cit]

"It is certainly not possible to find individual real numbers by their positions. You can find a real number that has a finite description [...]. But almost all real numbers have no finite description since the set of finite descriptions is countable." [Wilhelm, loc cit]

"But they have a magnitude and a position on the number line that is absolute. We just don't have a symbolic way of representing they still have a position in the space." [Jeffery Thompson, loc cit]

"The chance to hit a real number on the number line is zero. Further we need not only to hit a number but we have to know the meaning of that number and to apply it in a meaningful way." [Wilhelm, loc cit]

"Zero probability does not mean impossibility. The magnitude of the real number defines it and this will have position." [Jeffery Thompson, loc cit]

"But it is hard to touch a selected number if the probability is zero. Further it is impossible to write a word containing two neighbouring numbers because neighbours are not defined on the real line. Nevertheless there are no gaps either. And finally you will not use numbers as letters without knowing what they mean. Therefore you need to learn this. That requires a finite description. But there are only countably many. In short: Until now I have not seen a method to use real numbers or other uncountable sets as alphabets." [Wilhelm, loc cit]

"On the other hand, there's no real problem with the application of 'uncountable alphabets'. [...] Still, the real line with its points (as keys) would do. On the other hand, to *use* this 'keyboard', you would need 'pointlike' fingertips. [F. Fritsche alias "Me" in "How to apply uncountable alphabets?", sci.logic (9 Jan 2019)]

"... and unlimited visual resolution and unlimited control of position and momentum – and a brain able to process uncountable sets of data, preferably in form of lists." [W. Mückenheim, loc cit]

"Yeah, it might prove difficult to hit the same 'key' twice. Or even a *certain* (well known) *single* key. Hence I'd say such a keyboard is impractical! [F. Fritsche, loc cit]

And uncountable languages?

The ω-regular languages are a class of ω-languages that generalize the definition of regular languages to infinite words. Büchi showed in 1962 that ω-regular languages are precisely the ones definable in a particular monadic second-order logic called S1S. ["Omega-regular language", Wikipedia]

"To consider uncountable languages we have to look at infinite strings in place of finite strings. (AFAIK, having an infinite alphabet is not very interesting and doesn't correspond to a realistic model of computation by itself.)" [Kaveh in "Is there any uncountable Turing decidable language?", Computer Science.StackExchange (19 Feb 2016)]

"Languages are collections of words. Words are finite strings. The input is always finite. That's how languages are defined in the context of computability." [Yuval Filmusin "Can a recursive language be uncountable?", Computer Science.StackExchange (29 Apr 2015)]

According to Wiktionary an uncountable alphabet has been applied already hundreds of years ago. See the Glagolitic alphabet.

The relativeness of the power set

A common property of all models of ZFC

When Skolem had proved his theorem that every first-order theory like ZFC has a countable model unless it is self-contradictory (cp. 3.4.1 "Skolem's first proof" in chapter III) he tried to take the sting out of it by proposing that there could be countable models of ZFC which were lacking the mapping from the set ω (which is indispensable in a model of ZFC) to the countably many elements of the power set of ω – by this trick preserving the uncountability of the power set in the model, because also this uncountability is indispensable in the model. From "outside" this model could be countable.

In the following we will present plenty of quotes stating that the power set is not absolute, usually without any supporting statements other than the big argument implicit to all reasoning: "If the uncountable power set is not relative but absolute, then set theory cannot be maintained."

> "Since the Skolem-Löwenheim 'paradox', namely, that a countable model of set theory exists which is representative of the stumbling blocks that a nonspecialist encounters, I would like to briefly indicate how it is proved. What we are looking for is a countable set *M* of sets, such that if we ignore all other sets in the universe, a statement in *M* is true precisely if the same statement is true in the true universe of all sets." [Paul J. Cohen: "The discovery of forcing", Rocky Mountain J. Math. 32,4 (2002) p. 1076]

There is a common property of all models of ZFC: Each one contains the axiom of infinity and therefore contains an inductive set ω.[1] It is irrelevant how to denote its elements. The names of the natural numbers 1, 2, 3, ... are convenient and appropriate. If a finite or infinite subset of ω were missing from the model then its elements *k*, *m*, *n*, ... (where *k* means 1 or 2 or 3 or ..., and so do *m* and *n*, and ...) were missing too. Hence no subset of ℕ must be missing. It could be argued that the model must not contain undefinable sets. With the same right however this could be claimed for the usual universe *V* of set theory. Then there is nothing uncountable because only countably many definitions do exist (see "The list of everything"). Therefore this argument fails. *From outside* the power set of ℕ turns out to be uncountable. If the model has only countably many subsets of ℕ, then it is clearly not a model of all theorems (including the axioms) of ZFC.

One page earlier already, Cohen had referred to Skolem's "explanation":

> "The paradox vanishes when we realize that, to say that a set is uncountable, is to say that there is no enumeration of the set. So the set in *M* which plays the role of an uncountable set in *M*, although countable, is uncountable when considered in *M* since *M* lacks any enumeration of that set." [Paul J. Cohen: "The discovery of forcing", Rocky Mountain J. Math. 32,4 (2002) p. 1075]

[1] For example, it turns out that every standard transitive model *M* of ZFC contains all the von Neumann ordinals as well as $\aleph_0$. [Timothy Y. Chow: "A beginner's guide to forcing", arXiv (8 May 2008)] – How "does it turn out"? This question remains as unanswered as the question why the uncountable power set is not absolute.

This argument appears over and over again. It is invalid. Not the lack of an enumeration is the important thing but the important thing is clearly the complete presence of all subsets of ℕ. In the complete set ℕ they are automatically present. And they are visible from "outside"!

> In Gödel's universe some subsets of a set in L are missing. Otherwise the model would be uncountable and not constructible by formulas, according to ZFC, also from "outside", i.e., in the mathematics of our world. "The classes and sets of the model Δ will form a certain sub-family of the classes and sets of our original system Σ [...] We call the classes and sets of Δ *constructible* [...] Constructible sets are those which can be obtained by iterated application of the operations given by axioms" [Kurt Gödel: "The consistency of the continuum hypothesis", Princeton University Press (1940) p. 35]

This is the usual trick to keep the model free of uncountability, although it is easily seen through. It fails; *if* ℕ is constructible, *then* every element of ℕ is constructible and hence every subset of ℕ is constructible too. None is missing. A concise and instructive version of Kurt Gödel's argument is presented by Hrbacek and Jech [Karel Hrbacek, Thomas Jech: "Introduction to set theory", 2nd ed., Marcel Dekker, New York (1984) p. 232]

> "Let now $X \subseteq \omega$. The Axiom of Constructibility guarantees that $X \in L_{\alpha+1}$ for some, possibly uncountable, ordinal α. This means that there is a property P such that $n \in X$ if and only if $P(n)$ holds in $(L_\alpha, \in)$. By the Skolem-Löwenheim Theorem, there is an *at most countable* set $B \subseteq L_\alpha$ such that $(B, \in)$ satisfies the same statements as $(L_\alpha, \in)$."

This is what the authors derive from the Skolem-Löwenheim Theorem, but they withhold the essential condition that this only holds for *consistent* theories. So their proof is only valid when consistency of ZFC is *assumed*. (Otherwise everything including CH could be proved too, but with much less effort.)

Uncountably many subsets of ℕ could not all be definable, that is true. But that does not hinder them to have to belong to every model – just as the undefinable well-ordering of all subsets of ℕ belongs to the standard universe of set theory. The contrary statements quoted below are given either thoughtlessly or explicitly with the aim of deceiving.

> "Let $\mathcal{B} \in \mathcal{M}$ be a complete Boolean algebra [...] which is complete *inside* $\mathcal{M}$, meaning that if $A \subseteq \mathcal{B}$ and $A \in \mathcal{M}$ then $\wedge A \in \mathcal{M}$ and $\vee A \in \mathcal{M}$. Note that $\mathcal{B}$ need not contain the joins and meets of *all* its subsets – just those that lie in $\mathcal{M}$." [Clive Newstead: "Boolean-valued models and forcing" (2012)]

Again we read the usual excuse to keep models free of uncountability. It fails. The axiom of infinity requires ω, and the axiom of power set requires *all* subsets of ω. Otherwise not all theorems of ZF are satisfied, and so the model is not a model of ZF. Here are some texts supporting this apology, claiming that missing subsets do not undermine a model of ZF – but without any justification.

"In particular, there exists some *countable* model of ZFC. How can this happen, when it is a theorem of ZFC that there exists an uncountable set, and all its elements must also belong to the model? If we view our countable model $\mathcal{M}$ as a subset of the canonical model of ZFC, and we let X be a set such that $\mathcal{M} \vDash X$ is uncountable, then there is a bijection $f: \omega \to X$ in V, but no such bijection is an element of $\mathcal{M}$. Thus, it is consistent both that the universe of $\mathcal{M}$ is a countable set, and that it models the statement 'there exists an uncountable set.'" [Rowan Jacobs: "Forcing" (2011) p. 2]

"There are some sets that every transitive model of ZFC will contain. For instance, the Axiom of Infinity ensures that the natural numbers $\mathbb{N}$ (alternately called ω when viewed as an ordinal number) will be in every transitive model. In particular, not only does every model $\mathcal{M}$ have some interpretation of $\mathbb{N}$, but all of these interpretations are necessarily the same set. By contrast, every transitive model of ZFC must contain some interpretation of $\mathbb{R}$, by taking the power set of $\mathbb{N}$. However, what subsets of $\mathbb{N}$ exist will vary from model to model, so the interpretations of $\mathbb{R}$ will not necessarily be identical. In general this will be true of all power set constructions – different models may disagree on which subsets of some given set exist, so the power sets of that set may be different." [Rowan Jacobs: "Forcing" (2011) p. 3]

"A crucial counterexample is the powerset of $\aleph_0$, denoted by $2^{\aleph_0}$. Naively, one might suppose that the powerset axiom[1] of ZFC guarantees that $2^{\aleph_0}$ must be a member of any standard transitive model *M*. But let us look more closely at the precise statement of the powerset axiom. Given that $\aleph_0$ is in *M*, the powerset axiom guarantees the existence of y in *M* with the following property: For every z in *M*, $z \in y$ if and only if every w in *M* satisfying $w \in z$ also satisfies $w \in \aleph_0$. Now, does it follow that y is precisely the set of all subsets of $\aleph_0$?

No. First of all, it is not even immediately clear that z is a subset of $\aleph_0$; the axiom does not require that *every* w satisfying $w \in z$ also satisfies $w \in \aleph_0$; it requires only that *every w in M* satisfying $w \in z$ satisfies $w \in x$. However, under our assumption that *M* is *transitive,* every $w \in z$ is in fact in *M*, so indeed z is a subset of $\aleph_0$.

More importantly, though, y does not contain *every* subset of $\aleph_0$; it contains *only those subsets of x that are in M.* So if, for example, *M* happens to be *countable* (i.e., *M* contains only countably many elements), then y will be countable, and so a fortiori y cannot be equal to $2^{\aleph_0}$, since $2^{\aleph_0}$ is uncountable. The set y, which we might call the *powerset of* $\aleph_0$ *in M,* is not the same as the 'real' powerset of $\aleph_0$, a.k.a. $2^{\aleph_0}$; many subsets of $\aleph_0$ are 'missing' from y. This is a subtle and important point." [Timothy Y. Chow: "A beginner's guide to forcing", arXiv (2008) p. 6]

[1] Every set x has a so-called power set $y = \mathcal{P}(x)$. This is expressed formally as $\forall x\ \exists y\ \forall z: z \in y \Leftrightarrow z \subseteq x$. Compare also section 2.12 "ZFC-Axioms of set theory".

Even more subtle and important is the fact, that the axiom of power set defines the "real" power set of *every* set in the model. If ℕ is in the model, then there must not be missing any of its (in "reality" uncountably many) subsets. Or the other way round: If any subset of ℕ is missing in *M*, then *M* is not a model of the whole ZFC (including the "real" power set axiom). Therefore a countable model of ZFC and a model of whole ZFC are mutually incompatible. Skolem's proof implies that ZFC cannot have *any* model.

> "More generally, one says that a concept in *V* is *absolute* if it coincides with its counterpart in *M*. For example, 'the empty set', 'is a member of', 'is a subset of', 'is a bijection', and '$\aleph_0$' all turn out to be absolute for standard transitive models. On the other hand, 'is the powerset of' and 'uncountable' are *not* absolute. For a concept that is not absolute, we must distinguish carefully between the concept 'in the real world' (i.e., in *V*) and the concept in *M*. A careful study of ZFC necessarily requires keeping track of exactly which concepts are absolute and which are not. [...] the majority of basic concepts are absolute, except for those associated with taking powersets and cardinalities." [Timothy Y. Chow: "A beginner's guide to forcing", arXiv (2008) p. 6f]

If any non-empty subset $S \subseteq \mathbb{N}$ is missing in *M*, then the elements of this subset *S*, and hence these elements of ℕ, must be missing in *M*. (Otherwise subset *S* is not missing.) Then *M* is not even a model of ZF for violating the axiom of infinity.

> "Let us look carefully at what the Powerset Axiom really states. It says that for every *x* in *M*, there exists a *y* in *M* with the following property: if *z* is a member of *M* such that every *w* in *M* satisfying $w \in z$ also satisfies $w \in x$, then $z \in y$. We see now that even if $\in$ is interpreted as membership [...], it does not follow that *y* is the set of *all* subsets of *z*. [...] Now, it turns out that because *M* is transitive, the *z*'s *are* in fact subsets of *x*. What 'goes wrong' is the rest of the axiom: *y* does not contain *every subset of x*; it only contains *those subsets of x that are in M*. So it is perfectly possible that this 'powerset' of *x* is countable.
>
> What should we call *y*? Calling it the 'powerset of *x*' is potentially confusing; I prefer to reserve this term for the actual set of all subsets of *x*. The usual jargon is to call *y* 'the powerset of *x* in *M*'.
>
> As an exercise in understanding this concept, consider Cantor's famous theorem that the powerset of ω is uncountable. Cantor's proof can be mimicked using the axioms of ZFC to produce a formal theorem of ZFC. This yields: 'The powerset of ω in *M* is uncountable in *M*.' In order to see more clearly what this is saying, let us expand this out more fully to obtain: 'There is no bijection in *M* between ω and the powerset of ω in *M*,' where a bijection is a certain set (in fact, a certain set of ordered pairs, where an 'ordered pair' $\langle x, y\rangle$ is defined set-theoretically as $\{\{x\}, \{x, y\}\}$). So even though the powerset of ω in *M* is countable, and we can construct a bijection in the 'real world' between ω and the powerset of ω in *M*, it turns out that this bijection is *not a member of M*. There is therefore no contradiction between being 'uncountable in *M*' and being countable (this is known as 'Skolem's paradox').
>
> Once you grasp this point that appending 'in *M*' is crucial and can change the meaning of a term or a sentence dramatically, you may start to worry about all

> kinds of things in the preceding paragraphs. For example, shouldn't we distinguish between 'ω' and 'ω in *M*'? This is a legitimate worry. Fortunately, the transitivity of *M* implies that a lot of things, including 'is a subset of', 'is a function', 'ω', and other basic concepts, are *absolute*, meaning that they mean the same whether or not you append 'in *M*'. A complete treatment of forcing must necessarily include a careful discussion of which concepts are absolute and which are not. However, this is a rather tedious affair, so we will gloss over it. Instead, we will simply warn the reader when something is not absolute." [Tim Chow: "Forcing for dummies", sci.math.research (10 Mar 2001)]

If "is a subset of" is absolute, then no subset can be missing. Then its absence is in contradiction with the axiom of power set which requires that all subsets exist (cp. "The axiom of power set" in chapter II). Therefore there is no model of ZFC that contains less than all subsets of the inductive set. Therefore there is no model of ZFC that is countable from any perspective.

Most advocates of ZFC claim that the notions of "contained as an element" or of set or subset are undefined as such and need a model to become defined. That is a clear indication of fraud. The only meaning that has to be observed by all models of ZFC is given by the axioms of ZFC (cp. 2.12 "ZFC-Axioms of set theory") like: There *exists* an infinite set that *contains* the empty set.

There are two possible ways to build a model. The first one is by "iterated application of the operations given by axioms". By the axiom of infinity the model contains all elements of ω. But then it is unavoidable, by the same argument, that by the axiom of power set the model contains all subsets of ω – not only some of them. The second alternative does not apply the axioms but constructs every element. Then, of course, all elements of ω have to be constructed, for instance as { }, {{ }}, {{{ }}}, and so on, or briefly 111... where a 1 shows that its index is in the model. But if this is possible, then it is also possible to go through the sequence and to drop some of the 1, indicating the presence of its index, and to replace it by 0, indicating the absence of its index.

What Skolem proved in fact is that every model of any theory is countable (or better: not uncountable). Nowhere in his proof Skolem proves the existence of an uncountable model. Apparently he simply accepted what Cantor seemingly had "proven". In fact the set of all finitely definable elements is countable (or better: not uncountable), cp. section "The list of everything".

> "'All' just turns out not to mean what you think it means." [G. Greene in "Countable model of ZFC", sci.logic (2 Apr 2017)]

How does this happen to "turn out"? By assuming Cantor's finished infinity? If there are only these two alternatives: "infinity is finished and all means not all", or "all means all and infinity is not finished", every sober mind will know what to choose.

Comments on Cohen's legacy

The following text printed in blue is taken from the last paragraph, titled "*Some observations of a more subjective nature*" by Paul Cohen, the inventor of forcing (see section 3.2.2 "The continuum hypothesis cannot be proved in ZFC") [Paul J. Cohen: "The discovery of forcing", Rocky Mountain Journal of Mathematics 32,4 (2002) 1099f] I have added some subjective remarks.

Everyone agrees that, whether or not one believes that set theory refers to an existing reality, there is a beauty in its simplicity and in its scope. Someone who rejects that sets exist as "completed wholes" swimming in an ethereal fluid beyond all direct human experience has the formidable task of explaining from whence this beauty derives.

That is a religious argument. Further it is wrong. There are people who think that set theory is an unscientific and ugly construct, for instance myself.

On the other hand, how can one assert that something like the continuum exists when there is no way one could even in principle search it, or even worse, search the set of all subsets, to see if there was a set of intermediate cardinality? Faced with these two choices, I choose the first. The only reality we truly comprehend is that of our own experience. But we have a wonderful ability to extrapolate. The laws of the infinite are extrapolations of our experience with the finite.

Most of these extrapolations, starting with the use of the bijection as a measure for infinite sets, are absolutely unscientific because (1) really scientific results will never depend on "clever choice" of indices, and (2) universal quantification over infinite sets shows that *every element is followed by infinitely many elements*, proving that universal quantification fails per se. *Analytical* extrapolations however, like the never decreasing wealth of Scrooge McDuck or the never decreasing number of undefiled intervals when enumerating the rational numbers (cp. the sections "Scrooge McDuck" and "Not enumerating all positive rational numbers" are refused by set theorists because they show that the necessary assumption of finishing infinity is untenable.

If there is something infinite, perhaps it is the wonderful intuition we have which allows us to sense what axioms will lead to a consistent and beautiful system such as our contemporary set theory. The ultimate response to CH must be looked at in human, almost sociological terms.

I would be very interested to know whether possibly existing foreign civilizations have gone astray in this respect as much as humans.

We will debate, experiment, prove and conjecture until some picture emerges that satisfies this wonderful taskmaster that is our intuition.

Cohen repeatedly cites intuition. But intuition is furiously exorcized by set theorists if it sheds doubt on the results of set theory.

I think the consensus will be that CH is false.

I think the consensus will be that CH is meaningless.

{{Gödel}} told me that it was unthinkable that our intuition would not eventually discover an axiom that would resolve CH.

We need no further axioms. We have to recognize merely the fact that infinity is infinite in every respect. Infinity is potential or, to use an expression coined by Cantor, infinity is absolute.

Comments on Chow's beginner's guide to forcing

The following text printed in blue is taken from Chow's essay [Timothy Y. Chow: "A beginner's guide to forcing", arXiv (2008)] which was frequently cited in chapter III. I have added some critical remarks which were inappropriate in chapter III but which I found hard to restrain there.

Naively, one might suppose that the powerset axiom of ZFC guarantees that $2^{\aleph_0}$ must be a member of any standard transitive model *M*.

This is unscientific polemics. It is not naive to take an axiom word by word by what it says.

But let us look more closely at the precise statement of the powerset axiom. Given that $\aleph_0$ is in *M*

Probably the text means that the *set* $\mathbb{N}$ of natural numbers is in *M* or briefly ω is in *M*. If so, why not say so? In fact some proponents of modern set theory identify ω and $\aleph_0$. Even Cantor is reported to have done so in his later years. But should the use of $\aleph_0$ veil the fact that all natural numbers are in *M*? Anyhow, $\mathbb{N}$ or $\aleph_0$ has to be in *M*. This is not, as it sounds, a possible premise but unavoidable when *M* is a model of ZF and therefore of the axiom of infinity.

the powerset axiom guarantees the existence of *y* in *M* with the following property: For every *z* in *M*, $z \in y$ if and only if every *w* in *M* satisfying $w \in z$ also satisfies $w \in \aleph_0$. Now, does it follow that *y* is precisely the set of all subsets of $\aleph_0$? No. First of all, it is not even immediately clear that *z* is a subset of $\aleph_0$ [...] However, under our assumption that *M* is *transitive*, every $w \in z$ is in fact in *M*, so indeed *z* is a subset of $\aleph_0$. More importantly

Why "more" importantly? The former argument is admittedly invalid, so it is not in the least important and "more importantly" sounds more important than it could be.

The set *y*, which we might call the *powerset of* $\aleph_0$ *in M*, is not the same as the "real" powerset of $\aleph_0$, a.k.a. $2^{\aleph_0}$; many subsets of $\aleph_0$ are "missing" from *y*. This is a subtle and important point, so let us explore it further.

This is not subtle – and important is it at most as a tool to deceive the naive reader. The powerset axiom requires $x \subseteq \mathbb{N} \Leftrightarrow x \in \mathcal{P}(\mathbb{N})$. In words: either a model of ZF contains *all* subsets of $\mathbb{N}$ or it is not a model of ZF. Therefore it guarantees the *real* power set of $\mathbb{N}$ in every set *M* that contains $\mathbb{N}$. This set is *uncountable from outside*. Therefore there is no countable model of ZF. According to Skolem's result, this fact proves the inconsistency of ZF.

Two contradictory statements about Gödel's constructible model Δ

From section 2.17.2 we obtain the two following statements:

• Since by the axiom of infinity the model Δ must contain the minimal inductive set ω, we could have started also from there, $L^* = \omega$ or $L^* = \mathbb{N}$, instead of $L_0 = \emptyset$.

• "Not all concepts can be proved to be absolute; for example $\mathfrak{P}$ {{the power set}} and V {von Neumann's universe with all subsets of its sets}} cannot be proved to be absolute." [Kurt Gödel: "The consistency of the continuum hypothesis", Princeton University Press (1940) p. 44] That means, all finite sets and $\mathbb{N}$ are in V as well as in L but not all subsets of $\mathbb{N}$ are in L.

That is impossible. *The ZF-axioms are model-independent.* That means the possible subsets do not at all depend on Δ but only on the axioms of infinity and of power set (see 2.12) quoted here in Zermelo's original version[1]:

Axiom VII. The domain contains at least a set Z which contains the null-set as an element and is such that each of its elements a is related to another element of the form $\{a\}$, or which with each of its elements a contains also the related set $\{a\}$ as an element. (Axiom of the infinite.)

Axiom IV. Every set T is related to a second set $\mathfrak{U}(T)$ (the '"*power set*" of T), which contains all subsets of T and only those as elements. (Axiom of power set.)

A model of a theory is a structure that satisfies the sentences of that theory, in particular its axioms. Therefore every model of *this* theory contains $Z = \{\{\ \}, \{\{\ \}\}, \{\{\{\ \}\}\}, ...\}$ as well as all its subsets. This holds for all its possible models. We have always $S \subseteq Z \Rightarrow S \in \mathcal{P}(Z)$.

"All" in the axiom of power set means with no doubt that *all* collections of the elements of $\mathbb{N}$ (or ω or Z) have to exist in the model; no subset must be missing. So we get an absolute power set $\mathcal{P}(\mathbb{N})$ which, according to Hessenberg's proof (cp. 2.4), is uncountable and hence not in a model that is "countable from outside".

So the corrected statement reads: Since by the axioms of infinity and of power set the model Δ must contain all sets and all subsets of the inductive set ω, we could have started also from there, $L^* = \mathcal{P}(\omega)$ or $L^* = \mathcal{P}(\mathbb{N})$, instead of $L_0 = \emptyset$.

In a less formal way we can argue: If *all* $n \in \mathbb{N}$ can be constructed individually, but not *all* subsets of $\mathbb{N}$, then this means that deleting some n is harder than constructing them. That's hard to

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[1] Axiom VII. Der Bereich enthält mindestens eine Menge Z, welche die Nullmenge enthält und so beschaffen ist, daß jedem ihrer Elemente a ein weiteres Element der Form $\{a\}$ entspricht, oder welche mit jedem ihrer Elemente a auch die entsprechende Menge $\{a\}$ als Element enthält. (Axiom des Unendlichen.) Axiom IV. Jeder Menge T entspricht eine zweite Menge $\mathfrak{U}T$ (die "*Potenzmenge*" von T), welche alle Untermengen von T und nur solche als Elemente enthält. (Axiom der Potenzmenge.) [E. Zermelo: "Untersuchungen über die Grundlagen der Mengenlehre I", Math. Ann. 65 (1908) pp. 265ff]

believe. For instance the Binary Tree is said to be constructible. But with it automatically all subsets of $\mathbb{N}$ (and all real numbers) are constructed as (limits of) its paths. The only conclusion of the inconstructibility of all subsets of $\mathbb{N}$ is that not all elements of $\mathbb{N}$ can be constructed.

By the way, the so-called axiom of constructibility, abbreviated by $V = L$, claims that every set of V is constructible.

There is no countable model of ZF(C)

A model of a theory is a structure that satisfies the sentences of that theory.

A model of the Peano axioms for instance is every not repeating sequence with first term 1 (or 0) like the sequences $(1/n)$ or (2^n) or (x^n), even with $x = 1$, namely if not the distinctiveness of numerical values but of the structures of written symbols is concerned – a question that is left undecided by the Peano axioms. The only fixed member of a Peano model is 1 (or 0).

A model of the group axioms is the set of integers with addition ($\mathbb{Z}$, +) or the set of positive rational numbers with multiplication ($\mathbb{Q}_+$, ·) or the set of 2-vectors with vector addition ($\mathbb{R}^2$, +) or the set of invertible 2×2-matrices with matrix multiplication (M_2, •) or the set of bijective mappings of a set on its permutations with concatenation (f, o). The neutral element defined by the group axioms looks very different for every model in the given cases; it is 0, 1, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and f_{id}, respectively. Obviously here *no* fixed member belonging to all groups is specified.

In the following we will understand by ZA Zermelo's original axiomatization [E. Zermelo: "Untersuchungen über die Grundlagen der Mengenlehre I", Math. Ann. 65 (1908) pp. 107-128]. ZA includes the axioms:

- Axiom I. Every set is determined by its elements.
- Axiom II. There exists an (improper) set, the "*null-set*" 0, which does not contain any element at all.
- Axiom IV. Every set T is related to a second set $\mathfrak{U}(T)$ (the "'*power set*" of T), which contains all subsets of T and only those as elements {{in the following denoted by $\mathcal{P}(T)$}}.
- Axiom VII. The domain contains at least one set Z which contains the null-set as an element and has the property that every element a of it corresponds to another one of the form $\{a\}$, or which with each of its elements a contains also the corresponding set $\{a\}$ as an element.

These axiom are the same in ZF(C). Therefore the results obtained here are vaild too for ZF(C).

Contrary to the Peano or group axioms, ZA specifies the set $Z_0 = \{0, \{0\}, \{\{0\}\}, ...\}$ of elements of the form $\{...\{\{\{0\}\}\}...\}$, the minimum inductive set. As Zermelo explicitly states Z_0 is the common part of all possible sets satisfying axiom VII, the intersection of the sets Z, and therefore Z_0 belongs to every domain of ZA (and of ZF(C)). That implies in particular that *all its elements*, all pairs, all triples, *and all combinations of its elements* are the same in every domain of a model satisfying ZA (and ZF(C)). This implication does not necessitate any further axioms. Axiom IV is only required to guarantee that the set of all subsets is a set again.

Now it is easy to see from axiom I that all combinations of elements of Z_0 determine sets which are subsets of Z_0. By axiom IV these subsets form a set, the power set $\mathcal{P}(Z_0)$, and by Cantor's theorem it is clear that $\mathcal{P}(Z_0)$ is uncountable.

Therefore: If the power set of Z_0 is uncountable in the universe V, then it is uncountable in every model, namely it is absolutely uncountable, i.e., uncountable for every model "from outside" too.

From this aspect it is not surprising that Zermelo did never accept Skolem's result.

Nobody knows a model of ZF(C), let alone a countable model. What a countable model of ZF(C) would prove however is that Z_0 need not have an uncountable power set in spite of the fact that axiom I guarantees the existence of all its subsets with no regard to definability or other restrictions of the model. How could this happen? Only by the understanding that a model which cannot define an undefinable subset of Z_0 does not contain that undefinable subset of Z_0. Alas this idea can only evolve from the inversion of the implication in the axiom of separation (see 2.12.2) which reads briefly stated: "*every* defined subset B of A exists". The inversion "*only* defined subsets B of A exist" is of course invalid.

First this is violating logic, and second the same would be true in the set theory of the universe V and everywhere else. Most subsets of Z_0 are undefinable in our set theory. No uncountable power set could exist within our set theory because almost all its elements are not definable there – and nowhere else. So, if undefinablility of sets makes them not existing, then there are only countable models. If however this is not the case, then there are only uncountable models.

Conclusion: Either there are no countable models of ZF(C) and Skolem's theorem proves the inconsistency of ZF(C). Or there are only countable models of ZF(C). Then this fact proves the inconsistency of all significant results of set theory. So either ZF(C) has no countable model and therefore it is inconsistent. Or ZF(C) has only countable models and therefore it is wrong.[1] The least we can say is that this theory is of no interest in mathematics and other sciences.

In other words: Every model M of a set theory containing the axioms of infinity and powerset contains all subsets of ω, where "all" refers to all subsets that are available for the "*outside* observer" – or the outside observer recognizes that M is not a model of these two axioms. Therefore uncountability is a self-contradictory notion.

[1] Note that when God at the end of times will consider our universe and its universe of sets, then he will be able to count all elements of all sets that ever have appeared, including all "antidiagonal numbers" and including all antidiagonal numbers he might have bothered to construct himself. Note that it is impossible to insert undefinable digit sequences in a Cantor-list. Note that it is as impossible to extract an undefinable antidiagonal number from a Cantor-list. But every list and every antidiagonal number are defined at least by their spatio-temporal coordinates. This proves that also in our real physical universe ZF(C) is countable from inside and from outside.

Topology

Covering all rational points by rational intervals

Construct countably many closed intervals I_n of measure 10^{-n} on the positive real axis such that I_n contains the rational point[1] q_n in its centre. The first three intervals, obtained using Cantor's enumeration (cp. section 2.1), are shown in this figure:

The infinitely many intervals following could be seen under a microscope only. But if each one had been marked by a notch of a femtometer width, no free space was left.

The intervals cover a measure of less than 1/9 (because many are overlapping). To demonstrate the relation between intervals and complement assume for definiteness that N points of the first part [0, 1000] of the real axis are covered by the first N intervals of total measure $\approx 1/10$. Then their average distance $10^3/N$ is 10000 times their average length $10^{-1}/N$; the complement is as empty as the matter surrounding us. Using smaller intervals or a larger part of the real axis, the emptiness of the universe could be undercut.

To establish the required measure, uncountably many irrational points ξ would have to populate the complement Ξ. But two irrational points of the complement always have a rational point between each other, that means an interval I_n, because there are no rationals outside of intervals. These intervals or their endpoints can be considered as structuring and enumerating a Cantor-list. The only difference is that the enumeration does not follow the natural order of $\mathbb{N}$. But the number of naturals n does not change by reordering. So we need uncountably many irrational numbers separated by countably many intervals. That is a contradiction similar to uncountably many terms in a sequence or uncountably many entries populating a Cantor-list.

But if all rational points $q_n \in (0, \infty)$ are covered by intervals $I_n = [s_n, t_n]$ of measure $|I_n| = 10^{-n}$ such that q_n is the centre of I_n, then the endpoints s_n and t_n are rational too. Thus intervals must overlap. An infinite set of overlapping and therefore path-connected intervals is called a *cluster*.

Since the intervals cannot cover the whole positive real axis, every cluster Q_k must have ends. These ends cannot be rational points because all rationals are covered by intervals. Therefore the endpoints can only be irrational limits, where the end σ_k at left-hand side of Q_k is the limit of a sequence (s_{n_k}) and the end τ_k at right-hand side of Q_k is the limit of a sequence (t_{n_k}). But these limits, a countable set, are not sufficient to yield the infinite measure of the complement Ξ. Where are the missing irrational points ξ?

According to set theory they form a "completely scattered space", the so-called "Cantor dust", such that the missing irrational points – limits of infinite sequences of clusters or limits of

[1] We do not distinguish between a real *number* x and a *point* with real coordinate x, because we adopt Cantor's axiom uniquely relating real numbers and points of the real line.

arbitrary, imagined infinite sequences – are existing, separated from each other. Separated by what? As we will see, this is incompatible with real analysis and basic topology.

It does not make sense to talk about adjacent points or adjacent closed intervals of the real axis. But every point x is adjacent to two open intervals, namely $(-\infty, x)$ and (x, ∞). These intervals often can be partitioned; a limit point σ for instance is adjacent to its cluster Q. That means in every open ε-neighbourhood of σ there are infinitely many points of Q and therefore also infinitely many rational points $q \in Q$. If however $\xi \in \Xi$ is not a limit point, then it has an open ε-neighbourhood not containing rational points. (It is impossible that an open ε-neighbourhood contains only a finite set of rationals). Therefore it makes sense to talk about the endpoints σ and τ of a cluster and about the clusters next to or adjacent to an endpoint. In principle, given a fixed and complete enumeration of the rationals, set theory should allow to calculate every cluster Q and its adjacent limits σ and τ.

Now assume that $\xi \in \Xi$ is in the complement (however it may have been defined) but not a limit σ or τ of a cluster. Then it has, like every point on the real axis, two open intervals around itself. But there is an open ε-neighbourhood without points of any cluster (otherwise ξ is a limit). So this neighbourhood must be an interval of the complement Ξ. This is a contradiction because intervals with only irrational points cannot exist on the real axis.

The claim of set theory that $\xi \in \Xi$ is a degenerate interval (i.e., a point) without adjacent intervals is not acceptable because there are no gaps and no disconneted points on the real axis.

[W. Mückenheim in "Cantor / Abbildung N auf R", de.sci.mathematik (30 Nov 2008). W. Mückenheim: "Matheology §019", sci.math (27 May 2012). Ganzhinterseher: "How can Clusters in Cantor dust absorb enough irrational points?", MathOverflow (31 Dec 2018)]

Covering all rational points by irrational intervals

Instead of handling clusters the rational points of the real axis may be covered in a simpler way by closed intervals with irrational endpoints, $\Phi_n = [q_n - 10^{-n}\cdot\sqrt{2}, q_n + 10^{-n}\cdot\sqrt{2}]$ for instance. All intervals together have measure $(2/9)\cdot\sqrt{2}$ or less. The open complement Ξ of infinite measure does not contain any rational point because they are confined within intervals. All endpoints of the intervals are endpoints of the open complement Ξ. The further structure of the complement is not of interest. There are only three important facts:

- Cantor's axiom: Every real number determines a point on the real line.
- The rational points can be included in intervals of arbitrary small total measure.
- Between two irrational numbers $\alpha < \beta$ there is a rational number q with $\alpha < q < \beta$.

Topology of the real axis supplies with $\mathbb{X} = \mathbb{R}\backslash\mathbb{Q}$:

$$\forall\alpha \in \mathbb{X}\ \forall q \in \mathbb{Q}\ \exists\beta \in \mathbb{X}\backslash\{\alpha\}: |\alpha - \beta| < |\alpha - q| \qquad (1)$$
$$\forall\alpha \in \mathbb{X}\ \forall\beta \in \mathbb{X}\backslash\{\alpha\}\ \exists q \in \mathbb{Q}: |\alpha - q| < |\alpha - \beta|\ . \qquad (2)$$

For every point $\alpha \in \Xi$ however only (1) is satisfied: For every rational point $q \in \mathbb{R}$ there is an irrational point $\beta \neq \alpha$ closer to α – but not vice versa.

But can an irrational $\alpha \in \Xi$ exist at all? Since in the natural order there cannot exist two irrational numbers without a rational number between them, there is no irrational point in the complement either. The remainder of the real axis of infinite length is empty – if $\aleph_0$ is a sensible notion and if there are only $\aleph_0$ rational points on the real axis. Moreover even without further irrational numbers in the complement, the endpoints alone would provide an existence proof for irrational numbers without rational numbers between them. And if all Φ_n were extended by a factor 1.01, the irrational endpoints would disappear.

[W. Mückenheim: "Matheology §019", sci.math (1 Jun 2012). Hans Hinterseher: "How can the following three principles be reconciled?", MathOverflow (29 Nov 2018). W. Mückenheim: "I will enjoy your attempts.", sci.math (5-8 Dec 2018)]

Symmetry considerations

Removing N disconnected intervals from a circle leaves N disconnected intervals in the complement – if N is finite. Harnack had extrapolated this to the case of $\aleph_0$ intervals. [Axel Harnack: "Ueber den Inhalt von Punktmengen", Math. Annalen 25 (1885) pp. 241-250] For a simple proof see section "Ants moving notches". But this result shows either that the measure of the circle is zero or that the notion of countability is wrong. The latter is forbidden to think in set theory (even stronger than the former). Therefore a countable number of intervals removed from the real axis, must leave an uncountable number of separated points in the complement.[1]

We can replace the intervals by their midpoints without reducing the complement or changing the structure of separation. And the example does not depend on the rational character of points, since we could shift the whole system by π. After replacing the lengthy expression "cluster or separated (degenerate) interval or point of Cantor-dust" by "atom" we get the theorem:

(1) Removing $\aleph_0$ atoms separates $2^{\aleph_0}$ atoms in the complement.

(2) Removing $2^{\aleph_0}$ atoms separates $\aleph_0$ atoms in the complement.

(2) follows by symmetry, for instance when removing what in (1) has remained. Therefore we can state the general rule: Removing N atoms leaves N or 2^N or $\log_2 N$ atoms, depending on N. If we stop however in (2) after having removed $\aleph_0$ atoms, we have, according to (1), separated more atoms in the complement, namely $2^{\aleph_0}$, than after finishing.

[W. Mückenheim "Clusters and Cantor dust", sci.logic (4 Jan 2019). W. Mückenheim: "Warum gibt es so viele verschiedene Gegendarstellungen?", de.sci.mathematik (10 Dec 2018).]

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[1] These points must be separated because the set of intervals could contain all rational numbers, and irrational numbers cannot exist side by side, i.e., in a common interval without rational numbers. Their behaviour resembles protons and neutrons.

Ants moving notches

(1) Define a sequence (p_n) of points $p_n = 1/n$ in the unit interval. These points define subintervals $A_n = [1/(n+1), 1/n]$ for odd n and $B_n = [1/(n+1), 1/n]$ for even n. The intervals of sort A, marked by ===, and of sort B, marked by ---, are alternating. If the points are indicated by n, we have something like the following configuration (drawing not to scale):

...7--6==5---4===3-------2===========1 .

<u>Theorem</u> If two neighbouring points p_n and p_{n+1} are exchanged, the number of intervals remains the same, for example:

...7--6==5---3===4-------2===========1 .

The intervals remain alternating. In particular, the number of intervals cannot increase.

(2) Define a set of intervals I_m in the unit interval such that interval I_m has length $|I_m| = 10^{-m}$ and covers the rational number q_m of a suitable enumeration of all rational numbers of the unit interval. Then the union of all I_m has measure 1/9 or less (if intervals overlap). The remaining part of the unit interval has measure 8/9 or more and is allegedly split into uncountably many singletons. A sketch of the intervals I_m ~~~ is given here:

... *a b~~~~~~~~~c~~~~~~~~~~d~~~~~~e f~~~g~~~h i~~~j k~~~l* .

We cannot exclude intervals lying within intervals like *c~~~d* within *b~~~e* or, alternatively, overlapping intervals like *b~~~d* and *c~~~~e* and also adjacent intervals like *f~~~g* and *g~~~h*.

(3) Let the endpoints p_n of the configuration described in (1) move in an arbitrary way, say powered by little ants moving notches. Then it cannot be excluded that the p_n and the endpoints of the I_m of (2) will coincide (no particular order is required)

... *a b~~~~~~~~~c~~~~~~~~~~d~~~~~~e f~~~g~~~h i~~~j k~~~l*
... 3=7------------11=========5--------12=4---2===9-8==10-6==1 .

As our theorem shows, there will be not more than $\aleph_0$ intervals in the end position. This includes the set of I_m and the set of intervals in the complement. In case that intervals fall into intervals, the complete number can be reduced. In no case it can grow. Therefore the assertion of uncountably many degenerate intervals (so-called singletons – but there cannot exist irrational singletons without rational numbers separating them) in the complement has been contradicted.

(4) The same may occur in the unit interval bent to a circle, i.e., on a ring of circumference 1. In the first step construct $\aleph_0$ pairs of endpoints p_n. Then let the endpoints slide in an arbitrary way. They could in principle reach the configuration of the intervals I_m covering the rationals q_m with length 10^{-m}. This is not excluded as a final state – if the I_m can be brought to existence at all. In mathematics the $\aleph_0$ complementary intervals will never become uncountably many singletons during this continuous process.

[W. Mückenheim: "Matheology § 030", sci.logic (10 Jul 2012)]

Arguing over "scattered space"

"*As it is included into the irrational endpoints it cannot contain irrational numbers either.* The flaw is here." [fedja in "How can the following three principles be reconciled[1]?", MathOverflow & Math.StackExchange (29 Nov 2018)] This comment is merely a counterfactual claim.

"One thing you will observe is that some of your larger intervals contain many of the smaller intervals, so you have a redundant cover. Another thing to observe is this is the wrong forum for this question.[2]" [Gerhard Paseman, loc cit] This answer explains, but in the wrong direction, even decreasing the number of disconnected parts or atoms of the complement.

"You claim that (3) is violated[1] (disabled, to use your terminology) because of the statement from fedja. This is not the case. Try to provide a pair of different irrationals which violate that statement." [Thomas, loc cit] This is impossible *because the assumption*, namely $\aleph_0$ intervals confining the rational points, *is wrong*.

"The total length of intervals is finite and ℝ has infinite length so there are points in the complement and they are irrational numbers since all rational numbers have been covered." [Piotr Hajlasz, loc cit] This is not explaining anything but claiming a contradiction.

The most common and frequent answer, avoiding any further thought, says "as soon as you have an infinite number (even countable), weird things can (and do) happen." [Guillaume Brunerie in "Covering the Rationals – A Paradox?" MathOverflow (12 Feb 2012)]. But if the number of intervals can explode "in the infinite", the number of slots in Cantor-lists can explode too.

Another frequent answer "a point of the complement could have on each side an infinite *sequence* of removed intervals approaching it rather than having a directly adjacent interval" [Mike Terry in "Covering all rational points by irrational intervals", sci.logic, (24 Nov 2018)] claims one point per infinite clustersequence. That is a yield of $1/\infty$ whereas $\infty/0.1$ would be required. (Note that there are no limits x without some terms of the sequence converging to x and only to x.)

It has been claimed that in spite of symmetry between removed and remaining parts (see section "Symmetry considerations") $\aleph_0$ finite intervals let uncountably many atoms in the complement: "*Yes they do.*" [George Greene, loc cit (27 Dec 2018)]

And even completely impossible things have been claimed: "There are points in the complement C that are neither endpoints of intervals nor interior points of C." [Jürgen Rennenkampff in "Clusters and Cantor dust", sci.logic (5 Jan 2019)] So there are points in C that are not in C?

Finally the truth comes to light: "the question intended is *not* 'find me any sub-interval of ℝ that is adjacent to x'. It is 'find me one of the intervals in the given partition that is adjacent to x'." [Mike Terry loc cit (6 Jan 2019)] That is a clever move! Simply forget and drop what could raise disturbing conflicts, investigate only a given partition. But that's not mathematics – it's a fraud.

[1] For the three principles see section "Covering all rational points by irrational intervals".
[2] The "Research Community" of MathOverflow mercilessly deletes stuff that does not hit their level, in particular if this level is surpassed as here is obvious from the quoted answer.

The Kronecker-plane (I)

We are standing at the edge of the infinite Kronecker-plane. Nothing is living here, no trees, no shrubs, not even a blade of grass; only hot air is wafting over the dry desert landscape. Sometimes the tempest drives a ripped off shrub of wormwood or a torn plastic bag or an empty yoghurt cup across our path. In the far distance a huge mountain range rises discouragingly, the Kronecker-wall. If you want to cross the plane, you will unavoidably encounter the obstacle and you will have to climb over, because for every way j that you choose (see figure next page)

$$\sum_{k \in \mathbb{N}} \delta_{jk} = 1 \ .$$

This merciless law is the fundamental principle of Cantor's lifework, the *Grund-Satz* which he has concocted of his teacher's work: To run from south to north through the plane, i.e., to pass through all natural numbers, is impossible without crossing the Kronecker-wall.

Unless we go west as far as possible and choose the way at $j = 0$. Because for both, Kronecker and Cantor, zero was not a natural number, and therefore

$$\sum_{k \in \mathbb{N}} \delta_{0k} = 0 \ .$$

And yet there is another way for the knowledgeable scout. Far, far in the east, there were ω resides and the sun rises, the radiation of which however is scorching and burning the unprotected wayfarer, there is

$$\sum_{k \in \mathbb{N}} \delta_{\omega k} = 0$$

and the brave man can draw his course and run free. Alas this is an empty promise! Who will reach the edge?

But stop! There is even another way! The limit is taken always before ω, that means in the finite domain before the strip of ω. For every k there is a j that allows to avoid the wall – at least always during the next step. Before reaching $k = j$ traverse and switch the way to $j = k + 1$ (see figure next page). It allows you to remain in the plane.

So the Kronecker-wall does not cover the whole plane? Not at all! It covers hardly the smallest part! This is simply shown by the fact that there are infinitely many ways to circumvent it, for instance $j = k + 1$, $j = k + 2$, $j = k + 3$, ... and many, many more, if you apply fractions, products, logarithms, roots, or powers. By this technique you will not miss any natural number k and (not reach the north though, because it does not exist, but) advance as far as you like in northern direction.

[W. Mückenheim: "Das Kalenderblatt 101121", de.sci.mathematik (20 Nov 2010)]

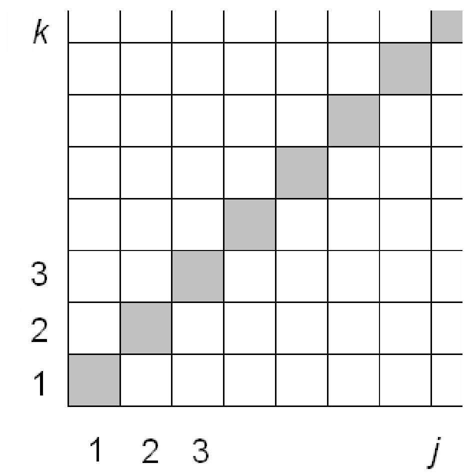
The Kronecker-plane (II)

The fundamental principle of Cantor's diagonal argument is this: Every digit sequence of the Cantor-list has a digonal digit. If we run through the ordered set of all natural numbers, then we hit each one exactly once. It is impossible to circumvent the diagonal because

$$\sum_{k=1}^{\infty} \delta_{jk} = 1 \ .$$

In order to scrutinize this principle we define the Kronecker-wall

$$\delta_{jk} = \begin{cases} 1 \text{ if } j = k \\ 0 \text{ else} \end{cases} .$$



For every natural number k we have

$$\lim_{j\to\infty} \delta_{jk} = 0$$

and we get the remarkable result

$$\lim_{j\to\infty} \sum_{k=1}^{\infty} \delta_{jk} = \lim_{j\to\infty} 1 = 1 \neq 0 = \sum_{k=1}^{\infty} 0 = \sum_{k=1}^{\infty} \lim_{j\to\infty} \delta_{jk}$$

because the limit of the sequence 1, 1, 1, ... is 1, and the infinite sum 0 + 0 + 0 + ... is 0.

It is impossible to circumvent the Kronecker-wall in the Kronecker-plane. It would be possible beyond the outmost right edge at $j = \infty$, alas ∞ is not a part of the plane.

This arguing, however, is false because contrary to sequences like $1/k$ every sequence (δ_{jk}) assumes its limit $\lim_{j\to\infty} \delta_{jk} = 0$ already for *finite argument* j, i. e., *inside* of the Kronecker-plane. That means it is possible to run through the ordered set of all natural numbers *within* the Kronecker-plane and *without* clashing with the Kronecker-wall. The way defined by $j = k + 1$, for instance, never collides with the Kronecker-wall but contains all natural numbers j and k. Obviously there are many such ways.

Thus, Cantor's *Grund-Satz* is invalid. Not every infinite way through the plane from bottom to top crosses the diagonal. But such a way is only possible if the set of natural numbers and with it the Kronecker-plane and its top do not exist as an actually finished infinity.

Kronecker certainly would have agreed.

[W. Mückenheim: "Die Geschichte des Unendlichen", 7th ed., Maro, Augsburg (2011) p. 116. W. Mückenheim: "Das Kalenderblatt 110414", de.sci.mathematik (13 Apr 2011)]

An analytical equivalent is this: Consider the sequence of functions $f_n(x)$ with $n = 1, 2, 3, ...$

$$f_n(x) = \begin{cases} 1 \text{ for } n-1 \leq x \leq n \\ 0 \text{ else} \end{cases}.$$

Then we obtain

$$\lim_{n\to\infty}\int_0^\infty f_n(x)\,\mathrm{d}x = \lim_{n\to\infty}\int_{n-1}^n \mathrm{d}x = \lim_{n\to\infty} 1 = 1 \neq 0 = \int_0^\infty 0\,\mathrm{d}x = \int_0^\infty \lim_{n\to\infty} f_n(x)\,\mathrm{d}x .$$

This difference between the limit of the integral and the integral over the limit is often cited as evidence that for sequences of sets the limit of the sequence of cardinal numbers need not be identical with the cardinal number of the set limit as has been discussed in section "Limit of sequences of cardinal numbers". This however is a misleading analogy. The limit of the sequence of sets should be one and the same set with compatible properties describing the elements and their cardinality. In the above formula however, we treat two completely different cases, namely the integral over a track within the plane, starting and arriving at a finite number *n*, and the track over the margin at ∞, i.e., outside of the plane, a track which in fact does not exist, since ∞ is not a constant.

My meadow saffron dream

I had a dream. I saw an infinite real line with its infinitely many unit intervals and in every unit interval there stood a purple meadow saffron, covering about one hundredth of the interval. And I thought that these flowers can be re-ordered such that they become dense, covering all rational numbers and squashing themselves. No uncovered part of the real line is any longer visible and between every two flowers there is another one.

Then I awoke and looked around me. And I realized, yes, I was only dreaming. And those who can believe in that dream in the waking state must be set theorists.

And it became as clear as never before to me: People want to be deceived, but Cantor, unwillingly of course, has accomplished the biggest fraud in history of science.

[W. Mückenheim: "I had a dream", sci.math (2 Sep 2017)]

On continuity-preserving manifolds

Cantor's essay on continuity-preserving manifolds [G. Cantor: "Über unendliche lineare Punktmannichfaltigkeiten" (3), Math. Ann. 20 (1882) pp. 113-121] contains a proof showing that the manifold $\mathbb{R}^n$ (with $n \geq 2$) remains continuous if the set of points with purely algebraic coordinates is taken off. According to Cantor's interpretation this is *a peculiar property of countable sets*. In fact, this property does not only hold for all countable sets but it is the same for many uncountable sets too. It becomes immediately clear, by simplifying the proof, that the continuity of $\mathbb{R}^n$ is also preserved, if the uncountable set of points with purely transcendental coordinates is removed.

In the following the expressions "manifold" and "continuum" denote the n-dimensional Euclidean space. They are used synonymously with the set of points, each of which is determined by a tuple of n coordinates, which the n-dimensional Euclidean space is isomorphic to.

By defining origin and axes of a coordinate system the points of a manifold are subdivided into three sets: the countable set $\mathbb{AA}$ of those points with purely algebraic numbers as coordinates, the uncountable set $\mathbb{TT}$ of those points with purely transcendental numbers as coordinates, and the uncountable set $\mathbb{AT}$ of the remaining points with mixed coordinates, i. e., with at least one algebraic number and at least one transcendental number serving as a coordinate. Of course these properties do not belong to a point itself because the type of coordinate system as well as its origin and its axes can be chosen in an arbitrary way. But once the system has been fixed, the bijective mapping of the points N of the continuum on the n-tuples $(x_1, x_2, ..., x_n)$ is fixed too:

$$N \leftrightarrow (x_1, x_2, ..., x_n) \in \mathbb{R}^n = \mathbb{AA} \cup \mathbb{TT} \cup \mathbb{AT} \,. \qquad (1)$$

A line or curve l connecting two points of $\mathbb{R}^n$ may contain infinitely many points of $\mathbb{AA}$. If the latter set is taken off, l is no longer continuous in the remaining manifold $(\mathbb{R}^n \setminus \mathbb{AA})$. But $\mathbb{R}^n$ (with $n \geq 2$) remains continuous even if the set of points with purely algebraic coordinates is taken off. This means between two of the remaining points, with not purely algebraic coordinates, which Cantor called N and N', one can always find a continuous linear connection of the same character, which Cantor called l'. In short

$$N, N' \in (\mathbb{R}^n \setminus \mathbb{AA}) \quad \Rightarrow \quad \exists l'(N, N'): N, N' \in l' \subset (\mathbb{R}^n \setminus \mathbb{AA}) \,. \qquad (2)$$

<u>Cantor's proof of the continuity of $(\mathbb{R}^n \setminus \mathbb{AA})$</u>

The set $\mathbb{AA}$ is countable. Hence, any interval of the uncountable set l contains points belonging to the uncountable set $(\mathbb{R}^n \setminus \mathbb{AA})$. We consider a finite set of them $\{N_1, N_2,..., N_k\}$. Between any pair of these points a part of a circle can be found which connects these points but contains no point of $\mathbb{AA}$. This is shown for two points, N and N_1, as follows: The centres of circles which on their circumference contain at least one point of $\mathbb{AA}$ form a countable set. The centres of circles containing on their circumference N and N_1 belong to a straight line (i.e., an uncountable set). This line contains at least one point which is centre of a circle containing on its circumference N

and N_1 but not any point of $\mathbb{AA}$. As this can be exemplified for any pair of the finite set of points $\{N, N_1, N_2, ..., N_k, N'\}$ the proof is complete (see Fig. (a) for the two-dimensional case).

<u>Simplified proof of the continuity of ($\mathbb{R}^n \setminus \mathbb{AA}$)</u>

This proof would work as well, if only the original pair of points, N and N', had been considered. But even straight lines can serve as connections. We apply Cartesian coordinates to show this. At least one coordinate, say x_ν, of the not purely algebraic point

$$N = (x_1, ..., x_{\nu-1}, x_\nu, x_{\nu+1}, ..., x_n) \tag{3}$$

is transcendental. Let this be constant while all the other x_μ (with $\mu = 1, ..., \nu-1, \nu+1, ..., n$) are continuously changed until they reach the values of the coordinates of N'

$$(x'_1, ..., x'_{\nu-1}, x_\nu, x'_{\nu+1}, ..., x'_n)\ . \tag{4}$$

The x_μ (with $\mu = 1, ..., \nu-1, \nu+1, ..., n$) define a hyper plane $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ within which we can choose an arbitrary way. If at least one of the final coordinates x'_μ is transcendental, we finish the proof by changing x_ν to x'_ν without leaving the set $\mathbb{TT} \cup \mathbb{AT}$ (see Fig. (b) for the two-dimensional case). If none of the final coordinates x'_μ is transcendental, we stop the process of continuously changing the x_μ for one of those coordinates, x_ρ, at the transcendental value x''_ρ before the final algebraic value x'_ρ of (4) is reached (or we re-adjust x''_ρ afterwards). Then, staying always in the set $\mathbb{TT} \cup \mathbb{AT}$, we let x_ν approach x'_ν, which in this case must be transcendental, and finally we complete the process by changing x_ρ from its intermediate transcendental value x''_ρ to its final value x'_ρ (see Fig. (d) for the two-dimensional case).

This method can also be applied to any pair of points of $\{N, N_1, N_2, ..., N_k, N'\}$ of l belonging to the set $\mathbb{TT} \cup \mathbb{AT}$. Although the complete length of the connection l' remains unchanged, the deviation of any of its points from the straight line l can be made as small as desired (see Fig. (c) for the two-dimensional case).

<u>Proof of the continuity of ($\mathbb{R}^n \setminus \mathbb{TT}$)</u>

Cantor considered the preserved continuity of ($\mathbb{R}^n \setminus \mathbb{AA}$) a peculiar property of countable sets (cp. section 4.2.2 "Physical space"). He obviously overlooked that taking off the uncountable set $\mathbb{TT}$ leads to a continuous manifold too, similar to that remaining after taking off $\mathbb{AA}$. This fact becomes immediately clear from the proof given in the preceding section but remains veiled in Cantor's original version.

Obviously any pair of points with at least one algebraic coordinate can be connected by a continuous linear subset of the same character

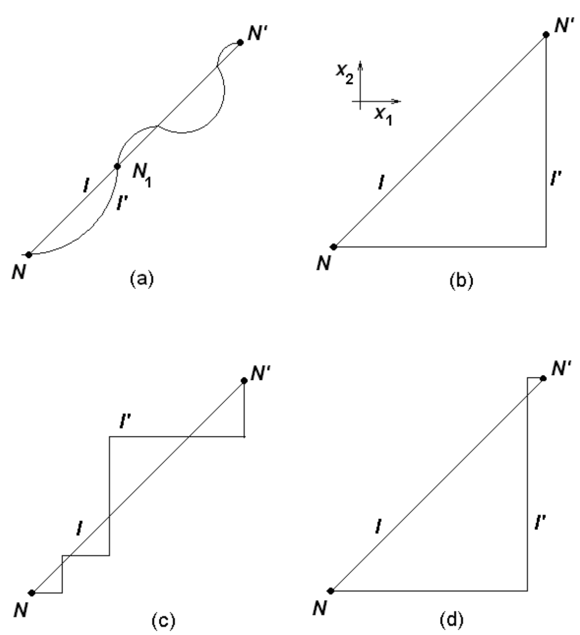
$$N, N' \in (\mathbb{R}^n \setminus \mathbb{TT}) \;\; \Rightarrow \;\; \exists l'(N, N'): N, N' \in l' \subset (\mathbb{R}^n \setminus \mathbb{TT})\ . \tag{5}$$

The proof runs precisely as demonstrated above, with the only difference that those coordinates which there and in the caption of the Figure are prescribed as transcendental, now have to be algebraic. We can even go further and take off all points with purely non-rational coordinates or even all points with purely non-natural coordinates, so that there remains at least one $x_\nu \in \mathbb{N}$ of point N and at least one $x'_\mu \in \mathbb{N}$ of point N'. It is obvious then that N and N' have a continuous connection as depicted in Fig. (b) or in Fig. (d) along the "grid lines". In fact there are infinitely many of these connections.

As an example we consider points $N = (n, \xi)$ and $N' = (n', \xi')$ with $n, n' \in \mathbb{N}$ and $\xi, \xi' \in \mathbb{R}$ in the two-dimensional Cartesian coordinate system $\mathbb{R}^2$. After taking off all points except those with at least one coordinate being a natural number, we have in the remaining manifold the connection by changing coordinates as described in (6). First, moving along the grid line $x_1 = n$, change ξ, the possibly non-natural x_2-coordinate of N, to an intermediate coordinate $x''_2 = m$, choosing any natural number m. Then change $x_1 = n$ to $x'_1 = n'$, moving along grid line $x''_2 = m$. Finally change $x''_2 = m$ to $x'_2 = \xi'$, moving along grid line $x'_1 = n'$, briefly

$$N = (x_1, x_2) = (n, \xi) \to (n, m) \to (n', m) \to (n', \xi') = (x'_1, x'_2) = N' . \qquad (6)$$

The connection does not contain any point with purely non-natural coordinates.



(a) Connection l' circumventing points of $\mathbb{A}\mathbb{A}$ between $N = (x_1, x_2)$ and $N' = (x'_1, x'_2)$ proposed by Cantor, **(b)** l' of the present proof in case x_2 and x'_1 are transcendental, **(c)** same as (b) in case a smaller deviation of l' from the straight line l is requested, **(d)** l' of the present proof in case only x_2 and x'_2 being transcendental.

[W. Mückenheim: "On Cantor's important proofs", arXiv (2003)]

Is it possible that mathematics is fundamentally wrong?

Mathematics *is* fundamentally wrong – at least what many modern mathematicians consider the fundament of mathematics: transfinite set theory. It is easy to prove (see below) that the set of rational numbers is not equinumerous with the set of natural numbers, because the so-called set-limit is required to show that no rational numbers remain without natural index. This set limit also "proves" that Scrooge McDuck can get bankrupt when he receives 10 $ per day and spends 1 $ per day.

Proof: For every *given* positive rational number a natural number can be *given* and vice versa. Usually it is claimed that this proves the eqinumerosity of the set ℕ of natural numbers and the set ℚ of positive rational numbers. But since the same can be claimed for the set ℕ of natural numbers and the set ℝ of positive real numbers too, this fact is not relevant.

The latter is not accepted as a proof of equinumerosity of ℕ and ℝ because there is a counter proof by the diagonal argument. In the former case of ℕ and ℚ however there is also a counter proof, namely from calculus.

Every fixed natural number n belongs to a finite initial segment $\{1, 2, 3, ..., n\}$ which is followed by infinitely many natural numbers $\{n+1, n+2, n+3, ...\}$, i.e., by nearly all natural numbers. Same holds for the rational numbers. In order to prove equinumerosity we have to show that these huge remainders are equinumerous. This however can be contradicted. In every interval $[0, n]$ on the real line there are infinitely many rational numbers not indexed by the first n natural numbers. Their number grows by an infinity with every step from n to $n+1$. Therefore we have a function $f(n)$ of positive rational numbers less than n which are not indexed by a natural number between 1 and n. According to calculus this function has the (improper) limit infinite.

Usually set theorists do not accept this argument because they blindly depend on the bijection between *given* numbers. This is insufficient because, as said above, the bijection between *given* numbers covers only a vanishing part of all numbers.

[W. Mückenheim in "Is it possible that mathematics is fundamentally wrong?", Quora (25 Dec 2016)]

Collected paradoxes

Strange things happen "*at* infinity". As we have seen transfinite set theory requires the belief in the following paradoxical results:

• There exist undefinable "real" numbers, i.e., undefinable subsets of ℕ. See sections "The list of everything" and "A common property of all models of ZFC".
• Alternatively every real number can be "defined" by an actually infinite sequence of digits, but without defining the last digit and the digit next to the last one and the digit next to the next and so on – in infinity. This leads to dark numbers. See section "Dark numbers in set theory".
• Fractions can become irrational "*at* infinity". See section "Sequences and limits".
• Well-ordering of objects that cannot be identified and distinguished is possible. See section "Choosing and well-ordering undefinable elements?".
• After ω unions in vain the union $\omega + 1$ reaches the aim ω or ℕ. See section "$\omega + 1$ unions".
• There are strictly increasing series that contain their limit. The sequence of all finite initial segments of ℕ does not contain ℕ, but when unioned it yields ℕ which is impossible if ℕ was not a term of the inclusion-monotonic sequence. See section "Contradicting inclusion monotony".
• Diverging sequences of sets can have empty limits. See sections "Dark numbers in set theory", "Scrooge McDuck", and "Disappearing sequences (I)".
• There are different irrational numbers without a rational number between them. See section "Covering all rational points by irrational intervals".

Some things seem to turn to the contrary when infinity is finished.

Endorsers – or the naïveté of contemporary set theorists

In section 2.13 we have learnt about the early, naïve approach of well-ordering reported by Hausdorff: Counting to ω and beyond. This method is unfeasible because of two reasons. Firstly, the method would supply a way to obtain a definable well-ordering of the real numbers which is known to be impossible, and secondly, the first ordinals are the natural numbers which cannot be exhausted in a step-by-step procedure without violating Peano's successor axiom.

Nevertheless there are some contemporary logicians who persist to endorse this method. In MathOverflow[1] Emil Jeřábek counterfactually claimed: "This does not violate any Peano axioms. It is a perfectly valid and commonly used construction. [...] Peano axioms are axioms of natural numbers. The sequence here is not indexed by natural numbers, but by ordinals, so Peano axioms are irrelevant." And Joel David Hamkins boasted: "I endorse this method."

Cast in the same mould, Cohen suggests "we repeat this process countably many times" (cp. section 3.4.3). And Hrbacek and Jech ponder, in order to increase the cardinality of a model, "if this procedure is iterated $\aleph_2$ times" (cp. section 2.18.3). How can $\aleph_0$ be surpassed in order to trick Hilbert's hotel that otherwise excludes an increase of cardinality?

[1] The remarks were comments to the question: [Wilhelm: "Endorsers of the method of well-ordering reported by Hausdorff?", MathOverflow (11 Mar 2018)] This question meanwhile has been deleted.

Opinions of scholars associated with David Hilbert

"{{Cantor's}} theory of *transfinite numbers*; this appears to me as the most admirable blossom of mathematical spirit and really one of the supreme achievements of purely intellectual human activity. [...] No one shall drive us from the paradise which Cantor has created for us. [...] Obviously it is only possible to reach these aims if we succeed in obtaining the complete elucidation about the essence of the infinite." [D. Hilbert: "Über das Unendliche", Mathematische Annalen 95 (1925) pp. 167 & 170]

Hermann Weyl who 1930 became Hilbert's successor at the university of Göttingen said: "The possible combinations of finitely many letters form a countable set, and since every determined real number must be definable by a finite number of words, there can exist only countably many real numbers – in contradiction to Cantor's classical theorem and its proof." [H. Weyl: "Das Kontinuum", Veit, Leipzig (1918) p. 18]

Paul Bernays, 1934-1939 Hilbert's co-author of "Grundlagen der Mathematik (Foundations of Mathematics)" said: "If we pursue the thought that each real number is defined by an arithmetical law, the idea of the totality of real numbers is no longer indispensable, and the axiom of choice is not at all evident." [P. Bernays: "On Platonism in mathematics" (1934) p. 5ff]

Wilhelm Ackermann, 1938 Hilbert's co-author of "Grundzüge der theoretischen Logik (Principles of mathematical logic)" said: "The reviewer however cannot follow the author when he speaks of the possibility of a more than countable set of primitive symbols since such a system of names cannot exist." [W. Ackermann: "Review of Leon Henkin: 'The completeness of the first-order functional calculus'", J. Symbolic Logic 15,1 (1950) p. 68]

Kurt Schütte, 1933 Hilbert's last doctoral student, said: "If we define the real numbers in a strictly formal system, where only finite derivations and fixed symbols are permitted, then these real numbers can certainly be enumerated because the formulas and derivations on the basis of their constructive definition are countable." [K. Schütte: "Beweistheorie", Springer (1960)]

Even David Hilbert himself gave a remarkable summary after carefully scrutinizing the infinite: "Finally we will return to our original topic and draw the conclusions of all our investigations about the infinite. On balance the result is this: The infinite is nowhere realized; it is neither present in nature nor admissible as the foundation of our rational thinking – a remarkable harmony of being and thinking." [D. Hilbert: "Über das Unendliche", Math. Annalen 95 (1925) p. 190]

Comments on papers by Bertrand Russell by P. Ehrlich

In his paper *Recent Work On The Principles of Mathematics*, which appeared in 1901, Bertrand Russell reported that the three central problems of traditional mathematical philosophy – the nature of the infinite, the nature of the infinitesimal, and the nature of the continuum – had all been "completely solved" [1901, p. 89]. Indeed, as Russell went on to add: "The solutions, for those acquainted with mathematics, are so clear as to leave no longer the slightest doubt or difficulty" [1901, p. 89]. According to Russell, the structure of the infinite and the continuum were completely revealed by Cantor and Dedekind, and the concept of an infinitesimal had been found to be incoherent and was "banish[ed] from mathematics" through the work of Weierstrass and others [1901, pp. 88, 90]. These themes were reiterated in Russell's often reprinted *Mathematics and the Metaphysician* [1918] and further developed in both editions of Russell's *The Principles of Mathematics* [1903; 1937], the works which perhaps more than any other helped to promulgate these ideas among historians and philosophers of mathematics. In the two editions of the latter work, however, the banishment of infinitesimals that Russell spoke of in 1901 was given an apparent theoretical urgency. No longer was it simply that "nobody could discover what the infinitely little might be" [1901, p. 90], but rather, according to Russell, the kinds of infinitesimals that had been of principal interest to mathematicians were shown to be either "mathematical fictions" whose existence would imply a contradiction [1903, p. 336; 1937, p. 336] or, outright "self-contradictory", as in the case of an infinitesimal line segment [1903, p. 368; 1937, p. 368]. In support of these contentions Russell could cite no less an authority than Georg Cantor, the founder of the theory of infinite sets. [...] But ... "The German logician Abraham Robinson, who invented what is known as *non-standard analysis*, thereby eventually conferred sense on the notion of an infinitesimal greater than 0 but less than any finite number." [A.W. Moore: "The infinite", Routledge, London (1990) p. 69]

B. Russell: "Recent work on the principles of mathematics", Int. Monthly 4 (1901) pp. 83-101.
B. Russell: "The principles of mathematics", Cambridge University Press (1903).
B. Russell: "Mathematics and the metaphysicians" in "Mysticism and logic", London (1918).
B. Russell: "Principles of mathematics", 2nd ed., Norton, New York (1937).

[P. Ehrlich: "The rise of non-Archimedean mathematics and the roots of a misconception I: The emergence of non-Archimedean systems of magnitudes", Arch. Hist. Exact Sci. 60 (2006) p. 1f]

Comments on a paper by Bertrand Russell

[B. Russell: "Mathematics and the metaphysicians" in "Mysticism and logic", Allen & Unwin, London (1918) pp. 57-74]

1) His {{Boole's}} book was in fact concerned with formal logic, and this is the same thing as mathematics. [p. 57]

"Kant already has taught [...] that mathematics includes a solid contents that is independent of logic and therefore cannot be substantiated by logic." [D. Hilbert: "Über das Unendliche", Math. Annalen 95 (1925)]

"Gradually the pendulum swung in the direction of pure logic and abstraction, actually so much that a dangerous separation between "pure" mathematics and essential realms of application emerged. [...] But it seems and it is to hope that this period of isolation has ended." [R. Courant, H. Robbins: "Was ist Mathematik?", Springer, Berlin (1962)]

"To create mathematics from pure logic has not yet been managed." [Gerhard Hessenberg: "Grundbegriffe der Mengenlehre", Vandenhoeck & Ruprecht, Göttingen (1906)]

2) Pure mathematics consists entirely of assertions to the effect that, if such and such proposition is true of *anything*, then such and such another proposition is true of that thing. [p. 57]

Of course every statement can be formulated as an implication – but need not. Pure mathematics consists of statement like "1 + 1 = 2", "1 < 2", "2 is a prime number", "$2m^2 = n^2$ has no solution in integers", "π and e are transcendental", "a point is what cannot be separated into parts", "the cube, one of five Platonic solids, has 8 vertices, 12 edges, and 6 sides" – without ifs and buts!

3) These rules of inference constitute the major part of the principle of formal logic. [p. 58]

True but irrelevant. In mathematics formal logic is merely applied as an auxiliary tool, in particular if the mental capacity of mathematicians otherwise would be stretched too far, and crutches are required to pass through difficult landscape.

4) Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true. [p. 58]

That may be true for Russell's vision of mathematics but not for real mathematics.

5) Now the fact is that, though there are indefinables and indemonstrables in every branch of applied mathematics, there are none in pure mathematics except such as belong to general logic. [p. 58]

In the following Russell praises transfinite set theory. Obviously he had not yet realized in 1917 what Cantor and König knew in 1906 already, namely the countability of the set of definitions. If the set of all real numbers was uncountable, as Russell believed, then almost all "real" numbers would lack any definition or demonstration.

6) For instance, nothing is plainer than that a whole always has more terms than a part, or that a number is increased by adding one to it. But these propositions are now known to be usually false. Most numbers are infinite. [p. 60]

So the laudable pure mathematics without indefinables and indemonstrables collapses into nothing.

7) He {{Leibniz}} was prevented from succeeding by respect for the authority of Aristotle, whom he could not believe guilty of definite, formal fallacies; but the subject which he desired to create now exists, in spite of the patronising contempt with which his schemes have been treated by all superior persons. [p. 61]

Patronizing contempt is the customary means to defend an out-dated credo.

8) The solutions, for those acquainted with mathematics, are so clear as to leave no longer the slightest doubt or difficulty. This achievement is probably the greatest of which our age has to boast. [p. 63]

Rather it is the greatest intellectual fraud of all times, as the present chapter shows.

9) The proofs favourable to infinity, on the other hand, involved no principle that had evil consequences. [p. 66]

How that? Russell himself later mentions Tristram Shandy [p. 69f]. And before that he explicitly says The fundamental numbers are not ordinal but are what is called *cardinal* [p. 67]. Nevertheless he does not become aware of the fact that in the Tristram Shandy case only the sequence of cardinal numbers has mathematical meaning? "This is an instance of the amazing power of desire in blinding even very able men to fallacies which would otherwise be obvious at once." [Bertrand Russell: "What I believe" in "Why I am not a Christian and other essays on religion and related subjects", Paul Edwards (ed.), Allen & Unwin, London (1957)]

10) There are exactly as many fractions as whole numbers. [p. 68]

In mathematics this is provably wrong, cp. "Not enumerating all positive rational numbers".

11) There is a greatest of all infinite numbers, which is the number of things altogether, of every sort and kind. It is obvious that there cannot be a greater number than this, because, if everything has been taken, there is nothing left to add. Cantor has a proof that there is no greatest number, and if this proof were valid, the contradictions of infinity would reappear in a sublimated form. But in this one point, the master has been guilty of a very subtle fallacy, which I hope to explain in some future work. [p. 68]

Of course a limit ordinal (like ω in case of $\mathbb{N}$) is always needed for actual or "finished" infinity. But Cantor defers potential infinity only from the domain of natural numbers to the domain of transfinite ordinals where he then calls it the absolute infinite (cp. section 1.2). All problems the potentially infinite is (unjustly) blamed for would reappear there. When writing this Russell must have been in a state with formative influence of rational thinking.

12) Note added in 1917: Cantor was not guilty of a fallacy in this point. [p. 68]

What a shame. Rational influence has ceased.

13) If Achilles were to overtake the tortoise, he would have been in more places than the tortoise, but we saw that he must, in any period, be in exactly as many places as the tortoise. [p. 69]

Achilles will reach and overtake the tortoise like the hour hand will reach 12 o'clock. But that has nothing to do with an infinite subdivision of the distance. For proof try to start at the other end (with an infinitely small step). There is no infinite sum of 1/2 + 1/4 + 1/8 + ..., there is only a limit, approached but never reached by any term of the sequence of partial sums.

14) This paradoxical but perfectly true proposition depends upon the fact that the number of days in all time is no greater than the number of years. [p. 70]

False claim. The factor is about 365 for every interval of time that can be verified. Results that cannot be verified belong to dirty mathematics, not to pure mathematics.

15) But nowadays the limit is defined quite differently, and the series which it limits may not approximate to it at all. This improvement is also due to Cantor, and it is one which has revolutionised mathematics. [p. 71]

Yes, it has transformed clear and consistent mathematics into self-contradictory matheology. In the present case: If all natural numbers are infinitely far distant from the limit ω:

What makes up this distance? How does this distance come about? What, besides unprovable matheological belief, does accomplish it?

16) But the unavoidable technicalities of this subject render it impossible to explain to any but professed mathematicians. [p. 71]

Strange. Euler recommended that one should check the own understanding by explaining the subject to a cobbler's apprentice.

17) In the best books there are no figures at all. [p. 72]

Well, that makes it clear why matheologians are unable to explain their theory to others, let alone to cobblers' apprentices. But with certainty we can say that those books are not the best ones. Rather those books should have been written better – or better should not have been written at all. Only poor thinkers dispense with most human means of expression and communication except the only language that they believe to be able to master.

18) But it is certain that his {{Euclid's}} propositions do not follow from the axioms which he enunciates. [...] he uses two circles which are assumed to intersect. But no explicit axiom assures us that they do so, and in some kinds of spaces they do not always intersect. It is quite doubtful whether our space belongs to one of these kinds or not. Thus Euclid fails entirely to prove his point in the very first proposition. [...] Under these circumstances, it is nothing less than a scandal that he should still be taught to boys in England. [p. 73]

That is hair-raising! When in the endpoints of a straight line of length l two circles with radius l are centred, then they intersect – in every plane that is possible in whatever kind of space of our universe. On the other hand, Russell praises Peano to the skies ("the science which Peano has perfected") and does not recognize that Peano's celebrated axioms of natural numbers don't even touch them (except 1 or 0) but define only sequences without repetitions starting at 1 (or 0) like (0,) 1, 4, 9, ... or (0,) 1, 1/2, 1/3, ... or (0,) x^0, x^1, x^2, x^3, ... (where x can even be put as 1, namely if we understand by "non-repeating" the shape of the symbol and not its numerical value. This possible interpretation is nowhere excluded by the Peano axioms because by "successors" the character of number is not sufficiently cleared.) Of course one can make the natural numbers of these sequences – but not without additional definitions or axioms. Besides their infinity the basic property of natural numbers is their constant distance. It is impossible to define the natural numbers without that notion, i.e., without first having the addition of 1. And no Peano axiom defines that.

19) A book should have either intelligebility or correctness; to combine the two is impossible, but to lack both is to be unworthy of such a place as Euclid has occupied in education. [p. 73]

Then first of all transfinite set theory has to be erased from all curricula.

20) The proof that all pure mathematics, including Geometry, is nothing but formal logic, is a fatal blow to the Kantian philosophy. [...] Kant's theory also has to be abandoned. The whole doctrine of *a priori* intuitions, by which Kant explained the possibility of pure mathematics, is wholly inapplicable to mathematics in its present form. [p. 74]

There is no such proof and there cannot be such a proof because the claim which has been plucked out of thin air is false. But it explains why Russell and Cantor were in such a good harmony. Cantor mentioned "just this abominable mummy, as Kant is" [G. Cantor, letter to B. Russell (19 Sep 1911)]. It is overdue to abandon what they have tried to sell as "mathematics in its present form".

Psychological aspects of set theory

Every scientific theory is more or less depending on human psychology. In case of set theory psychological and religious aspects are intimately intermingled and of special weight, as Cantor's approach clearly shows (cp. section 4.1 "Cantor on theology"). In the present section we will scrutinize these psychological aspects.

A psychological argument

> Cantor knew about the true Father of Christ, the true Author of Shakespeare's Writings, the true Jakob Böhme, and the true Infinity. Nowadays his disciples endorse only one of his four findings.

More than 1000 students of Engineering, Informatics, and Design have attended my lecture series on the History of the Infinite starting in 2003 and presented until today. Nearly everyone could accurately answer examination questions like:

• Why is an enumeration of "all fractions" impossible?
• Why is Cantor's diagonal argument mistaken?
• Show by the game "We conquer the Binary Tree" that the set of infinite paths in the Binary Tree is not uncountable.

What makes the difference to mathematicians the overwhelming majority of whom are of different opinion? Have my students failed to understand transfinite set theory? Are they less intelligent than average mathematicians? Of course not. Those mathematicians who believe that it requires a lot of intelligence to comprehend Cantor's simple ideas cannot be very bright. As Cantor already stated himself (cp. section 4.3 "Cantor on the ease of his theory") it does not require mathematical training to understand the basics of his theory. The enumeration of a countable set and the exclusion of the antidiagonal number from the "list" are really not difficult to understand. The only difference that I can figure out is this: While my students learn these things in lesson 11 only few days before they learn the contradictions in lesson 12 for mathematicians this time interval extends over years or decades, often filled with much time and effort invested into set theory. This erects a psychological block which is hard to remove.

It is clear, for instance, that the complete infinite Binary Tree cannot be diagonalized (since it already contains every infinite bit sequence). Nevertheless this is not a problem for set theorists. They simply claim it could be done (cp. the discussion to section "Colour the Binary Tree", in particular the statement by A. Blass) or they refuse to think at all about that case.

So they will remain believing or pretending to "understand" that infinity somehow can be finished, that they can simply count beyond the infinite, that irrational numbers can be represented by (decimal) fractions, that translation invariance is violated with finite strings, that set limits are somehow more convincing than cardinality limits of analysis, that set theory requires a lot of brightness, and that only they have enough. There appears to be no remedy.

Matheology

> A matheologian is a man or, in rare cases, a woman who believes in thoughts that nobody can think, except, perhaps, a God or, in rare cases, a Goddess.

I have often been asked why I have coined the word matheology and in what sense I use it. Here is my answer: Humans consist of molecules. Theologians recognize more, namely something that exists beyond and independently of the molecules and that persists even when the body has dispersed, namely what they call the soul. Set theorists argue in a similar way: They believe in "real numbers" that cannot be addressed in mathematical discourse in analogy to souls that cannot be analyzed using physical instruments. But these "real numbers" have properties. For instance they can be well-ordered, that means they must have features to distinguish them although no human is able to do so. Further, according to the actual infinity of set theory, the infinite Binary Tree with its countably many nodes and edges can be constructed such that all nodes and edges are there and none is missing. These components I would like to call allegorically the "molecules of the tree". Set theorists recognize more, namely uncountably many paths of infinite length. The molecules of the tree do not confirm this. For every infinite path P, there is a node where it deviates from a fixed path P_0. But there is no node by which P_0 deviates from all other paths P. So nodes (at finite levels – and others do not exist) do never individualize a path (like digits do never individualize a real number). On the other hand, the antidiagonal number of the Cantor-list is defined by its digits only. If the paths of the Binary Tree are not individualized by their nodes, then also the antidiagonal number of the Cantor-list is not individualized and cannot be recognized as distinct from all entries of the list.

Another facet of matheology is the belief that different ways of representing one and the same thing generate different things. A simple example is the sequence of partial sums when written in a condensed form like 3/1, 31/10, 314/100, ... = 3, 3.1, 3.14, ... = 3.14... Here we have changed nothing mathematically but only introduced an abbreviation and yet another abbreviation. But matheology makes an "irrational number" from these sequences of fractions.

The allusion to theology also suggests itself because the "souls" of matheology go back to the work of a very religious strict antidarwinist. The punch line is however that Cantor never used these "souls". His original diagonal argument, the cause of all matheological delusions, used exclusively "molecules", namely strings of bits (indicated by the letters *m* and *w*) existing at finite places without any limits defined at all. And Cantor determinedly denied the existence of undefinable numbers or other undefinable objects of mathematics; cp. the letter from Cantor to Hilbert of 8 August 1906, in chapter II.

By the way, there is another striking analogy between theology and set theory: In early religious belief God resided in a distance infinitely far from men. Modern theology however teaches that He is always around in zero distance, invisible though. According to Cantor ω is infinitely far from all natural numbers n: "$\omega - n$ is always equal to ω" [Cantor, p. 395]. But according to modern set theory ω is in zero distance from all n; after von Neumann ω *is* $\mathbb{N}$. "The distance from n to $\mathbb{N}$ is neither $\aleph_0$ nor ω [...]. It is 0." [F. Jeffries in "Comments on 20 sentences from Bertrand Russell's 'Mathematics and the metaphysicians'", sci.math (8 May 2016)]

More can be found in a series of 555 essays on transfinite set theory that appeared under the title "Matheology" from 9 May 2012 to 5 December 2014 in sci.math.

Parallel evolution in theology and set theory

In the beginning there were many Gods living on Mount Olympus. Later it turned out that there is no God on top of this mountain. So Gods retired into the heaven above the sky. Meanwhile cosmonauts and astronauts reported them missing there too. So they retired to invisibility in deep space or into another dimension.

In the beginning Cantor proclaimed uncountably many *real* numbers. Later it turned out that so many numbers cannot exist in a mathematics where everything has to be finitely defined. So the real numbers had to retire to infinite digit strings. Meanwhile we know from the Binary Tree that at most countably many continuous paths starting from the root node can be distinguished. Now the real numbers have retired into indistinguishability, where they exist even well-ordered.

That's why this kind of "mathematics" should be called matheology and become divorced and separated and put in quarantine from any kind of serious science.

Déjà vu

If there were a scientific proof of Copernicanism, Bellarmine conceded in his letter, then the passages in Scripture should be reconsidered, since "we should rather have to say that we do not understand them than to say something is false which had been proven". But since no such proof "has been shown to me", he continued, one must stick to the manifest meaning of Scripture and the "common agreement of the holy fathers". All of these agreed that the sun revolves around the Earth. [Amir Alexander: "Infinitesimal", Oneworld, London (2015) p. 84]

Why does this paragraph remind me strongly of modern mathematics?

Modern mathematics as religion (by Norman J. Wildberger)

Modern mathematics *doesn't make complete sense*. The unfortunate consequences include difficulty in deciding what to teach and how to teach it, many papers that are logically flawed, the challenge of recruiting young people to the subject, and an unfortunate teetering on the brink of irrelevance.

If mathematics made complete sense it would be a lot easier to teach, and a lot easier to learn. Using flawed and ambiguous concepts, hiding confusions and circular reasoning, pulling theorems out of thin air to be justified 'later' (i.e. never) and relying on appeals to authority don't help young people, they make things more difficult for them.

If mathematics made complete sense there would be higher standards of rigour, with fewer but better books and papers published. That might make it easier for ordinary researchers to be confident of a small but meaningful contribution. If mathematics made complete sense then the physicists wouldn't have to thrash around quite so wildly for the right mathematical theories for quantum field theory and string theory. Mathematics that makes complete sense tends to parallel the real world and be highly relevant to it, while mathematics that doesn't make complete sense rarely ever hits the nail right on the head, although it can still be very useful.

So where exactly are the logical problems? The troubles stem from the *consistent refusal by the Academy to get serious about the foundational aspects of the subject*, and are augmented by the twentieth centuries' *whole hearted and largely uncritical embrace of Set Theory*.

Most of the problems with the foundational aspects arise from mathematicians' erroneous belief that they properly understand the content of public school and high school mathematics, and that further clarification and codification is largely unnecessary. Most (but not all) of the difficulties of Set Theory arise from the insistence that there exist 'infinite sets', and that it is the job of mathematics to study them and use them.

In perpetuating these notions, modern mathematics takes on many of the aspects of a *religion*. It has its essential creed – namely Set Theory, and its unquestioned assumptions, namely that mathematics is based on 'Axioms', in particular the Zermelo-Fraenkel 'Axioms of Set Theory'. It has its anointed priesthood, the *logicians*, who specialize in studying the *foundations of mathematics*, a supposedly deep and difficult subject that requires years of devotion to master. Other mathematicians learn to invoke the official mantras when questioned by outsiders, but have only a hazy view about how the elementary aspects of the subject hang together logically.

Training of the young is like that in secret societies – immersion in the cult involves intensive undergraduate memorization of the standard thoughts before they are properly understood, so that comprehension often follows belief instead of the other (more healthy) way around. A long and often painful graduate school apprenticeship keeps the cadet busy jumping through the many required hoops, discourages critical thought about the foundations of the subject, but then gradually yields to the gentle acceptance and support of the brotherhood. The ever-present demons of inadequacy, failure and banishment are however never far from view, ensuring that most stay on the well-trodden path.

The large international conferences let the fellowship gather together and congratulate themselves on the uniformity and sanity of their world view, though to the rare outsider that sneaks into such events the proceedings no doubt seem characterized by jargon, mutual incomprehensibility and irrelevance to the outside world. The official doctrine is that all views and opinions are valued if they contain truth, and that ultimately only elegance and utility decide what gets studied. The reality is less ennobling – the usual hierarchical structures reward allegiance, conformity and technical mastery of the doctrines, elevate the interests of the powerful, and discourage dissent.

There is no evil intent or ugly conspiracy here – the practice is held in place by a mixture of well-meaning effort, inertia and self-interest. We humans have a fondness for believing what those around us do, and a willingness to mold our intellectual constructs to support those hypotheses which justify our habits and make us feel good.

[Norman J. Wildberger: "Set theory: Should you believe?" (2005)]

Opinion 68 (by Doron Zeilberger)

Herren Geheimrat Hilbert und Prof. Dr. Cantor, I'd like to be Excused from your "Paradise": It is a Paradise of Fools, and besides feels more like Hell

David Hilbert famously said:

"No one shall expel us from the paradise that Cantor has created for us."

Don't worry, dear David and dear Georg, I am not trying to kick you out. But, it won't be quite as much fun, since you won't have the pleasure of my company. *I* am leaving on my own volition.

For many years I was sitting on the fence. I knew that it was a paradise of fools, but so what? We humans are silly creatures, and it does not harm anyone if we make believe that $\aleph_0$, $\aleph_1$, etc. have independent existence. Granted, some of the greatest minds, like Gödel, were fanatical platonists and believed that infinite sets existed independently of us. But if one uses the name-dropping rhetorics, then one would have to accept the veracity of Astrology and Alchemy, on the grounds that Newton and Kepler endorsed them. An equally great set theorist, Paul Cohen, knew that it was *only a game* with axioms. In other words, Cohen is a sincere formalist, while Hilbert was just using formalism as a rhetoric sword against intuitionism, and deep in his heart he genuinely believed that Paradise was real.

My mind was made up about a month ago, during a wonderful talk (in the INTEGERS 2005 conference in honor of Ron Graham's 70th birthday) by MIT (undergrad!) Jacob Fox (whom I am sure you would have a chance to hear about in years to come), that meta-proved that the answer to an extremely concrete question about coloring the points in the plane, has two completely different answers (I think it was 3 and 4) depending on the axiom system for Set Theory one uses. What is the *right* answer?, 3 or 4? Neither, of course! The question was meaningless to begin with, since it talked about the *infinite* plane, and infinite is just as fictional (in fact, much more so) than white unicorns. Many times, it works out, and one gets seemingly reasonable answers, but Jacob Fox's example shows that these are flukes.

It is true that the Hilbert-Cantor Paradise was a *practical* necessity for many years, since humans did not have computers to help them, hence lots of combinatorics was out of reach, and so they had to cheat and use abstract nonsense, that Paul Gordan rightly criticized as *theology*. But, hooray!, now we have computers and combinatorics has advanced so much. There are lots of challenging *finitary* problems that are just as much fun (and to my eyes, much more fun!) to keep us busy.

Now, don't worry all you infinitarians out there! You are welcome to stay in your Paradise of fools. Also, lots of what you do is *interesting*, because if you cut-the-semantics-nonsense, then you have beautiful combinatorial structures, like John Conway's surreal numbers that can "handle" "infinite" ordinals (and much more beyond). But as Conway showed so well (literally!) it is "only" a (finite!) game.

While you are welcome to stay in your Cantorian Paradise, you may want to consider switching to my kind of Paradise, that of finite combinatorics. No offense, but most of the infinitarian lore is sooo boring and the Bourbakian abstract nonsense leaves you with such a bitter taste that it feels more like Hell.

But, if you decide to stick with Cantor and Hilbert, I will still talk to you. After all, eating meat is even more ridiculous than believing in the (actual) infinity, yet I still talk to carnivores, (and even am married to one).

[Doron Zeilberger: "Opinion 68" (23 Nov 2005)]

Cantor's Theory: Mathematical creationism (by David Petry)

Cantor's theory (classical set theory) has the same relationship to the mathematical sciences as Creationism theory has to the physical sciences. They are similar in content and similar in origin. Cantor's theory is essentially a creation myth.

Both Cantor's theory and Creationism theory are founded on the proposition that we must acknowledge the existence of some abstract infinite entity lying beyond what we can observe in order to understand the reality that we do observe.

Furthermore, both have religious origins, and both try to hide their religious origins. Creationism comes from ancient Jewish religious teachings about the origin of the universe, and Cantor's theory of the infinite has its origins in Medieval Jewish religious/mystical teachings known as Kabbalah, wherein the world of the infinite is a higher level of existence.

Both Cantor's theory and Creationism theory are pseudoscience. Both the Creationists and the Cantorians impose upon their disciples a world view in which people must modify their thinking to incorporate certain axioms handed down from higher authority, and they are then compelled to accept any "logical" conclusions derived from those axioms. Anyone who dares to suggest that those axioms and the conclusions derived from those axioms don't pass reality checks, is demonized as an idiot, imbecile, crackpot, heretic, or some other kind of subhuman, and excluded from the community.

Both theories do interfere with scientific, technological and social progress.

A new world view, and a new paradigm for mathematics, have emerged from the computer revolution. This new world view strips away the mysticism from our *understanding and theories* of the mind ...

We now think of the brain as a computer, and the mind as the software running on the computer. Mathematics is a tool invented by the mind to help it understand the world in a precise, quantitative way. The brain and the mind (and mathematics) have co-evolved, and this evolution can be explained without recourse to abstract entities lying outside the world we observe.

Furthermore, due to the existence of computers which are clearly distinct from the human brain, we are forced to admit that there is something about the virtual world that has an objective existence. From a mathematical perspective, we can think of the computer as a microscope which helps us peer into a world of computation, and then mathematics itself is the science which studies the phenomena observed in that (virtual) world. The world of computation can be accepted as a given, just as the physical world is accepted as a given in the physical sciences. The fundamental objects living in the world of computation are data structures and algorithms, and a foundation for mathematics can be built on those objects. We study mathematics because the phenomena observed in the world of computation can serve as a model for phenomena observed in the physical world.

For those who accept this new world view, it is quite absurd to think that the mind, which lives in the world of computation, can "prove" the existence of a super-infinite world which has no connection to the phenomena observed in the world of computation. The explanation for Cantor's theory lies in the ability of the mind to delude itself.

Footnote 1: Everyone in the United States knows what Creationism is, but perhaps other people don't. The Creationists take the biblical creation myth as literal scientific truth, and they want the public schools to teach this theory as an alternative to evolution.

Footnote 2: One interesting difference between the Cantorians and the Creationists is a political difference. The Creationists have strong connections to Christian/conservative politics, and the Cantorians have connections to Humanistic/liberal politics.

Footnote 3: Debunking pseudoscience is a noble endeavor.

[David Petry: "Cantor's theory: Mathematical creationism", sci.math (22 Nov 2004)]

VII MatheRealism

MatheRealism[1] as a philosophical foundation of mathematics is based on the alliterative Materialism and the undisputable fact that mathematics is not independent of physical constraints of reality. Mathematics as monologue, dialogue, and discourse needs tools for expressing and communicating ideas. MatheRealism denies the existence of entities which, in principle, can never be observed or communicated – in particular thoughts that no-one can think. MatheRealism distinguishes between *numbers* which can be determined exactly and *ideas* which can not. MatheRealism leads to the *elimination of any actual infinity* from mathematics. [W. Mückenheim: "MatheRealism", PlanetMath.Org (3 May 2007)]

Mathematics and reality

"How is it possible that mathematics, which is a product of human thinking independent of all experience, fits reality in such an excellent way?" [A. Einstein: "Geometrie und Erfahrung", Festvortrag, Berlin (1921), reprinted in A. Einstein: "Mein Weltbild", C. Seelig (ed.), Ullstein, Frankfurt (1966) p. 119]

Without mental images from sensory impressions and experience thinking is impossible. Without reality (which includes the apparatus required for thinking as well as the objects of thinking – we never think of an abstractum "number 3" but always of three things or the written 3 or the spoken word or any materialization which could have supplied the abstraction) mathematics could not have evolved like a universe could not have evolved without energy and mass. Therefore real mathematics agrees with reality in the excellent way it does.

Einstein answers his question in a relativizing way: "In so far the theorems of mathematics concern reality they are not certain, and in so far as they are certain they do not concern reality." [A. Einstein: "Geometrie und Erfahrung", Festvortrag, Berlin (1921), reprinted in A. Einstein: "Mein Weltbild", C. Seelig (ed.), Ullstein, Frankfurt (1966) p. 119f]

He states a contraposition $(R \Rightarrow \neg C) \Leftrightarrow (C \Rightarrow \neg R)$. Both statements are equivalent. Both statements are false. To contradict them a counterexample is sufficient. A theorem of mathematics is the law of commutation of addition of natural numbers $a + b = b + a$. It can be proven in every case in the reality of a wallet with two pockets.

[1] Please do not mix up MatheRealism with so-called Realism in the current philosophy of mathematics which, in fact, is merely an idealism without any roots in reality.

Kolmogorov complexity

> "Ultrafinitists don't believe that really large natural numbers exist. The hard part is getting them to name the first one that doesn't." [John Baez: "The inconsistency of arithmetic", The n-Category Café (30 Sep 2011)] "The problem is not the *size* of the number but its information contents. On a simple pocket calculator, you can multiply 10^{30} by 10^{50}, but you cannot add or multiply two numbers with more than 10 digits there. In real life, you can do superexponentiation but you cannot use a sequence of more than 10^{100} digits that lack a finite expansion rule like 0.101010… or $\Sigma 1/n^2$." [W. Mückenheim, loc cit]

"In algorithmic information theory, the notion of Kolmogorov complexity is named after the famous mathematician Andrey Kolmogorov even though it was independently discovered and published by Ray Solomonoff a year before Kolmogorov. Li and Vitanyi, in 'An Introduction to Kolmogorov Complexity and Its Applications' (p. 84), write: Ray Solomonoff [...] introduced [what is now known as] 'Kolmogorov complexity' in a long journal paper in 1964. [...] This makes Solomonoff the first inventor and raises the question whether we should talk about Solomonoff complexity." ["Matthew effect", Wikipedia]

A string x of bits with $|x| = n$ bit is incompressible, if no string p of bits with less than n bits exists, which defines or generates the string x (for instance via a computer program).

"The idea is that a string is random if it cannot be compressed. That is, if it has no short description. Using {{Kolmogorov complexity}} $C(x)$ we can formalize this idea via the following.

Theorem 1.2. For all n, there exists some x with $|x| = n$ such that $C(x) \geq n$. Such x are called (Kolmogorov) *random*.

Proof. Suppose not. Then for all x, $C(x) < n$. Thus for all x there exists some p_x such that $g(p_x) = x$ and $|p_x| < n$. Obviously, if $x \neq y$ then $p_x \neq p_y$.

But there are $2^n - 1$ programs of length less than n, and 2^n strings of length n. {{Compare the finite paths up to level $n - 1$ in the Binary Tree and the paths wit n nodes, i.e., those with one nth node beyond the level $n - 1$}}. By the pigeonhole principle, if all strings of length n have a program shorter than n, then there must be some program that produces two different strings. Clearly this is absurd, so it must be the case that at least one string of length n has a program of length at least n." [Lance Fortnow: "Kolmogorov complexity" (2000)]

In order to simplify this notion (since it has turned out to confuse some readers) let us introduce the *Text Complexity* (TC) which is the measure for the least number of symbols by which a notion like a real number can be expressed on a common keyboard. "One hundred" for example has TC 3 since "100" or "10^2" need three symbols. Most natural numbers will be expressed best by their digits. 123123123123123123123123123123123123123123123123123123123123, here expressed with 60 symbols, however has TC = 27, which is obtained from the 27 symbols of "20_strings_123_concatenated". Of course these complexities depend on the used language, in particular on the basic equipment of words. For instance we could introduce an abbreviation for "20" and for "string" and for "concatenated" or use Latin C for 100. But in order to understand and to apply the corresponding argument (cp. section "An upper bound for cardinal numbers") a rough measure is sufficient. For further discussion of this topic see [W. Mückenheim in "Infinities and infinitesimals", sci.math.research (12 Jun 2007)].

Kronecker confirmed

"Mathematics cannot be outwitted, but mathematicians have been outwitted to an alarming degree. As the often praised book by Bolzano proves, this deception has been accomplished by the roughest means. Bolzano wants to prove that a function of *x* which is positive in a point of the domain of continuity and negative in another, necessarily is zero in between. And always it is concluded in this way: Either there exists a value at which the function is zero or there is none. Bolzano's only shrewdness is that he continues not by the argument but on the curve. But for that sake one should know, from a given point, *how long* the function is still positive, how long it is still negative. But considering functions like $\Sigma^{\infty}_{n=1} \sin(n^2x)/n^2$ we can see immediately that we cannot see it. Bolzano's explanations cannot even be applied to square roots of whole functions. And that is the best proof of their falsehood." ["Über den Begriff der Zahl in der Mathematik", Public lecture in summer semester 1891 – Kronecker's last lecture. "Sur le concept de nombre en mathematique" Cours inédit de Leopold Kronecker à Berlin (1891) Retranscrit et commenté par Jacqueline Boniface et Norbert Schappacher: Revue d'histoire des mathématiques 7 (2001) p. 269f]

In fact in MatheRealism Bolzano's proof is invalid. A zero with Kolmogorov complexity of more than 10^{100} bit cannot exist.

Does the infinitely small exist in reality?

Quarks are the smallest elementary particles presently known. Down to 10^{-19} m there is no structure detectable. Many physicists including the late W. Heisenberg are convinced that there is no deeper structure of matter. On the other hand, the experience with the step-wise recognition of molecules, atoms, and elementary particles suggests that these physicists may be in error and that matter may be further divisible. However, it is not divisible in infinity. There is a clear-cut limit.

Lengths which are too small to be handled by material meter sticks can be measured in terms of wavelengths λ of electromagnetic waves with

$$\lambda = c/\nu \qquad (c = 3\cdot 10^8 \text{ m/s}) .$$

The frequency ν is given by the energy *E* of the photon

$$\nu = E/h \qquad (h = 6{,}6\cdot 10^{-34} \text{ Js}) .$$

A photon cannot contain more than all the energy $E = mc^2$ of the universe which has a mass of about $m = 5\cdot 10^{55}$ g (including the dark matter). This yields the complete energy $E = 5\cdot 10^{69}$ J. So the absolute minimum of distance is $4\cdot 10^{-95}$ m.

[W. Mückenheim: "The infinite in sciences and arts", Proceedings of the 2nd International Symposium of Mathematics and its Connections to the Arts and Sciences (MACAS 2), B. Sriraman, C. Michelsen, A. Beckmann, V. Freiman (eds.), Centre for Science and Mathematics Education, University of Southern Denmark, Odense (2008) pp. 265-272 & arXiv]

Does the infinitely large exist in reality?

Modern cosmology teaches us that the universe has a beginning and is finite. But even if we do not trust in this wisdom, we know that relativity theory is as correct as human knowledge can be. According to relativity theory, the accessible part of the universe is a sphere of $R = 50\cdot10^9$ light years radius containing a volume of 10^{80} m^3. (This sphere is growing with time but will remain finite forever.) "Warp" propulsion, "worm hole" traffic, and other science fiction (and scientific fiction) does not work without time reversal. Therefore it will remain impossible to leave (and to know more than) this finite sphere. Modern quantum mechanics has taught us that entities which are not measurable in principle, do not exist. Therefore, also an upper bound (which is certainly not the supremum) of 10^{365} for the number of elementary spatial cells in the universe can be calculated from the minimal length of 10^{-95} m estimated above.

A similar estimation has been performed by Krauss and Starkman: "The physical limits to computation have been under active scrutiny over the past decade or two, as theoretical investigations of the possible impact of quantum mechanical processes on computing have begun to make contact with realizable experimental configurations. We demonstrate here that the observed acceleration of the Universe can produce a universal limit on the total amount of information that can be stored and processed in the future, putting an ultimate limit on future technology for any civilization, including a time-limit on Moore's Law {{chip performance will double every 1 or 2 years}}. The limits we derive are stringent, and include the possibilities that the computing performed is either distributed or local. A careful consideration of the effect of horizons on information processing is necessary for this analysis, which suggests that the total amount of information that can be processed by any observer is significantly less than the Hawking-Bekenstein entropy {{$k\cdot\pi(R/L_{\text{Planck}})^2$}} associated with the existence of an event horizon in an accelerating universe. [...] We first consider the total amount of energy that one can harvest centrally. [...] one finds $E_{\text{max}} \approx 3.5\cdot10^{67}$ J, comparable to the total rest-mass energy of baryonic matter within today's horizon. This total accessible energy puts a limit on the maximum amount of information that can be registered and processed at the origin in the entire future history of the Universe. [...] Dividing the total energy by this value yields a limit on the number of bits that can be processed at the origin for the future of the Universe: Information Processed [...] = $1.35\cdot10^{120}$. [...] It is remarkable that the effective future computational capacity for any computer in our Universe is finite, although, given the existence of a global event horizon, it is not surprising. Note that if the equation of state parameter w for dark energy is less than -1, implying that the rate of acceleration of the Universe increases with time, then similar although much more stringent bounds on the future computational capacity of the universe can be derived. In this latter case, distributed computing is more efficient than local computing (by a factor as large as 10^{10} for $w = -1.2$, for example), because the Hawking-Bekenstein temperature increases with time, and thus one gains by performing computations earlier in time. [...] On a more concrete level, perhaps, our limit gives a physical constraint on the length of time over which Moore's Law can continue to operate. [...] Our estimate for the total information processing capability of any system in our Universe implies an ultimate limit on the processing capability of any system in the future, independent of its physical manifestation and implies that Moore's Law cannot continue unabated for more than 600 years for any technological civilization." [Lawrence M. Krauss, Glenn D. Starkman: "Universal limits on computation", arXiv (2004)]

Therefore it is not only theoretically wrong that a process can always be completed when every single step can, but it is already practically impossible to perform a step the identification of which requires more than 10^{120} bits.

"The *entanglement phenomenon* involving light (or electrons), discovered by Albert Einstein, 1935, John Stewart Bell, 1964, and Alain Aspect, 1982, demonstrates that the *Continuum of light* is interiorly somehow "*tightly interlaced*", so that the distance, even enormous, between its "*entangled*" points becomes unimportant; this signifies that the Continuum is not a set, a "bag of points", but that the points on it appear as the consequence of *our activities*. [...] The Continuum C (or the field ℝ) appears as a *numerical approximation* to a complex reality of observations. – It is not *a set* in the original sense of traditional set theory. In particular, the *power set axiom* cannot be applied to it." [Edouard Belaga: "From traditional set theory – that of Cantor, Hilbert, Gödel, Cohen – to its necessary quantum extension", Institut des Hautes Études Scientifiques (Jun 2011) pp. 28 & 30]

How long lasts eternity?

We know that the universe has been expanding for about $14 \cdot 10^9$ years. This process, depending on the mass density of the universe, will probably continue in eternity. Strictly speaking the universe will remain expanding without bound if its over-all total energy

$$E = T + V$$

is positive. For a rough estimation consider a small mass element m on the surface of a homogeneous sphere of radius R and mass density ρ that contains the whole mass M of the universe. The potential energy is given by

$$V = -G\frac{mM}{R} = -Gm\rho\frac{4}{3}\pi R^2$$

where G is the constant of gravitation. As the velocity v of the mass element we put the product of radius R and Hubble constant H so that the kinetic energy is given by

$$T = \frac{m}{2}v^2 = \frac{m}{2}(RH)^2 .$$

The total energy has the sign of

$$E = mR^2\left(\frac{H^2}{2} - \frac{4\pi G\rho}{3}\right)$$

being independent of m and R. The universe will expand forever if

$$\rho < \rho_c = \frac{3H^2}{8\pi G} \ .$$

The critical density $\rho_c = 5 \cdot 10^{-27}$ kg/m^3 is about the mass of three hydrogen atoms per cubic meter.

According to newer astronomic results the over-all energy density is very close to zero, suggesting a Euclidean space, an expansion with asymptotic velocity zero but without bound. Eternity, however, will never be completed. So time like space has a potentially infinite character. Both are of unbounded size but always finite.

Will intelligent creatures survive in eternity? One constraint is the limited supply of free energy which is necessary for any form of life. This problem could be solved however, according to Dyson [F.J. Dyson: "Time without end: Physics and biology in an open universe", Rev. Mod. Phys. 51 (1979) pp. 447-460], by living for shorter and shorter intervals interrupted by long phases of hibernation. By means of series like $1/2 + 1/4 + 1/8 + ... \to 1$, a limited amount of energy could then last forever – and with it intelligent life.

Alas, there is the risk of sudden death of a creature by an accident. If we assume that in our civilisation one out of 200 lives ends in an unnatural way, then we can calculate that the risk to die by external cause during this very minute is at least 10^{-10} because there are roughly

$$200 \cdot 80 \text{ a} \cdot 365 \text{ d/a} \cdot 24 \text{ h/d} \cdot 60 \text{ min/h} = 8.4 \text{ billion minutes}$$

in 200 lives of 80 years average duration, one of which ends by external cause. The risk that 7,000,000,000 people will die during this very minute is then $10^{-70,000,000,000}$ (not taking into account cosmic catastrophes, epidemic diseases etc.). This is a very small but positive probability. And even if in future the risk of accidental death can be significantly reduced while the population may be enormously increased[1], the risk of a sudden end of all life within a short time interval is not zero and, therefore, will occur before eternity is finished.

So, after a while, nobody will be present to measure time – and it may well be asked if an entity does exist which in principle cannot be measured.

[W. Mückenheim: "The infinite in sciences and arts", Proceedings of the 2nd International Symposium of Mathematics and its Connections to the Arts and Sciences (MACAS 2), B. Sriraman, C. Michelsen, A. Beckmann, V. Freiman (eds.), Centre for Science and Mathematics Education, University of Southern Denmark, Odense (2008) pp. 265-272 & arXiv. W. Mückenheim: "Die Mathematik des Unendlichen", Shaker-Verlag, Aachen (2006) p. 135]

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[1] but at most to 10^{80} individuals because this is the number of atoms in the accessible universe, and life without at least one atom seems impossible.

Is the heaven infinite?

The existence of a creator has become more and more improbable in history of mankind. Copernicus, Bruno, and Darwin contributed to remove mankind from the centre of the universe, the position chosen by God for his creatures. The development of the character of God himself closely reflects the social development of human societies. G.C. Lichtenberg noted: "God created man according to his image? That means probably, man created God according to his."

The discovery of foreign cultures, in America and Australia, revealed that people there had not been informed in advance about the God of Jews, Christians, or Moslems – a highly unfair state of affairs in case belief in this God was advantageous before or after death.

The results of neurology and cerebral surgery show that characteristic traits and behaviour usually retraced to the human soul can be arbitrarily manipulated by electric currents, drugs, or surgery while an immortal soul cannot be localized.

Of course it is impossible to prove or to disprove the existence of one or more Gods, but it is easy to disprove the absolutum, as Cantor called it, i.e., the infinity of every property of a God. Medieval scholastics already asked, whether God could make a stone that heavy that he himself was incapable of lifting it. God cannot know the complete future unless the universe is deterministic. But in this case, there could be no free will and no living creature could prove itself suitable or unsuitable to enter paradise or hell – and the whole creation was meaningless.

Therefore, actual infinity, as being inherent to theological items, cannot be excluded but is at best problematic; it is certainly not due to every property of God.

[W. Mückenheim: "The infinite in sciences and arts", Proceedings of the 2nd International Symposium of Mathematics and its Connections to the Arts and Sciences (MACAS 2), B. Sriraman, C. Michelsen, A. Beckmann, V. Freiman (eds.), Centre for Science and Mathematics Education, University of Southern Denmark, Odense (2008) pp. 265-272 & arXiv]

What are numbers?

Numbers are special properties of objects like colour or size, perhaps a bit more precise, but not different in principle. They define how many parts the object has or can have.

An early herdsman may have reported a loss of cattle to his master by saying: There were as many lions as I have eyes. They have taken as many cows as I have fingers. In some languages the word for 5 is the same as that for hand [M. Cantor: "Vorlesungen über Geschichte der Mathematik", Vol. 1, 3rd ed., Teubner, Leipzig (1907)]. These sentences are similar to other comparative descriptions: The lions were larger than big dogs. Their skin was as yellow as the sun. Their roar was as loud as thunder. They were as strong as bears. I fled as swift as the wind.

Platonism is a special form of theology, perhaps a bit more precise, but not different in principle. It has no legitimacy in MatheRealism.

Which numbers exist?

(For a related definition of the existence of natural numbers cp. section "An upper bound for cardinal numbers" below.)

- If the definition of a number exists in a memory (such that it can be retrieved), then the defined number exists in that memory.
- If the definition of a number can exist in a memory (such that it could be retrieved), then the defined number can exist in that memory.
- If the definition of a number cannot exist in a memory (because too much memory space is required), then the defined number cannot exist in that memory.
- If the definition of a number cannot exist anywhere, then this very number does not exist. Because "in mathematics description and object are equivalent" [Wittgenstein].

The question about the existence of numbers rarely can be answered by yes or no. The number 1 exists, for instance here on the screen, but also in every abstraction of a singleton set. The number 1/727 exists here. Its decimal representation with a period of 726 digits

0.[00137551581843191196698762035763411279229711141678129298486932599724896836313
617606602475928473177441540577716643741403026134800550206327372764786795048143
053645116918844566712517193947730398899587345254470426409903713892709766162310
866574965612104539202200825309491059147180192572214580467675378266850068775790
921595598349381017881705639614855570839064649243466299862448418156808803301237
964236588720770288858321870701513067400275103163686382393397524071526822558459
422283356258596973865199449793672627235213204951856946354883081155433287482806
052269601100412654745529573590096286107290233837689133425034387895460797799174
690508940852819807427785419532324621733149931224209078404401650618982118294360
385144429160935350756533 7]

perhaps has never existed in a human brain and presumably never will, because of insignificance, unless someone would learn it by heart or apply a suitable algorithm for sporting reasons. But the decimal representation exists here on the screen and in every simple calculation program – latently though, but with a rather high degree of existence.

Nelson's number (cp. "Edward Nelson" in chapter V) existed already before somebody had calculated it. Did it exist before Nelson formulated it? Perhaps its degree of existence had been 0.1 before, 0.9 afterwards, and is now 1.0? That is difficult to quantify. A number that requires 10^{10} steps to be calculated has certainly a much lower degree of existence, and a number that requires more steps than are available, does not exist. Its degree of existence is 0 and that will never change.

Even summoning all support of the accessible universe we could not calculate all numbers between 1 and $10^{10^{100}}$. But each of these numbers x can be paired with another number y to yield the sum $x + y = 10^{10^{100}} + 1$ such that the sum $(10^{10^{100}} + 1)10^{10^{100}}/2$ of all of them is known.

An upper bound for cardinal numbers

In MatheRealism we consider what the mathematicians of the 19^{th} century could not yet know, and what those of the 20^{th} century seem to have pushed out of their minds: The universe contains less than 10^{80} protons and certainly less than 10^{100} particles which can store bits. It is, however, a prerequisite of set theory that an element of a set must differ from any other element of that set by at least one property. For this sake one would need at least one bit per element. Therefore, an upper limit of the number of elements of all sets is 10^{100}. The supremum is certainly less. We have to revise the idea of actually existing Cantorian sets. Even the smallest one, the set $\mathbb{N}$ of *all* natural numbers does not exist, let alone the set of all real numbers. At least two notions are to be distinguished in MatheRealism with respect to natural numbers.

1) Only those natural numbers exist which are *available*, i.e., which can be used by someone for calculating purposes. This proposition, including the "someone", is left vague on purpose. The existence of a natural number has "relativistic" aspects: The question of its existence can be answered differently by different individuals and at different times. As an example consider a poem which exists for the poet who just has written it but not yet for anybody else. Another example is the set of prime numbers to be discovered in the year 2050. It does not yet exist. It is unknown how many elements it will have, even whether it will have elements at all.

The set $\mathbb{N}^*$ of all natural numbers which exist relative to an observer can increase or decrease like the set of all known primes. Therefore it is difficult to determine its cardinal number $|\mathbb{N}^*|$. But obviously $|\mathbb{N}^*| < 10^{100}$ is not infinite.

2) All natural numbers which have ever existed and which ever will exist do not form a set in the sense of set theory because not all of them are simultaneously available and distinguishable. Some of them do not exist yet, others have ceased to exist. The number of all those numbers of this collection is *potentially* infinite because it is not bounded by any threshold, but in no case any cardinal number can become the actual infinity $\aleph_0$ let alone exceed it.

We can conclude: The set $\mathbb{N} = \{1, 2, 3, ... \}$ of *all* natural numbers in the sense of Cantorian set theory does not exist. It is simply not available. But it is difficult to recognize that, because usually only some comparatively small numbers are chosen as examples and in calculations, followed by the dubious symbol "...". The natural numbers, in general thoughtlessly imagined as an unbroken sequence, do not come along like the shiny wagons of a long train. Their sequence has gaps.

As a result, we find in MatheRealism that infinite sets consisting of distinct elements cannot exist, neither in the brain of any intelligent being nor elsewhere in the universe. 10^{100} is an upper limit for any cardinal number of existing ideas, i.e., thought objects. Therefore, all the paradoxes of set theory, not easy to be circumvented otherwise, vanish.

[W. Mückenheim: "Das Unendliche", de.sci.mathematik (5 Jan 2005). W. Mückenheim: "Physical constraints of numbers", Proceedings of the First International Symposium of Mathematics and its Connections to the Arts and Sciences, A. Beckmann, C. Michelsen, B. Sriraman (eds.), Franzbecker, Berlin (2005) pp. 134-141 & arXiv]

How to realize numbers?

All sciences need and many arts apply mathematics whereas mathematics seems to be independent of all of them, only based upon logic. This conservative opinion however need to be revised because, contrary to Platonic idealism (frequently called "realism" by mathematicians), mathematical ideas, notions, and, in particular, numbers are not at all independent of physical laws and prerequisites.

The existence of all natural numbers 1, 2, 3, ... is not guaranteed by their definition alone. In order to discuss this topic in detail we would need some formalized concept of existence. In its full generality the meaning of existence is a difficult philosophical question. The responses to it span the wide range between materialism and solipsism. But we need not consider this problem in great depth. What are the essentials of an idea? The answer to this question is comparatively easy. An idea exists on its own, if it is uniquely identifiable, i.e., if it can be distinguished from any other idea. A poem not yet written but existing in its author's mind is an idea as well as a mental image of a landscape or a mathematical problem. The result of the latter often is a number. While numbers belong to the set of ideas, they have to satisfy an even stronger criterion in order to be considered as really existing. Like an ordinary idea a number n can be individualised by a mere name. But its existence is certainly not yet established by labels like "2" or "π". "Number" is a patent of nobility, not issued unless its value, i.e., its ratio with respect to the unit, $n/1$, can be fixed precisely or at least to any desired precision.

The basic and most secure method to establish the reality of a number is to form its fundamental set. The roman numerals are reminiscent of this method. While "2" is a name, "II" is both, a name and a part of that number, namely of the fundamental set

2 = {all pairs like: II, you & me, mum & dad, sun & moon} .

2 is the property that all its subsets have in common. Of course this realization of 2 presupposes some a priori knowledge about 2. But here we are concerned with the mere realization. The same method can be applied in case of 3

3 = {all triples like: III, sun & moon & earth, father & son & holy spirit} .

Although IIII is easily intelligible, and pigeons are able to distinguish up to IIIIII slots at first glance, it is impractical to realize larger numbers in this way. The Romans ceased at IV already. And we would have great difficulties to identify numbers beyond 20 in this construction.

Position systems, decimal or binary or other n-adic systems remedy this problem and have the advantage to accomplish both identify a number and put it in order with other numbers by economical consumption of symbols.

Does the fundamental set of 4711 exist? We don't know. It did at least in Cologne at the beginning of the 19th century. A set of 10^{1000} elements does definitely not exist in the accessible universe. Nevertheless 4711 and 10^{1000} are natural numbers. Their values, i.e., their ratios with respect to the unit are exactly determined.

It is impossible, however, to satisfy this condition for *all* natural numbers. It would require an unlimited amount of resources. But the universe is finite – at least that part available to us. Here is a simple means to realise the implications: First find out how many different natural numbers can be stored on a 10 GB hard disk. Then, step by step, expand the horizon to the 10^{11} neurons of your brain, to the 10^{28} atoms of your body, to the 10^{50} atoms of our earth, to the 10^{68} atoms of our galaxy, and finally to the 10^{78} protons within the universe. In principle, the whole universe could be turned into a big computer, but with far fewer resources than is usually expected without a thought be given to it.

Does a number exist in spite of the fact that we do not know and *cannot* know much more about it but that it should be a natural number? Does a poem exist, if nobody knows anything about it, except that it consists of 80 characters? The attempt to label every 80-digits number like

12345678901234567890123456789012345678901234567890123456789012345678901234567890

by a single proton only, would already consume more protons than the universe can supply. What you see is a number with no doubt. *Each* of those 80-digits strings can be noted on a small piece of paper. But *all* of them cannot even exist as *individual ideas* simultaneously, let alone as numbers with definite values. Given that photons and leptons can be recruited to store bits and given that the mysterious dark matter consists of particles or other means which can be used for that purpose too, it is nevertheless quite impossible to encode the values of 10^{100} natural numbers in order to have them simultaneously available. (An advanced estimation, based on the Planck-length $1.6 \cdot 10^{-35}$ m, leads to an upper limit of 10^{205} but the concrete number is quite irrelevant. So let us stay with 10^{100}, the Googol.)

Even some numbers smaller than $2^{10^{100}}$ can never be stored, known, or thought of. In short they do not exist.[1] (It must not be forgotten: Also one's head, brain, mind, and all thoughts belong to the interior of the universe.) To avoid misunderstanding: Of course, there is no largest natural number. By short cuts like $10^{10^{10^{\cdots}}}$ we are able to express numbers with precisely known values surpassing any desired magnitude. But it will never be possible, by any means of future technology and of mathematical techniques, to know that natural number P which consists of the first 10^{100} decimal digits of π (given π is a normal irrational without any pattern appearing in its n-adic expansion). This P will never be fully available. But what is a natural number the digits of which will never be known? P is an idea but it is not a number and, contrary to any 80-digits string, it will never be a number. It is even impossible to distinguish it from that number P', which is created by replacing the last digit of P by, say, 5. It will probably never be possible to decide, whether $P < P'$ or $P = P'$ or $P > P'$. But if none of these relations can ever be verified, then we can conclude, adopting a realistic philosophical position, that none of them is true.

A method to compute hexadecimal digits of π without knowing the previous digits [D.H. Bailey, P.B. Borwein, S. Plouffe: "On the rapid computation of various polylogarithmic constants", Math. Comput. 66 (1997) pp. 903-913] fails in this domain too because of finite precision and lacking memory space. But even if the last digit of P could be computed, we would need to know

[1] Compare a commercial pocket calculator which can express and add numbers like 10^{40} and 10^{50} but not 123456789012345 and 1213141516171819.

all the other digits to distinguish P from all similar natural numbers 314...d... where d means the nth digit with $1 \leq n \leq 10^{100}$. And why should we stop at the comparatively small Googol?

It is obvious how to apply the aforementioned ideas to the collection of rational numbers. Rational numbers can be defined as equivalence classes of pairs of natural numbers. By multiplying a rational number by its denominator, we obtain a natural number. Natural numbers measure values based on the unit, rational numbers measure values depending on their denominator. All rational numbers have some terminating n-adic expansions. In other representations they have periods. In case the terminating expansion or the period is not too long, these numbers can be identified and hence they do exist. A rational number however which approximates π better than $1/2^{10^{100}}$, i.e., to 10^{100} bits, does not exist.

[W. Mückenheim: "Physical constraints of numbers", Proceedings of the First International Symposium of Mathematics and its Connections to the Arts and Sciences, A. Beckmann, C. Michelsen, B. Sriraman (eds.), Franzbecker, Berlin (2005) pp. 134-141 & arXiv. W. Mückenheim: "Die Geschichte des Unendlichen", 7th ed., Maro, Augsburg (2011) ch. XII.4]

Are the real numbers really real?

Cauchy, Weierstrass, Cantor, and Dedekind attempted to give meaning to irrational numbers, being well aware that there was more to do than to find suitable names. According to Cantor, $\sqrt{3}$ is not a number but only a symbol: "$\sqrt{3}$ ist also nur ein *Zeichen* für eine Zahl, welche erst noch gefunden werden soll, nicht aber deren Definition. Letztere wird jedoch in meiner Weise etwa durch (1.7, 1.73, 1.732, ...) befriedigend gegeben" [Cantor, p. 114].

It is argued that $\sqrt{3}$ does exist, because *it can be approximated to any desired precision* by some sequence (a_n) such that for any positive ε we can find a natural number n_0 such that the distance $|a_n - \sqrt{3}| < \varepsilon$ for $n \geq n_0$. It is clear that Cantor and his contemporaries could not perceive the *principle* limits of their approach. But every present-day scientist should know that it is *in principle* impossible to approximate any irrational like $\sqrt{3}$ to $\varepsilon < 1/2^{10^{100}}$. (An exception are irrationals like $\sqrt{3}/2^{10^{100}}$. But that does not establish their existence.) Therefore, the condition to achieve *any desired precision* fails in decimal and binary and any other fixed n-adic representation.

What about continued fractions, sequences, series, or modular identities determining irrational numbers x with "arbitrary precision"? All these methods devised to compare x with 1 must necessarily fail, because the *uninterrupted* sequence of natural numbers up to "arbitrary magnitude" required to calculate the terms and to store the rational approximations is not available. As a result we find that irrational numbers do not exist other than as ideas. In MatheRealism irrational numbers simply are not numbers but only ideas.

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MatheRealism in geometry

A *real* straight line consists of physical points, namely its molecules. Their number will rarely exceed 10^{20}. Same holds for areas and bodies, often containing some more molecules though.

An *ideal* straight line (or area or body) consists of points which are identified by their coordinates. It is only imagined. Imagined points that are not imagined do not exist. If there are less than 10^{100} indices, then less than 10^{100} points can be imagined.

Intervals, closed and open, are identified by their endpoints. The number of points inside is a matter of taste. Either we consider how many points inside *have been defined* or how many points inside *can be defined*. It is obvious that there is no point next to another one or next to the endpoint. Therefore open intervals have a very frayed character – when looking closely.

Epilogue: About mathematics and reality and this book[1]

Mathematics serves to provide an overview and facilitates organizing and recognizing reality. For this purpose mathematical objects like numbers, figures, symbols, or structures have been created, their properties have been investigated and expressed in statements which, proved by means of logic, have become theorems. All this happens in a language as clear as possible. In order to define the meaning of a word however we need other words the meaning of which is known already. To avoid a circulus vitiosus a basis of words is required which cannot be analyzed further. Statements containing only those words and appearing so evident that further proof is not necessary have been introduced into ancient geometry already by Euclid and have been called axioms.

It is controversial whether mathematics is a natural science. Many European and American universities house a faculty of mathematics and natural sciences, revealing the close connection between both. On the other hand according to modern view the choice of an axiom system is no longer bound to reality but completely arbitrary, as long as internal contradictions are avoided and the accepted mathematical theorems can be proved. This latter point however shows that the decision about the acceptance of mathematical theorems is not left to the axiom system. On the contrary, in the end reality decides whether an axiom system is acceptable.

Unfortunately modern axiom systems lead to several unrealistic consequences. Therefore realistic mathematics should be developed from *those* foundations from which it originally arised, namely counting of units and drawing of lines. *Mathematics owes its origin to the abstraction from observations of reality*. A statement like "I + I = II" need not be derived from a proof covering many pages. This statement itself is a much more natural foundation of arithmetic than any axiom devised for that purpose. It can be proved by means of an abacus much more strictly than by any sophisticated chain of logical conclusions.

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[1] Form the preface of my book [W. Mückenheim: "Mathematik für die ersten Semester", 4th ed., De Gruyter, Berlin (2015)] which here is addressed as "this book".

The edifice of present mathematics has been erected by means of an axiom system based on Georg Cantor's teaching of the *transfinite numbers*. According to his own statement he developed it in order to describe the completed infinities which he surmised "in nature and in every extended part of space how small ever" [1]. In the light of modern realization of nature it has turned out however that reality does not contain anything to apply transfinite numbers upon. "In the intellectual general picture of our century the actual infinite appears quite bluntly as an anachronism" [2]. On the other hand the finity of the accessible universe leads to the recognition that mathematics like every other science is forced to get by on finite means. But without infinite means there are no infinite results. A set of numbers consists of numbers that somehow must be distinguishable, i.e., that have distinct expressions. A number can be expressed by a name, by a definition, by a string of digits, or by other marks. If the number of all marks is bounded, then the set of distinguishable elements is bounded too. The universe with its 10^{78} protons, let alone every part that really can be utilized for thinking and calculating, has a finite capacity for storing information. Therefore the number of distinct marks is limited by purely material reasons. If only significantly less than 10^{100} units of information or *digits* can be stored, then it is evidently impossible to distinguish more than 10^{100} *numbers*. But what cannot be marked and cannot be thought, that cannot be a number – what never has been thought and never can be thought does not belong to the set of thoughts. Without diving into a discussion about the meaning of existence we should agree that it is impossible and forever will remain impossible to apply "numbers" that cannot be marked in any way as individuals. They do not belong to mathematics as far as mathematics belongs to reality.

The finity of every set of numbers however does not imply the existence of a largest number, as often is assumed erroneously. The number 10^{100} and also much larger numbers like 10^{1000} can be named and identified, for instance here on this sheet or in the mind of the reader.[1] But many numbers the representation of which would require 10^{100} different digits, cannot be defined and thus can not be utilized. It is impossible to count from 1 till $10^{10^{100}}$ – and this is independent of the time available. "The sequence of natural numbers does not appear perfect like an intercity train. It has gaps" [3]. And these gaps are increasing with the size of the numbers. Therefore it is not meaningful to talk of an actually infinite number sequence, and in the present book it is not attempted to postulate the existence of actually infinite sets or to compute with transfinite numbers. The most important theorems of a mathematics oriented towards reality can be proved in good approximation by means of experiments, performed mainly on efficient computers. Computers are the telescopes and microscopes of mathematicians. They improve the perspective and allow to distinguish details that cannot be seen otherwise. The notions "infinite set" and "set of all numbers with a certain property" are used in this book, but here we mean sets that have no actual existence, sets that cannot be overlooked and that therefore are infinite in the *literal* sense. In contrast to an actually infinite set the number of elements of a *potentially infinite* set can neither be determined nor be surpassed, since the set is never completed. "Numbers are free creations of the human mind" [4]. Their number is finite and always will remain so. "A construction does not exist until it is made; when something new is made, it is something new and not a selection from a pre-existing collection" [5]. Therefore sets of numbers are not fixed. "The natural numbers of today are not the same as the natural numbers of yesterday" [6]. "The infinite is nowhere realized; it is neither present in nature nor admissible as the basis of our rational thinking – a remarkable harmony between being and thinking" [7].

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[1] In order to immediately comprehend the argument consider a pocket calculator with a ten-digit display. This device can process and display the number 10^{50} but not the number 123456789012345.

The finity of every set makes also the set of all indexes of digits of a number finite. The common tacitly accepted assumption that every real number can be approximated with "arbitrary precision" has not unrestricted validity. The number axis is not free of gaps; the notions of continuity, convergence, and other basics of calculus become problematic; the intermediate value theorem or the fundamental theorem of algebra "suffer from exceptions".

No-one can find a remedy! Mathematics is not existing outside of reality. It is of little use to require the existence of actually infinite sets and so to "prove" the completeness of real numbers. This does not better rectify the shortage than a merchant who adds some zeros to his balance – as Immanuel Kant emphasized in an analogous context [8]. The really accessible "continuum" has a granular structure. The grain size depends on the computing capacity. The mathematician who is merely equipped with an abacus has access to integers only. Fortunately the grainy structure is usually fine enough not to cause disadvantages. Just like the quantization of the earth's orbit has no relevance to astronomical problems and the molecular structure of butter does not noticeably limit its spreadability, the principle uncertainty of numbers will not impair their suitability for the purpose of organizing and recognizing reality. In general the 10-digit-precision of pocket calculators or the 100-digit-precision of simple computing programs are sufficient. The knowledge of the first 10^{100} digits will only be strived for on extremely rare occasions and will never be reached with irrational numbers [9].

But at the most this lack will become perceptible in fundamental research. And even for that realm the inventor of non-standard analysis has asserted: "Infinite totalities do not exist in any sense of the word (i.e., either really or ideally). More precisely, any mention, or purported mention, of infinite totalities is, literally, meaningless. Nevertheless, we should continue the business of Mathematics 'as usual', i.e., we should act as if infinite totalities really existed" [10]. Without suppressing the knowledge about this shortage and blocking it out of our consciousness we can and will apply mathematics for organizing and recognizing reality as if infinite sets existed.

[1 E. Zermelo (ed.): "Georg Cantor: Gesammelte Abhandlungen mathematischen und philosophischen Inhalts", Springer, Berlin (1932) p. 399]
[2 P. Lorenzen: "Das Aktual-Unendliche in der Mathematik", Phil. naturalis 4 (1957) p. 3]
[3 W. Mückenheim: "Physical constraints of numbers", Proceedings of the First International Symposium of Mathematics and its Connections to the Arts and Sciences, A. Beckmann, C. Michelsen, B. Sriraman (eds.), Franzbecker, Berlin (2005) p. 141]
[4 R. Dedekind: "Was sind und was sollen die Zahlen?", Vieweg, Braunschweig (1888) p. III]
[5 E. Nelson: "Predicative arithmetic", Princeton University Press, Princeton (1986) p. 2]
[6 D. Isles: "What evidence is there that 2^65536 is a natural number?", Notre Dame Journal of Formal Logic 33,4 (1992) p. 478]
[7 D. Hilbert: "Über das Unendliche", Math. Ann. 95 (1925) p. 190]
[8 I. Kant: "Critik der reinen Vernunft", 1st ed., Hartknoch, Riga (1781) chapter 102]
[9 W. Mückenheim: "Die Mathematik des Unendlichen", Shaker, Aachen (2006) p. 137]
[10 W.A.J. Luxemburg, S. Koerner (eds.): "A. Robinson: Selected papers", Vol. 2, North Holland, Amsterdam (1979) p. 507]