Complex Analysis, Exam 1

Dr. R. Howell, 2/16/2018

$\overline{}$
100

Name: _____

Favorite Ice Cream Flavor: _____

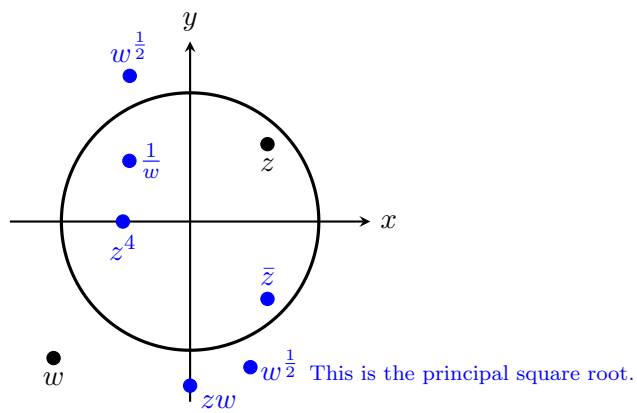
"Whatever your hand finds to do, do it with your might ..." Ecclesiastes 9:10 (ESV)

___ 1 **1.** What is the best subject in all of mathematics?

Answer: **C** O M P L E X **A** N A L Y S I S !

___ 12 **2.** Given the unit circle, and points z and w as shown, plot accurately and label the following points:
a. z^4; **b.** $w^{\frac{1}{2}}$ (show both points); **c.** zw; **d.** $\bar{z}$; **e.** $\frac{1}{w}$.

As the values for z and w are not given, you should do this problem by appealing to geometry, and without any computations.



___ 15 **3.** This question relates to derivatives of complex functions.

a. Show that the function $f(z) = \bar{z}$ is nowhere differentiable. (10 points)

For $z_0 = x_0 + iy_0$ we will show that $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ does not exist.

First, approach z_0 along a line parallel to the x-axis (so $z = x + iy_0$):

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{(x+iy_0) \to (x_0+iy_0)} \frac{(x - iy_0) - (x_0 - iy_0)}{(x + iy_0) - (x_0 + iy_0)} = \lim_{(x+iy_0) \to (x_0+iy_0)} \frac{x - x_0}{x - x_0} = 1.$$

Next, approach z_0 along a line parallel to the y-axis (so $z = x_0 + iy$):

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{(x_0+iy) \to (x_0+iy_0)} \frac{(x_0 - iy) - (x_0 - iy_0)}{(x_0 + iy) - (x_0 + iy_0)} = \lim_{(x_0+iy) \to (x_0+iy_0)} \frac{-i(y - y_0)}{i(y - y_0)} = -1.$$

The two limits are different, so $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ does not exist.

b. Define what it means for a function f to be analytic at z_0. (5 points)

It means that there is some positive number r such that f is differentiable at every point in the open disk $D_r(z_0) = \{z : |z - z_0| < r\}$.

____ 20

4. This question deals with complex roots and the history of complex numbers.

a. Let $\zeta = -1 - i$. Evaluate (3 points each)

i. $|\zeta| =$ $\underline{\quad \sqrt{2} \quad}$.

ii. $\mathrm{Arg}(\zeta) =$ $\underline{\quad -\frac{3\pi}{4} \quad}$.

Note: the next part will require the above two answers. You may obtain them for 3 points each.

b. Find all complex numbers z such that $z^3 = -1 - i$. Put the principal cube root in the Cartesian form $x + iy$. Put the other roots in exponential form. (9 points)

Any solution of the form $z = re^{i\theta}$ must satisfy the equation $z^3 = r^3 e^{i3\theta} = -1 - i = \sqrt{2}e^{-i\frac{3\pi}{4}}$. The equation $r^3 e^{i3\theta} = \sqrt{2}e^{-i\frac{3\pi}{4}}$ implies that $r^3 = \sqrt{2}$, and $3\theta = -\frac{3\pi}{4} + 2k\pi$, where $k \in \mathbb{Z}$. Solving for r and θ gives $r = \sqrt[6]{2}$ and $\theta = -\frac{\pi}{4} + \frac{2k\pi}{3}$, where $k \in \mathbb{Z}$.

The principal cube root is obtained by setting $k = 0$, yielding $z_0 = \sqrt[6]{2}e^{-i\frac{\pi}{4}}$. In Cartesian form, we have $z_0 = \sqrt[6]{2}\left(\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}\right) = \frac{\sqrt[6]{16}}{2} - i\frac{\sqrt[6]{16}}{2}$.

The remaining distinct roots are obtained by setting $k = 1$ and 2, yielding (in exponential form) $z = \sqrt[6]{2}e^{i\frac{5\pi}{12}}$, and $z = \sqrt[6]{2}e^{i\frac{13\pi}{12}}$.

c. Explain why cubic equations, rather than quadratic equations, played a pivotal role in helping to obtain the acceptance of the idea of complex numbers by mathematicians. (5 points)

When complex numbers arose as solutions to quadratic equations (such as $x^2 + 1 = 0$) they were easily dismissed as meaningless. Rafael Bombelli (1526–1572), however, showed that complex numbers are indispensable in obtaining *real* solutions to certain cubic equations. For example, the depressed cubic $x^3 - 15x - 4 = 0$ clearly has $x = 4$ as a solution, but Bombelli demonstrated that the only way to get this answer (without guessing) is to take a detour into the realm of complex numbers with the calculation $\sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i} = (2 + i) + (2 - i) = 4$. With this discovery the utility of complex numbers could not be ignored.

____ 10

5. Find the radius of convergence of the series $\displaystyle\sum_{n=1}^{\infty} n4^n z^n$.

We compute the limit of the absolute value of the $(n+1)^{\text{st}}$ term divided by the nth term:

$$\lim_{n\to\infty}\left|\frac{(n+1)4^{n+1}z^{n+1}}{n4^n z^n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\right||4z| = |4z|.$$

By the ratio test, the series converges absolutely if $|4z| < 1$, that is, if $|z| < \frac{1}{4}$. The radius of convergence is thus $\frac{1}{4}$.

14 **6.** Find the image of the line $2x + 4y = 1$ under the reciprocal transformation $w = f(z) = \frac{1}{z}$.

Designate $A = \{(x, y) : 2x + 4y = 1\}$, and set $B = f(A)$. Then

$$\begin{aligned} w = u + iv \in B &\iff f^{-1}(w) = \frac{1}{w} \in A \\ &\iff \frac{u}{u^2+v^2} + i\frac{-v}{u^2+v^2} \in A \\ &\iff 2\left(\frac{u}{u^2+v^2}\right) + 4\left(\frac{-v}{u^2+v^2}\right) = 1 \qquad (1) \\ &\implies 2u - 4v = u^2 + v^2 \qquad (2) \\ &\iff (u-1)^2 + (v+2)^2 = 5. \qquad (3) \end{aligned}$$

Equation (3) describes $C_{\sqrt{5}}(1 - 2i)$, a circle with radius $\sqrt{5}$ that is centered at $1 - 2i$. As the notation indicates, however, Equations (1) and (2) are not equivalent: Equation (2) is satisfied when $(u, v) = (0, 0)$, but Equation (1) is undefined at that point. Technically, then, the image of A under f is $C_{\sqrt{5}}(1-2i)$ *except* the point $0 = (0, 0)$. To avoid this unpleasant situation we can lift our domain to the Riemann Sphere, where the point at infinity gets mapped by f to 0, *i.e.*, $f(\infty) = 0$.

14 **7.** Let M designate the Mandelbrot set, and $A = \{c \in \mathbb{C} : |c| \leq \frac{1}{4}\}$. Prove that $A \subseteq M$.
I have a hint that will help you get started. You can get it for the asking, but it will cost one point.

Let $c \in A$, and let $\{a_k\}_{k=0}^{\infty}$ be the orbit of zero generated by $f_c(z) = z^2 + c$. We will show via mathematical induction that, for all $k \geq 0$, $|a_k| \leq \frac{1}{2}$.

For the base case, note that $|a_0| = |0| \leq \frac{1}{2}$.

Next, suppose that, for some $k \geq 0$, $|a_k| \leq \frac{1}{2}$. Then

$$|a_{k+1}| = |f_c(a_k)| = |a_k^2 + c| \leq |a_k^2| + |c| = |a_k|^2 + |c| \leq \left(\frac{1}{2}\right)^2 + \frac{1}{4} = \frac{1}{2}.$$

The above calculations show that the sequence $\{a_k\}_{k=0}^{\infty}$ is bounded, so $A \subseteq M$.

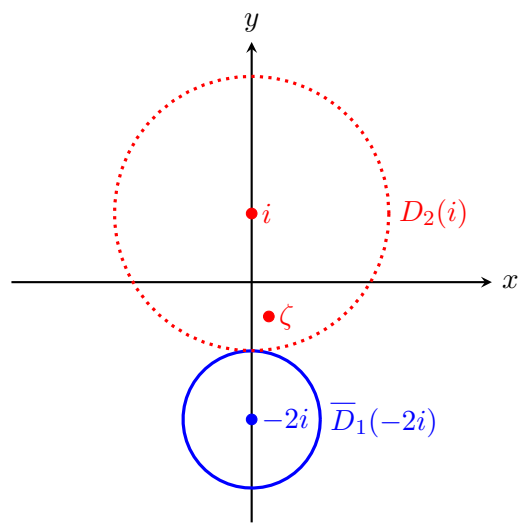
Note: a slightly more detailed proof is given in the text for this class (Example 4.11, page 141).

___ 14

8. Suppose ζ is a fixed complex number such that $|\zeta - i| < 2$.

a. Explain by appealing to a geometric picture why it must then be the case that $|\zeta + 2i| > 1$.

If $|\zeta - i| < 2$, then ζ is within two units of the point i. Referring to the figure to the right, ζ must be somewhere *within* the red open disk of radius two centered at i. Therefore, ζ can never get to the boundary of the closed blue disk, which describes the set of points whose distance from $-2i$ is less than or equal to one unit. In other words, ζ must be *more than* one unit from the point $-2i$. In summary, if ζ is *within* $D_2(i)$, then ζ must lie *outside* $\overline{D}_1(-2i)$, which is what the inequality $|\zeta + 2i| > 1$ asserts.



b. The triangle inequality can be used to prove that, for all z_1, $z_2 \in \mathbb{C}$,

$$|z_1 + z_2| \geq |z_1| - |z_2|. \tag{1}$$

Use Inequality (1) to prove mathematically that, if $|\zeta - i| < 2$, then $|\zeta + 2i| > 1$.
A hint that will help you get started is available for one point.

Clearly, $|\zeta - i| < 2$ implies $-|\zeta - i| > -2$. Combining this fact with Inequality (1) gives

$$\begin{aligned} |\zeta + 2i| &= |3i + (\zeta - i)| \\ &\geq |3i| - |\zeta - i| \\ &> 3 - 2 \\ &= 1, \quad \text{QED.} \end{aligned}$$

Extra Credit Problems (put your work on the back of the previous page)

1. Use the triangle inequality to prove that $|z_1 + z_2| \geq |z_1| - |z_2|$ for all z_1, $z_2 \in \mathbb{C}$.

2. Find a closed-form expression that gives the value of the series in problem 5.
Hint: ask yourself how the series in problem 5 might come from another series via differentiation.

Solution to Extra Credit Problems

1. Use the triangle inequality to prove that $|z_1 + z_2| \geq |z_1| - |z_2|$ for all z_1, $z_2 \in \mathbb{C}$.

By the triangle inequality,

$$\begin{aligned} |z_1| &= |-z_2 + (z_1 + z_2)| \\ &\leq |-z_2| + |z_1 + z_2| \\ &= |z_2| + |z_1 + z_2|. \end{aligned}$$

Subtracting $|z_2|$ from both sides of the inequality gives the desired result.

2. Find a closed-form expression that gives the value of the series in problem 5.
Hint: Ask yourself how the series you see might come from another series via differentiation.

Begin with the identity

$$\sum_{n=0}^{\infty} (4z)^n = \frac{1}{1-4z}, \text{ for } |4z| < 1, \text{ or } |z| < \frac{1}{4}.$$

Differentiating both sides gives

$$\sum_{n=1}^{\infty} n(4z)^{n-1}(4) = \sum_{n=1}^{\infty} n4^n z^{n-1} = \frac{4}{(1-4z)^2}, \text{ for } |z| < \frac{1}{4}.$$

Multiplying both sides by z yields

$$\sum_{n=1}^{\infty} n4^n z^n = \frac{4z}{(1-4z)^2}, \text{ for } |z| < \frac{1}{4}.$$