Chapter 4

Complex Analysis

4.1 Complex Differentiation

Recall the definition of differentiation for a real function $f(x)$:

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}.$$

In this definition, it is important that the limit is the same whichever direction we approach from. Consider $|x|$ at $x = 0$ for example; if we approach from the right ($\delta x \to 0^+$) then the limit is $+1$, whereas if we approach from the left ($\delta x \to 0^-$) the limit is -1. Because these limits are different, we say that $|x|$ is not differentiable at $x = 0$.

Now extend the definition to complex functions $f(z)$:

$$\boxed{f'(z) = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}.}$$

Again, the limit must be the same whichever direction we approach from; but now there is an infinity of possible directions.

Definition: if $f'(z)$ exists and is continuous in some region R of the complex plane, we say that f is *analytic* in R. If $f(z)$ is analytic in some small region around a point z_0, then we say that $f(z)$ is *analytic at* z_0. The term *regular* is also used instead of analytic.

Note: the property of analyticity is in fact a surprisingly strong one! For example, two consequences include:

(i) If a function is analytic then it is differentiable infinitely many times. (Cf. the existence of real functions which can be differentiated N times but no more, for any given N.)

(ii) If a function is analytic and bounded in the whole complex plane, then it is constant. (Liouville's Theorem.)

The Cauchy–Riemann Equations

Separate f and z into real and imaginary parts:

$$f(z) = u(x, y) + iv(x, y)$$

where $z = x + iy$ and u, v are real functions. Suppose that f is differentiable at z. We can take δz in any direction; first take it to be real, $\delta z = \delta x$. Then

$$\begin{aligned}
f'(z) &= \lim_{\delta x \to 0} \frac{f(z + \delta x) - f(z)}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{u(x + \delta x, y) + iv(x + \delta x, y) - u(x, y) - iv(x, y)}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{u(x + \delta x, y) - u(x, y)}{\delta x} + i \lim_{\delta x \to 0} \frac{v(x + \delta x, y) - v(x, y)}{\delta x} \\
&= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}.
\end{aligned}$$

Now take δz to be pure imaginary, $\delta z = i\delta y$. Then

$$\begin{aligned}
f'(z) &= \lim_{\delta y \to 0} \frac{f(z + i\delta y) - f(z)}{i\delta y} \\
&= \lim_{\delta y \to 0} \frac{u(x, y + \delta y) + iv(x, y + \delta y) - u(x, y) - iv(x, y)}{i\delta y} \\
&= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.
\end{aligned}$$

The two values for $f'(z)$ are the same since f is differentiable, so

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

$$\implies \qquad \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

– the *Cauchy–Riemann equations*. It is also possible to show that if the Cauchy–Riemann equations hold at a point z, then f is differentiable there (subject to certain technical conditions on the continuity of the partial derivatives).

If we know the real part u of an analytic function, the Cauchy–Riemann equations allow us to find the imaginary part v (up to a constant), and vice versa. For example, if $u(x, y) = x^2 - y^2$ then

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x \qquad \implies \qquad v = 2xy + g(x)$$

for some function $g(x)$; so

$$-2y = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -2y - g'(x) \quad \Longrightarrow \quad g'(x) = 0 \quad \Longrightarrow \quad g = \text{const.} = \alpha, \text{ say.}$$

Hence

$$f(z) = x^2 - y^2 + 2ixy + i\alpha = (x + iy)^2 + i\alpha = z^2 + i\alpha.$$

Examples of Analytic Functions

(i) $f(z) = z$ is analytic in the whole of $\mathbb{C}$. Here $u = x$, $v = y$, and the Cauchy–Riemann equations are satisfied ($1 = 1$; $0 = 0$).

(ii) $f(z) = z^n$ (n a positive integer) is analytic in $\mathbb{C}$. Here we write $z = r(\cos\theta + i\sin\theta)$ and by de Moivre's theorem, $z^n = r^n(\cos n\theta + i\sin n\theta)$. Hence $u = r^n\cos n\theta$ and $v = r^n\sin n\theta$: we can check the Cauchy–Riemann equations (using $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(y/x)$). The derivative is nz^{n-1}, as we might expect!

(iii) $f(z) = e^z = e^x e^{iy} = e^x(\cos y + i\sin y)$. So

$$\frac{\partial u}{\partial x} = e^x\cos y = \frac{\partial}{\partial y}(e^x\sin y) = \frac{\partial v}{\partial y}; \qquad \frac{\partial u}{\partial y} = -e^x\sin y = -\frac{\partial v}{\partial x}.$$

The derivative is

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x\cos y + ie^x\sin y = e^z,$$

again as expected.

(iv) $f(z) = 1/z$: check that this is analytic with derivative $-1/z^2$ in any region R which does *not* include the origin.

(v) Any rational function – i.e., $f(z) = P(z)/Q(z)$ where P and Q are polynomials – is analytic *except* at points where $Q(z) = 0$. For instance, $f(z) = (z-i)/(z+i)$ is analytic except at $z = -i$.

(vi) Many standard functions obey the usual rules for their derivatives; e.g.,

$$\frac{\mathrm{d}}{\mathrm{d}z}\sin z = \cos z, \qquad \frac{\mathrm{d}}{\mathrm{d}z}\sinh z = \cosh z,$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\cos z = -\sin z, \qquad \frac{\mathrm{d}}{\mathrm{d}z}\cosh z = \sinh z,$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\log z = \frac{1}{z} \quad \text{(when } \log z \text{ is defined as later).}$$

The product, quotient and chain rules hold in exactly the same way as for real functions.

Examples of Non-Analytic Functions

(i) $f(z) = \mathrm{Re}(z)$. Here $u = x$, $v = 0$, but $1 \neq 0$. $\mathrm{Re}(z)$ is *nowhere* analytic.

(ii) $f(z) = |z|$; here $u = \sqrt{x^2 + y^2}$, $v = 0$. Ditto!

(iii) $f(z) = \bar{z} = x - iy$ (complex conjugate, also denoted z^*). Here $u = x$, $v = -y$, so $\partial u/\partial x = 1 \neq -1 = \partial v/\partial y$. Ditto!

(iv) $f(z) = |z|^2 = x^2 + y^2$. The Cauchy–Riemann equations are only satisfied at the origin, so f is only differentiable at $z = 0$. However, it is not analytic there because there is no small region containing the origin within which f is differentiable.

Harmonic Functions

Suppose $f(z) = u + iv$ is analytic. Then

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) \\ &= \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2}.\end{aligned}$$

Hence

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

i.e., u satisfies Laplace's equation in two dimensions. Similarly, v does too. Such functions u and v are said to be *harmonic*.

4.2 Zeros of Complex Functions

The zeros of $f(z)$ are the points z_0 where $f(z_0) = 0$. A zero is of order n if

$$0 = f'(z_0) = f''(z_0) = \cdots = f^{(n-1)}(z_0), \quad \text{but } f^{(n)}(z_0) \neq 0.$$

A zero of order one (i.e., one where $f'(z_0) \neq 0$) is called a *simple* zero. Examples:

(i) $f(z) = z$ has a simple zero at $z = 0$.

(ii) $f(z) = (z - i)^2$ has a zero of order two at $z = i$.

(iii) $f(z) = z^2 - 1 = (z - 1)(z + 1)$ has two simple zeros at $z = \pm 1$.

(iv) $f(z) = (z - w)^N g(z)$, where w is a complex constant, N a positive integer and $g(z)$ an analytic function satisfying $g(w) \neq 0$, has a zero of order N at $z = w$.

(v) Where are the zeros of $f(z) = \sinh z$? We know there is a simple zero at $z = 0$. The others are where

$$0 = \sinh z = \frac{e^z - e^{-z}}{2} \quad \iff \quad e^z = e^{-z} \quad \iff \quad e^{2z} = 1 \quad \iff \quad 2z = 2n\pi i,$$

where n is an integer. (Check that $e^{x+iy} = 1 \Leftrightarrow x = 0$ and $y = 2n\pi$.) So the zeros are on the imaginary axis at $\dots, -2\pi i, -\pi i, 0, \pi i, 2\pi i, 3\pi i, \dots$, and they are all simple.

Another way of defining the order of a zero is by the first non-zero power of $(z - z_0)$ in its Taylor series. For example, consider the zero of $\sinh^3 z$ at $z = \pi i$. Now $\sinh z = -\sinh(z - \pi i) = -\sinh\zeta$ where $\zeta = z - \pi i$, and close to $z = \pi i$ the Taylor series for $\sinh z$ is therefore

$$-(\zeta + \tfrac{1}{3!}\zeta^3 + \cdots).$$

Hence the Taylor series for $\sinh^3 z$ at $z = \pi i$ is

$$-(\zeta + \tfrac{1}{3!}\zeta^3 + \cdots)^3 = -(z - \pi i)^3 - \tfrac{1}{2}(z - \pi i)^5 + \cdots.$$

The zero is therefore of order 3.

4.3 Laurent Expansions

Suppose that $f(z)$ is analytic at z_0. Then we can expand f in a Taylor Series about z_0:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for suitable complex constants a_n.

Example: e^z has a Taylor Series about $z = i$ given by

$$e^z = e^i e^{z-i} = e^i \sum_{n=0}^{\infty} \frac{(z-i)^n}{n!},$$

so $a_n = e^i/n!$.

Now consider an $f(z)$ which is *not* analytic at z_0, but for which $(z - z_0)f(z)$ *is* analytic. (E.g., $f(z) = e^z/(z - z_0)$.) Then, for suitable b_n,

$$\begin{aligned}
(z - z_0)f(z) &= \sum_{n=0}^{\infty} b_n (z - z_0)^n \\
\implies \qquad f(z) &= \frac{b_0}{z - z_0} + b_1 + b_2(z - z_0) + b_3(z - z_0)^2 + \cdots \\
&= \sum_{n=-1}^{\infty} a_n (z - z_0)^n
\end{aligned}$$

where $a_n = b_{n+1}$. Generalising this, if $(z-z_0)^N f(z)$ is analytic at z_0 then for suitable a_n,

$$f(z) = \sum_{n=-N}^{\infty} a_n (z-z_0)^n.$$

But what if however large N is, $(z-z_0)^N f(z)$ is still not analytic at z_0? We might try taking N to be "infinite", and in fact this does always work.

Formally, it is possible to show that if $f(z)$ is analytic in an annulus $a < |z-z_0| < b$ for some a, b (regardless of whether f is analytic at z_0 itself) then f has a unique *Laurent expansion*

$$\boxed{f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n}$$

in the annulus.

Examples:

(i) $f(z) = e^z/z^3 = \sum_{n=0}^{\infty} z^{n-3}/n!$ about $z_0 = 0$ has $a_n = 0$ for $n < -3$ and $a_n = 1/(n+3)!$ for $n \geq -3$.

(ii) $f(z) = e^z/(z^2-1)$ about $z_0 = 1$ (where it has a singularity). Here we write everything in terms of $\zeta = z - z_0$, so

$$\begin{aligned} f(z) &= \frac{e^\zeta e^{z_0}}{\zeta(\zeta+2)} = \frac{e^{z_0}}{2\zeta} e^\zeta (1 + \tfrac{1}{2}\zeta)^{-1} \\ &= \frac{e}{2\zeta}(1 + \zeta + \tfrac{1}{2!}\zeta^2 + \cdots)(1 - \tfrac{1}{2}\zeta + \cdots) \\ &= \frac{e}{2\zeta}(1 + \tfrac{1}{2}\zeta + \cdots) \\ &= \frac{e}{2}\left(\frac{1}{z-z_0} + \frac{1}{2} + \cdots\right). \end{aligned}$$

Hence $a_{-1} = e/2$, $a_0 = e/4$, etc.

(iii) $f(z) = \exp(1/z)$ about $z_0 = 0$ has

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots,$$

so that $a_n = 1/(-n)!$ for $n \leq 0$.

(iv) This doesn't seem to work for $f(z) = z^{-1/2}$ – why? We shall see later that it is impossible to find an annulus around $z_0 = 0$ in which $z^{-1/2}$ is analytic.

If $f(z)$ is in fact analytic at $z = z_0$, then its Laurent expansion about z_0 is just its Taylor series.

4.4 Classification of Singularities

Suppose that f has a singularity at $z = z_0$, but is analytic within some circle $|z - z_0| < r$ except at z_0 itself. Such a singularity is called an *isolated singularity.* Choosing any annulus inside the circle (e.g., $r/2 < |z - z_0| < r$), we see that f has a Laurent expansion about z_0. We can use the coefficients a_n of the expansion to classify the singularity; there are three possible cases.

Essential Isolated Singularities

If there is no integer N such that $a_n = 0$ for all $n < -N$ – i.e., if however far n goes towards $-\infty$ there are always some non-zero a_n's – then f is said to have an *essential isolated singularity.* Examples:

(i) $\exp(1/z)$ has an essential isolated singularity at $z = 0$, because all the a_n's are non-zero for $n \le 0$ (we showed above that $a_n = 1/(-n)!$).

(ii) $\sin(1/z)$ also has an essential isolated singularity at $z = 0$, because

$$a_n = \begin{cases} (-1)^{(n+1)/2}/(-n)! & n \text{ negative and odd,} \\ 0 & n \text{ positive or even.} \end{cases}$$

However negative n is, there are some non-zero a_n's for still more negative n.

Near an essential isolated singularity of a function $f(z)$, it can be shown that f takes *all* possible complex values (bar at most one). For example, $\sin(1/z)$ takes all possible complex values near the origin; $\exp(1/z)$ takes all except zero.

Poles

If $a_n = 0$ for all $n < -N$ (where N is some specific positive integer) but $a_{-N} \neq 0$, then f is said to have a *pole* of order N. (If $N = 1$, then we call this a *simple pole.*) This is the most common, and the most important, of the three cases. Examples:

(i) $1/(z - i)$ has a simple pole at $z = i$.

(ii) $(\cos z)/z$ (which has Laurent expansion $z^{-1} - \frac{1}{2!}z + \frac{1}{4!}z^3 - \cdots$) has a simple pole at $z = 0$.

(iii) $1/(z^2 + 1)$ has two simple poles, at $z = \pm i$.

(iv) $z^2/\{(z-3)^3(z-i)^2\}$ has a pole of order 2 at $z=i$ and a pole of order 3 at $z=3$.

To show formally that, for instance, there is a pole of order 2 at $z=i$, notice first that $z^2/(z-3)^3$ is analytic there so has a Taylor series

$$b_0 + b_1(z-i) + b_2(z-i)^2 + \cdots .$$

Hence

$$\frac{z^2}{(z-3)^3(z-i)^2} = \frac{b_0}{(z-i)^2} + \frac{b_1}{(z-i)} + b_2 + \cdots .$$

(v) If $g(z)$ has a zero of order m at $z=z_0$ then $1/g(z)$ has a pole of order m there (and vice versa). Hence $\cot z$ has a simple pole at the origin, because $\tan z$ has a simple zero there.

To prove this, note that $g(z)=(z-z_0)^m G(z)$ for some function $G(z)$ satisfying $G(z_0)\neq 0$. The function $1/G(z)$ is analytic at $z=z_0$, so it has a Taylor series $c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots$. Hence

$$1/g(z) = c_0(z-z_0)^{-m} + c_1(z-z_0)^{-m+1} + c_2(z-z_0)^{-m+2} + \cdots$$

as required.

Removable Singularities

If $a_n = 0$ for all $n<0$ (so that the Laurent expansion is just $a_0 + a_1(z-z_0) + \cdots$), then f is said to have a *removable singularity.* By redefining $f(z_0)=a_0$ we can remove the singularity completely. Examples:

(i) (Somewhat contrived.)

$$f(z) = \begin{cases} 0 & z = 0 \\ 1 & z \neq 0 \end{cases}$$

has a singularity at $z=0$. Because the origin is not part of any annulus around itself, so that $f(z)=1$ everywhere in the annulus, the Laurent expansion has $a_0=1$ and all other $a_n=0$, so f has a removable singularity; by redefining $f(0)=1$ we obtain an analytic function.

(ii) $f(z)=(\sin z)/z$ is *not defined* at $z=0$, but by defining $f(0)=1$ we obtain an analytic function.

(iii) A rational function $f(z)=P(z)/Q(z)$ (where P and Q are polynomials) has a singularity at any point z_0 where $Q(z_0)=0$; but if $P(z_0)=0$ as well then the singularity is removable by redefining $f(z_0)=P'(z_0)/Q'(z_0)$, assuming that $Q'(z_0)\neq 0$.

Note: the singularity of $z^{-1/2}$ at $z=0$ is not classified under this scheme (though it might look as if it has a pole), as it has no Laurent expansion. (See §4.7 for further explanation: in fact $z=0$ is a *branch point singularity.*)

4.5 Residues

We shall see in Chapter 5 that it is important to be able to calculate the coefficient a_{-1} of the Laurent expansion of a function $f(z)$ about a pole at z_0. This coefficient is called the *residue* of the pole, which we shall denote by $\operatorname*{res}_{z=z_0} f(z)$.

At a simple pole, the residue is given by $a_{-1} = \lim_{z\to z_0}\{(z-z_0)f(z)\}$, because:

$$\begin{aligned}\lim_{z\to z_0}\{(z-z_0)f(z)\} &= \lim_{z\to z_0}\left\{(z-z_0)\left(\frac{a_{-1}}{z-z_0}+a_0+a_1(z-z_0)+\cdots\right)\right\}\\ &= \lim_{z\to z_0}\{a_{-1}+a_0(z-z_0)+a_1(z-z_0)^2+\cdots\}\\ &= a_{-1}.\end{aligned}$$

In general, at a pole of order N, the residue is given by the useful formula

$$\boxed{a_{-1} = \lim_{z\to z_0}\left\{\frac{1}{(N-1)!}\frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}}\big((z-z_0)^N f(z)\big)\right\}}$$

which can easily be proved by first writing down the Laurent expansion of $f(z)$ and then evaluating the right-hand side of the above formula.

4.6 The Point at Infinity

In the complex plane, we can reach the "point at infinity" by going off in any direction. Conceptually, we may use the *Riemann Sphere*, which is a sphere resting on the complex plane with its "South Pole" at $z=0$.

For any point in $\mathbb{C}$, drawing a line through the "North Pole" of the sphere to the point, and noting where this line intersects the sphere, specifies an equivalent point on the sphere. Then the point at infinity is equivalent to the "North Pole" of the sphere itself.

We can use all the concepts of §4.4 on the point at infinity by using the transformation $\zeta = 1/z$. Let $g(\zeta) = f(1/\zeta)$ and find the Laurent expansion of g at $\zeta=0$. Any type of singularity that g has at $\zeta=0$ is also said to apply to f "at infinity". Examples:

(i) $f(z) = z^n$ has a pole of order n at ∞ (because $g(\zeta) = f(1/\zeta) = \zeta^{-n}$ which has a pole of order n at $\zeta = 0$).

(ii) $f(z) = e^z$ has an essential singularity at ∞.

The *residue at infinity* of f is, similarly, defined to be the residue of $g(\zeta)$ at $\zeta = 0$; so, for example, $f(z) = 2z$ has a simple pole at ∞ (because $g(\zeta) = 2/\zeta$) with residue 2.

4.7 Multi-Valued Functions

In this section we shall use as our archetypal example $\log z$, the natural logarithm function. For a complex number $z = re^{i\theta}$, we define $\log z = \log r + i\theta$. There are thus infinitely many values, or "branches", of $\log z$, for θ may take an infinity of values. For example,

$$\log i = \frac{\pi i}{2} \text{ or } \frac{5\pi i}{2} \text{ or } -\frac{3\pi i}{2} \text{ or}\ldots,$$

depending on which choice of θ we make.

Branch Points

Consider the three curves shown in the diagram. On C_1, we could choose θ to be always in the range $(0, \frac{\pi}{2})$, and then $\log z$ would be continuous and single-valued going round C_1. On C_2, we could choose $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$ and $\log z$ would again be continuous and single-valued. But for C_3, which encircles the origin, there is no such choice; whatever we do, $\log z$ cannot be made continuous around C_3 (it must either "jump" somewhere or be multi-valued). A *branch point* of a function – here, the origin – is a point which it is impossible to encircle with a curve upon which the function is continuous and single-valued. The function is said to have a *branch point singularity* at that point.

Examples:

(i) $\log(z - a)$ has a branch point at $z = a$.

(ii) $\log(z^2 - 1) = \log(z + 1) + \log(z - 1)$ has two branch points, at ± 1.

(iii) $z^{1/2} = \sqrt{r}\, e^{i\theta/2}$ has a branch point at the origin. (Useful exercise: verify this.) The same is true of $z^\alpha = r^\alpha e^{i\alpha\theta}$ when α is any non-integer.

Branch Cuts

If we wish to make $\log z$ continuous and single-valued, therefore, we must stop any curve from encircling the origin. We do this by introducing a *branch cut* from $-\infty$ on the real axis to the origin. No curve is allowed to cross this cut. We can then decide to fix on values of θ lying in the range $-\pi < \theta \le \pi$ *only*, and we have defined a *branch* of $\log z$ which is single-valued and continuous on any curve (which doesn't cross the cut). This branch is analytic everywhere (with $\frac{\mathrm{d}}{\mathrm{d}z} \log z = 1/z$) *except* on the negative real axis.

(If a curve *did* cross the cut, from above it to below it say, then θ would jump from $+\pi$ to $-\pi$, and $\log z$ would be discontinuous and therefore not analytic. This is not allowed.)

This cut is the canonical (i.e., "standard") branch cut for $\log z$, and the resulting value of $\log z$ is called the principal value of the logarithm.

What are the values of $\log z$ just above and below the branch cut? Consider a point on the negative real axis, $z = x$, $x < 0$. Just above the cut, at $z = x + i0^+$, $\theta = +\pi$, so $\log z = \log|x| + i\pi$. Just below it, at $z = x + i0^-$, $\log z = \log|x| - i\pi$.

Note that many different branch cuts are possible: *any cut which stops curves wrapping round the branch point will do*. In diagram (a), we could choose $-3\pi/2 < \theta \le \pi/2$; the exact choice is more difficult to write down in case (b), but this is an equally valid cut.

Exactly the same choices of branch cut can be made for z^α (when α is not an integer). Note that this implies that neither $\log z$ nor z^α have Laurent expansions about the origin: for any annulus $a < |z| < b$ would have to be crossed by a branch cut, so the function would not be analytic in the annulus.

Multiple Branch Cuts

When there is more than one branch point, we may need more than one branch cut. For $f(z) = \{z(z-1)\}^{1/3}$ there are two branch points, at 0 and 1.

We need two branch cuts; a possibility is shown in the diagram. Then no curve can wrap round either 0 or 1. For any z, we write $z = r_0 e^{i\theta_0}$ where $-\pi < \theta_0 \le \pi$ and $z - 1 = r_1 e^{i\theta_1}$ where $0 \le \theta_1 < 2\pi$. Then we define

$$\{z(z-1)\}^{1/3} = \sqrt[3]{r_0 r_1}\, e^{i(\theta_0+\theta_1)/3}.$$

This is continuous so long as we don't cross either cut.

The value of $f(z)$ just above the cut on the positive real axis at $z = x$ is $\sqrt[3]{x(x-1)}$ (since $\theta_0 = \theta_1 = 0$ there); just below it is $\sqrt[3]{x(x-1)}\, e^{2\pi i/3}$ ($\theta_0 = 0$, $\theta_1 = 2\pi$). For the cut on the negative real axis we have $\sqrt[3]{|x|(|x|+1)}\, e^{2\pi i/3}$ just above and $\sqrt[3]{|x|(|x|+1)}$ just below.

Example: consider

$$\Phi(x, y) = \operatorname{Im}\left(\frac{2T_0}{\pi} \log \frac{z+a}{z-a}\right) + 2T_0$$

where $z = x + iy$, a is a real constant and we use the branch cuts shown below to make log analytic.

Then

$$\Phi = \frac{2T_0}{\pi}(\theta_0 - \theta_1) + 2T_0$$

where $-\pi < \theta_0 \le \pi$ and $0 \le \theta_1 < 2\pi$. On the circle $|z| = a$, what is Φ? Above the real axis, we know that $\theta_0 + (\pi - \theta_1) = \pi/2$ (property of circles), so $\Phi = +T_0$. Below the real axis, $(-\theta_0) + (\theta_1 - \pi) = \pi/2$ (same property), so $\Phi = -T_0$. We also note that everywhere in the circle $|z| < a$, $\nabla^2\Phi = 0$ as Φ is the imaginary part of an analytic function.

Hence Φ is the steady-state temperature distribution in a cylinder heated on one side to $+T_0$ and on the other to $-T_0$; we solved this problem using separation of variables in Chapter 2. To see the connection between the solutions, write $\theta_0 = \tan^{-1}\bigl(y/(x+a)\bigr)$ and $\theta_1 = \pi + \tan^{-1}\bigl(y/(x-a)\bigr)$; then construct the Fourier Series for

$$\frac{2T_0}{\pi}\left(\tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x-a}\right).$$
