Trig Equations with Half-Angles and Multiple Angles

What follows are illustrations of dealing with trig equations with multiple angles.

Equation with a Half-angle

Example: Solve $2\sqrt{3}\sin\frac{\theta}{2} = 3$ over the interval $[0°, 360°)$.

Solution: Write the interval $[0°, 360°)$ as an inequality

$$0° \le \theta < 360°$$
$$0° \le \frac{\theta}{2} < 180°$$

and set up the equation

$$2\sqrt{3}\sin\frac{\theta}{2} = 3$$
$$\sin\frac{\theta}{2} = \frac{3}{2\sqrt{3}}$$
$$\sin\frac{\theta}{2} = \frac{\sqrt{3}}{2}$$
$$\frac{\theta}{2} = 60°, 120°$$
$$\theta = 120°, 240°$$

and write the solution set

$$S.S. = \{120°, 240°\}$$

Equation with a Double Angle

Example: Solve $\cos 2x = \frac{\sqrt{3}}{2}$ over the interval $[0, 2\pi)$.

Solution: Write the interval

$$[0, 2\pi)$$

as the inequality

$$0 \le x < 2\pi$$

and then multiply by 2 to obtain the interval for $2x$:

$$0 \le 2x < 4\pi$$

Using radian measure we find all numbers in this interval with cosine value $\frac{\sqrt{3}}{2}$. These are $\frac{\pi}{6}$, $\frac{11\pi}{6}$, $\frac{13\pi}{6}$, and $\frac{23\pi}{6}$. So

$$2x = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}$$
$$x = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$$

Write the solution set

$$S.S. = \left\{\frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}\right\}$$

Solving an Equation Using a Double Angle Identity

Example: Solve $\cos 2x + \cos x = 0$ over the interval $[0, 2\pi)$.

Solution: To solve this we must change $\cos 2x$ using a double-angle identity (see the formula list)

$$\begin{aligned} \cos 2x + \cos x &= 0 \\ 2\cos^2 x - 1 + \cos x &= 0 \\ 2\cos^2 x + \cos x - 1 &= 0 \\ (2\cos x - 1)(\cos x + 1) &= 0 \end{aligned}$$

Now divide the the problem into two parts

$$\begin{array}{lll} 2\cos x - 1 = 0 & \text{or} & \cos x + 1 = 0 \\ \cos x = \dfrac{1}{2} & \text{or} & \cos x = -1 \\ x = \dfrac{\pi}{3} \text{ or } x = \dfrac{5\pi}{3} & \text{or} & x = \pi \end{array}$$

The solution set is

$$S.S. = \left\{ \frac{\pi}{3}, \pi, \frac{5\pi}{3} \right\}$$

Example: Solve $1 - \sin\theta = \cos 2\theta$ over the interval $[0°, 360°)$.

Solution: Replace $\cos 2\theta$ using a double-angle identity.

$$\begin{aligned} 1 - \sin\theta &= \cos 2\theta \\ 1 - \sin\theta &= 1 - 2\sin^2\theta \\ 2\sin^2\theta - \sin\theta &= 0 \\ \sin\theta(2\sin\theta - 1) &= 0 \end{aligned}$$

Divide the problem into two parts

$$\begin{array}{lll} \sin\theta = 0 & \text{or} & 2\sin\theta - 1 = 0 \\ \theta = 0° \text{ or } 180° & \text{or} & \sin\theta = \dfrac{1}{2} \\ & & \theta = 30° \text{ or } 150° \end{array}$$

The solution set is

$$S.S. = \{0°, 30°, 150°, 180°\}$$

Solving an Equation Using a Multiple-Angle Identity

Solve $4\sin\theta\cos\theta = \sqrt{3}$ over the interval $[0°, 360°)$.

$$\begin{aligned} 4\sin\theta\cos\theta &= \sqrt{3} \\ 2(2\sin\theta\cos\theta) &= \sqrt{3} \\ 2\sin 2\theta &= \sqrt{3} \\ \sin 2\theta &= \frac{\sqrt{3}}{2} \end{aligned}$$

From the given interval $0° \le \theta < 360°$, the interval for 2θ is $0° \le 2\theta < 720°$.

$$2\theta = 60°, 120°, 420°, 480°$$
$$\theta = 30°, 60°, 210°, 240°$$
$$S.S. = \{30°, 60°, 210°, 240°\}$$

Since the period of $\sin 2\theta$ is $\pi = 180°$, we can represent *all* solutions in this way:

$$S.S. = \{30° + 180°n, 60° + 180°n, \text{ where } n \text{ is any integer}\}$$

Solving an Equation with a Multiple Angle

Solve $\tan 3x + \sec 3x = 2$ over the interval $[0, 2\pi)$.

Solution: Since we have tangents and secants, squaring both sides will let us express everything in terms of tangent:

$$\tan 3x + \sec 3x = 2$$
$$\sec 3x = 2 - \tan 3x$$
$$\sec^2 3x = (2 - \tan 3x)^2 \quad \text{square both sides}$$
$$1 + \tan^2 3x = 4 - 4\tan 3x + \tan^2 3x$$
$$-3 = -4\tan 3x$$
$$\tan 3x = \frac{3}{4}$$
$$3x = 0.6435 \text{ or [Quadrant I]}$$
$$3x = .6435 + \pi = 3.7851 \text{ [Quadrant III]}$$

The solution for $3x$ must be Quadrants I and III. Since $0 \le x < 2\pi$, we have $0 \le 3x < 6\pi$, and

$$3x = .6435 + (n)2\pi, \text{ where } n = 0, 1, 2 \text{ or}$$
$$3x = 3.7851 + (n)2\pi, \text{ where } n = 0, 1, 2$$
$$x = 0.2145, 2.3089, 4.4033 \text{ or}$$
$$x = 1.2617, 3.3561, 5.4505$$

We must test each of these proposed solutions, because they were produced by squaring both sides of the equation, and extraneous roots are possible. The cosine function has period 2π, a multiple of the period of the tangent function (π). It is enough, then, to test $x = .2145$ and $x = 1.2617$. You can check these approximations with the calculator to obtain

$$\tan(3 * 0.2145) + 1/\cos(3 * .2145) = 1.999997228$$

but

$$\tan(3 * 0.1.2617) + 1/\cos(3 * .1.2617) = -.4999961015$$

We conclude that, rounded off to four decimal places,

$$S.S. = \{.2145, 2.3089, 4.4033\}$$

You can see this by graphing $Y_1 = \tan(3x) + 1/\cos(3x) - 2$ on your calculator in the Window $[0, 2\pi] \times [-1, 1]$

with Xscl = 1 and note where the graph (TI-84) crosses the x-axis.