Lesson 1: Special Triangles and the Unit Circle

Student Outcomes

- Students determine the values of the sine, cosine, and tangent functions for rotations of $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$ radians.
- Students use the unit circle to express the values of the sine, cosine, and tangent functions for $\pi - \theta$, $\pi + \theta$, and $2\pi - \theta$ for real-numbered values of θ.

Lesson Notes

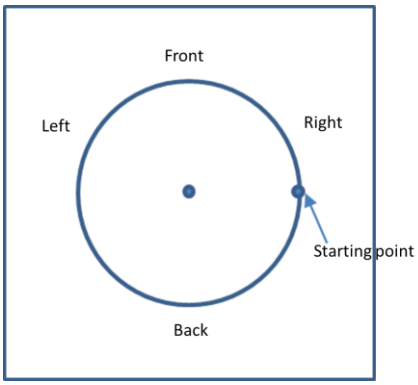
In Algebra II Module 2, students were introduced to the unit circle and the trigonometric functions associated with it. This lesson reviews these concepts, including the history behind the development of trigonometry. Students apply their knowledge of the unit circle and right triangles to find the values of sine, cosine, and tangent for rotations of $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$ radians. They also examine the relationship between the sine, cosine, and tangent of θ and its relationship to sine, cosine, and tangent for $\pi - \theta$, $\pi + \theta$, and $2\pi - \theta$, allowing them to evaluate the trigonometric functions for values of θ in all four quadrants of the coordinate plane.

Classwork

Opening (7 minutes)

MP.4

In Algebra II Module 2, students modeled a Ferris wheel using a paper plate. In this lesson, they model a carousel, which they can use to help them recall their previous knowledge about the unit circle. Students should complete the task in pairs, and each pair should be given a paper plate, a brass fastener, and a sheet of cardstock that is large enough to be visible once the paper plate has been affixed to it. Students fasten the center of the paper plate to the cardstock using the brass fastener, which serves as the center of the carousel. Students then label the cardstock to indicate directionality (front, back, left, and right of the center). On their paper plates, students should also indicate the starting point for the ride, which should be a point on the plate directly to the right of the center of the carousel. This point represents the rider.



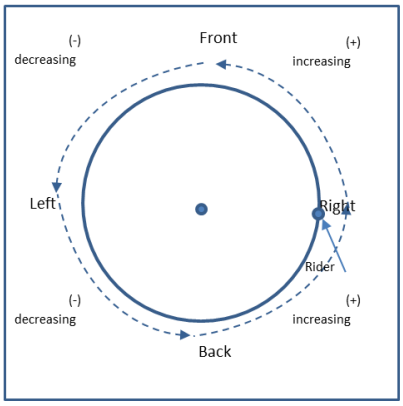
Each pair of students should be assigned to either group 1 or group 2. Students in group 1 should answer prompt 1, and students in group 2 should answer prompt 2.

1. Over the course of one complete turn, describe the position of the rider with respect to whether he is positioned to the left or to the right of the center of the carousel. Use as much detail as you can.
2. Over the course of one complete turn, describe the position of the rider with respect to whether she is positioned to the front or to the back of the center of the carousel. Use as much detail as you can.

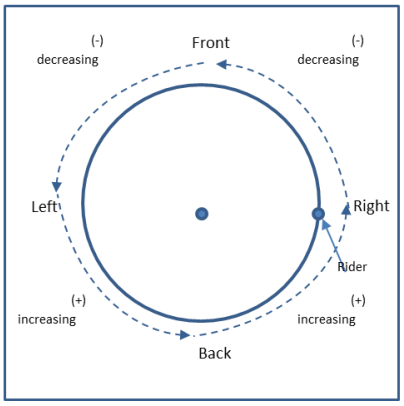
After a few minutes, several pairs should share their findings, which could be displayed on the board. Alternatively, a volunteer could record students' findings regarding the position of the rider on two charts (one for front/back and one for right/left). Students may or may not reference trigonometric functions, which is addressed explicitly later in the lesson. Likely student responses are shown:

- *The rider begins with a front/back position that is the same as the center of the carousel.*
- *As the carousel rotates counterclockwise, the rider moves in front of the center to a maximum value at a one-quarter turn. The rider then remains in front of the center point but decreases until she is again level with the center at one-half turn. As the carousel continues to rotate, the rider's position is behind the center, and the front/back value reaches its minimum at three-quarters of a turn. The rider continues to be behind the center, but the front/back position increases until the rider is again level at a full rotation.*
- *The maximum front/back distance from the center is the same and is equal to the radius of the carousel.*
- *The pattern of the front/back position of the rider repeats with every full turn. In other words, at one and one-quarter turns, the position of the rider is the same as it is for a one-quarter turn.*
- *The starting position of the rider is a maximum distance to the right of the center of the carousel. This distance is equal to the radius of the carousel.*
- *As the carousel rotates counterclockwise, the rider's position remains to the right of the center until, at a one-quarter turn, the rider is equidistant from the left and the right of the center. As the carousel continues to rotate, the rider's position is to the left of the center until he is again level with the center at a three-quarters turn. As the carousel continues to rotate, the rider's position is to the right of the center when it reaches a full turn.*
- *If "to the right" and "in front" are defined as positive directions (+) and "to the left" and "back" are defined as negative directions (–), the motions can be summarized in the diagrams:*

Summary of Front/Back Position of Rider



Summary of Left/Right Position of Rider

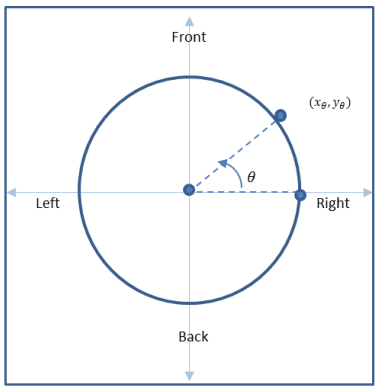


Discussion (8 minutes): Review of the Unit Circle and History of Trigonometry

This brief discussion recounts what students learned in Algebra II about the unit circle and origins of trigonometric functions. This review helps students recall the properties of the unit circle, which they need to apply to find the value of trigonometric functions applied to specific rotation values.

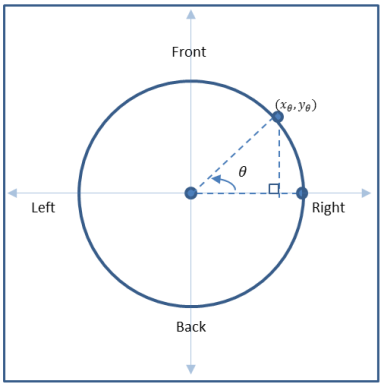
- When you created the carousel models, why do you think we defined the rider's starting position as immediately to the right of the center and the direction of rotation to be counterclockwise?
  - *Answers will vary but may address that when they learned about the unit circle, the starting point was the point* $(1,0)$ *on the positive* x*-axis and a positive rotation was defined to be counterclockwise.*
- You were introduced to the origins of trigonometry in Algebra II. Hundreds of years ago, scientists were interested in the heights of the sun and other stars. Before we knew about the Earth's rotation about its axis and its orbit around the sun, scientists assumed that the sun rotated around the Earth in a motion that was somewhat circular. Why would they have defined the sun's motion as counterclockwise?
  - *Answers will vary, but some students might recall that the sun rises in the east and sets in the west, appearing to trace a counterclockwise path in the sky if the observer is facing north.*
- So our unit circle models the apparent counterclockwise rotation of the sun about the Earth, with the observer on Earth representing the center of the circle, the sun's position as it rises at the horizon as the initial position, and the radius of the circle defined as 1. We define θ as the rotation of the initial ray, which passes through the origin and the point $(1,0)$, to end up at the terminal ray. What units do we use to measure the rotation θ?
  - *It is measured using degrees or radians.*
- And how do we define degrees and radians?
  - *A degree is one three hundred sixtieth of a full rotation; a radian represents the amount of rotation that* θ *undergoes so the length of the path traced by the initial ray from the positive* x*-axis to its terminal location is equal to the radius of the circle.*
- What is the radius of the unit circle?
  - 1
- How many radians are contained in a full rotation?
  - 2π
- Because of the direct relationship between the radius of the unit circle and radians, we use radians as our primary means of measuring rotations θ.

- Now that we have discussed θ as the amount of rotation that the initial ray undergoes in a counterclockwise direction, let's assign the radius of the carousel as 1 unit. Simulate a rotation of θ by marking a point on your plate that represents the position of the rider after a rotation of θ. Create a sketch on your carousel model to represent the rotation. For ease of notation, let's imagine that our carousel is superimposed on a coordinate plane, where the center is the origin and for now, rotation by θ produces an image point in the first quadrant. We can label the position of our rider given a rotation of θ as (x_θ, y_θ):



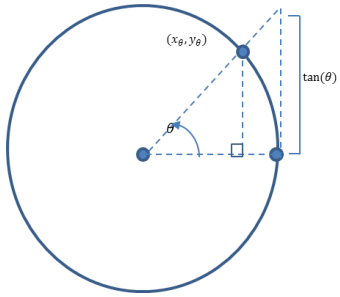
MP.7

- How could we represent the front/back distance between the center of the carousel and our rider given the amount of rotation, θ? Sketch this on your carousel model.
  - *Answers may vary but should indicate that a vertical line segment could be drawn from the rider's position to the segment passing through the center and the starting position, and the distance represents the rider's distance in front of the carousel's center.*
- And what is this distance?
  - y_θ



- Ancient scientists used the abbreviation "jhah" to refer to this distance. This abbreviation was converted from Sanskrit into the Arabic term "jiab" and then rewritten as the term "jaib," which was translated as the English term for cove, or sinus. This term was abbreviated into the term sine, which we are familiar with as a trigonometric function. Explain how our understanding of right triangle trigonometry demonstrates that $\sin(\theta) = y_\theta$.
  - *Using right triangle trigonometry,* $\sin(\theta) = \frac{y_\theta}{r}$, *and since* $r = 1$, $\sin(\theta) = y_\theta$.

- Now, Western scholars studying the height of the sun defined the segment representing the horizontal displacement of the sun as the "companion side" of the sine, which was shortened to the term cosine. How can we represent this distance on our model?
  - *It is x_θ.*
- And how does our understanding of right triangle trigonometry confirm that $\cos(\theta) = x_\theta$?
  - *Using right triangle trigonometry, $\cos(\theta) = \frac{x_\theta}{r}$, and since $r = 1$, $\cos(\theta) = x_\theta$.*
- Now you probably recall one additional core trigonometric function, the tangent function. The name is derived from the length of the line segment that has a point of tangency with the unit circle at the point $(1,0)$ and has as its end points the point $(1,0)$ and its point of intersection with the secant that passes from the origin through the terminal location of the ray with rotation, θ, as shown in the diagram.



- Use this diagram to determine the length $\tan(\theta)$. Share your solution with a partner.
  - *Answers may vary but should address that the smaller and larger right triangles in the diagram are similar, and as such, the ratios of their corresponding leg lengths are equal: $\frac{\tan(\theta)}{1} = \frac{y_\theta}{x_\theta}$.*
- How does our understanding of right triangle trigonometry flow from this definition of $\tan(\theta)$?
- In right triangle trigonometry, we defined

$$\tan(\theta) = \frac{\text{length of opposite side}}{\text{length of adjacent side}} = \frac{y_\theta}{x_\theta}.$$

- Now we have defined the unit circle, measurements of rotation about the circle, and the trigonometric functions associated with it. Let's use the unit circle to find the position of points on the unit circle for specific rotation values of θ.

Example 1 (6 minutes)

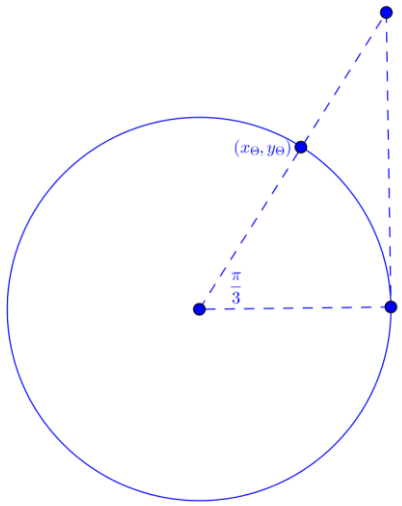
This example applies what students have learned about the unit circle to determine the values of the primary trigonometric functions for $\theta = \frac{\pi}{3}$. Students use a similar procedure to determine the trigonometric values for $\theta = \frac{\pi}{6}$ and $\theta = \frac{\pi}{4}$. They apply these results to determine the value of trigonometric functions for additional values of θ, including those outside Quadrant I.

Scaffolding:

- Encourage students to sketch the situation. Prompt them to draw in the altitude to the equilateral triangle in Example 1.
- Post an anchor chart with a right triangle and the definitions of $\sin(\theta)$ as the quotient of the lengths of the opposite side and the hypotenuse, $\cos(\theta)$ as the quotient of the lengths of the adjacent side and the hypotenuse, and $\tan(\theta)$ as the quotient of the lengths of the opposite and adjacent sides.

MP.7

- The diagram depicts the center of the carousel, the starting point of the rider, and the final position of the rider after rotating by $\frac{\pi}{3}$ radians, which is $60°$. How can we find the values of the x- and y-coordinates of the rider's final position?

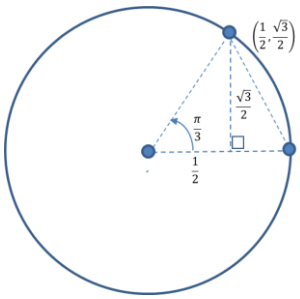


  - *We can use what we know about triangles and trigonometry to find the coordinates of the rider's final position.*
- What type of triangle is formed by the origin, the starting point, and the point representing the final position of the rider? How can you tell?
  - *These three points form an equilateral triangle. Two of the sides represent radii of the unit circle, so their lengths are* 1. *The base angles theorem can be applied to determine that each angle of the triangle has a measure of* $60°$.
- How can knowing the triangle formed by our three points is equilateral help us determine the values of the trigonometric functions?
  - *If we draw the altitude of the triangle, we create two* $30°-60°-90°$ *triangles. We know that the horizontal line segment is bisected, so the value of* $x_\theta = \frac{1}{2}$.
- And how do we find the value of y_θ?
  - *We use the Pythagorean theorem, where we have a known leg length of* $\frac{1}{2}$ *and hypotenuse length of* 1.

MP.2

Example 1

Find the following values for the rotation $\theta = \frac{\pi}{3}$ around the carousel. Create a sketch of the situation to help you. Interpret what each value means in terms of the position of the rider.



a. $\sin(\theta)$

$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

For the rotation $\theta = \frac{\pi}{3}$, the rider is located $\frac{\sqrt{3}}{2}$ units in front of the center of the carousel.

b. $\cos(\theta)$

Based on the diagram shown in part (a), $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$.

For the rotation $\theta = \frac{\pi}{3}$, the rider is located $\frac{1}{2}$ unit to the right of the center of the carousel.

c. $\tan(\theta)$

Based on the diagram shown in part (a), $\tan\left(\frac{\pi}{3}\right) = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$.

For the rotation $\theta = \frac{\pi}{3}$, the ratio of the front/back position to the right/left position relative to the center of the carousel is $\sqrt{3}$.

Exercise 1 (6 minutes)

Students should continue to use their paper plates to model the situations described in the exercise. They should be assigned to complete either part (a) or part (b) in their pairs from the opening activity. After a few minutes, each pair should explain their response to a pair assigned to a different part from them. Then, a few selected groups could share their results in a whole-class setting.

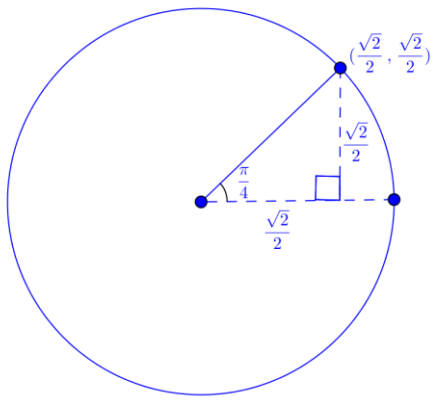
Exercise 1

Assume that the carousel is being safety tested, and a safety mannequin is the rider. The ride is being stopped at different rotation values so technicians can check the carousel's parts. Find the sine, cosine, and tangent for each rotation indicated, and explain how these values relate to the position of the mannequin when the carousel stops at these rotation values. Use your carousel models to help you determine the values, and sketch your model in the space provided.

Scaffolding:

Prompt students to draw a vertical line segment from the stopping point to the line segment representing the initial ray, and ask, "What are the measures of the angles of the resulting triangle?"

a. $\theta = \frac{\pi}{4}$

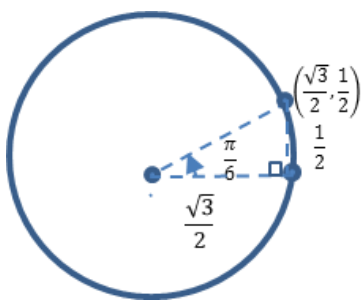


Since $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, *the rider is approximately* $\frac{\sqrt{2}}{2}$ *units in front of the carousel's center when it stops.*

Since $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, *the rider is approximately* $\frac{\sqrt{2}}{2}$ *units to the right of the carousel's center when it stops.*

Since $\tan\left(\frac{\pi}{4}\right) = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$, *the front/back distance of the rider is equal to its right/left distance when it stops.*

b. $\theta = \frac{\pi}{6}$



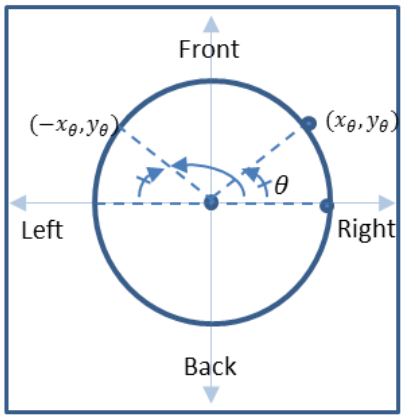
Since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, *the rider is approximately* $\frac{1}{2}$ *unit in front of the carousel's center when it stops.*

Since $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, *the rider is approximately* $\frac{\sqrt{3}}{2}$ *units to the right of the carousel's center when it stops.*

Since $\tan\left(\frac{\pi}{6}\right) = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$, *the front/back to right/left ratio of the rider is* $\frac{\sqrt{3}}{3}$ *when it stops.*

Discussion (5 minutes): Trigonometric Functions in All Four Quadrants

- We have just determined the results of applying the sine, cosine, and tangent functions to specific values of θ. All of these values, though, were restricted to the first quadrant. Let's see if our observations from the beginning of the lesson can help us expand our understanding of the effects of applying the trigonometric functions in the other quadrants. What do you recall about the front/back position of the rider as the carousel rotates counterclockwise?
  - *Answers will vary but might address that the rider's front/back position is 0 initially, increases to a maximum value (that we have defined as 1 unit forward) at one-quarter turn, and then decreases until it returns to 0 at one-half turn. It then becomes increasingly negative until it reaches a minimum position 1 unit behind the center of the carousel, and then it increases until it reaches a position of 0 after one full turn.*
- Refer to the sketch of θ in Quadrant I on your paper plate model. How do we represent the front/back position of a ray with rotation θ?
  - *y_θ represents the front/back position.*
- Approximate another location on the model where the front/back position of a ray is equal to y_θ. Sketch a ray from the origin to this location. Describe the rotation of our initial ray that lands us at this location.
  - *Answers may vary but should indicate that the new terminal ray is located somewhere in Quadrant II, and the clockwise rotation between the negative x-axis and our new terminal ray is the same as the counterclockwise rotation between the positive x-axis and our original terminal ray in Quadrant I.*



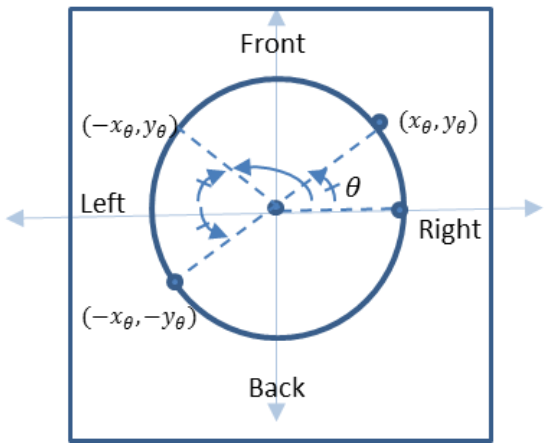
- Let's try to confirm this using transformations. Describe the image that results if we reflect the original terminal ray in Quadrant I over the positive y-axis.
  - *The reflection creates a new terminal ray located in Quadrant II where the angle made with the negative x-axis and the ray is θ, and the ray intersects the unit circle at $(-x_\theta, y_\theta)$.*
- Now how can we determine the amount of rotation of this image? Explain.
  - *The rotation of this image is $(\pi - \theta)$ because the rotation is θ less than a one-half turn, and a one-half turn is π radians.*

- Based on our definitions from earlier in this lesson, what conclusions can we draw about $\sin(\pi - \theta)$, $\cos(\pi - \theta)$, and $\tan(\pi - \theta)$? Explain how you know.
  - *Because the corresponding y-values are the same for θ and for $\pi - \theta$, $\sin(\pi - \theta) = y_\theta = \sin(\theta)$.*

    Because the corresponding x-values are opposites for θ and for $\pi - \theta$, $\cos(\pi - \theta) = -x_\theta = -\cos(\theta)$.

    Because the corresponding x-values are opposites but y-values are the same for θ and for $-\theta$,

    $\tan(\pi - \theta) = \frac{y_\theta}{-x_\theta} = -\tan(\theta)$.
- Reflect the image of the ray in Quadrant II over the negative x-axis. Describe the new image.
  - *The new image is a ray in Quadrant III where the measure of the angle between the image ray and the negative x-axis is θ. This new ray intersects the unit circle at $(-x_\theta, -y_\theta)$.*



- How can we designate the amount of rotation of this image? Explain.
  - *The rotation of this image is $(\pi + \theta)$ because the rotation is θ more than a one-half turn, which is π radians.*
- And what are the values of $\sin(\pi + \theta)$, $\cos(\pi + \theta)$, and $\tan(\pi + \theta)$?
  - $\sin(\pi + \theta) = -y_\theta = -\sin(\theta)$

    $\cos(\pi + \theta) = -x_\theta = -\cos(\theta)$

    $\tan(\pi + \theta) = \frac{-y_\theta}{-x_\theta} = \tan(\theta)$
- And what conjectures can we make about the values of $\sin(2\pi - \theta)$, $\cos(2\pi - \theta)$, and $\tan(2\pi - \theta)$? Explain.
  - *Rotation by $(2\pi - \theta)$ produces a reflection of the ray containing (x_θ, y_θ) over the positive x-axis, resulting in an image that intersects the unit circle at $(x_\theta, -y_\theta)$. This means that:*
  - $\sin(2\pi - \theta) = -y_\theta = -\sin(\theta)$

    $\cos(2\pi - \theta) = x_\theta = \cos(\theta)$

    $\tan(2\pi - \theta) = \frac{-y_\theta}{x_\theta} = -\tan(\theta)$

Example 2 (2 minutes)

This example demonstrates how students can apply their discoveries relating rotations in the four quadrants to find trigonometric function values for specific θ in all four quadrants. The example should be completed in a whole-class setting, with students writing their responses on paper or on individual white boards.

- In part (a), what rotation is represented by $-\frac{\pi}{3}$?
  - *It is a clockwise rotation by $\frac{\pi}{3}$ radians.*
- What positive rotation produces the same terminal ray as rotation by $-\frac{\pi}{3}$?
  - $2\pi - \frac{\pi}{3}$
- How can we verify the sign of $\sin\left(-\frac{\pi}{3}\right)$?
  - *The rotation results in a point on the unit circle in Quadrant IV, and the y-coordinates of points in Quadrant IV are negative.*
- How can we verify the sign of $\tan\left(\frac{5\pi}{4}\right)$?
  - *The rotation results in a point on the unit circle in Quadrant III, and the x- and y-coordinates in Quadrant III are negative, which means that ratio of the coordinates is positive.*

Scaffolding:

- Advanced students could compute the values without further prompting.
- Advanced students could be challenged to evaluate trigonometric functions for θ exceeding 2π. For example, they could evaluate $\tan\left(\frac{25\pi}{4}\right)$.

Example 2

Use your understanding of the unit circle and trigonometric functions to find the values requested.

a. $\sin\left(-\frac{\pi}{3}\right)$

$$\sin\left(-\frac{\pi}{3}\right) = \sin\left(2\pi - \frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

b. $\tan\left(\frac{5\pi}{4}\right)$

$$\tan\left(\frac{5\pi}{4}\right) = \tan\left(\pi + \frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1$$

Exercise 2 (4 minutes)

Students should complete the exercise independently. After a few minutes, they should verify their responses with a partner. At an appropriate time, selected students could share their answers.

Exercise 2

Use your understanding of the unit circle to determine the values of the functions shown.

a. $\sin\left(\frac{11\pi}{6}\right)$

$$\sin\left(\frac{11\pi}{6}\right) = \sin\left(2\pi - \frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$$

b. $\cos\left(\frac{3\pi}{4}\right)$

$\cos\left(\frac{3\pi}{4}\right) = \cos\left(\pi - \frac{\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

c. $\tan(-\pi)$

$\tan(-\pi) = \tan(\pi + 0) = \tan(0) = \frac{0}{1} = 0$

Closing (2 minutes)

Ask students to work with a partner to respond to the following statement:

- Anna says that for any real number θ, $\sin(\theta) = \sin(\pi - \theta)$. Is she correct? Explain how you know.
  - *Yes, Anna is correct. The point* $(\cos(\theta), \sin(\theta))$ *reflects across the* y*-axis to* $(\cos(\pi - \theta), \sin(\pi - \theta))$. *Since these two points have the same* y*-coordinate,* $\sin(\pi - \theta) = \sin(\theta)$.

Exit Ticket (5 minutes)

Name _____ Date _____

Lesson 1: Special Triangles and the Unit Circle

Exit Ticket

1. Evaluate the following trigonometric expressions, and explain how you used the unit circle to determine your answer.

   a. $\sin\left(\pi + \frac{\pi}{3}\right)$

   b. $\cos\left(2\pi - \frac{\pi}{6}\right)$

2. Corinne says that for any real number θ, $\cos(\theta) = \cos(\theta - \pi)$. Is she correct? Explain how you know.

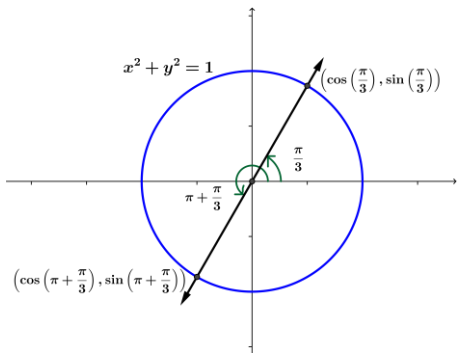
Exit Ticket Sample Solutions

1. **Evaluate the following trigonometric expressions, and explain how you used the unit circle to determine your answer.**

   a. $\sin\left(\pi+\frac{\pi}{3}\right)$

   $\sin\left(\pi+\frac{\pi}{3}\right)=-\sin\left(\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2}$

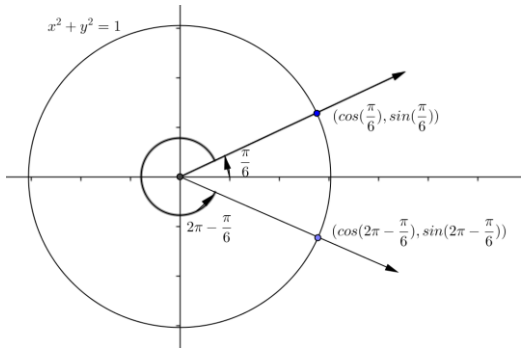
   Because the point $\left(\cos\left(\pi+\frac{\pi}{3}\right),\sin\left(\pi+\frac{\pi}{3}\right)\right)$ is directly opposite the point $\left(\cos\left(\frac{\pi}{3}\right),\sin\left(\frac{\pi}{3}\right)\right)$, we know that the values of $\sin\left(\frac{\pi}{3}\right)$ and $\sin\left(\pi+\frac{\pi}{3}\right)$ are opposites.



   b. $\cos\left(2\pi-\frac{\pi}{6}\right)$

   $\cos\left(2\pi-\frac{\pi}{6}\right)=\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$

   Because the point $\left(\cos\left(2\pi-\frac{\pi}{6}\right),\sin\left(2\pi-\frac{\pi}{6}\right)\right)$ is the reflection of the point $\left(\cos\left(\frac{\pi}{6}\right),\sin\left(\frac{\pi}{6}\right)\right)$ across the x-axis, we know that the values of $\cos\left(\frac{\pi}{6}\right)$ and $\cos\left(2\pi-\frac{\pi}{6}\right)$ are equal.



2. **Corinne says that for any real number θ, $\cos(\theta)=\cos(\theta-\pi)$. Is she correct? Explain how you know.**

   Yes, Corinne is correct. The point $(\cos(\theta),\sin(\theta))$ reflects across the y-axis to $(\cos(\pi-\theta),\sin(\pi-\theta))$. These two points have opposite x-coordinates, so $\cos(\pi-\theta)=\cos(\theta)$. Since the cosine function is an even function, $\cos(\pi-\theta)=\cos(-(\theta-\pi))=\cos(\theta-\pi)$.

   Thus, $\cos(\theta-\pi)=\cos(\theta)$.

Problem Set Sample Solutions

1. **Complete the chart below.**

θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\sin(\theta)$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos(\theta)$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan(\theta)$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

2. **Evaluate the following trigonometric expressions, and explain how you used the unit circle to determine your answer.**

   a. $\cos\left(\pi+\frac{\pi}{3}\right)$

   $\cos\left(\pi+\frac{\pi}{3}\right)=-\cos\left(\frac{\pi}{3}\right)=-\frac{1}{2}$

   The rotation was $\frac{\pi}{3}$ more than π bringing the ray to the third quadrant where cosine is negative.

   $\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$, *so* $-\cos\left(\frac{\pi}{3}\right)=-\frac{1}{2}$.

   b. $\sin\left(\pi-\frac{\pi}{4}\right)$

   $\sin\left(\pi-\frac{\pi}{4}\right)=\sin\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$

   The rotation was $\frac{\pi}{4}$ less than π bringing the ray to the second quadrant where sine is positive.

   $\sin\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$, *so* $\sin\left(\pi-\frac{\pi}{4}\right)=\sin\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$.

   c. $\sin\left(2\pi-\frac{\pi}{6}\right)$

   $\sin\left(2\pi-\frac{\pi}{6}\right)=-\sin\left(\frac{\pi}{6}\right)=-\frac{1}{2}$

   The rotation was $\frac{\pi}{6}$ less than 2π bringing the ray to the fourth quadrant where sine is negative.

   $\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$, *so* $\sin\left(2\pi-\frac{\pi}{6}\right)=-\sin\left(\frac{\pi}{6}\right)=-\frac{1}{2}$.

   d. $\cos\left(\pi+\frac{\pi}{6}\right)$

   $\cos\left(\pi+\frac{\pi}{6}\right)=-\cos\left(\frac{\pi}{6}\right)=-\frac{\sqrt{3}}{2}$

   The rotation was $\frac{\pi}{6}$ more than π bringing the ray to the third quadrant where cosine is negative.

   $\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$, *so* $-\cos\left(\frac{\pi}{6}\right)=-\frac{\sqrt{3}}{2}$.

e. $\cos\left(\pi - \frac{\pi}{4}\right)$

$\cos\left(\pi - \frac{\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

The rotation was $\frac{\pi}{4}$ less than π bringing the ray to the second quadrant where cosine is negative.

$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, *so* $-\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$.

f. $\cos\left(2\pi - \frac{\pi}{3}\right)$

$\cos\left(2\pi - \frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

The rotation was $\frac{\pi}{3}$ less than 2π bringing the ray to the fourth quadrant where cosine is positive.

$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, *so* $\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}$.

g. $\tan\left(\pi + \frac{\pi}{4}\right)$

$\tan\left(\pi + \frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1$

The rotation was $\frac{\pi}{4}$ more than π bringing the ray to the third quadrant where tangent is positive.

$\tan\left(\frac{\pi}{4}\right) = 1$, *so* $\tan\left(\frac{5\pi}{4}\right) = 1$.

h. $\tan\left(\pi - \frac{\pi}{6}\right)$

$\tan\left(\pi - \frac{\pi}{6}\right) = -\tan\left(\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$

The rotation was $\frac{\pi}{6}$ less than π bringing the ray to the second quadrant where tangent is negative.

$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$, *so* $-\tan\left(\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$.

i. $\tan\left(2\pi - \frac{\pi}{3}\right)$

$\tan\left(2\pi - \frac{\pi}{3}\right) = -\tan\left(\frac{\pi}{3}\right) = -\sqrt{3}$

The rotation was $\frac{\pi}{3}$ less than 2π bringing the ray to the fourth quadrant where tangent is negative.

$\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$, *so* $-\tan\left(\frac{\pi}{3}\right) = -\sqrt{3}$.

3. Rewrite the following trigonometric expressions in an equivalent form using $\pi + \theta$, $\pi - \theta$, or $2\pi - \theta$ and evaluate.

a. $\cos\left(\frac{\pi}{3}\right)$

$\cos\left(2\pi - \frac{\pi}{3}\right) = \frac{1}{2}$

b. $\cos\left(-\frac{\pi}{4}\right)$

$\cos\left(2\pi - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

c. $\sin\left(\frac{\pi}{6}\right)$

$\sin\left(\pi - \frac{\pi}{6}\right) = \frac{1}{2}$

d. $\sin\left(\frac{4\pi}{3}\right)$

$\sin\left(\pi + \frac{\pi}{3}\right) = \sin\left(2\pi - \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

e. $\tan\left(-\frac{\pi}{6}\right)$

$\tan\left(\pi - \frac{\pi}{6}\right) = \tan\left(2\pi - \frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}$

f. $\tan\left(-\frac{5\pi}{6}\right)$

$\tan\left(\pi + \frac{\pi}{6}\right) = \tan\left(2\pi + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$

4. Identify the quadrant of the plane that contains the terminal ray of a rotation by θ if θ satisfies the given conditions.

a. $\sin(\theta) > 0$ and $\cos(\theta) > 0$

Quadrant I

b. $\sin(\theta) < 0$ and $\cos(\theta) < 0$

Quadrant III

c. $\sin(\theta) < 0$ and $\tan(\theta) > 0$

Quadrant III

d. $\tan(\theta) > 0$ and $\sin(\theta) > 0$

Quadrant I

e. $\tan(\theta) < 0$ and $\sin(\theta) > 0$

Quadrant II

f. $\tan(\theta) < 0$ and $\cos(\theta) > 0$

Quadrant IV

g. $\cos(\theta) < 0$ and $\tan(\theta) > 0$

Quadrant III

h. $\sin(\theta) > 0$ and $\cos(\theta) < 0$

Quadrant II

5. Explain why $\sin^2(\theta) + \cos^2(\theta) = 1$.

For any real number θ the point $(\cos(\theta), \sin(\theta))$ lies on the unit circle with equation $x^2 + y^2 = 1$. Thus, we must have $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$. With the shorthand notation $(\sin(\theta))^2 = \sin^2(\theta)$ and $(\cos(\theta))^2 = \cos^2(\theta)$, this gives $\sin^2(\theta) + \cos^2(\theta) = 1$.

6. Explain how it is possible to have $\sin(\theta) < 0$, $\cos(\theta) < 0$, and $\tan(\theta) > 0$. For which values of θ between 0 and 2π does this happen?

Because $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$, if $\sin(\theta)$ and $\cos(\theta)$ are both negative, their quotient is positive. Thus, it is possible to have $\sin(\theta) < 0$, $\cos(\theta) < 0$, and $\tan(\theta) > 0$. This happens when the terminal ray of θ lies in the third quadrant, which is true for $\pi < \theta < \frac{3\pi}{2}$.

7. Duncan says that for any real number θ, $\tan(\theta) = \tan(\pi - \theta)$. Is he correct? Explain how you know.

No, Duncan is not correct. The terminal ray of rotation by θ and the terminal ray of rotation by $\pi - \theta$ are reflections of each other across the y-axis. Thus, $(\cos(\theta), \sin(\theta))$ is the reflection of $(\cos(\pi - \theta), \sin(\pi - \theta))$ across the y-axis. This means that $\cos(\pi - \theta) = -\cos(\theta)$ and $\sin(\pi - \theta) = \sin(\theta)$.

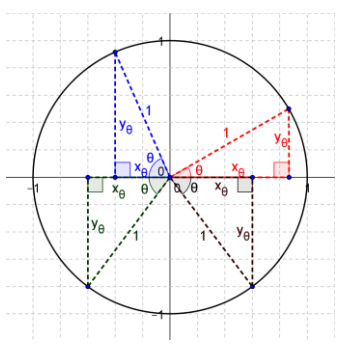
Thus,

$$\tan(\pi - \theta) = \frac{\sin(\pi - \theta)}{\cos(\pi - \theta)} = \frac{\sin(\theta)}{-\cos(\theta)} = -\tan(\theta).$$

We see that

$$\tan(\pi - \theta) \neq \tan(\theta).$$

8. Given the following trigonometric functions, identify the quadrant in which the terminal ray of θ lies in the unit circle shown below. Find the other two trigonometric functions of θ of $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$.



a. $\sin(\theta) = \frac{1}{2}$ and $\cos(\theta) > 0$

Quadrant I; $\theta = \frac{\pi}{6}$

$\sin(\theta) = \frac{1}{2}$, $\cos(\theta) = \frac{\sqrt{3}}{2}$, $\tan(\theta) = \frac{\sqrt{3}}{3}$

b. $\cos(\theta) = -\frac{1}{2}$ and $\sin(\theta) > 0$

Quadrant II; $\theta = \frac{\pi}{3}$

$\sin(\theta) = \frac{\sqrt{3}}{2}$, $\cos(\theta) = -\frac{1}{2}$, $\tan(\theta) = -\sqrt{3}$

c. $\tan(\theta) = 1$ and $\cos(\theta) < 0$

Quadrant III; $\theta = \frac{\pi}{4}$

$\sin(\theta) = -\frac{\sqrt{2}}{2}$, $\cos(\theta) = -\frac{\sqrt{2}}{2}$, $\tan(\theta) = 1$

d. $\sin(\theta) = -\frac{\sqrt{3}}{2}$ and $\cot(\theta) < 0$

Quadrant IV; $\theta = \frac{\pi}{3}$

$\sin(\theta) = -\frac{\sqrt{3}}{2}$, $\cos(\theta) = \frac{1}{2}$, $\tan(\theta) = -\sqrt{3}$

e. $\tan(\theta) = -\sqrt{3}$ and $\cos(\theta) < 0$

Quadrant II; $\theta = \frac{\pi}{3}$

$\sin(\theta) = \frac{\sqrt{3}}{2}$, $\cos(\theta) = -\frac{1}{2}$, $\tan(\theta) = -\sqrt{3}$

f. $\sec(\theta) = -2$ and $\sin(\theta) < 0$

Quadrant III; $\theta = \frac{\pi}{3}$

$\sin(\theta) = -\frac{\sqrt{3}}{2}$, $\cos(\theta) = -\frac{1}{2}$, $\tan(\theta) = \sqrt{3}$

g. $\cot(\theta) = \sqrt{3}$ and $\csc(\theta) > 0$

Quadrant I; $\theta = \frac{\pi}{6}$

$\sin(\theta) = \frac{1}{2}$, $\cos(\theta) = \frac{\sqrt{3}}{2}$, $\tan(\theta) = \frac{\sqrt{3}}{3}$

9. Toby thinks the following trigonometric equations are true. Use $\theta = \frac{\pi}{6}, \frac{\pi}{4}$, and $\frac{\pi}{3}$ to develop a conjecture whether or not he is correct in each case below.

a. $\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$

$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$

$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \cos\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$

$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{3}\right)$

Yes, he seems to be correct.

b. $\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right)$

$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} = \sin\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$

$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \sin\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$

$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right)$

Yes, he seems to be correct.

10. Toby also thinks the following trigonometric equations are true. Is he correct? Justify your answer.

a. $\sin\left(\pi - \frac{\pi}{3}\right) = \sin(\pi) - \sin\left(\frac{\pi}{3}\right)$

He is not correct because trigonometric functions are not linear.

$\sin\left(\pi - \frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\sin(\pi) - \sin\left(\frac{\pi}{3}\right) = 0 - \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$

b. $\cos\left(\pi - \frac{\pi}{3}\right) = \cos(\pi) - \cos\left(\frac{\pi}{3}\right)$

He is not correct because trigonometric functions are not linear.

$\cos\left(\pi - \frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$

$\cos(\pi) - \cos\left(\frac{\pi}{3}\right) = -1 - \frac{1}{2} = -\frac{3}{2}$

c. $\tan\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{6}\right)$

He is not correct because trigonometric functions are not linear.

$\tan\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$

$\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{6}\right) = \sqrt{3} - \frac{\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$

d. $\sin\left(\pi+\frac{\pi}{6}\right)=\sin(\pi)+\sin\left(\frac{\pi}{6}\right)$

He is not correct because trigonometric functions are not linear.

$\sin\left(\pi+\frac{\pi}{6}\right)=-\sin\left(\frac{\pi}{6}\right)=-\frac{1}{2}$

$\sin(\pi)+\sin\left(\frac{\pi}{6}\right)=0+\frac{1}{2}=\frac{1}{2}$

e. $\cos\left(\pi+\frac{\pi}{4}\right)=\cos(\pi)+\cos\left(\frac{\pi}{4}\right)$

He is not correct because trigonometric functions are not linear.

$\cos\left(\pi+\frac{\pi}{4}\right)=-\cos\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$

$\cos(\pi)+\cos\left(\frac{\pi}{4}\right)=-1+\frac{\sqrt{2}}{2}=\frac{-2+\sqrt{2}}{2}$