

Exercises and solutions  
for  
**Essentials of Programming in**  
***Mathematica***<sup>®</sup>

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## I

# Programming with *Mathematica*

### I.2 Getting started: exercises

1. Generate a random real number between one and one hundred. Then create a vector of twelve random numbers between one and one hundred. Finally, create a  $4 \times 4$  array of such numbers and then compute the determinant of that array.
2. Using `Table`, create a  $4 \times 4$  Hilbert matrix. The entry  $a_{ij}$  in row  $i$ , column  $j$  of the Hilbert matrix is given by  $\frac{1}{i+j-1}$ . Check your solution against the built-in `HilbertMatrix`.
3. Add the two lists,  $\{1, 2, 3, 4, 5\}$  and  $\{2, 4, 6, 8, 10\}$ . Then multiply them element-wise. Finally, multiply the two lists as vectors (dot product).
4. Generate a list of the first twenty-five integers in five different ways.
5. Add the integers one through one thousand in as many different ways as you can.
6. A  $2 \times 2$  matrix can be created using lists such as  $\{\{a, b\}, \{c, d\}\}$ . Define a  $2 \times 2$  numerical matrix and then find its inverse, determinant, transpose, and trace.
7. Create the following matrix using list notation:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Then find the inverse, determinant, and transpose of the matrix. Finally, compute the fifth matrix power of this matrix ( $m.m.m.m.m$ ).

### I.2 Solutions

1. First, here is a random real number between one and one hundred.

```
In[1]:= RandomReal[{1, 100}]
```

```
Out[1]= 83.1802
```

This gives a vector of twelve such random numbers.

```
In[2]:= RandomReal[{1, 100}, {12}]
```

```
Out[2]= {91.5353, 8.35684, 68.0464, 57.9997, 37.9279,
          97.2452, 62.8762, 36.5797, 4.61388, 87.2442, 58.6649, 45.0762}
```

A  $4 \times 4$  array of them.

```
In[3]:= RandomReal[{1, 100}, {4, 4}]
```

```
Out[3]= {{71.1387, 15.5669, 32.9543, 10.6077}, {8.89123, 59.9753, 5.55336, 92.4248},
          {31.9101, 49.9324, 51.6797, 26.3127}, {79.4064, 50.808, 4.1497, 82.4448}}
```

And the determinant of this array.

```
In[4]:= Det[%]
```

```
Out[4]= -3.65557 × 106
```

2. The entry  $a_{ij}$  entry of the Hilbert matrix is defined as  $\frac{1}{i+j-1}$ .

```
In[5]:= Table[ $\frac{1}{i+j-1}$ , {j, 4}, {i, 4}]
```

```
Out[5]= {{1,  $\frac{1}{2}$ ,  $\frac{1}{3}$ ,  $\frac{1}{4}$ }, { $\frac{1}{2}$ ,  $\frac{1}{3}$ ,  $\frac{1}{4}$ ,  $\frac{1}{5}$ }, { $\frac{1}{3}$ ,  $\frac{1}{4}$ ,  $\frac{1}{5}$ ,  $\frac{1}{6}$ }, { $\frac{1}{4}$ ,  $\frac{1}{5}$ ,  $\frac{1}{6}$ ,  $\frac{1}{7}$ }}
```

Check against the built-in function.

```
In[6]:= HilbertMatrix[4]
```

```
Out[6]= {{1,  $\frac{1}{2}$ ,  $\frac{1}{3}$ ,  $\frac{1}{4}$ }, { $\frac{1}{2}$ ,  $\frac{1}{3}$ ,  $\frac{1}{4}$ ,  $\frac{1}{5}$ }, { $\frac{1}{3}$ ,  $\frac{1}{4}$ ,  $\frac{1}{5}$ ,  $\frac{1}{6}$ }, { $\frac{1}{4}$ ,  $\frac{1}{5}$ ,  $\frac{1}{6}$ ,  $\frac{1}{7}$ }}
```

3. First, add the two lists.

```
In[7]:= {1, 2, 3, 4, 5} + {2, 4, 6, 8, 10}
```

```
Out[7]= {3, 6, 9, 12, 15}
```

To multiply the two lists, put a space between them.

```
In[8]:= {1, 2, 3, 4, 5} {2, 4, 6, 8, 10}
```

```
Out[8]= {2, 8, 18, 32, 50}
```

The dot product can be computed using either a built-in function or traditional mathematical notation.

```
In[9]:= Dot[{1, 2, 3, 4, 5}, {2, 4, 6, 8, 10}]
```

```
Out[9]= 110
```

```
In[10]:= {1, 2, 3, 4, 5} . {2, 4, 6, 8, 10}
```

```
Out[10]= 110
```

4. First the straightforward ways of generating the list of the first twenty-five integers.

```
In[11]:= Range[25]
```

```
Out[11]= {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,
          13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25}
```

Here are two alternate syntaxes for the Range function, a prefix form and a postfix form:

```
In[12]:= Range@25
```

```
Out[12]= {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,
          13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25}
```

```
In[13]:= 25 // Range
```

```
Out[13]= {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,
          13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25}
```

Using Table:

```
In[14]:= Table[i, {i, 1, 25}]
```

```
Out[14]= {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,
          13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25}
```

The coefficients of the linear term in the first 25 rows of the binomial expansion of  $(1 + x)^n$ :

```
In[15]:= Table[Binomial[n, 1], {n, 1, 25}]
```

```
Out[15]= {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,
          13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25}
```

A very roundabout (and not terribly efficient) way to do this: get the coefficients of the following series.

```
In[16]:= series = Series[ $\frac{1}{(1-x)^2}$ , {x, 0, 24}]
```

```
Out[16]= 1 + 2 x + 3 x^2 + 4 x^3 + 5 x^4 + 6 x^5 + 7 x^6 + 8 x^7 + 9 x^8 + 10 x^9 +
          11 x^10 + 12 x^11 + 13 x^12 + 14 x^13 + 15 x^14 + 16 x^15 + 17 x^16 + 18 x^17 +
          19 x^18 + 20 x^19 + 21 x^20 + 22 x^21 + 23 x^22 + 24 x^23 + 25 x^24 + O[x]^25
```

```
In[17]:= CoefficientList[Normal[series], x]
```

```
Out[17]= {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,
          13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25}
```

5. Several mathematical functions are built in that will compute the sum directly.

```
In[18]:= Sum[i, {i, 10^3}]
```

```
Out[18]= 500500
```

The same sum can be computed using familiar mathematical notation, templates for which are available from the Palettes menu.

```
In[19]:= 
$$\sum_i^{10^3} i$$

```

```
Out[19]= 500 500
```

In fact the sum can be done in parallel (assuming you are working on a multi-core machine) by using a parallelized version of Sum.

```
In[20]:= ParallelSum[i, {i, 103}]
```

```
Out[20]= 500 500
```

If you are already acquainted with an imperative style of programming, such as is found in procedural languages like C, FORTRAN, and JAVA, then the following looping constructs should be familiar.

```
In[21]:= i = 0;
Do[i = i + j, {j, 103}] ;
i
```

```
Out[23]= 500 500
```

```
In[24]:= res = 0;
i = 0;
For[i = 1, i ≤ 103, i++, res = i + res] ;
res
```

```
Out[27]= 500 500
```

```
In[28]:= res = 0;
i = 0;
While[i ≤ 103, res = i + res; i++];
res
```

```
Out[31]= 500 500
```

A declarative style of programming – including functional languages such as LISP, HASKELL, SCHEME, F# – is one in which functions are used to declare what the program should do, rather than giving explicit steps or actions to be performed. In such languages, computation is done by evaluating functions that operate on the appropriate inputs. In this case, the input is a list of the first thousand positive integers; the semicolon is used to suppress the display of the output of that particular input.

```
In[32]:= lis = Range[103];
```

```
In[33]:= Apply[Plus, lis]
```

```
Out[33]= 500 500
```

```
In[34]:= Fold[Plus, 0, lis]
```

```
Out[34]= 500 500
```

```

In[35]:= Last[Accumulate[lis]]
Out[35]= 500 500

In[36]:= Total[lis]
Out[36]= 500 500

```

Other approaches can be considered as well, including a recursive approach (Prolog, Haskell or Scheme), one using replacement rules, and another using sparse array arithmetic.

```

In[37]:= s[0] = 0;
          s[n_] := s[n] = s[n - 1] + n

In[39]:= s[1000]
Out[39]= 500 500

In[40]:= lis /. {x_, y__} -> x + y
Out[40]= 500 500

In[41]:= mat = SparseArray[{i_, i_} -> lis[[i]], {10^3, 10^3}];
          Tr[mat]
Out[42]= 500 500

```

One approach to computing the sum of the first  $n$  integers is to use the fact (known to Gauss at a young age) that the sum is equal to the binomial  $\binom{n+1}{2}$ .

```

In[43]:= n = 10^3;
          Binomial[n + 1, 2]
Out[44]= 500 500

```

Although all of these approaches give the same answer (they had better!), some are more efficient in terms of memory management and some are faster than others. We don't expect that these examples will all make sense to you at this point but after having read this book, you should be quite comfortable with each of these paradigms so that you can apply them to a wide variety of problems and choose the best approach for the programming problems you will encounter.

6. Here is a  $2 \times 2$  matrix filled with numbers.

```
In[45]:= mat = {{3, 5}, {1, 2}};
```

And here are the operations on that matrix.

```

In[46]:= Inverse[mat]
Out[46]= {{2, -5}, {-1, 3}}

In[47]:= Det[mat]
Out[47]= 1

```

```
In[48]:= Transpose[mat]
```

```
Out[48]= {{3, 1}, {5, 2}}
```

```
In[49]:= Tr[mat]
```

```
Out[49]= 5
```

7. Here is the list representation of the matrix given in the exercise.

```
In[50]:= mat = {{1, 1}, {1, 0}};
```

Here are the inverse, determinant, and transpose.

```
In[51]:= Inverse[mat]
```

```
Out[51]= {{0, 1}, {1, -1}}
```

```
In[52]:= Det[mat]
```

```
Out[52]= -1
```

```
In[53]:= Transpose[mat]
```

```
Out[53]= {{1, 1}, {1, 0}}
```

As for the matrix power, you could write it out explicitly.

```
In[54]:= mat.mat.mat.mat.mat
```

```
Out[54]= {{8, 5}, {5, 3}}
```

But that is tedious and wouldn't be sensible for the 100th power say. Another built-in function can be used for the matrix power. You should see the Fibonacci numbers lurking in the output, a fact which is explored in Section 5.4.

```
In[55]:= MatrixPower[mat, 5]
```

```
Out[55]= {{8, 5}, {5, 3}}
```

```
In[56]:= MatrixPower[mat, 100]
```

```
Out[56]= {{573 147 844 013 817 084 101, 354 224 848 179 261 915 075},
          {354 224 848 179 261 915 075, 218 922 995 834 555 169 026}}
```

```
In[57]:= Fibonacci[101]
```

```
Out[57]= 573 147 844 013 817 084 101
```

```
In[58]:= Clear[s, lis, mat, n, i, res]
```

## The *Mathematica* language

### 2.1 Expressions: exercises

1. Determine if each of the following are atomic expressions. If the expression is not atomic, find its head.
  - a.  $8 / 5$
  - b.  $8 / 5 + x$
  - c.  $\{\{a, b\}, \{c, d\}\}$
  - d.  $"8/5 + x"$
2. Give the full (internal) form of the expression  $a (b + c)$ .
3. What is the traditional representation of  $\text{Times}[a, \text{Power}[\text{Plus}[b, c], -1]]$ .
4. What is the part specification of the symbol  $b$  in the expression  $a x^2 + b x + c$ ?
5. What will be the result of evaluating each of the following? Use `FullForm` on the expressions to help you understand their structures.
  - a.  $(x^2 + y) z / w$  `[[2, 1, 2]]`
  - b.  $(a / b)$  `[[2, 2]]`
6. Use `Level` to find all the factors in the following expression. Then find all the terms inside the parentheses of the output.
 

```
In[1]:= expr = LegendreP[5, x]
```

```
Out[1]=  $\frac{1}{8} (15 x - 70 x^3 + 63 x^5)$ 
```
7. Explain why the following expression returns an integer instead of displaying the internal representation of the fraction.

```
In[2]:= FullForm[ $\frac{12}{4}$ ]
```

```
Out[2]//FullForm= 3
```

8. Modify the code for the one-dimensional random walk in this section to create two-dimensional random walks. In this case the step directions will be the vectors pointing in the compass directions,  $\{0, 1\}$ ,  $\{0, -1\}$ ,  $\{1, 0\}$ , and  $\{-1, 0\}$ .

## 2.1 Solutions

1. The functions `AtomQ`, `Head`, and `FullForm` will help answer these questions.

- a. The fraction  $8/5$  is atomic with head `Rational`.

```
In[1]:= {AtomQ[8/5], Head[8/5]}
```

```
Out[1]= {True, Rational}
```

- b. The expression  $8/5 + x$  is not atomic. It is a normal expression with head `Plus`.

```
In[2]:= {AtomQ[8/5 + x], Head[8/5 + x], FullForm[8/5 + x]}
```

```
Out[2]= {False, Plus, Plus[Rational[8, 5], x]}
```

- c. The list  $\{\{a, b\}, \{c, d\}\}$  is not atomic. It has head `List`.

```
In[3]:= {AtomQ[{{a, b}, {c, d}}], Head[{{a, b}, {c, d}}]}
```

```
Out[3]= {False, List}
```

- d. The string `"8/5 + x"` is atomic. It has head `String`.

```
In[4]:= {AtomQ["8/5+x"], Head["8/5+x"]}
```

```
Out[4]= {True, String}
```

2. The expression  $a(b+c)$  is given in full form as `Times[a, Plus[b, c]]`.  
 3. This is simply  $\frac{a}{b+c}$  as can be seen by evaluating the full form expression.

```
In[5]:= Times[a, Power[Plus[b, c], -1]]
```

```
Out[5]=  $\frac{a}{b+c}$ 
```

4. There are three elements in the expression;  $b x$  is the second element.

```
In[6]:= expr = a x^2 + b x + c;
```

```
In[7]:= FullForm[expr]
```

```
Out[7]//FullForm= Plus[c, Times[b, x], Times[a, Power[x, 2]]]
```

The first element of `Times[b, x]` is `b`, so the part specification is `2, 1`.



```
In[8]:= expr[[2]]
```

```
Out[8]= b x
```

```
In[9]:= expr[[2, 1]]
```

```
Out[9]= b
```

5. Looking at the internal representation of this expression with `FullForm` helps to unwind the part specification.

- a. Here is the internal representation of the expression:

```
In[10]:= FullForm[ $\frac{(x^2 + y) z}{w}$ ]
```

```
Out[10]//FullForm= Times[Power[w, -1], Plus[Power[x, 2], y], z]
```

```
In[11]:=  $\frac{(x^2 + y) z}{w}$ [[2, 1, 2]]
```

```
Out[11]= 2
```

- b. From the `FullForm` of `a / b`, you can see that the second part is `Power[b, -1]` and the second part of that is `-1`. Note the need for parentheses here as the `Part` function has higher precedence than `Power`. For more information on operator precedence, see the tutorial [Operator Input Forms \(WLDC\)](#)

```
In[12]:= FullForm[a/b]
```

```
Out[12]//FullForm= Times[a, Power[b, -1]]
```

```
In[13]:= (a/b)[[2, 2]]
```

```
Out[13]= -1
```

6. At first, you might try getting the expressions at level two as follows:

```
In[14]:= expr = LegendreP[5, x]
```

```
Out[14]=  $\frac{1}{8} (15 x - 70 x^3 + 63 x^5)$ 
```

```
In[15]:= Level[expr, 2]
```

```
Out[15]=  $\left\{ \frac{1}{8}, 15 x, -70 x^3, 63 x^5, 15 x - 70 x^3 + 63 x^5 \right\}$ 
```

But that gets all the parts down to level two. To extract just those parts *at* level two, use a slightly different syntax with `Level`.

```
In[16]:= Level[expr, {2}]
```

```
Out[16]=  $\{ 15 x, -70 x^3, 63 x^5 \}$ 
```

7. *Mathematica* evaluates arguments to functions before passing them up to the calling function. So the fraction first evaluates to 3 and the head of 3 is of course `Integer`. To get the internal representation of the expression *before* it is evaluated, use `Defer`.

```
In[17]:= Defer[FullForm[12/4]]
```

```
Out[17]= Times[12, Power[4, -1]]
```

8. The directions are the two-dimensional lists that can be thought of as vectors pointing in the compass directions north, south, east, and west.

```
In[18]:= dirs = {{0, 1}, {0, -1}, {1, 0}, {-1, 0}};
```

This picks a direction at random.

```
In[19]:= RandomChoice[dirs]
```

```
Out[19]= {1, 0}
```

This chooses five such.

```
In[20]:= RandomChoice[dirs, 5]
```

```
Out[20]= {{1, 0}, {0, 1}, {0, 1}, {-1, 0}, {0, -1}}
```

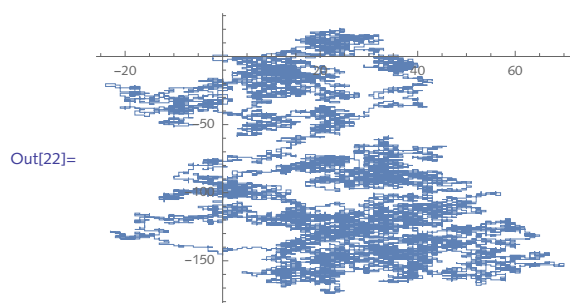
Here are the running sums, just like in the one-dimensional case.

```
In[21]:= Accumulate[%]
```

```
Out[21]= {{1, 0}, {1, 1}, {1, 2}, {0, 2}, {0, 1}}
```

And here is the nested code to create a 25 000-step two-dimensional random walk.

```
In[22]:= ListLinePlot[Accumulate[RandomChoice[dirs, 25000]], PlotStyle -> Thin]
```



## 2.2 Numbers: exercises

1. Is the expression  $2 + \pi$  a number? Is it numeric? What is the difference?
2. Convert the base 10 integer 65 to base 2. Then convert back to base 10.
3. Define a function `complexToPolar` that converts complex numbers to their polar representations. Then, convert the numbers  $3 + 3i$  and  $e^{\pi i/3}$  to polar form.

4. Use `NumberForm` to display an approximate number with exactly four precise digits and three digits to the right of the decimal. Then use `PaddedForm` to display the numbers in the following vector with precisely two digits to the right of the decimal:

```
In[1]:= vec = RandomReal[{0, 1}, 8]
```

```
Out[1]:= {0.897363, 0.629743, 0.657265, 0.959865, 0.681584, 0.706607, 0.995883, 0.111384}
```

5. Make a histogram of the frequencies of the first 100 000 digits of  $\pi$ . It is an open problem in number theory as to whether the digits are *normal*, meaning that each of the digits zero through nine occur with about the same frequency in the decimal expansion of  $\pi$ . See [Bailey et al. \(2012\)](#) for more information on normality and the digits of  $\pi$ .
6. Convert each of the characters in a string such as “Apple” to their eight-bit binary character code representation. For example, the character code for the letter A is 65:

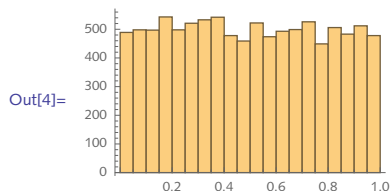
```
In[2]:= ToCharacterCode["A"]
```

```
Out[2]:= {65}
```

The eight-bit binary representation of 65 is 1000001, so your solution should return that base 2 number for the letter A. Binary representations of letters are used in certain ciphers such as the XOR cipher discussed in Exercise 5 of Section 7.1.

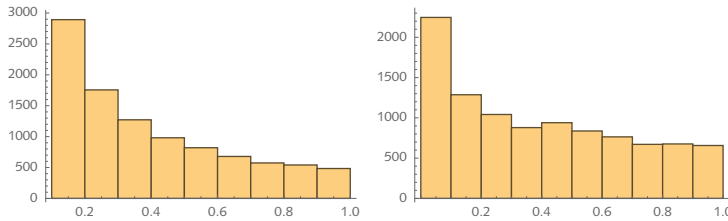
7. Graphs consist of a set of vertices and edges connecting some subset of those vertices. They are implemented in *Mathematica* with `Graph`, which takes two arguments: a list of vertices and a list of edges. Create a random graph on  $n$  vertices by choosing  $m$  edges from the  $\binom{n}{2}$  possible edges. Such random graphs are commonly specified as  $G(n, m)$  and are essentially the model upon which the built-in `RandomGraph` is based.
8. Extract the first 5000 digits in the decimal expansion of  $1/17$  or any other rational number. Then play them using `ListPlay`, which emits sound whose amplitude is given by the sequence of digits. Compare with the first 5000 digits of  $\pi$ .
9. `RandomReal` by default outputs numbers uniformly distributed on the interval  $[0, 1]$ .

```
In[3]:= data = RandomReal[{0, 1}, {10^4}];  
Histogram[data, 15]
```



Bias the list of random numbers toward the lower end of this interval, giving a histogram similar to those in Figure 2.1.

FIGURE 2.1. Distributions of random number data biased toward the lower end of the interval [0, 1].



10. Information theory, as conceived by Claude Shannon in the 1940s and 1950s, was originally interested in maximizing the amount of data that can be stored and retrieved over some channel such as a telephone line. Shannon devised a measure, now called *entropy*, that gives the theoretical maxima for such a signal. Entropy can be thought of as the average uncertainty of a single random variable and is computed by the following, where  $p(x)$  is the probability of event  $x$  over a domain  $X$ :

$$H(X) = -\sum_{x \in X} p(x) \log_2 p(x)$$

Generate a plot of the entropy (built into *Mathematica* as `Entropy`) as a function of success probability. You can simulate  $n$  trials of a coin toss with probability  $p$  using a Bernoulli distribution as follows:

```
RandomVariate[BernoulliDistribution[p], n]
```

## 2.2 Solutions

1. The expression  $2 + \pi$  is certainly numeric as both 2 and  $\pi$  are numeric.

```
In[1]:= NumericQ[2 + π]
```

```
Out[1]= True
```

But it is not an explicit number because  $\pi$  is not an explicit number.

```
In[2]:= NumberQ[2 + π]
```

```
Out[2]= False
```

2. This converts the base 2 representation of the number 65 to base 10.

```
In[3]:= 2^^1000001
```

```
Out[3]= 65
```

To go in the other direction, use `BaseForm`:

```
In[4]:= BaseForm[65, 2]
```

```
Out[4]//BaseForm= 10000012
```

3. This function gives the polar form as a list consisting of the magnitude and the polar angle.

```
In[5]:= complexToPolar[z_] := {Abs[z], Arg[z]}
```

```
In[6]:= complexToPolar[3 + 3 I]
```

```
Out[6]= {3 Sqrt[2], Pi/4}
```

Check against a built-in function (introduced in *Mathematica* 10.1):

```
In[7]:= AbsArg[3 + 3 I]
```

```
Out[7]= {3 Sqrt[2], Pi/4}
```

```
In[8]:= complexToPolar[E^(Pi I/3)]
```

```
Out[8]= {1, Pi/3}
```

```
In[9]:= AbsArg[E^(Pi I/3)]
```

```
Out[9]= {1, Pi/3}
```

4. Here is an approximation of  $\pi$  to 20-digit precision.

```
In[10]:= pi = N[Pi, 20]
```

```
Out[10]= 3.1415926535897932385
```

Display four precise digits with three digits to the right of the decimal point.

```
In[11]:= NumberForm[pi, {4, 3}]
```

```
Out[11]//NumberForm=
3.142
```

For the second part of the exercise, here is a vector of machine precision numbers.

```
In[12]:= vec = RandomReal[{0, 1}, 8]
```

```
Out[12]= {0.650946, 0.238017, 0.664332, 0.840782, 0.211514, 0.0455802, 0.612099, 0.228518}
```

This forces the display to use three digits in total with two digits to the right of the decimal.

```
In[13]:= PaddedForm[vec, {3, 2}]
```

```
Out[13]//PaddedForm=
{ 0.65, 0.24, 0.66, 0.84, 0.21, 0.05, 0.61, 0.23}
```

5. To get the digits of  $\pi$ , use `RealDigits`.

```
In[14]:= RealDigits[N[Pi]]
```

```
Out[14]= {{3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7, 9, 3}, 1}
```

What is returned is a list with the digits first, followed by the exponent, 1. We are only interested in the first sublist, so use `First`.

```
In[15]:= First[%]
```

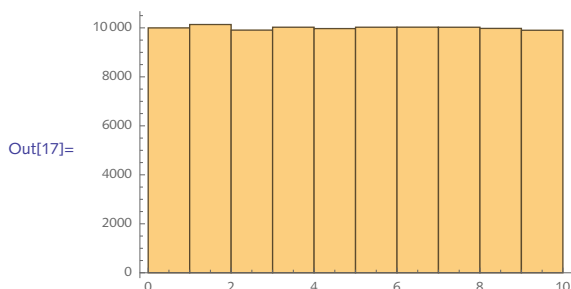
```
Out[15]= {3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7, 9, 3}
```

Here then, is a list of the first 100 000 digits of  $\pi$ , suppressing the display of the output using the semicolon.

```
In[16]:= pidigs = First[RealDigits[N[ $\pi$ , 105]]];
```

This histogram shows that each of the digits zero through nine appear with about the same frequency. This is referred to as the *normality* of the digits of  $\pi$  (see Bailey et al. 2012).

```
In[17]:= Histogram[pidigs]
```



6. First, here are the character codes of the letters in our test string, “Apple”.

```
In[18]:= ToCharacterCode["Apple"]
```

```
Out[18]= {65, 112, 112, 108, 101}
```

Here are the base two representation of each of the above character codes.

```
In[19]:= IntegerDigits[%, 2]
```

```
Out[19]= {{1, 0, 0, 0, 0, 0, 1}, {1, 1, 1, 0, 0, 0, 0},
           {1, 1, 1, 0, 0, 0, 0}, {1, 1, 0, 1, 1, 0, 0}, {1, 1, 0, 0, 1, 0, 1}}
```

Because the numbers are less than 128, the base two representation only have seven bits. To get the eight-bit representations, use a third argument to `IntegerDigits`.

```
In[20]:= IntegerDigits[ToCharacterCode["Apple"], 2, 8]
```

```
Out[20]= {{0, 1, 0, 0, 0, 0, 0, 1}, {0, 1, 1, 1, 0, 0, 0, 0},
           {0, 1, 1, 1, 0, 0, 0, 0}, {0, 1, 1, 0, 1, 1, 0, 0}, {0, 1, 1, 0, 0, 1, 0, 1}}
```

7. The built-in `Graph` function takes two arguments: a list of the vertex indices and a list of the edges. For a graph with  $n$  vertices, the vertex indices are simply the list of the integers one through  $n$ .

```
In[21]:= n = 5;
```

```
vertices = Range[5]
```

```
Out[22]= {1, 2, 3, 4, 5}
```

For the edges, we need a list of all possible edges in a graph with  $n$  vertices. `CompleteGraph[n]`

is a graph with all possible edges on  $n$  vertices so we can borrow that list from `CompleteGraph`.

```
In[23]:= edgesCG = EdgeList [CompleteGraph [n] ]
```

```
Out[23]= { 1 ↔ 2, 1 ↔ 3, 1 ↔ 4, 1 ↔ 5, 2 ↔ 3, 2 ↔ 4, 2 ↔ 5, 3 ↔ 4, 3 ↔ 5, 4 ↔ 5 }
```

To randomly choose  $m$  of them, use `RandomChoice`. `RandomChoice` chooses elements with replacement. Note that nothing prohibits what are referred to as multi-graphs, that is, graphs where multiple edges exist between pairs of vertices.

```
In[24]:= RandomChoice [edgesCG, 8]
```

```
Out[24]= { 1 ↔ 3, 4 ↔ 5, 2 ↔ 5, 2 ↔ 3, 1 ↔ 4, 2 ↔ 3, 1 ↔ 2, 1 ↔ 2 }
```

To avoid multi-edges, use `RandomSample` which chooses without replacement. But you will need to keep the desired number of edges,  $m$ , smaller than  $\binom{n}{2}$ .

```
In[25]:= RandomSample [edgesCG, 8]
```

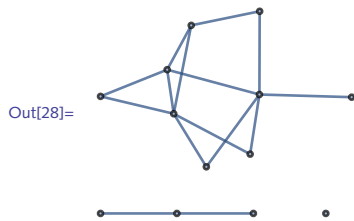
```
Out[25]= { 1 ↔ 5, 3 ↔ 4, 2 ↔ 3, 1 ↔ 2, 2 ↔ 4, 2 ↔ 5, 1 ↔ 3, 3 ↔ 5 }
```

This puts the pieces together and scales it up a bit. Repeated evaluation will cause different random graphs to be displayed, all with  $n$  vertices and  $m$  edges.

```
In[26]:= n = 13;
```

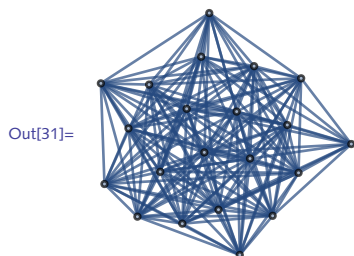
```
m = 15;
```

```
Graph [Range [n], RandomSample [EdgeList [CompleteGraph [n] ], m] ]
```



Also note that because we have used `CompleteGraph`, there are no self-edges, that is, an edge from a vertex to itself.

```
In[29]:= n = 21;
m = 167;
Graph[Range[n], RandomSample[EdgeList[CompleteGraph[n]], m]]
```



The built-in `RandomGraph` constructs only simple graphs – no multi-edges and no self loops, hence the total possible number of edges cannot exceed  $\binom{n}{2}$ .

```
In[32]:= RandomGraph[{21, 211}]
RandomGraph::args: RandomGraph[{21, 211}] called with invalid parameters. >>
```

```
Out[32]:= RandomGraph[{21, 211}]
```

```
In[33]:= Binomial[21, 2]
```

```
Out[33]:= 210
```

8. First, note that `RealDigits` returns a list with two elements, the digits and the exponent, in this case indicating that the first digit starts one place to the right of the decimal point.

```
In[34]:= RealDigits[N[1/17, 20]]
```

```
Out[34]:= {{5, 8, 8, 2, 3, 5, 2, 9, 4, 1, 1, 7, 6, 4, 7, 0, 5, 8, 8, 2}, -1}
```

To get only the digits, use `First`.

```
In[35]:= First[%]
```

```
Out[35]:= {5, 8, 8, 2, 3, 5, 2, 9, 4, 1, 1, 7, 6, 4, 7, 0, 5, 8, 8, 2}
```

Here then are the first 5000 digits of  $1/17$ .

```
In[36]:= digs = First[RealDigits[N[1/17, 5000]]];
```

And this plays them through the speakers of your computer.

```
In[37]:= ListPlay[digs]
```

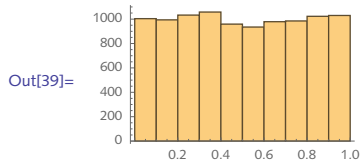




9. There are several ways that the random number sequences can be biased. First look at a picture of unbiased data. The random number generator uses a uniform probability distribution by default so we expect to see numbers uniformly distributed across the interval  $[0, 1]$ .

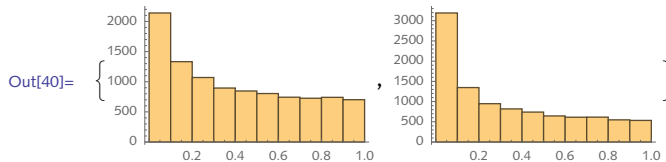
```
In[38]:= data = RandomReal[{0, 1}, {10^4}];
```

```
In[39]:= Histogram[data, 10]
```



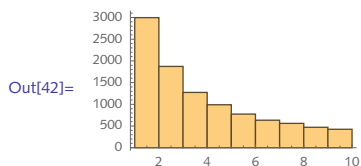
One way to bias the numbers toward zero is to transform them in such a way that they bunch around zero. Since these numbers are all less than one, raising them to a power will make them smaller.

```
In[40]:= {Histogram[data^1.5, 10, ImageSize -> 160], Histogram[data^2, 10, ImageSize -> 160]}
```



Of course, choosing from a different distribution biases the numbers in a sense.

```
In[41]:= data = RandomVariate[BenfordDistribution[10], {10^4}];
Histogram[data, 10]
```



In Chapters 3 and 9 we discuss listability, which will explain why we can raise every element in a vector to a power using the syntax above.

```
In[43]:= {a, b, c, d, e}^1.5
```

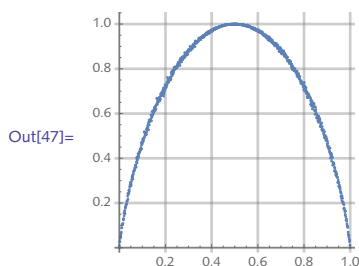
```
Out[43]= {a^1.5, b^1.5, c^1.5, d^1.5, e^1.5}
```

10. Run 10 000 trials with a range of probabilities from zero to one in increments of .001. The table creates pairs consisting of  $p$  together with the entropy (in base 2) for each trial.

```
In[44]:= trials = 10 000;
incr = 0.001;
info = Table[{p, Entropy[2, RandomVariate[BernoulliDistribution[p], trials]]},
  {p, 0, 1, incr}];
```

Make a plot.

```
In[47]:= ListPlot[info, AspectRatio -> 1, GridLines -> Automatic]
```



## 2.3 Definitions: exercises

1. Create a function `reciprocal  $\left[\frac{a}{b}\right]$`  that returns the reciprocal of the fraction  $a/b$ . Check your solution with numeric and symbolic fractions and with fractions containing zero in the numerator.
2. Using `Total`, create a function to sum the first  $n$  positive integers.
3. Create a function to compute the sum of the digits of any integer. Write an additional rule to give the sum of the base- $b$  digits of an integer. Then use your function to compute the *Hamming weight* of any integer: the Hamming weight of an integer is given by the number of ones in the binary representation of that number. It has wide use in computer science (modular exponentiation and hash tables), cryptography, and coding theory (Knuth 2011).
4. Write a function `sumsOfCubes  $[n]$`  that takes a positive integer argument  $n$  and computes the sums of cubes of the digits of  $n$  (Hayes 1992).
5. What rules are created by each of the following functions? Check your predictions by evaluating them and then querying *Mathematica* with `? function_name`.
  - a. `randLis1  $[n\_]$  := RandomReal[1, {n}]`
  - b. `randLis2  $[n\_]$  := (x = RandomReal[]; Table[x, {n}])`
  - c. `randLis3  $[n\_]$  := (x := RandomReal[]; Table[x, {n}])`
  - d. `randLis4  $[n_] = Table[RandomReal[], {n}]$`
6. Consider two functions `f` and `g`, which are identical except that one is written using an immediate assignment and the other using a delayed assignment.

```
In[1]:= f[n_] = Sum[(1 + x)^j, {j, 1, n}];
```

```
In[2]:= g[n_] := Sum[(1 + x)^j, {j, 1, n}]
```

Explain why the outputs of these two functions *look* so different. Are they in fact different?

```
In[3]:= f[2]
Out[3]= 
$$\frac{(1+x)(-1+(1+x)^2)}{x}$$

```

```
In[4]:= g[2]
Out[4]= 1 + x + (1 + x)^2
```

7. Write rules for a function `log` (note lowercase) that encapsulate the following identities:

$$\begin{aligned}\log(ab) &= \log(a) + \log(b); \\ \log\left(\frac{a}{b}\right) &= \log(a) - \log(b); \\ \log(a^n) &= n \log(a).\end{aligned}$$

8. Create a piecewise-defined function  $g(x)$  based on the following; then plot the function from  $-2$  to  $0$ .

$$g(x) = \begin{cases} -\sqrt{1-(x+2)^2} & -2 \leq x \leq -1 \\ \sqrt{1-x^2} & x < 0 \end{cases}$$

9. The built-in function `RotateRight` rotates the elements in a list one place to the right, with the last element swinging around to the front.

```
In[5]:= RotateRight[{a, b, c, d, e}]
Out[5]= {e, a, b, c, d}
```

Create a function `IntegerRotateRight[n]` that takes an integer  $n$  and returns an integer with the original digits rotated one place to the right. Use this function to first verify that 142 857 is a divisor of its right rotation and then find all such numbers less than one million

([Project Euler, Problem #168](#)).

10. The Champernowne constant is a famous number that is created by concatenating successive integers and interpreting them as decimal digits. For example, here are the first 31 digits of the base-10 Champernowne number:

```
In[6]:= N[ChampernowneNumber[10], 31]
Out[6]= 0.1234567891011121314151617181920
```

Concatenation can be used to generate integers also. Create a function to generate the  $n$ th Smarandache–Wellin number, formed by concatenating the digits of successive primes. The first such number is 2, then 23, then 235, followed by 2357, 235711, and so on. Numerous open questions exist about these numbers: for example, it is not known if an infinite number of them are prime; see [Crandall and Pomerance \(2005\)](#) and [Sloane \(A019518\)](#).

## 2.3 Solutions

1. Note that simply giving the reciprocal on the right-hand side does not work.

```
In[1]:= reciprocal[a_ / b_] := b / a
In[2]:= reciprocal[3 / 4]
Out[2]=  $\frac{4}{3}$ 
```

Look at the internal form to see why the pattern matcher failed to match  $3/4$  with  $a_ / b_$ .

```
In[3]:= FullForm[3 / 4]
Out[3]//FullForm= Rational[3, 4]
```

So a better approach is to use the internal form of such fractions.

```
In[4]:= reciprocal[Rational[a_, b_]] := Rational[b, a]
```

Alternatively, match with the head.

```
In[5]:= reciprocal[z_Rational] := Denominator[z] / Numerator[z]
```

This works for numeric and symbolic fractions.

```
In[6]:= reciprocal[3 / 4]
Out[6]=  $\frac{4}{3}$ 
In[7]:= reciprocal[z / (x + y)]
Out[7]=  $\frac{x + y}{z}$ 
```

But there is an issue with fractions containing zero in the numerator.

```
In[8]:= reciprocal[0 / 5]
Power::infy : Infinite expression  $\frac{1}{0}$  encountered. >>
Out[8]= ComplexInfinity
```

We will wait until Section 2.4 to resolve this issue.

2. Generate the list of integers 1 through  $n$ , then total that list.

```
In[9]:= sumInts[n_] := Total[Range[n]]
In[10]:= sumInts[100]
Out[10]= 5050
In[11]:= sumInts[1000]
Out[11]= 500500
```

We have not been careful to check that the arguments are positive integers here. See Section 4.1 for a discussion of patterns used to perform argument checking on your functions.

3. Once you have a list of the digits in any integer (`IntegerDigits`), simply total the list.

```
In[12]:= DigitSum[n_] := Total[IntegerDigits[n]]
```

```
In[13]:= DigitSum[10!]
```

```
Out[13]= 27
```

One rule can cover both parts of this exercise, using a default value of 10 for the base (see Section 5.4 for a discussion of default values).

```
In[14]:= DigitSum[n_, base_: 10] := Total[IntegerDigits[n, base]]
```

The Hamming weight of a number is the number of ones in its binary representation.

```
In[15]:= DigitSum[231 - 1, 2]
```

```
Out[15]= 31
```

Here is a comparison with a built-in function:

```
In[16]:= DigitCount[231 - 1, 2, 1]
```

```
Out[16]= 31
```

4. Here is the `sumsOfCubes` function.

```
In[17]:= sumsOfCubes[n_Integer] := Total[IntegerDigits[n]3]
```

```
In[18]:= sumsOfCubes[124]
```

```
Out[18]= 73
```

5. This exercise focuses on the difference between immediate and delayed assignments.

- a. This will generate a list of  $n$  random numbers.

```
In[19]:= randLis1[n_] := RandomReal[1, {n}]
```

```
In[20]:= randLis1[3]
```

```
Out[20]= {0.726437, 0.820623, 0.349356}
```

- b. Since the definition for `x` is an immediate assignment, its value does not change in `Table`. But each time `randLis2` is called, a new value is assigned to `x`.

```
In[21]:= randLis2[n_] := (x = RandomReal[]; Table[x, {n}])
```

```
In[22]:= randLis2[3]
```

```
Out[22]= {0.974798, 0.974798, 0.974798}
```

```
In[23]:= randLis2[3]
```

```
Out[23]= {0.621851, 0.621851, 0.621851}
```

- c. Because the definition for  $x$  is a delayed assignment, the definition for `randLis3` is functionally equivalent to `randLis1`.

```
In[24]:= randLis3[n_] := (x := RandomReal[]; Table[x, {n}])
```

```
In[25]:= randLis3[3]
```

```
Out[25]= {0.412708, 0.253301, 0.361384}
```

- d. In an immediate assignment, the right-hand side of the definition is evaluated first. But in this case,  $n$  does not have a value, so `Table` is not able to evaluate properly.

```
In[26]:= randLis4[n_] = Table[RandomReal[], {n}]
```

```
Out[26]= {0.138356, 0.374127, 0.873417, 0.769375, 0.0394529, 0.983081, 0.160601,
          0.982007, 0.191435, 0.744383, 0.761298, 0.311281, 0.461935, 0.525905,
          0.921668, 0.41622, 0.442365, 0.389952, 0.171867, 0.992726, 0.406445}
```

```
In[27]:= Clear[x, n]
```

6. The definition for  $f$  given in the exercise evaluates the sum first (immediate assignment), giving a symbolic expression for the general sum from 1 to  $n$ . When  $f[2]$  is evaluated, the argument 2 is then substituted into this expression for  $n$ . In the case of  $g$ , the value of  $n$  is substituted and then the sum is evaluated. Although the resulting expressions output by these two functions look different at first, expanding them gives the same result.

```
In[28]:= f[n_] = Sum[(1 + x)^j, {j, 1, n}]
```

```
Out[28]= (1 + x) (-1 + (1 + x)^n)
          x
```

```
In[29]:= g[n_] := Sum[(1 + x)^j, {j, 1, n}]
```

```
In[30]:= Expand[f[2]]
```

```
Out[30]= 2 + 3 x + x^2
```

```
In[31]:= Expand[g[2]]
```

```
Out[31]= 2 + 3 x + x^2
```

7. The rules for the logarithm function are as follows. Note, there is no need to program the division rule separately. Do you see why? (Look at `FullForm[x / y]`.)

```
In[32]:= log[a_*b_] := log[a] + log[b]
```

```
In[33]:= log[a_^n_] := n log[a]
```

```
In[34]:= log[x y^2 z^3]
```

```
Out[34]= log[x] + 2 log[y] + 3 log[z]
```

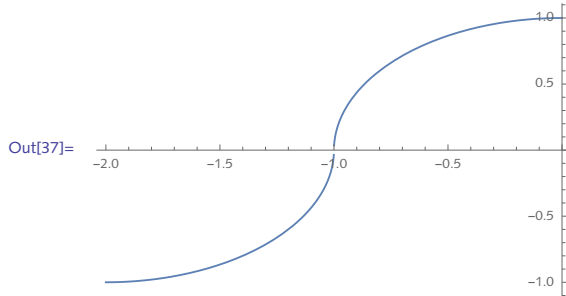
```
In[35]:= log[x / y]
```

```
Out[35]= log[x] - log[y]
```

8. Using `Piecewise`, we have:

```
In[36]:= g[x_] := Piecewise[{{{-1 Sqrt[1 - (x + 2)^2], -2 ≤ x ≤ -1}, {Sqrt[1 - x^2], x < 0}}}]
```

```
In[37]:= Plot[g[x], {x, -2, 0}]
```



9. First get a list of the digits of the integer.

```
In[38]:= IntegerDigits[142857]
```

```
Out[38]= {1, 4, 2, 8, 5, 7}
```

Next rotate the list using RotateRight.

```
In[39]:= RotateRight[IntegerDigits[142857]]
```

```
Out[39]= {7, 1, 4, 2, 8, 5}
```

Reconstruct the number from the list of digits using FromDigits.

```
In[40]:= FromDigits[RotateRight[IntegerDigits[142857]]]
```

```
Out[40]= 714285
```

Here is the function:

```
In[41]:= IntegerRotateRight[n_Integer] := FromDigits[RotateRight[IntegerDigits[n]]]
```

And this shows that 142857 is a divisor of its right rotation (third divisor from the end):

```
In[42]:= Divisors[IntegerRotateRight[142857]]
```

```
Out[42]= {1, 3, 5, 9, 11, 13, 15, 27, 33, 37, 39, 45, 55, 65, 99, 111, 117, 135,
143, 165, 185, 195, 297, 333, 351, 407, 429, 481, 495, 555, 585, 715,
999, 1221, 1287, 1443, 1485, 1665, 1755, 2035, 2145, 2405, 3663, 3861,
4329, 4995, 5291, 6105, 6435, 7215, 10989, 12987, 15873, 18315, 19305,
21645, 26455, 47619, 54945, 64935, 79365, 142857, 238095, 714285}
```

```
In[43]:= MemberQ[Divisors[IntegerRotateRight[142857]], 142857]
```

```
Out[43]= True
```

Here are all the integers less than one million that satisfy this property; most are palindromes (see Chapter 5 for discussion of Select and also pure functions).

```
In[44]:= Select[Range[10, 106], MemberQ[Divisors[IntegerRotateRight[#, #] &]
```

```
Out[44]= {11, 22, 33, 44, 55, 66, 77, 88, 99, 111, 222, 333, 444, 555, 666,
          777, 888, 999, 1111, 2222, 3333, 4444, 5555, 6666, 7777, 8888, 9999,
          11111, 22222, 33333, 44444, 55555, 66666, 77777, 88888, 99999,
          102564, 111111, 128205, 142857, 153846, 179487, 205128, 222222,
          230769, 333333, 444444, 555555, 666666, 777777, 888888, 999999}
```

10. One approach to creating these numbers is to use `IntegerDigits` to extract the digits of successive primes and then use `FromDigits` to concatenate this list of digits.

To prototype, get the digits from the first ten prime numbers.

```
In[45]:= Table[Prime[i], {i, 10}]
```

```
Out[45]= {2, 3, 5, 7, 11, 13, 17, 19, 23, 29}
```

```
In[46]:= IntegerDigits[%]
```

```
Out[46]= {{2}, {3}, {5}, {7}, {1, 1}, {1, 3}, {1, 7}, {1, 9}, {2, 3}, {2, 9}}
```

(As an aside, the above works because `IntegerDigits` is listable and hence automatically maps across lists.)

Next, flatten the output from `IntegerDigits` and turn that list into a number using `FromDigits`.

```
In[47]:= Flatten[%]
```

```
Out[47]= {2, 3, 5, 7, 1, 1, 1, 3, 1, 7, 1, 9, 2, 3, 2, 9}
```

```
In[48]:= FromDigits[%]
```

```
Out[48]= 2357111317192329
```

Here then is a function that creates the  $n$ th Smarandache-Wellin number.

```
In[49]:= SmarandacheWellin[n_] :=
          FromDigits[Flatten[IntegerDigits@Table[Prime[i], {i, n}]]]
```

And here are the first ten such numbers.

```
In[50]:= Table[SmarandacheWellin[i], {i, 10}]
```

```
Out[50]= {2, 23, 235, 2357, 235711, 23571113, 2357111317,
          235711131719, 23571113171923, 2357111317192329}
```

Which are prime numbers themselves?

```
In[51]:= Select[%, PrimeQ]
```

```
Out[51]= {2, 23, 2357}
```

And here is a really big Smarandache-Wellin prime.



```

In[52]:= SmarandacheWellin[1429] // N
Out[52]= 2.357111317192329 × 105718

In[53]:= PrimeQ[SmarandacheWellin[1429]]
Out[53]= True

```

In Section 7.2 we will look at an alternative approach to constructing these numbers using string functions.

## 2.4 Predicates and Boolean operations: exercises

1. Create a predicate function that returns a value of `True` if its argument is between  $-1$  and  $1$ .
2. Define a predicate function `StringCharacterQ[str]` that returns `True` if its argument `str` is a single string character, and returns `False` otherwise.
3. Write a predicate function `NaturalQ[n]` that returns a value of `True` if  $n$  is a natural number and a value of `False` otherwise, that is, `NaturalQ[n]` gives `True` if  $n$  is among  $0, 1, 2, 3, \dots$
4. Create a predicate function, `SquareNumberQ[n]` that returns `True` if  $n$  is a square number, such as  $1, 4, 9, 16, \dots$
5. Create a predicate function `TriangularNumberQ[t]` that returns `True` whenever its argument  $t$  is a triangular number. The  $n$ th triangular number  $T_n$  is given by the formula

$$T_n = \sum_{k=1}^n k = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2} = \binom{n+1}{2}$$

6. Based on the solution to the two previous exercises, create a predicate function `SquareTriangularNumberQ[n]` that returns a value of `True` if  $n$  is *both* a square number and a triangular number. Then use this predicate to find all square triangular numbers less than one million.
7. Create a predicate function `RealPositiveQ[x]` that returns a value of `True` if  $x$  is a positive real number (“real” in the mathematical sense, i.e.,  $x \in \mathbb{R}$ ). Add a second rule that accepts vectors as arguments and returns `True` if every element of the vector argument is a positive real number.
8. The built-in function `CoprimeQ[a, b]` returns `True` if  $a$  and  $b$  are relatively prime (share no common factors other than  $1$ ) and returns `False` otherwise. Use `ArrayPlot` to visualize pairs of relatively prime numbers from  $1$  to  $100$ . Use `Boole` to convert the table of `True/False` values returned by `CoPrimeQ` to zeros and ones.

9. An undirected graph  $gr$  is considered *dense* if the number of edges in  $gr$  is close to the maximum number of edges. The maximum for a graph with  $n$  edges occurs when *every* pair of vertices is connected by an edge and, assuming no self-loops and no multi-edges, is given by the number of two-element subsets of  $n$  objects,  $\binom{n}{2}$ . The density  $\mathcal{D}$  of a graph can be defined as

$$\mathcal{D} = \frac{|E|}{|V|(|V| - 1)}$$

where  $|E|$  is the number of edges and  $|V|$  is the number of vertices (given by `EdgeCount` and `VertexCount`, respectively). A graph with all possible edges has a density of 1 and a graph with no edges has density 0. Although there are differences of opinion as to where the cutoff is, assume that a graph is dense if its density is greater than or equal to 0.5.

Define a function `DenseGraphQ[gr]` that returns a value of `True` if  $gr$  is dense in the above sense and returns a value of `False` otherwise. As tests, `DenseGraphQ` should give `True` for `CompleteGraph[n]` for any  $n$  and it should return `False` for `RandomGraph[BernoulliGraphDistribution[n, pr]]` for small probabilities  $pr$ .

## 2.4 Solutions

1. There are several ways to define this function, either using the relational operator for less than, or with the absolute value function.

```
In[1]:= f[x_] := -1 < x < 1
```

```
In[2]:= f[x_] := Abs[x] < 1
```

```
In[3]:= f[4]
```

```
Out[3]= False
```

```
In[4]:= f[-0.35]
```

```
Out[4]= True
```

2. The requirements here are that the argument be both a string (`StringQ`) and have length (`StringLength`) one.

```
In[5]:= StringCharacterQ[ch_] := StringQ[ch] && StringLength[ch] == 1
```

```
In[6]:= StringCharacterQ["v"]
```

```
Out[6]= True
```

```
In[7]:= StringCharacterQ["vi"]
```

```
Out[7]= False
```

```
In[8]:= StringCharacterQ[v]
```

```
Out[8]= False
```

3. A number  $n$  can be considered a natural number if it is both an integer and greater than or equal to zero. There is some historical precedent for not including zero, but most mathematicians and computer scientists now include it and so for our purposes, we will adopt the convention that zero is a natural number.

```
In[9]:= NaturalQ[n_] := IntegerQ[n] && n ≥ 0
```

```
In[10]:= NaturalQ[0]
```

```
Out[10]= True
```

```
In[11]:= NaturalQ[-4]
```

```
Out[11]= False
```

4. To check that a number is a perfect square, it is sufficient to see if its square root is an integer.

```
In[12]:= SquareNumberQ[n_] := IntegerQ[Sqrt[n]]
```

```
In[13]:= SquareNumberQ[10]
```

```
Out[13]= False
```

```
In[14]:= Select[Range[100], SquareNumberQ]
```

```
Out[14]= {1, 4, 9, 16, 25, 36, 49, 64, 81, 100}
```

5. Since any triangular number  $T$  is equal to  $n(n+1)/2$  for some  $n$ , a bit of algebraic manipulation gives:

$$2T = n^2 + n$$

$$8T = 4n^2 + 4n$$

Completing the square gives:

$$\begin{aligned} 8T + 1 &= 4n^2 + 4n + 1 \\ &= (2n + 1)^2 \end{aligned}$$

So,  $T$  is triangular if and only if  $8T + 1$  is an odd perfect square. Here then is the test.

```
In[15]:= TriangularNumberQ[t_] := OddQ[Sqrt[8t + 1]] && SquareNumberQ[8t + 1]
```

```
In[16]:= TriangularNumberQ[6]
```

```
Out[16]= True
```

```
In[17]:= TriangularNumberQ[55]
```

```
Out[17]= True
```

```
In[18]:= TriangularNumberQ[56]
```

```
Out[18]= False
```

6. Combine the previous two solutions with a conjunction.

```
In[19]:= SquareTriangularNumberQ[n_] := TriangularNumberQ[n] && SquareNumberQ[n]
```

```
In[20]:= Select[Range[1 000 000], SquareTriangularNumberQ]
```

```
Out[20]= {1, 36, 1225, 41 616}
```

7. First, create a predicate that checks if a single number is real and positive.

```
In[21]:= RealPositiveQ[n_?NumericQ] := Im[n] == 0 && Positive[n]
```

```
In[22]:= Select[{-4, 23, 3 + 4 I,  $\pi$ }, RealPositiveQ]
```

```
Out[22]= {23,  $\pi$ }
```

Now, add a rule that checks if a vector consists entirely of numbers that are real and positive. AllTrue[*expr*, *test*] applies the test to each of the elements in *expr* and returns True if all of them are true.

```
In[23]:= RealPositiveQ[vec_?VectorQ] := AllTrue[vec, RealPositiveQ]
```

```
In[24]:= RealPositiveQ[{-4, 23, 3 + 4 I,  $\pi$ }]
```

```
Out[24]= False
```

Actually we can make this code a bit more compact by using the two-argument form of VectorQ. The second argument is a predicate that tests each element of the vector calling the one argument form of RealPositiveQ.

```
In[25]:= Clear[RealPositiveQ];
```

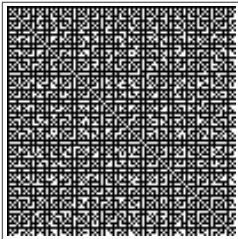
```
RealPositiveQ[n_?NumericQ] := Im[n] == 0 && Positive[n]
```

```
In[27]:= RealPositiveQ[vec_] := VectorQ[vec, RealPositiveQ]
```

8. Wrapping the table produced by CoprimeQ in Boole converts the true/false values to zeros and ones, which is what ArrayPlot needs.

```
In[28]:= ArrayPlot[Boole[Table[CoprimeQ[a, b], {a, 100}, {b, 100}]]]
```

```
Out[28]=
```

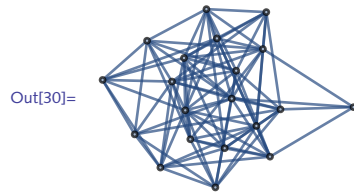


9. Given the definition of graph density in the exercise, here is an implementation that takes a graph as an argument.

```
In[29]:= DenseGraphQ[gr_Graph] :=
```

```
2 EdgeCount[gr] / (VertexCount[gr] (VertexCount[gr] - 1)) ≥ 0.5
```

```
In[30]:= gr = RandomGraph [ {20, 90} ]
```

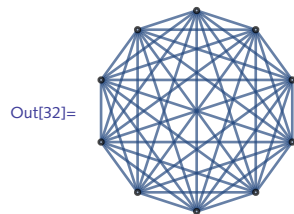


```
In[31]:= DenseGraphQ [ gr ]
```

```
Out[31]= False
```

Complete graphs are dense as they have all possible edges.

```
In[32]:= CompleteGraph [10]
```



```
In[33]:= DenseGraphQ [%]
```

```
Out[33]= True
```

Actually, there is a built-in function that gives the density explicitly.

```
In[34]:= GraphDensity [CompleteGraph [10] ]
```

```
Out[34]= 1
```

```
In[35]:= GraphDensity [gr]
```

```
Out[35]= 9/19
```

So an simpler definition would just use that.

```
In[36]:= DenseGraphQ [gr_Graph] := GraphDensity [gr] ≥ 0.5
```

## 2.5 Attributes: exercises

- I. Ordinarily, when you define a function, it has no attributes. The arguments are evaluated before being passed up to the calling function. So, in the following case,  $2 + 3$  is evaluated before it is passed to  $g$ .

```
In[1]:= g [x_ + y_] := x2 + y2
```

```
In[2]:= g[2 + 3]
Out[2]= g[5]
```

Use one of the `Hold` attributes to give `g` the property that its argument is not evaluated first. The resulting output should look like this:

```
In[3]:= g[2 + 3]
Out[3]= 13
```

2. Define a function that takes each number,  $x$ , in a vector of numbers and returns  $x$  if it is within a certain interval, say  $-0.5 < x < 0.5$ , and returns  $\sqrt{x}$  otherwise. Then make your function listable so that it can operate on vectors (lists) directly.
3. The definitions used in the solution to Exercise 1 of Section 2.3 for the reciprocal function failed to properly deal with the special case of zero in the numerator.

```
In[4]:= reciprocal[Rational[a_, b_]] := Rational[b, a]
In[5]:= reciprocal[0/5]
Out[5]= reciprocal[0]
```

Correct this problem by giving `reciprocal` the appropriate attribute.

## 2.5 Solutions

1. First clear any definitions and attributes that might be associated with `g`.

```
In[1]:= ClearAll[g]
```

Then set the `HoldAll` attribute to prevent initial evaluation of the argument of this function.

```
In[2]:= SetAttributes[g, HoldAll]
In[3]:= g[x_ + y_] := x^2 + y^2
In[4]:= g[a + b]
Out[4]= a^2 + b^2
In[5]:= g[2 + 3]
Out[5]= 13
```

2. Here is a small list of random numbers to use.

```
In[6]:= vec = RandomReal[{-1, 1}, 10]
Out[6]= {-0.798986, -0.84542, -0.747004, -0.44054, 0.117601,
         -0.243944, -0.227587, 0.994751, -0.107316, 0.692977}
```

The function could be set up to take two arguments, the number and the bound.

```
In[7]:= fun[x_?NumberQ, bound_] := If[-bound < x < bound, x, Sqrt[x]]
```

Make fun listable.

```
In[8]:= SetAttributes[fun, Listable]
```

```
In[9]:= fun[vec, 0.5]
```

```
Out[9]= {0. + 0.89386 i, 0. + 0.919467 i, 0. + 0.864294 i, -0.44054,
        0.117601, -0.243944, -0.227587, 0.997372, -0.107316, 0.832452}
```

3. First, here is the original definition for reciprocal.

```
In[10]:= reciprocal[z_Rational] := Denominator[z]/Numerator[z]
```

```
In[11]:= reciprocal[0/5]
```

```
Power::infy: Infinite expression  $\frac{1}{0}$  encountered. >>
```

```
Out[11]= ComplexInfinity
```

Give it the HoldAll attribute to prevent fractions such from first evaluating and being reduced.

```
In[12]:= SetAttributes[reciprocal, HoldAll]
```

```
In[13]:= reciprocal[z_Rational] := Denominator[z]/Numerator[z]
```

```
In[14]:= reciprocal[0/5]
```

```
Power::infy: Infinite expression  $\frac{1}{0}$  encountered. >>
```

```
Out[14]= ComplexInfinity
```

We have resolved one problem, but a new one arises. The unevaluated form of  $0/5$  is in fact not a Rational. It will become Rational once it is evaluated.

```
In[15]:= FullForm[HoldForm[0/5]]
```

```
Out[15]//FullForm= HoldForm[Times[0, Power[5, -1]]]
```

So, one more rule is needed to cover this situation.

```
In[16]:= reciprocal[Times[a_, Power[b_, -1]]] := b/a
```

```
In[17]:= reciprocal[0/5]
```

```
Power::infy: Infinite expression  $\frac{1}{0}$  encountered. >>
```

```
Out[17]= ComplexInfinity
```

```
In[18]:= reciprocal[2/3]
```

```
Out[18]=  $\frac{3}{2}$ 
```

In fact, this last rule now handles more complicated rational expressions such as the following.

```
In[19]:= reciprocal[x / (y + z) ]
```

```
Out[19]= 
$$\frac{y + z}{x}$$

```



## 3

# Lists and associations

### 3.1 Creating and displaying lists: exercises

1. Create a list of the multiples of five less than or equal to one hundred.
2. Create a list of the reciprocals of the powers of two as the powers go from zero to sixteen.
3. Generate the list  $\{\{\emptyset\}, \{\emptyset, 2\}, \{\emptyset, 2, 4\}, \{\emptyset, 2, 4, 6\}, \{\emptyset, 2, 4, 6, 8\}\}$  in two different ways using the Table function.
4. Generate both of the following arrays using the Table function:

`In[1]:= Array[f, 5]`

`Out[1]= {f[1], f[2], f[3], f[4], f[5]}`

`In[2]:= Array[f, {3, 4}]`

`Out[2]= {{f[1, 1], f[1, 2], f[1, 3], f[1, 4]},  
{f[2, 1], f[2, 2], f[2, 3], f[2, 4]}, {f[3, 1], f[3, 2], f[3, 3], f[3, 4]}}`

5. Use Table to create an  $n \times n$  matrix consisting of the positive integers one through  $n^2$  arranged such that the first row is the list  $\{1, 2, 3, \dots, n\}$ , the second row is the list  $\{1 + n, 2 + n, 3 + n, \dots, 2n\}$ , and in general the  $k$ th row is the list  $\{1 + (k - 1)n, 2 + (k - 1)n, 3 + (k - 1)n, \dots, n^2\}$ . For example, for  $n = 4$ , you should have

$\{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{9, 10, 11, 12\}, \{13, 14, 15, 16\}\}$

In matrix form, this would be:

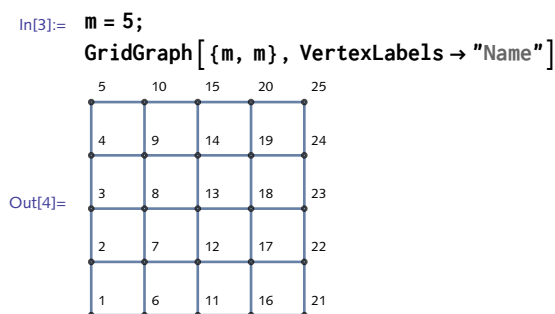
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$$

6. Using `Table`, create a symmetric matrix of the binomial coefficients for any  $n$  similar to that in Table 3.1.

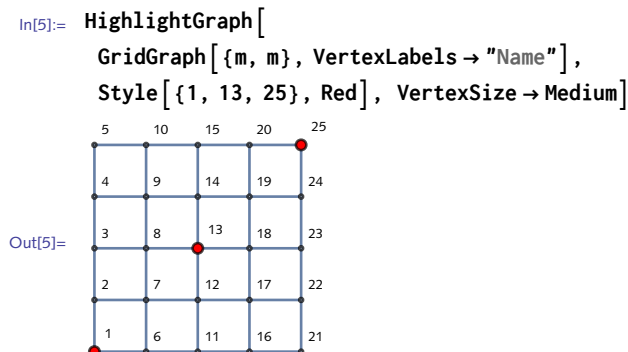
TABLE 3.1. *Pascal's matrix for  $n = 5$*

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{pmatrix}$$

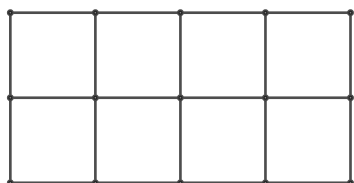
7. Given an  $m \times m$  square lattice like the grid graph below, color all vertices on the bottom red, and on the top white. Your solution should be as general as possible, so that changing the size of the lattice (changing the value of  $m$ ) will still work to color the lattice.



To change the property of select vertices in a graph, use `HighlightGraph`. For example, the following colors vertices 1, 13, and 25 red.



8. Import six images, resize them to the same dimensions, then display them inside a  $3 \times 2$  grid using options for `Grid` to format the output.
9. Construct an integer lattice graphic like in Figure 3.1. Start by creating a list of the pairs of coordinate points. Then connect the appropriate pairs of coordinates with lines (use `Graphics[Line[...]]`). Add points with `Graphics[Point[...]]`. See Chapter 8 for details about creating plots from graphics primitives. Consider the function `CoordinateBoundsArray` to get the list of integer coordinates.

FIGURE 3.1. A  $5 \times 3$  rectangular lattice.

### 3.1 Solutions

1. Using Table, here are the multiples of 5.

```
In[1]:= Table[5 j, {j, 1, 100 / 5}]
```

```
Out[1]= {5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95, 100}
```

This can also be done with Range.

```
In[2]:= Range[5, 100, 5]
```

```
Out[2]= {5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95, 100}
```

2. Here are the reciprocals of the powers of two.

```
In[3]:= Table[1/2^i, {i, 0, 16}]
```

```
Out[3]= {1, 1/2, 1/4, 1/8, 1/16, 1/32, 1/64, 1/128, 1/256, 1/512,
1/1024, 1/2048, 1/4096, 1/8192, 1/16384, 1/32768, 1/65536}
```

```
In[4]:= 1 / (2 ^ Range[0, 16])
```

```
Out[4]= {1, 1/2, 1/4, 1/8, 1/16, 1/32, 1/64, 1/128, 1/256, 1/512,
1/1024, 1/2048, 1/4096, 1/8192, 1/16384, 1/32768, 1/65536}
```

3. You can take every other element in the iterator list, or encode that in the expression  $2 j$ .

```
In[5]:= Table[j, {i, 0, 8, 2}, {j, 0, i, 2}]
```

```
Out[5]= {{0}, {0, 2}, {0, 2, 4}, {0, 2, 4, 6}, {0, 2, 4, 6, 8}}
```

```
In[6]:= Table[2 j, {i, 0, 4}, {j, 0, i}]
```

```
Out[6]= {{0}, {0, 2}, {0, 2, 4}, {0, 2, 4, 6}, {0, 2, 4, 6, 8}}
```

4. These lists can be generated with Table, using two iterators for the second example.

```
In[7]:= Table[f[i], {i, 5}]
```

```
Out[7]= {f[1], f[2], f[3], f[4], f[5]}
```

```
In[8]:= Table[f[i, j], {i, 3}, {j, 4}]
Out[8]= {{f[1, 1], f[1, 2], f[1, 3], f[1, 4]},
          {f[2, 1], f[2, 2], f[2, 3], f[2, 4]}, {f[3, 1], f[3, 2], f[3, 3], f[3, 4]}}
```

5. There are numerous ways to create this array. Here is one approach:

```
In[9]:= n = 4;
        Table[1 + i + j, {j, 0, n^2 - 1, n}, {i, 0, n - 1}]
Out[10]= {{1, 2, 3, 4}, {5, 6, 7, 8}, {9, 10, 11, 12}, {13, 14, 15, 16}}
```

But in fact, a more direct implementation using the statement of the problem, gives the following:

```
In[11]:= n = 4;
         Table[j + k n, {k, 0, n - 1}, {j, 1, n}]
Out[12]= {{1, 2, 3, 4}, {5, 6, 7, 8}, {9, 10, 11, 12}, {13, 14, 15, 16}}
```

```
In[13]:= MatrixForm[%]
Out[13]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$$

```

6. The binomial coefficients can be generated with Binomial. For example here are the coefficients in the expansion of  $(1 + x)^5$ :

```
In[14]:= Table[Binomial[5, k], {k, 0, 5}]
Out[14]= {1, 5, 10, 10, 5, 1}
```

```
In[15]:= Expand[(1 + x)^5]
Out[15]= 1 + 5 x + 10 x^2 + 10 x^3 + 5 x^4 + x^5
```

So to get all coefficients for exponent  $n$ , as  $n$  runs from one to six say, we need a second iterator for Table. A bit of thought is needed to determine which iterator list comes first and to make  $j$  dependent upon  $n$ .

```
In[16]:= n = 6;
         Table[Binomial[j, i], {i, 0, n - 1}, {j, i, n + i - 1}]
Out[17]= {{1, 1, 1, 1, 1, 1}, {1, 2, 3, 4, 5, 6}, {1, 3, 6, 10, 15, 21},
          {1, 4, 10, 20, 35, 56}, {1, 5, 15, 35, 70, 126}, {1, 6, 21, 56, 126, 252}}
```

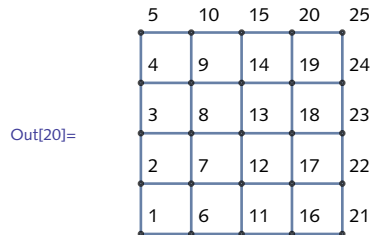
```
In[18]:= MatrixForm[%]
Out[18]//MatrixForm=

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 6 & 10 & 15 & 21 \\ 1 & 4 & 10 & 20 & 35 & 56 \\ 1 & 5 & 15 & 35 & 70 & 126 \\ 1 & 6 & 21 & 56 & 126 & 252 \end{pmatrix}$$

```

7. First, here is the grid graph:

```
In[19]:= m = 5;
          gg = GridGraph[{m, m}, VertexLabels -> "Name"]
```



Since this is a square grid, the vertex numbers on the top of the lattice are multiples of  $m$  up to  $m^2$ .

```
In[21]:= top = Range[m, m^2, m]
```

Out[21]= {5, 10, 15, 20, 25}

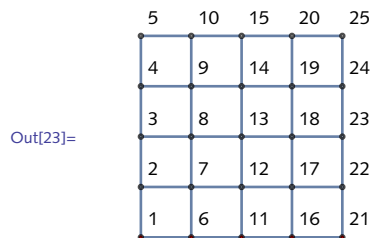
The bottom vertices range from one to  $m^2 - m + 1$  in increments of  $m$ .

```
In[22]:= bot = Range[1, m^2 - m + 1, m]
```

Out[22]= {1, 6, 11, 16, 21}

To change the properties of a set of vertices, use `Style`.

```
In[23]:= HighlightGraph[GridGraph[{m, m}, VertexLabels -> "Name"],
                        {Style[bot, Red], Style[top, White]}]
```

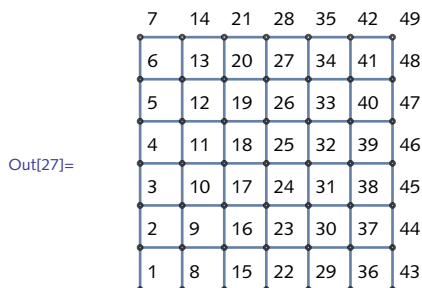


Changing the size of the lattice should not trigger a need to change the code.

```

In[24]:= m = 7;
top = Range[m, m^2, m];
bot = Range[1, m^2 - m + 1, m];
HighlightGraph[GridGraph[{m, m}, VertexLabels -> "Name"],
  {Style[bot, Red], Style[top, White]}]

```



8. Here are some sample images to work with.

```

In[28]:= images = Map[ExampleData, {{ "TestImage", "Girl12"}, {"TestImage", "Peppers"},
  {"TestImage", "Aerial"}, {"TestImage", "Moon"}, {"TestImage", "Tank2"},
  {"TestImage", "Ruler"} }];

```

Get their dimensions.

```

In[29]:= Map[ImageDimensions, images]

```

```

Out[29]= {{256, 256}, {512, 512}, {256, 256}, {256, 256}, {512, 512}, {512, 512}}

```

Resize the larger images.

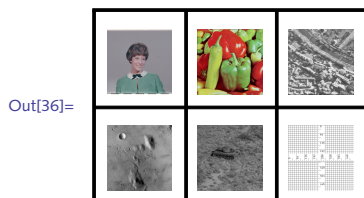
```

In[30]:= img1 = images[[1]];
img2 = ImageResize[images[[2]], 256];
img3 = images[[3]];
img4 = images[[4]];
img5 = ImageResize[images[[5]], 256];
img6 = ImageResize[images[[6]], 256];

```

Finally, put in a grid with some formatting.

```
In[36]:= Grid[{
  {img1, img2, img3},
  {img4, img5, img6}
}, Frame → All, Spacings → {1, 1}, ItemSize → {3, 3}]
```



9. A bit of thought is needed to get the iterators right using Table.

```
In[37]:= xmin = -2; xmax = 2; ymin = -1; ymax = 1;
hlines = Table[{ {xmin, y}, {xmax, y} }, {y, ymin, ymax}]
```

```
Out[38]= {{ {-2, -1}, {2, -1}}, {{-2, 0}, {2, 0}}, {{-2, 1}, {2, 1}}}
```

```
In[39]:= vlines = Table[{ {x, ymin}, {x, ymax} }, {x, xmin, xmax}]
```

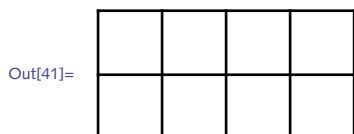
```
Out[39]= {{ {-2, -1}, {-2, 1}}, {{-1, -1}, {-1, 1}},
  {{0, -1}, {0, 1}}, {{1, -1}, {1, 1}}, {{2, -1}, {2, 1}}}
```

Join the two sets of lines and then flatten to remove one set of braces.

```
In[40]:= pairs = Flatten[{hlines, vlines}, 1]
```

```
Out[40]= {{ {-2, -1}, {2, -1}}, {{-2, 0}, {2, 0}},
  {{-2, 1}, {2, 1}}, {{-2, -1}, {-2, 1}}, {{-1, -1}, {-1, 1}},
  {{0, -1}, {0, 1}}, {{1, -1}, {1, 1}}, {{2, -1}, {2, 1}}}
```

```
In[41]:= Graphics[Line[pairs]]
```



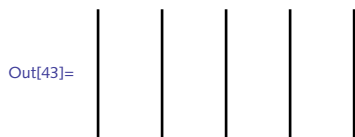
Actually, this can be done more compactly. First create the coordinate points for a  $5 \times 3$  lattice.

```
In[42]:= coords = Table[{i, j}, {i, 1, 5}, {j, 1, 3}]
```

```
Out[42]= {{ {1, 1}, {1, 2}, {1, 3}}, {{2, 1}, {2, 2}, {2, 3}},
  {{3, 1}, {3, 2}, {3, 3}}, {{4, 1}, {4, 2}, {4, 3}}, {{5, 1}, {5, 2}, {5, 3}}}
```

This gives the vertical lines:

```
In[43]:= Graphics[Line[coords]]
```

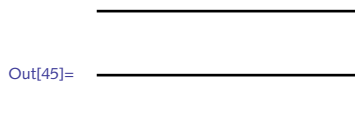


Transposing the coordinates gives a list that can be used for the horizontal lines. Look carefully at the structure of `coords` to understand what exactly is being transposed.

```
In[44]:= Transpose@coords
```

```
Out[44]= {{1, 1}, {2, 1}, {3, 1}, {4, 1}, {5, 1}},
          {{1, 2}, {2, 2}, {3, 2}, {4, 2}, {5, 2}},
          {{1, 3}, {2, 3}, {3, 3}, {4, 3}, {5, 3}}}
```

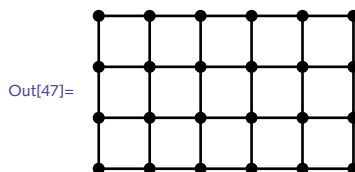
```
In[45]:= Graphics[Line[Transpose@coords]]
```



This puts everything together, adding points at each coordinate. `Module` is a localization construct, discussed in Section 6.1.

```
In[46]:= Lattice[{xdim_, ydim_}] := Module[{coords},
  coords = Table[{i, j}, {i, 1, xdim}, {j, 1, ydim}];
  Graphics[{
    Line[coords], Line[Transpose[coords]],
    PointSize[Medium], Point[Flatten[coords, 1]]
  }]]
```

```
In[47]:= Lattice[{6, 4}]
```



Alternatively, you could use `CoordinateBoundsArray`, new in *Mathematica* 10.1:



In[48]:= ?CoordinateBoundsArray

`CoordinateBoundsArray[{ $x_{min}$ ,  $x_{max}$ }, { $y_{min}$ ,  $y_{max}$ }, ...]` generates an array of  
 $\{x, y, \dots\}$  coordinates with integer steps in each dimension.  
`CoordinateBoundsArray[{ $xrange$ ,  $yrange$ , ...},  $d$ ]`  
 uses step  $d$  in each dimension.  
`CoordinateBoundsArray[{ $xrange$ ,  $yrange$ , ...}, { $dx$ ,  $dy$ , ...}]`  
 uses steps  $dx$ ,  $dy$ , ... in successive dimensions.  
`CoordinateBoundsArray[{ $xrange$ ,  $yrange$ , ...}, Into[ $n$ ]]`  
 divides into  $n$  equal steps in each dimension.  
`CoordinateBoundsArray[{ $xrange$ ,  $yrange$ , ...}, steps, offsets]`  
 specifies offsets to use for each coordinate point.  
`CoordinateBoundsArray[{ $xrange$ ,  $yrange$ , ...}, steps, offsets,  $k$ ]`  
 expands the array by  $k$  elements in every direction. >>

So the following would simplify the computation:

In[49]:= `coords = CoordinateBoundsArray[{1, 5}, {1, 3}]`


Out[49]= `{{{1, 1}, {1, 2}, {1, 3}}, {{2, 1}, {2, 2}, {2, 3}},  
 {{3, 1}, {3, 2}, {3, 3}}, {{4, 1}, {4, 2}, {4, 3}}, {{5, 1}, {5, 2}, {5, 3}}}`

This gives five triples of coordinates and if you look carefully, these can be used to get the vertical lines.

In[50]:= `Dimensions[coords]`

Out[50]= `{5, 3, 2}`

In[51]:= `Graphics[Line[coords]]`

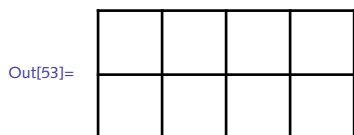
Out[51]= 

The horizontal lines can be obtained by transposing the coordinates returned by `CoordinateBoundsArray`.

In[52]:= `Transpose[coords]`

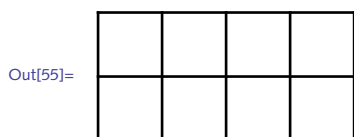
Out[52]= `{{{1, 1}, {2, 1}, {3, 1}, {4, 1}, {5, 1}},  
 {{1, 2}, {2, 2}, {3, 2}, {4, 2}, {5, 2}},  
 {{1, 3}, {2, 3}, {3, 3}, {4, 3}, {5, 3}}}`

```
In[53]:= Graphics[{Line[coords], Line[Transpose[coords]]}]
```



```
In[54]:= Lattice[{xmin_, xmax_}, {ymin_, ymax_}] := Module[{coords},
  coords = CoordinateBoundsArray[{{xmin, xmax}, {ymin, ymax}}];
  Graphics[{Line[coords], Line[Transpose[coords]]}]
]
```

```
In[55]:= Lattice[{1, 5}, {1, 3}]
```



### 3.2 Testing and measuring lists: exercises

1. What is the length of the following list? What are its dimensions? What is the position of g?

```
{ {a, b}, {c, d}, {e, f}, {g, h} }
```

2. The following input generates a list of 10 000 zeros and ones weighted heavily toward the ones. Determine if there are any zeros in this list and if there are, find how many.

```
In[1]:= lis = RandomChoice[ {.0001, .9999} → {0, 1}, {10 000}];
```

3. Given a list of integers such as the following, count the number of zeros. Find a way to count all those elements of the list which are not ones.

```
In[2]:= ints = RandomInteger[{-5, 5}, 30]
```

```
Out[2]= {1, 4, -1, 2, -2, 4, 4, -5, 5, 3, 4, -3, -3, 1,
 0, 4, -1, -2, 2, 0, -5, 1, -5, -4, 3, -5, -2, -3, 3, 0}
```

4. Given the list `{{{1, a}, {2, b}, {3, c}}, {{4, d}, {5, e}, {6, f}}}`, determine its dimensions. Use the `Dimensions` function to check your answer.
5. Find the positions of the nines in the following list. Confirm using `Position`.

```
{ {2, 1, 10}, {9, 5, 7}, {2, 10, 4}, {10, 1, 9}, {6, 1, 6} }
```

6. Determine if there are any prime numbers in the interval `[4 302 407 360, 4 302 407 713]`. Once you have a list of the integers that you want to test for primality, use `Position` (see Section 4.1) and `Extract` to return the explicit primes.

## 3.2 Solutions

1. The following list has length four because it has four elements, the four pairs.

```
In[1]:= lis = {{a, b}, {c, d}, {e, f}, {g, h}};
```

```
In[2]:= Length[lis]
```

```
Out[2]= 4
```

Its dimensions are  $4 \times 2$ ; that is, it has four rows and two columns when thought of as a rectangular array.

```
In[3]:= Dimensions[lis]
```

```
Out[3]= {4, 2}
```

```
In[4]:= MatrixForm[lis]
```

```
Out[4]//MatrixForm=
```

$$\begin{pmatrix} a & b \\ c & d \\ e & f \\ g & h \end{pmatrix}$$

The element *g* is in the fourth row first column.

```
In[5]:= Position[lis, g]
```

```
Out[5]= {{4, 1}}
```

2. Each time you evaluate the following input you will get a different list of 10 000 zeros and ones. Seeding the random number generator will give repeatable results.

```
In[6]:= SeedRandom[1];
```

```
lis = RandomChoice[ {.0001, .9999} -> {0, 1}, {10000}];
```

On the computer on which this was run, this seed gives some zeros in the list.

```
In[8]:= FreeQ[lis, 0]
```

```
Out[8]= False
```

```
In[9]:= Count[lis, 0]
```

```
Out[9]= 2
```

3. Here is the list of integers to use.

```
In[10]:= ints = RandomInteger[{-5, 5}, 30]
```

```
Out[10]= {0, 2, 5, -4, 3, -2, 5, -4, 1, -5, -5, 3, -3, -3,
-4, -2, -2, -4, 2, -5, -2, -5, 1, 4, 4, -4, 1, 0, -2, 2}
```

Count all elements that match 0.

```
In[11]:= Count[ints, 0]
```

```
Out[11]= 2
```

Count all integers in `ints` that do not match `1`.

```
In[12]:= Count[ints, Except[1]]
```

```
Out[12]= 27
```

4. From the top level, there are two lists, each consisting of three sublists, each sublist consisting of two elements.

```
In[13]:= Dimensions[{{1, a}, {2, b}, {3, c}}, {{4, d}, {5, e}, {6, f}}]
```

```
Out[13]= {2, 3, 2}
```

5. The `Position` function tells us that the `9`s are located in the second sublist, first position, and in the fourth sublist, third position.

```
In[14]:= Position[{{2, 1, 10}, {9, 5, 7}, {2, 10, 4}, {10, 1, 9}, {6, 1, 6}}, 9]
```

```
Out[14]= {{2, 1}, {4, 3}}
```

6. Here is the interval we are interested in:

```
In[15]:= ints = Range[4 302 407 360, 4 302 407 713];
```

Using some pattern matching (look ahead to Section 4.1), this gives the positions of the integers in `ints` that pass the `PrimeQ` test.

```
In[16]:= pos = Position[ints, p_?PrimeQ]
```

```
Out[16]= {{354}}
```

`Extract` and `Position` work well together. The positions given by `Position` can be given directly to `Extract` to get those elements in `ints` that are specified by the positions in `Position`.

```
In[17]:= Extract[ints, pos]
```

```
Out[17]= {4 302 407 713}
```

```
In[18]:= PrimeQ[%]
```

```
Out[18]= {True}
```

### 3.3 Operations on lists: exercises

- I. Given a list of coordinate pairs such as the following:

```
{{x1, y1}, {x2, y2}, {x3, y3}, {x4, y4}, {x5, y5}}
```

separate the `x` and `y` components to get

```
{x1, x2, x3, x4, x5}, {y1, y2, y3, y4, y5}
```

- Use the `Part` function to extract the elements of a list that are in the even-indexed positions. So in the list below, the even-indexed elements are {3, 8, 3, 4, 2, 13}. Then extract all those elements in the odd-indexed positions.

```
In[1]:= lis = RandomInteger[{1, 20}, {12}]
```

```
Out[1]= {5, 3, 3, 8, 17, 3, 3, 4, 20, 2, 11, 13}
```

- Given the following list of integers, find the five largest numbers in the list. Then find the five smallest numbers in the list.

```
In[2]:= nums = RandomInteger[{-100, 100}, {25}]
```

```
Out[2]= {38, 6, -15, -31, 44, 11, -100, 47, -14, 26, 50,
         48, -72, 24, 12, 66, 10, 31, 78, 85, 11, -67, 23, 63, -45}
```

- Use `Table` to create the following matrix. Once created, use `Table` again to add all the elements on and above the diagonal.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{pmatrix}$$

- Rearrange the list of numbers one through ten so that any adjacent numbers (for example, 1 and 2, 2 and 3, and so on) are not adjacent in the output.
- Create a list of all prime numbers less than 100. Repeat for a list of primes less than 1000. Consider using the functions `Prime` and `PrimePi`.
- Take the partitioned list of integers from the solution to Exercise 5 in Section 3.1 and use `Grid` to display the partitioned list in a grid similar to that in Figure 3.2.

FIGURE 3.2. A  $6 \times 6$  integer grid.

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

- Make a histogram of the frequencies of the leading digit in the first 10 000 Fibonacci numbers. The resulting distribution is an instance of Benford's law, which concerns the frequency of the leading digits in many kinds of data. The phenomenon, whereby a 1 occurs about 30% of the time, a 2 occurs about 17.6% of the time, and so on, has been shown to occur in well-known numerical sequences, population counts, death rates, Fibonacci numbers, and has even been used to detect corporate and tax fraud.

9. Given a matrix, use list component assignment to swap any two rows.
10. Create a function `AddColumn[mat, col, pos]` that inserts a column vector `col` into the matrix `mat` at the column position given by `pos`. For example:

```
In[3]:= mat = RandomInteger[9, {4, 4}];
        MatrixForm[mat]
Out[4]//MatrixForm=

$$\begin{pmatrix} 4 & 8 & 5 & 9 \\ 9 & 4 & 1 & 9 \\ 1 & 9 & 5 & 5 \\ 9 & 5 & 6 & 7 \end{pmatrix}$$

In[5]:= AddColumn[mat, {a, b, c, d}, 3] // MatrixForm
Out[5]//MatrixForm=

$$\begin{pmatrix} 4 & 8 & a & 5 & 9 \\ 9 & 4 & b & 1 & 9 \\ 1 & 9 & c & 5 & 5 \\ 9 & 5 & d & 6 & 7 \end{pmatrix}$$

```

11. How would you perform the same task as `Prepend[{x, y}, z]` using the `Join` function?
12. Starting with the lists `{1, 2, 3, 4}` and `{a, b, c, d}`, create the list `{2, 4, b, d}`. Then create the list `{1, a, 2, b, 3, c, 4, d}`.
13. Many lotteries include games that require you to pick several numbers and match them against the lottery's random number generator. The numbers are independent, so this is essentially random sampling with replacement. The built-in `RandomChoice` does this. For example, here are five random samples from the integers zero through nine:

```
In[6]:= RandomChoice[Range[0, 9], 5]
Out[6]= {5, 4, 1, 1, 3}
```

Write your own function `randomChoice[lis, n]` that performs a random sampling with replacement, where `n` is the number of elements being chosen from the list `lis`. Here is a typical result using a list of symbols:

```
In[7]:= randomChoice[{a, b, c, d, e, f, g, h}, 12]
Out[7]= {e, c, h, b, g, c, d, c, b, c, b, f}
```

14. Given two lists, find all those elements that are not common to the two lists. For example, starting with the lists `{a, b, c, d}` and `{a, b, e, f}`, your answer would return the list `{c, d, e, f}`.
15. One of the tasks in computational linguistics involves statistical analysis of text using what are called *n*-grams – sequences of *n* adjacent letters or words. Their frequency distribution in a body of text can be used to predict word usage based on the previous history or usage.

Import a file consisting of some text and find the twenty most frequently occurring word combinations. Pairs of words that are grouped like this are called *bigrams*, that is, *n*-grams for *n* = 2.

Use `TextWords` (new in *Mathematica* 10.1) to split long strings into a list of words that can then

be operated on with the list manipulation functions.

```
In[8]:= TextWords["Use StringSplit to split long strings into words."]
```

```
Out[8]= {Use, StringSplit, to, split, long, strings, into, words}
```

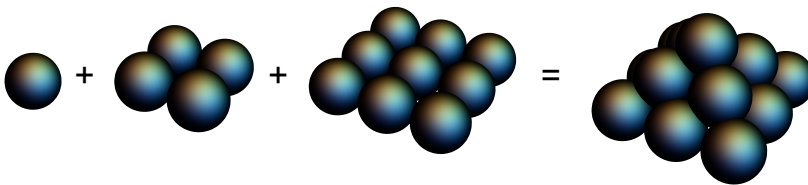
16. Based on the previous exercise, create a function `NGrams[str, n]` that takes a string of text and returns a list of  $n$ -grams, that is, a list of the  $n$  adjacent words. For example:

```
In[9]:= NGrams["Use StringSplit to split long strings into words.", 3]
```

```
Out[9]= {{Use, StringSplit, to}, {StringSplit, to, split}, {to, split, long},
         {split, long, strings}, {long, strings, into}, {strings, into, words}}
```

17. Write your own user-defined functions using the `Characters` and `StringJoin` functions to perform the same operations as `StringInsert` and `StringDrop`.
18. Use `ToCharacterCode` and `FromCharacterCode` to perform the same operations as the built-in `StringJoin` and `StringReverse` functions.
19. Compute the first ten square-pyramidal numbers in three different ways. The first few square-pyramidal numbers are  $1^2 = 1$ ,  $1^2 + 2^2 = 5$ ,  $1^2 + 2^2 + 3^2 = 14$ , .... The number of stacked spheres with a square base (Figure 3.3) are represented by these numbers. In addition, they give a solution to the problem of counting squares in an  $n \times n$  grid.

FIGURE 3.3. Graphical representation of square-pyramidal numbers.



### 3.3 Solutions

1. This is a straightforward use of the `Transpose` function.

```
In[1]:= Transpose[{{x1, y1}, {x2, y2}, {x3, y3}, {x4, y4}, {x5, y5}}]
```

```
Out[1]= {{x1, x2, x3, x4, x5}, {y1, y2, y3, y4, y5}}
```

2. The most direct way to extract the even (or odd) indexed elements is to use the `Span` function as a second argument to `Part`.

```
In[2]:= lis = {5, 3, 3, 8, 17, 3, 3, 4, 20, 2, 11, 13};
```

```
In[3]:= Part[lis, 2 ;; -1 ;; 2]
```

```
Out[3]= {3, 8, 3, 4, 2, 13}
```

Here is the shorthand notation:

```
In[4]:= lis[2 ;; -1 ;; 2]  
Out[4]= {3, 8, 3, 4, 2, 13}
```

And here are the elements in the odd positions.

```
In[5]:= lis[1 ;; -1 ;; 2]  
Out[5]= {5, 3, 17, 3, 20, 11}
```

3. Here is the list of numbers:

```
In[6]:= nums = RandomInteger[{-100, 100}, {25}]  
Out[6]= {-45, -24, 9, 38, -82, -60, -21, -66, 23, 70, -67, -13,  
-98, -68, -97, -55, 10, -11, 25, -13, 7, -100, 88, 36, -86}
```

Sorting gives:

```
In[7]:= Sort[nums]  
Out[7]= {-100, -98, -97, -86, -82, -68, -67, -66, -60, -55,  
-45, -24, -21, -13, -13, -11, 7, 9, 10, 23, 25, 36, 38, 70, 88}
```

The smallest five:

```
In[8]:= Take[Sort[nums], 5]  
Out[8]= {-100, -98, -97, -86, -82}
```

The largest five:

```
In[9]:= Take[Sort[nums], -5 ;; -1]  
Out[9]= {25, 36, 38, 70, 88}
```

Alternatively, use the built-in functions **TakeLargest** and **TakeSmallest**:

```
In[10]:= TakeLargest[nums, 5]  
Out[10]= {88, 70, 38, 36, 25}  
  
In[11]:= TakeSmallest[nums, 5]  
Out[11]= {-100, -98, -97, -86, -82}
```

4. There are several ways to create the matrix. One way is to use **Table** with two iterators.

```
In[12]:= mat = Table[i j, {i, 1, 4}, {j, 1, 4}]  
Out[12]= {{1, 2, 3, 4}, {2, 4, 6, 8}, {3, 6, 9, 12}, {4, 8, 12, 16}}
```



```
In[13]:= MatrixForm[mat]
```

```
Out[13]//MatrixForm=
```

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{pmatrix}$$

Another way is to use `Outer` (discussed in Section 5.1).

```
In[14]:= Outer[Times, Range[4], Range[4]]
```

```
Out[14]= {{1, 2, 3, 4}, {2, 4, 6, 8}, {3, 6, 9, 12}, {4, 8, 12, 16}}
```

To add only those elements on or above the diagonal in `mat`, the following will pick out only those elements.

```
In[15]:= Table[mat[[i, j]], {i, 1, 4}, {j, 1, i}]
```

```
Out[15]= {{1}, {2, 4}, {3, 6, 9}, {4, 8, 12, 16}}
```

Then flatten the nested lists.

```
In[16]:= Flatten[%]
```

```
Out[16]= {1, 2, 4, 3, 6, 9, 4, 8, 12, 16}
```

And finally, use `Total` to sum this list.

```
In[17]:= Total[%]
```

```
Out[17]= 65
```

- There are many possible approaches to this problem including a brute-force approach that creates all permutations and selects on those that meet the criteria. Another approach is to split the list and shuffle:

```
In[18]:= lis = Range[10]
```

```
Out[18]= {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
```

```
In[19]:= p = Partition[lis, 5]
```

```
Out[19]= {{1, 2, 3, 4, 5}, {6, 7, 8, 9, 10}}
```

```
In[20]:= Riffle[p[[1]], p[[2]]]
```

```
Out[20]= {1, 6, 2, 7, 3, 8, 4, 9, 5, 10}
```

Once you are familiar with the `Apply` function (Chapter 5), this is done more compactly as follows:

```
In[21]:= Apply[Riffle, p]
```

```
Out[21]= {1, 6, 2, 7, 3, 8, 4, 9, 5, 10}
```

- First, note that `PrimePi[n]` returns the *number* of primes less than  $n$ .

```
In[22]:= PrimePi[100]
```

```
Out[22]= 25
```

So to list all the primes less than 100, we want the first  $\text{PrimePi}[100] = 25$  primes.

```
In[23]:= Table[Prime[n], {n, PrimePi[100]}]
```

```
Out[23]= {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31,
          37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97}
```

7. An  $n \times n$  grid with rows of length  $n$  will have  $n^2$  elements in total. Starting with the list of integers one through  $n^2$ , partition it into sublists of length  $n$ . For example, for  $n = 4$ :

```
In[24]:= n = 4;
lis = Partition[Range[n^2], n]
```

```
Out[25]= {{1, 2, 3, 4}, {5, 6, 7, 8}, {9, 10, 11, 12}, {13, 14, 15, 16}}
```

And then put the list into a grid:

```
In[26]:= Grid[lis, Frame → All]
```

```
Out[26]=
```

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Alternatively, you can use `Multicolumn`:

```
In[27]:= n = 4;
lis = Multicolumn[Range[n^2], n, Appearance → "Horizontal", Frame → All]
```

```
Out[28]=
```

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

```
In[29]:= Clear[n]
```

8. To extract the leading digit of any number, use `IntegerDigits` to generate a list of the digits in a number, and then take the first element in that list. For example, here are the digits in the 50th Fibonacci number.

```
In[30]:= IntegerDigits[Fibonacci[50]]
```

```
Out[30]= {1, 2, 5, 8, 6, 2, 6, 9, 0, 2, 5}
```

And this gives the first digit from that list.

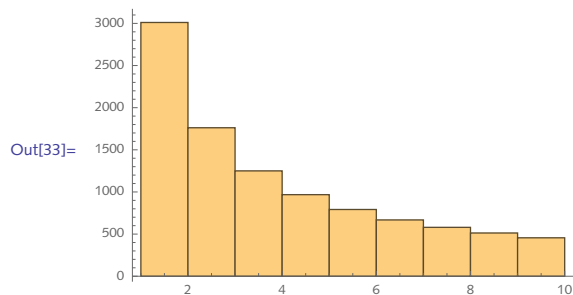
```
In[31]:= First[%]
```

```
Out[31]= 1
```

To do this for the first 100 000 Fibonacci numbers, use `Table`.

```
In[32]:= digits = Table[First[IntegerDigits[Fibonacci[i]]], {i, 10^4}];
```

```
In[33]:= Histogram[digits]
```



9. The standard procedural approach is to use a temporary variable to do the swapping.

```
In[34]:= mat = RandomInteger[9, {4, 4}];
MatrixForm[mat]
```

Out[35]//MatrixForm=

$$\begin{pmatrix} 4 & 1 & 2 & 6 \\ 1 & 6 & 2 & 8 \\ 0 & 7 & 2 & 4 \\ 1 & 7 & 6 & 9 \end{pmatrix}$$

```
In[36]:= temp = mat[[1]];
mat[[1]] = mat[[2]];
mat[[2]] = temp;
MatrixForm[mat]
```

Out[39]//MatrixForm=

$$\begin{pmatrix} 1 & 6 & 2 & 8 \\ 4 & 1 & 2 & 6 \\ 0 & 7 & 2 & 4 \\ 1 & 7 & 6 & 9 \end{pmatrix}$$

But you can use parallel assignments to avoid the temporary variable.

```
In[40]:= mat = RandomInteger[9, {4, 4}];
MatrixForm[mat]
```

Out[41]//MatrixForm=

$$\begin{pmatrix} 2 & 8 & 2 & 8 \\ 1 & 2 & 7 & 8 \\ 0 & 0 & 7 & 7 \\ 4 & 2 & 7 & 2 \end{pmatrix}$$

```
In[42]:= {mat[[2]], mat[[1]]} = {mat[[1]], mat[[2]]};
MatrixForm[mat]
```

Out[43]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & 7 & 8 \\ 2 & 8 & 2 & 8 \\ 0 & 0 & 7 & 7 \\ 4 & 2 & 7 & 2 \end{pmatrix}$$

In fact you can make this a bit more compact.

```
In[44]:= mat = RandomInteger[9, {4, 4}];
MatrixForm[mat]
```

```
Out[45]//MatrixForm=
```

$$\begin{pmatrix} 3 & 5 & 8 & 7 \\ 1 & 2 & 1 & 2 \\ 5 & 1 & 1 & 9 \\ 5 & 6 & 2 & 8 \end{pmatrix}$$

```
In[46]:= mat[[{2, 1}]] = mat[[{1, 2}]];
MatrixForm[mat]
```

```
Out[47]//MatrixForm=
```

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 3 & 5 & 8 & 7 \\ 5 & 1 & 1 & 9 \\ 5 & 6 & 2 & 8 \end{pmatrix}$$

A key point to notice is that in this exercise, the matrix `mat` was overwritten in each case; in other words, these were destructive operations. Section 6.1 discusses how to handle row and column swapping properly so that the original matrix remains untouched.

10. We prototype with a small matrix.

```
In[48]:= mat = RandomInteger[10, {3, 3}];
MatrixForm[mat]
```

```
Out[49]//MatrixForm=
```

$$\begin{pmatrix} 0 & 3 & 6 \\ 3 & 6 & 8 \\ 2 & 9 & 6 \end{pmatrix}$$

Using `Mean` on a matrix produces a vector consisting of column means.

```
In[50]:= Mean[mat]
```

```
Out[50]= {5/3, 6, 20/3}
```

To subtract the means from their respective columns in `mat`, operate on the transposed matrix.

```
In[51]:= centeredMat = Transpose[Transpose[mat] - Mean[mat]]
```

```
Out[51]= {{-5/3, -3, -2/3}, {4/3, 0, 4/3}, {1/3, 3, -2/3}}
```

```
In[52]:= Mean[centeredMat]
```

```
Out[52]= {0, 0, 0}
```

Alternatively, you could create a centering matrix which, when multiplied by the original matrix, centers it.

```
In[53]:= CenteringMatrix[n_] := IdentityMatrix[n] - ConstantArray[1/n, {n, n}]
```

```
In[54]:= CenteringMatrix[3].mat
```

```
Out[54]= {{-5/3, -3, -2/3}, {4/3, 0, 4/3}, {1/3, 3, -2/3}}
```

As an aside, the centering matrix is symmetric and idempotent.

```
In[55]:= SymmetricMatrixQ[CenteringMatrix[3]]
```

```
Out[55]= True
```

```
In[56]:= CenteringMatrix[3].CenteringMatrix[3] == CenteringMatrix[3]
```

```
Out[56]= True
```

This can also be accomplished with the built-in function `Standardize`.

```
In[57]:= Standardize[mat, Mean, 1 &]
```

```
Out[57]= {{-5/3, -3, -2/3}, {4/3, 0, 4/3}, {1/3, 3, -2/3}}
```

II. You need to first transpose the matrix to operate on the columns as rows.

```
In[58]:= mat = RandomInteger[9, {4, 4}];
MatrixForm[mat]
```

```
Out[59]//MatrixForm=
```

$$\begin{pmatrix} 8 & 2 & 2 & 9 \\ 3 & 7 & 9 & 7 \\ 1 & 2 & 7 & 9 \\ 0 & 2 & 6 & 5 \end{pmatrix}$$

```
In[60]:= Transpose[mat]
```

```
Out[60]= {{8, 3, 1, 0}, {2, 7, 2, 2}, {2, 9, 7, 6}, {9, 7, 9, 5}}
```

Now insert the column vector at the desired position. Then transpose back.

```
In[61]:= Insert[Transpose[mat], {a, b, c, d}, 3] // MatrixForm
```

```
Out[61]//MatrixForm=
```

$$\begin{pmatrix} 8 & 3 & 1 & 0 \\ 2 & 7 & 2 & 2 \\ a & b & c & d \\ 2 & 9 & 7 & 6 \\ 9 & 7 & 9 & 5 \end{pmatrix}$$

```
In[62]:= Transpose@Insert[Transpose[mat], {a, b, c, d}, 3] // MatrixForm
```

```
Out[62]//MatrixForm=
```

$$\begin{pmatrix} 8 & 2 & a & 2 & 9 \\ 3 & 7 & b & 9 & 7 \\ 1 & 2 & c & 7 & 9 \\ 0 & 2 & d & 6 & 5 \end{pmatrix}$$

Here then is the function, with some basic argument checking to make sure the number of elements in the column vector is the same as the number of rows of the matrix.

```
In[63]:= AddColumn[mat_, col_, pos_] /; Length[col] == Length[mat] :=
Transpose[Insert[Transpose[mat], col, pos]]
```

12. Join expects lists as arguments.

```
In[64]:= Join[{z}, {x, y}]
```

```
Out[64]= {z, x, y}
```

13. Joining the two lists and then using `Part` with `Span` is the most direct way to do this.

```
In[65]:= expr = Join[{1, 2, 3, 4}, {a, b, c, d}]
```

```
Out[65]= {1, 2, 3, 4, a, b, c, d}
```

```
In[66]:= expr[[2 ;; -1 ;; 2]]
```

```
Out[66]= {2, 4, b, d}
```

For the second part of this exercise, the function `Riffle` is perfect for this task.

```
In[67]:= Riffle[{1, 2, 3, 4}, {a, b, c, d}]
```

```
Out[67]= {1, a, 2, b, 3, c, 4, d}
```

This can also be done in two steps by first transposing the two lists and then flattening.

```
In[68]:= Transpose[{1, 2, 3, 4}, {a, b, c, d}]
```

```
Out[68]= {{1, a}, {2, b}, {3, c}, {4, d}}
```

```
In[69]:= Flatten[%]
```

```
Out[69]= {1, a, 2, b, 3, c, 4, d}
```

14. One way to do this is to take the list and simply pick out elements at random locations. The right-most location in the list is given by `Length[lis]`, using `Part` and `RandomInteger`.

```
In[70]:= randomChoice[lis_, n_] := lis[[RandomInteger[{1, Length[lis]}, {n}]]]
```

```
In[71]:= randomChoice[{a, b, c, d, e, f, g, h}, 12]
```

```
Out[71]= {h, b, a, g, b, c, f, a, g, f, a, d}
```

15. This is another way of asking for all those elements that are in the union but not the intersection of the two sets.

```
In[72]:= A = {a, b, c, d};
```

```
B = {a, b, e, f};
```

```
In[74]:= Complement[A ∪ B, A ∩ B]
```

```
Out[74]= {c, d, e, f}
```

```
In[75]:= Complement[Union[A, B], Intersection[A, B]]
```

```
Out[75]= {c, d, e, f}
```

16. We will use Darwin's *On the Origin of Species* text, built into *Mathematica* via `ExampleData`.

```
In[76]:= darwin = ExampleData[{"Text", "OriginOfSpecies"}];
words = TextWords[darwin]
```

```
Out[77]= {INTRODUCTION, When, on, board, H.M.S., Beagle, as,
          naturalist, I, was, much, ... 149 960 ..., most, beautiful,
          and, most, wonderful, have, been, and, are, being, evolved}
```

large output

show less

show more

show all

set size limit...

First, partition the list of words into pairs with overlap one.

```
In[78]:= par = Partition[words, 2, 1]
```

```
Out[78]= {{INTRODUCTION, When}, {When, on}, {on, board},
          {board, H.M.S.}, {H.M.S., Beagle}, ... 149 971 ..., {have, been},
          {been, and}, {and, are}, {are, being}, {being, evolved}}
```

large output

show less

show more

show all

set size limit...

Then tally them and sort by the frequency count, given by the last element in each sublist. Finally, take the last twenty expressions, those bigrams occurring the most frequently.

```
In[79]:= tally = SortBy[Tally[par], Last];
Take[tally, -20]
```

```
Out[80]= {{has, been}, 190}, {{each, other}, 192}, {{species, of}, 201},
          {{of, life}, 227}, {{with, the}, 230}, {{natural, selection}, 234},
          {{it, is}, 237}, {{and, the}, 238}, {{by, the}, 242}, {{from, the}, 256},
          {{in, a}, 256}, {{to, be}, 267}, {{of, a}, 268}, {{have, been}, 432},
          {{that, the}, 434}, {{on, the}, 498}, {{to, the}, 582},
          {{the, same}, 715}, {{in, the}, 1024}, {{of, the}, 1993}}
```

Here are the next twenty most frequently occurring bigrams.

```
In[81]:= Take[tally, -40 ;; -21]
```

```
Out[81]= {{to, have}, 120}, {{which, are}, 121}, {{conditions, of}, 124},
          {{between, the}, 125}, {{do, not}, 128}, {{and, in}, 132},
          {{the, case}, 133}, {{can, be}, 137}, {{will, be}, 138}, {{as, the}, 140},
          {{would, be}, 142}, {{number, of}, 144}, {{all, the}, 150}, {{at, the}, 157},
          {{the, most}, 160}, {{the, other}, 166}, {{for, the}, 170},
          {{the, species}, 172}, {{I, have}, 185}, {{may, be}, 189}
```

This can be done in one step using WordCounts (new in *Mathematica* 10.1) with a second argu-

ment to count bigrams only.

```
In[82]:= WordCounts[darwin, 2]
```

```
Out[82]= <| {of, the} → 1993, {in, the} → 1023, {the, same} → 715,
           {to, the} → 582, {on, the} → 497, ... 47 987 ..., {a, barrier} → 1,
           {a, barnacle} → 1, {a, bare} → 1, {a, bar} → 1, {a, bank} → 1 |>
```

large output

show less

show more

show all

set size limit...

```
In[83]:= Take[%, 20]
```

```
Out[83]= <| {of, the} → 1993, {in, the} → 1023, {the, same} → 715, {to, the} → 582,
           {on, the} → 497, {that, the} → 434, {have, been} → 431, {of, a} → 268,
           {to, be} → 267, {in, a} → 256, {from, the} → 256, {by, the} → 240,
           {and, the} → 238, {it, is} → 235, {natural, selection} → 233, {with, the} → 228,
           {of, life} → 227, {species, of} → 200, {each, other} → 192, {has, been} → 190 |>
```

17. This is a straightforward extension of the previous exercise.

```
In[84]:= NGrams[text_, n_] := Partition[TextWords[text], n, 1]
```

```
In[85]:= sentence = "Use StringSplit to split long strings into words.";
NGrams[sentence, 3]
```

```
Out[86]= {{Use, StringSplit, to}, {StringSplit, to, split}, {to, split, long},
           {split, long, strings}, {long, strings, into}, {strings, into, words}}
```

18. Here is our user-defined stringInsert.

```
In[87]:= stringInsert[str1_, str2_, pos_] := StringJoin@Join[
           Take[Characters[str1], pos - 1],
           Characters[str2],
           Drop[Characters[str1], pos - 1]
         ]
```

```
In[88]:= stringInsert["Joy world", "to the ", 5]
```

```
Out[88]= Joy to the world
```

```
In[89]:= stringDrop[str_, pos_] := StringJoin[Drop[Characters[str], pos]]
```

```
In[90]:= stringDrop["ABCDEF", -2]
```

```
Out[90]= ABCD
```

The idea in these two examples is to convert a string to a list of characters, operate on that list using list manipulation functions like Join, Take, and Drop, then convert back to a string. More efficient approaches use string manipulation functions directly (see Chapter 7).

19. First, here is how we might write our own StringJoin.



```
In[91]:= FromCharCode[Join[
      ToCharCode["To be, "], ToCharCode["or not to be"]
    ]]
Out[91]= To be, or not to be
```

And here is a how we might implement a StringReverse.

```
In[92]:= FromCharCode[Reverse[ToCharCode["never odd or even"]]]
Out[92]= neve ro ddo reven
```

20. First we could use the Sum function to add  $k^2$  as  $k$  goes from 1 to  $n$ .

```
In[93]:= Sum[k2, {k, 1, 10}]
Out[93]= 385
```

Here is a list of the first ten square pyramidal numbers.

```
In[94]:= Table[Sum[k2, {k, 1, n}], {n, 1, 10}]
Out[94]= {1, 5, 14, 30, 55, 91, 140, 204, 285, 385}
```

A closed form expression will be much faster.

```
In[95]:= Sum[k2, {k, 1, n}]
Out[95]=  $\frac{1}{6} n (1 + n) (1 + 2 n)$ 
```

```
In[96]:= Table[ $\frac{1}{6} n (1 + n) (1 + 2 n)$ , {n, 1, 10}]
Out[96]= {1, 5, 14, 30, 55, 91, 140, 204, 285, 385}
```

Another way of looking at the computation is that we are squaring each of the first  $n$  numbers and then adding those squares together. Think dot product of the list of the first  $n$  integers with itself.

```
In[97]:= Range[10].Range[10]
Out[97]= 385
```

```
In[98]:= Table[Range[n].Range[n], {n, 1, 10}]
Out[98]= {1, 5, 14, 30, 55, 91, 140, 204, 285, 385}
```

Or, more directly, square each of the integers one through ten, then add them up.

```
In[99]:= Total[Range[10]2]
Out[99]= 385
```

### 3.4 Associations: exercises

1. Create an association consisting of several songs in your music library. Include keys for song title, artist, release date, album cover. Once the association is defined, convert all the album covers to smaller images using Thumbnail.
2. Modify the MakeRef function to display the year of publication at the end of the line for authors. Adjust the style so that the year displays in a bold font.


```
In[1]:= MakeRef[hamming1950]
```

Hamming, Richard (**1950**)  
*Error detecting and error correcting codes*  
<https://archive.org/details/bstj29-2-147>

### 3.4 Solutions

1. Here is a small association consisting of information on three albums.

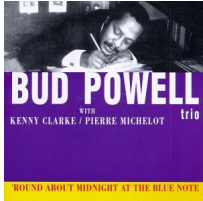
```
In[1]:= alb1 = Association[{
  "SongTitle" → "Desvairada",
  "AlbumArtist" → "Paulo Bellinati",
  "AlbumTitle" → "The Guitar Works of Garoto",
  "Year" → "1991",
  "Cover" →
```



```

}];
```

```
In[2]:= alb2 = Association[{
  "SongTitle" → "Monk's Mood",
  "AlbumArtist" → "Bud Powell",
  "AlbumTitle" → "Round About Midnight At The Blue Note",
  "Year" → "1962",
  "Cover" →
```



```

}];
```

```
In[3]:= alb3 = Association[{
  "SongTitle" → "Dounia",
  "AlbumArtist" → "Rokia Traoré",
  "AlbumTitle" → "Tchamantché",
  "Year" → "2008",
```

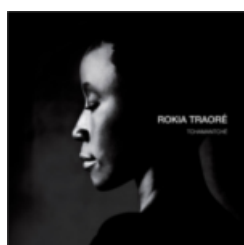
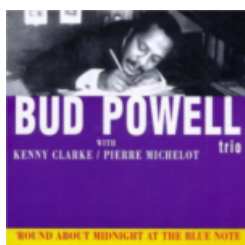
```
  "Cover" →
```



```
  }];
```

```
In[4]:= Map[Thumbnail, {alb1["Cover"], alb2["Cover"], alb3["Cover"]}]]
```

```
Out[4]= {
```



2. It is first necessary to modify the MakeRef function so that newline characters do not appear between all items. We want the author text to be followed by the date and then a newline. So instead of using a default separator in Row, we will manually put them where we want them.

```
In[5]:= makeAuthor[ref_] := Style[ref["Author"], "TR"]
```

```
In[6]:= makeTitle[ref_] := Style[ref["Title"], "TI"]
```

```
In[7]:= makeLink[ref_] := Hyperlink[Style[ref["Url"], "TR"], ref["Url"]]
```

```
In[8]:= makeDate[ref_] :=
  Row[{Style[" (", "TR"], Style[ref["Year"], "TB"], Style[" )", "TR"]}]
```

```
In[9]:= MakeRef[ref_] := CellPrint@TextCell[Row[{
  makeAuthor[ref], makeDate[ref], "\n",
  makeTitle[ref], "\n",
  makeLink[ref]
}], "Text", ShowStringCharacters → False]
```

```
In[10]:= art1 = Association@{  
    "Author" → "Hamming, Richard W.",  
    "Title" → "Error detecting and error correcting codes",  
    "Journal" → "The Bell System Technical Journal",  
    "Year" → 1950,  
    "Volume" → 29,  
    "Issue" → 2,  
    "Pages" → "147-160",  
    "Url" → "https://archive.org/details/bstj29-2-147"  
}
```

```
Out[10]= <| Author → Hamming, Richard W.,  
    Title → Error detecting and error correcting codes,  
    Journal → The Bell System Technical Journal, Year → 1950, Volume → 29,  
    Issue → 2, Pages → 147-160, Url → https://archive.org/details/bstj29-2-147 |>
```

```
In[11]:= MakeRef[art1]
```

Hamming, Richard W. (1950)  
*Error detecting and error correcting codes*  
<https://archive.org/details/bstj29-2-147>

---

## 4 Patterns and rules

### 4.1 Patterns: exercises

1. Explain why the following pattern match fails. Then find two different patterns that correctly match a complex number such as  $3 + 4i$ .

```
In[1]:= MatchQ[3 + 4 I, a_ + b_ I]
```

```
Out[1]= False
```

2. Use conditional patterns to find all those numbers in a list of integers that are divisible by 2 or 3 or 5.
3. Write down four conditional patterns that match the expression  $\{4, \{a, b\}, "g"\}$ .
4. Explain why the expression  $1 / y$  is not matched by the pattern  $a_ / b_$ .

```
In[2]:= MatchQ[1 / y, a_ / b_]
```

```
Out[2]= False
```

Determine the correct pattern that can be used to match the symbolic expression  $x / y$ .

```
In[3]:= MatchQ[x / y, Power[a_, -1]]
```

```
Out[3]= False
```

5. Write a function `Collatz` that takes an integer  $n$  as an argument and returns  $3n + 1$  if  $n$  is an odd integer and returns  $n/2$  if  $n$  is even.
6. Write the `Collatz` function from the above exercise, but this time you should also check that the argument to `Collatz` is positive.
7. Use alternatives to write a function `abs[x]` that returns  $x$  if  $x \geq 0$ , and  $-x$  if  $x < 0$ , whenever  $x$  is an integer or a rational number. Whenever  $x$  is complex, `abs[x]` should return  $\sqrt{\text{re}(x)^2 + \text{im}(x)^2}$ .

8. Create a function `swapTwo[lis]` that returns `lis` with only its first two elements interchanged; for example, the input `swapTwo[{a, b, c, d, e}]` should return `{b, a, c, d, e}`. If `lis` has fewer than two elements, `swapTwo` just returns `lis`. Write `swapTwo` using three clauses: one for the empty list, one for one-element lists, and one for all other lists. Then write it using two clauses: one for lists of length zero or one and another for all longer lists.
9. Explain the different results from the following three pattern matches:

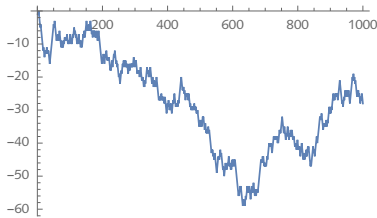
```
In[4]:= MatchQ[{4, 6, 8}, x_ /; Length[x] > 4]
Out[4]= False

In[5]:= MatchQ[{4, 6, 8}, {x___} /; Length[x] > 4]
Length::argx : Length called with 3 arguments; 1 argument is expected. >
Out[5]= False

In[6]:= MatchQ[{4, 6, 8}, {x___} /; Plus[x] > 10]
Out[6]= True
```

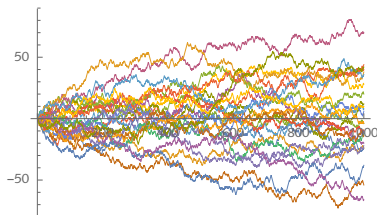
10. Write a rule for the one-dimensional case of `showWalk` described in this section. Then write an additional rule to handle multiple one-dimensional random walks.

```
In[7]:= Needs["EPM`RandomWalks`"]
In[8]:= walk = RandomWalk[1000, Dimension -> 1];
showWalk[walk]
```



```
Out[9]=
```

```
In[10]:= walks = Table[RandomWalk[1000, Dimension -> 1], {25}];
showWalk[walks]
```

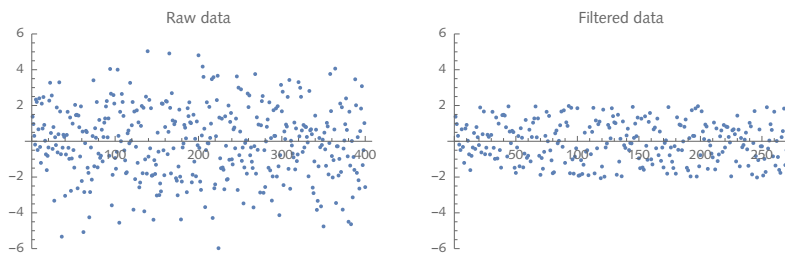


```
Out[11]=
```

- II. Given a set of data like that in Figure 4.I, remove all outliers, defined here by being greater than two standard deviations from the mean of the data.

```
In[12]:= rawData = RandomVariate[NormalDistribution[0, 2], {200}];
```

FIGURE 4.1. Scatter plots of original data and data with outliers removed.



## 4.1 Solutions

1. Look at the internal representation of the complex number.

```
In[1]:= FullForm[3 + 4 I]
```

```
Out[1]/FullForm= Complex[3, 4]
```

Although this is an equivalent representation to the traditional notation  $3 + 4i$ , it is not the same syntactically and the pattern matcher is a syntactic creature. Instead, match on head `Complex`.

```
In[2]:= MatchQ[3 + 4 I, _Complex]
```

```
Out[2]= True
```

Although giving different information, you could also check that the expression is a number.

```
In[3]:= MatchQ[3 + 4 I, _?NumberQ]
```

```
Out[3]= True
```

2. Start by creating a list of integers with which to work.

```
In[4]:= lis = RandomInteger[1000, {20}]
```

```
Out[4]= {270, 885, 466, 194, 374, 237, 249, 228, 184,
        704, 636, 922, 10, 619, 806, 497, 290, 329, 463, 168}
```

`IntegerQ` is a predicate; it returns `True` or `False`, so we need to use the logical OR to separate clauses here.

```
In[5]:= Cases[lis, n_ /; IntegerQ[n/2] || IntegerQ[n/3] || IntegerQ[n/5]]
```

```
Out[5]= {270, 885, 466, 194, 374, 237, 249, 228, 184, 704, 636, 922, 10, 806, 290, 168}
```

This is a bit more compact and direct.

```
In[6]:= Cases[lis, n_ /; Mod[n, 2] == 0 || Mod[n, 3] == 0 || Mod[n, 5] == 0]
```

```
Out[6]= {270, 885, 466, 194, 374, 237, 249, 228, 184, 704, 636, 922, 10, 806, 290, 168}
```

Once you are familiar with pure functions (Section 5.5), you can also do this with `Select`.

```
In[7]:= Select[lis, Mod[#, 2] == 0 || Mod[#, 3] == 0 || Mod[#, 5] == 0 &]
Out[7]= {270, 885, 466, 194, 374, 237, 249, 228, 184, 704, 636, 922, 10, 806, 290, 168}
```

3. FullForm should help to guide you.

```
In[8]:= FullForm[{4, {a, b}, "g"}]
Out[8]//FullForm= List[4, List[a, b], "g"]

In[9]:= MatchQ[{4, {a, b}, "g"}, x_List /; Length[x] == 3]
Out[9]= True

In[10]:= MatchQ[{4, {a, b}, "g"}, {_, y_, _} /; y[[0]] == List]
Out[10]= True

In[11]:= MatchQ[{4, {a, b}, "g"}, {x_, y_, z_} /; AtomQ[z]]
Out[11]= True

In[12]:= MatchQ[{4, {a, b}, "g"}, {x_, _, _} /; EvenQ[x]]
Out[12]= True
```

4. Look at the internal representation of these two expressions:

```
In[13]:= FullForm[1 / x]
Out[13]//FullForm= Power[x, -1]

In[14]:= FullForm[a_ / b_]
Out[14]//FullForm= Times[Pattern[a, Blank[]], Power[Pattern[b, Blank[]], -1]]
```

So  $1 / x$  is a special case of the pattern  $a_ / b_$  but it is one that *Mathematica* simplifies to an expression with head `Power`.

$x / y$  can be matched by the pattern  $a_ / b_$ .

```
In[15]:= MatchQ[x / y, a_ / b_]
Out[15]= True
```

5. The Collatz function has a direct implementation based on its definition. There is no need to check explicitly that the argument is an integer since `OddQ` and `EvenQ` handle that.

```
In[16]:= Collatz[n_?OddQ] := 3 n + 1
In[17]:= Collatz[n_?EvenQ] :=  $\frac{n}{2}$ 
```

Here we iterate the Collatz function fifteen times starting with an initial value of 23.

```
In[18]:= NestList[Collatz, 23, 15]
Out[18]= {23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1}
```

Check for arguments that do not match the patterns above.



```
In[19]:= Collatz[24.0]
```

```
Out[19]= Collatz[24.]
```

6. Here again is the Collatz function, but this time using a condition on the right-hand side of the definition.

```
In[20]:= Clear[Collatz]
```

```
In[21]:= Collatz[n_] := 3 n + 1 /; OddQ[n] && Positive[n]
```

```
In[22]:= Collatz[n_] :=  $\frac{n}{2}$  /; EvenQ[n] && Positive[n]
```

```
In[23]:= Collatz[4.3]
```

```
Out[23]= Collatz[4.3]
```

```
In[24]:= Collatz[-3]
```

```
Out[24]= Collatz[-3]
```

```
In[25]:= NestList[Collatz, 22, 15]
```

```
Out[25]= {22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1}
```

You could also put the conditions inside the pattern on the left-hand side if you prefer.

```
In[26]:= Clear[Collatz]
```

```
In[27]:= Collatz[n_ /; OddQ[n] && Positive[n]] := 3 n + 1
```

```
In[28]:= Collatz[n_ /; EvenQ[n] && Positive[n]] :=  $\frac{n}{2}$ 
```

```
In[29]:= NestList[Collatz, 22, 15]
```

```
Out[29]= {22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1}
```

7. Using alternatives, this gives the definition for real, integer, or rational arguments.

```
In[30]:= abs[x_Real | x_Integer | x_Rational] := If[x ≥ 0, x, -x]
```

Here is the definition for complex arguments.

```
In[31]:= abs[x_Complex] :=  $\sqrt{\text{Re}[x]^2 + \text{Im}[x]^2}$ 
```

Note that these rules are not invoked for symbolic arguments.

```
In[32]:= Map[abs, {-3, 3 + 4 I,  $\frac{-4}{5}$ , a}]
```

```
Out[32]= {3, 5,  $\frac{4}{5}$ , abs[a]}
```

8. We first have to consider the base cases. Given a list with no elements, swapTwo should return the empty list. And, given a list with one element, swapping should give that one element back.

```
In[33]:= swapTwo[{}] := {}
        swapTwo[{x_}] := {x}
```

Now, we use the triple-blank to indicate that *r* could be a sequence of zero or more elements.

```
In[35]:= swapTwo[{x_, y_, r___}] := {y, x, r}
```

```
In[36]:= swapTwo[{}]
```

```
Out[36]= {}
```

```
In[37]:= swapTwo[{a}]
```

```
Out[37]= {a}
```

```
In[38]:= swapTwo[{a, b, c, d}]
```

```
Out[38]= {b, a, c, d}
```

Notice in this second definition for `swapTwo` that the second clause covers both the situation where the argument is the empty list and when it contains only one element.

```
In[39]:= swapTwo2[{x_, y_, r___}] := {y, x, r}
        swapTwo2[x_] := x
```

```
In[41]:= swapTwo[{}]
```

```
Out[41]= {}
```

```
In[42]:= swapTwo[{a}]
```

```
Out[42]= {a}
```

```
In[43]:= swapTwo[{a, b, c, d}]
```

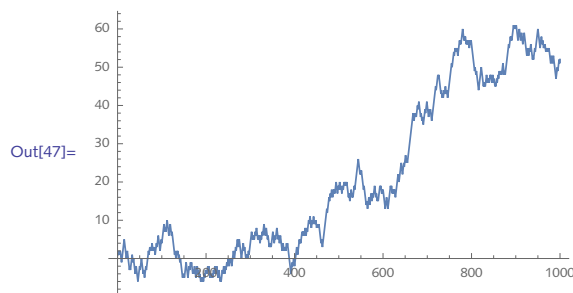
```
Out[43]= {b, a, c, d}
```

9. In the first example, *x* was associated with the entire list `{4, 6, 8}`; since the length of the list `{4, 6, 8}` is not greater than 4, the match failed. In the second example, *x* became the sequence 4, 6, 8 so that the condition was `Length[4, 6, 8] > 4`; but `Length` can only have one argument, hence the error. In the last example, *x* was again associated with 4, 6, 8, but now the condition was `Plus[4, 6, 8] > 10`, which is perfectly valid syntax, and true.
10. To write the one-dimensional rule, you can simply use `ListLinePlot` on the right hand side. The pattern on the left hand side though needs to match a one-dimensional list of numbers. `VectorQ` can be used to check for this.

```
In[44]:= showWalk[coords_?VectorQ] := ListLinePlot[coords]
```

```
In[45]:= Needs["EPM`RandomWalks`"]
```

```
In[46]:= walk = RandomWalk[1000, Dimension -> 1];
showWalk[walk]
```



The case of multiple walks is a bit trickier. First note the structure of a list of multiple one-dimensional random walks. Here we have 25 ten-step walks.

```
In[48]:= walks = Table[RandomWalk[10, Dimension -> 1], {25}];
Dimensions[walks]
```

Out[49]= {25, 10}

In situations where you are not sure what the pattern needs to be to match a complicated expression, it oftentimes helps to look at some representative examples.

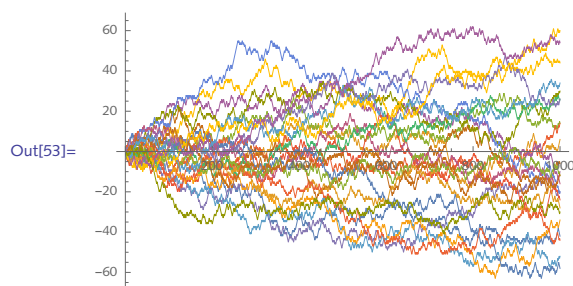
```
In[50]:= Short[Take[walks, 5], 2]
```

Out[50]//Short= {{-1, -2, -1, -2, -3, -4, -3, -4, -5, -4},  
<<3>>, {-1, -2, -3, -2, -3, -2, -3, -4, -5, -4}}

So the structure is  $\{\{n\text{-steps}\}, \{n\text{-steps}\}, \dots, \{n\text{-steps}\}\}$  and in this specific example we have 25 repeats or trials. We don't know how long each walk will be ahead of time so we need a double blank to accommodate walks of length one or more. And we will check that each walk list consists of numbers. Here is the rule. We have also added a style directive to thin out the lines.

```
In[51]:= showWalk[coords : {{__?NumberQ} ..}] :=
ListLinePlot[coords, PlotStyle -> Directive[Thin]]
```

```
In[52]:= walks = Table[RandomWalk[1000, Dimension -> 1], {25}];
showWalk[walks]
```



II. The mean and standard deviation of the data can be obtained with built-in functions.

```
In[54]:= data = RandomVariate[NormalDistribution[0, 2], {200}];
```

```
In[55]:= {μ, σ} = {Mean[data], StandardDeviation[data]}
```

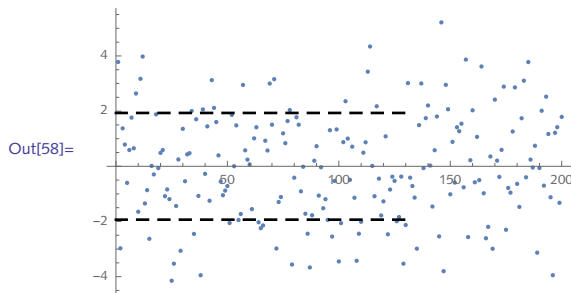
```
Out[55]= {-0.00968766, 1.93378}
```

This extracts all those elements of data that are within one standard deviation of the mean.

```
In[56]:= filtered = Cases[data, p_ /; Abs[μ - p] < σ];  
len = Length[filtered]
```

```
Out[57]= 132
```

```
In[58]:= ListPlot[data, Epilog -> {  
    Dashed, Line[{0, σ}, {len, σ}], Line[{0, -σ}, {len, -σ}]}]
```



## 4.2 Transformation rules: exercises

1. Here is a rule designed to switch the order of each pair of expressions in a list. It works fine on the first example, but fails on the second.

```
In[1]:= {{a, b}, {c, d}, {e, f}} /. {x_, y_} -> {y, x}
```

```
Out[1]= {{b, a}, {d, c}, {f, e}}
```

```
In[2]:= {{a, b}, {c, d}} /. {x_, y_} -> {y, x}
```

```
Out[2]= {{c, d}, {a, b}}
```

Explain what has gone wrong and rewrite this rule to correct the situation, that is, so that the second example returns  $\{\{b, a\}, \{d, c\}\}$ .

2. Given a  $3 \times 3$  matrix, here is a rule intended to swap the elements in the second and third columns:

```
In[3]:= mat = {{a, b, c}, {d, e, f}, {g, h, i}};
```

```
In[4]:= mat /. {x_, y_, z_} -> {z, y, x} // MatrixForm
```

```
Out[4]//MatrixForm= 
$$\begin{pmatrix} a & b & c \\ g & h & i \\ d & e & f \end{pmatrix}$$

```

Explain what has gone wrong and rewrite the rule so that it correctly swaps columns two and three.

- Use pattern matching to extract all negative solutions of the following polynomial:

$$x^9 + 3.4 x^6 - 25 x^5 - 213 x^4 - 477 x^3 + 1012 x^2 + 111 x - 123$$

Then extract all real solutions; that is, those which are not complex.

- Create a rewrite rule that uses a repeated replacement to “unnest” the nested lists within a list.

```
In[5]:= unNest[{{α, α, α}, {α}, {{β, β, β}, {β, β}}, {α, α}}]
```

```
Out[5]= {α, α, α, α, β, β, β, β, α, α}
```

- Define a function using pattern matching and repeated replacement to sum the elements of a list such as that produced by `Range[100]`.
- Using the built-in function `ReplaceList`, write a function `cartesianProduct` that takes two lists as input and returns the Cartesian product of these lists.

```
In[6]:= cartesianProduct[{x1, x2, x3}, {y1, y2}]
```

```
Out[6]= {{x1, y1}, {x1, y2}, {x2, y1}, {x2, y2}, {x3, y1}, {x3, y2}}
```

- Write a function to count the total number of multiplications in any polynomial expression. For example, given a power, your function should return one less than the exponent.

```
In[7]:= MultiplyCount[t^5]
```

```
Out[7]= 4
```

```
In[8]:= MultiplyCount[a x y t]
```

```
Out[8]= 3
```

```
In[9]:= MultiplyCount[a x y t^4 + w t]
```

```
Out[9]= 7
```

- Create six graphical objects, one each to represent the faces of a standard six-sided die. `Dice[n]` should display the face of the appropriate die, as below. Then use the `Dice` function to create a function `RollDice[]` that “rolls” two dice and displays them side-by-side. Create an additional rule, `RollDice[n]`, that rolls a pair of dice  $n$  times and displays the result in a list or row.

```
In[10]:= Table[Dice[n], {n, 1, 6}]
```

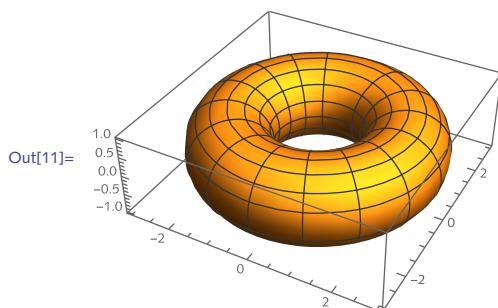


One way to approach this problem is to think of a die face as a grid of nine elements, some of which are turned on (white) and some turned off (blue above). Then create one set of rules for each of the six die faces. Once your rules are defined, you could use something like the following graphics code (a bit incomplete as written here) to create your images:

```
Dice[n_] := GraphicsGrid[
  Map[Graphics, Partition[Range[9], 3] /. rules[[n]], {2}]]
```

9. Make a scatter plot of the points used to construct the polygons in a torus, which is given parametrically as follows:

```
In[11]:= ParametricPlot3D[{(2 + Cos[v]) Sin[u], (2 + Cos[v]) Cos[u], Sin[v]},
  {u, 0, 2 π}, {v, 0, 2 π}]
```



## 4.2 Solutions

- i. The problem here is that the pattern is too general and has been matched by the entire expression, which has the form  $\{x_, y_\}$ , where  $x$  is matched by  $\{a, b\}$  and  $y$  is matched by  $\{c, d\}$ . To fix this, use patterns to restrict the expressions that match.

```
In[1]:= {{a, b}, {c, d}} /. {x_Symbol, y_Symbol} -> {y, x}
```

```
Out[1]= {{b, a}, {d, c}}
```

```
In[2]:= {{a, b}, {c, d}, {e, f}} /. {x_Symbol, y_Symbol} -> {y, x}
```

```
Out[2]= {{b, a}, {d, c}, {f, e}}
```

2. Here is the matrix.

```
In[3]:= mat = {{a, b, c}, {d, e, f}, {g, h, i}};
MatrixForm[mat]
```

```
Out[4]//MatrixForm=
```

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

This rule swaps the second and third rows, not the columns.

```
In[5]:= mat /. {x_, y_, z_} => {x, z, y} // MatrixForm
```

```
Out[5]//MatrixForm=
```

$$\begin{pmatrix} a & b & c \\ g & h & i \\ d & e & f \end{pmatrix}$$

It appears as if the pattern matcher starts at the outer list expression rather than at the bottom of the nested expression. Starting at the top, the rule matches the entire matrix with the first row matching the pattern `x_`, the second row matching `y_` and the third row matching `z_`. Some additional constraints will help here.

```
In[6]:= mat /. {x_Symbol, y_Symbol, z_Symbol} => {x, z, y} // MatrixForm
```

```
Out[6]//MatrixForm=
```

$$\begin{pmatrix} a & c & b \\ d & f & e \\ g & i & h \end{pmatrix}$$

3. First, get the solutions to this polynomial.

```
In[7]:= soln = Solve[x^9 + 3.4 x^6 - 25 x^5 - 213 x^4 - 477 x^3 + 1012 x^2 + 111 x - 123 == 0, x]
```

```
Out[7]= {{x -> -2.80961}, {x -> -1.85186 - 2.15082 I}, {x -> -1.85186 + 2.15082 I},
{x -> -0.376453}, {x -> 0.323073}, {x -> 1.06103 - 3.12709 I},
{x -> 1.06103 + 3.12709 I}, {x -> 1.30533}, {x -> 3.13931}}
```

The pattern needs to match an expression consisting of a list with a rule inside where the value on the right-hand side of the rule should pass the `Negative` test.

```
In[8]:= Cases[soln, {x_ -> _?Negative}]
```

```
Out[8]= {{x -> -2.80961}, {x -> -0.376453}}
```

Here are two solutions for the noncomplex roots.

```
In[9]:= Cases[soln, {_ -> _Real}]
```

```
Out[9]= {{x -> -2.80961}, {x -> -0.376453}, {x -> 0.323073}, {x -> 1.30533}, {x -> 3.13931}}
```

```
In[10]:= DeleteCases[soln, {_ -> _Complex}]
```

```
Out[10]= {{x -> -2.80961}, {x -> -0.376453}, {x -> 0.323073}, {x -> 1.30533}, {x -> 3.13931}}
```

4. The transformation rule unnests lists within a list.

```
In[11]:= unNest[lis_] := Map[(# // . {x__} => x &), lis]
```

```
In[12]:= unNest[{{a, a, a}, {a}, {{b, b, b}, {b, b}}, {a, a}}]
```

```
Out[12]= {a, a, a, a, b, b, b, b, b, a, a}
```

5. Note the need to put  $y$  in a list on the right-hand side of the rule. Also, an immediate rule is required here.

```
In[13]:= sumList[lis_] := First[lis /. {x_, y___} → x + {y}]
```

```
In[14]:= sumList[{1, 5, 8, 3, 9, 3}]
```

```
Out[14]= 29
```

6. The triple blank is required both before and after the variables  $x$  and  $y$ .

```
In[15]:= cartesianProduct[lis1_, lis2_] :=  
  ReplaceList[{lis1, lis2}, {{___, x_, ___}, {___, y_, ___}} → {x, y}]
```

We could also have a rule for an argument consisting of the empty list.

```
In[16]:= cartesianProduct[{}] := {}
```

```
In[17]:= Clear[x, y, z, a, b, c, d]
```

```
In[18]:= cartesianProduct[{a, b, c}, {x, y, z}]
```

```
Out[18]= {{a, x}, {a, y}, {a, z}, {b, x}, {b, y}, {b, z}, {c, x}, {c, y}, {c, z}}
```

```
In[19]:= cartesianProduct[{}]
```

```
Out[19]= {}
```

7. For an expression of the form  $\text{Power}[a, b]$ , the number of multiplies is  $b - 1$ .

```
In[20]:= Cases[{x^4}, Power[_, exp_] → exp - 1]
```

```
Out[20]= {3}
```

For an expression of the form  $\text{Times}[a, b, c, \dots]$ , the number of multiplications is given by one less than the number of arguments.

```
In[21]:= Cases[{a b c d e}, fac_Times → Length[fac] - 1]
```

```
Out[21]= {4}
```

For a mix of terms of these two cases, we will need to total up the counts from the respective terms. Here is a function that puts this all together. Use `Infinity` as a third argument to `Cases` to make sure the search goes all the way down the expression tree.

```
In[22]:= MultiplyCount[expr_] :=  
  Total@Cases[{expr}, Power[_, exp_] → exp - 1, Infinity] +  
  Total@Cases[{expr}, fac_Times → Length[fac] - 1, Infinity]
```

```
In[23]:= MultiplyCount[a b^2 c d^5]
```

```
Out[23]= 8
```

```
In[24]:= poly = Expand[(x + y - z)^3]
```

```
Out[24]= x^3 + 3 x^2 y + 3 x y^2 + y^3 - 3 x^2 z - 6 x y z - 3 y^2 z + 3 x z^2 + 3 y z^2 - z^3
```



```
In[25]:= MultiplyCount[poly]
```

```
Out[25]= 28
```

Ideally we should check that the expression passed to this function is a polynomial first. That is addressed in Exercise 5, Section 6.2.

8. First, we create a grid of the nine locations on the die.

```
In[26]:= lis = Partition[Range[9], 3];
          Grid[lis]
```

```
Out[27]=  1  2  3
          4  5  6
          7  8  9
```

Next, use graphics primitives to indicate if a location on the grid is colored (on) or not (off).

```
In[28]:= off = {Red, Disk[]};
          on = {White, Disk[]};
```

Here are the rules for a five.

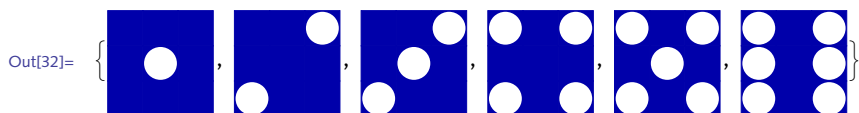
```
In[30]:= GraphicsGrid[Map[Graphics,
                          lis /. {1 → on, 2 → off, 3 → on, 4 → off, 5 → on, 6 → off, 7 → on, 8 → off, 9 → on},
                          {2}], Background → Red, Spacings → 10, ImageSize → 40]
```

```
Out[30]= 
```

The five other rules are straightforward. Here then is a function that wraps up the code. Note the use of the Background option to GraphicsGrid to pick up the color from the value of off.

```
In[31]:= Dice[n_] := Module[{rules, off = {Darker@Blue, Disk[]}, on = {White, Disk[]}},
  rules = {
    {1 → off, 2 → off, 3 → off, 4 → off, 5 → on, 6 → off, 7 → off, 8 → off, 9 → off},
    {1 → off, 2 → off, 3 → on, 4 → off, 5 → off, 6 → off, 7 → on, 8 → off, 9 → off},
    {1 → off, 2 → off, 3 → on, 4 → off, 5 → on, 6 → off, 7 → on, 8 → off, 9 → off},
    {1 → on, 2 → off, 3 → on, 4 → off, 5 → off, 6 → off, 7 → on, 8 → off, 9 → on},
    {1 → on, 2 → off, 3 → on, 4 → off, 5 → on, 6 → off, 7 → on, 8 → off, 9 → on},
    {1 → on, 2 → off, 3 → on, 4 → on, 5 → off, 6 → on, 7 → on, 8 → off, 9 → on}
  };
  GraphicsGrid[Map[Graphics,
    Partition[Range[9], 3] /. rules[[n]],
    {2}], Background → First[off], Spacings → 10, ImageSize → 40]
]
```

```
In[32]:= Table[Dice[n], {n, 1, 6}]
```

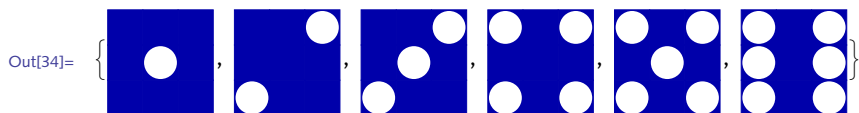


This can be done a bit more compactly by using a  $3 \times 3$  matrix of zeros and ones.

```
In[33]:= Dice[n_, OptionsPattern[]] := Module[{rules, color, grid},
  color = OptionValue[Color];
  rules = {0 -> {color, Disk[]}, 1 -> {White, Disk[]}};
  grid = {
    {0, 0, 0, 0, 1, 0, 0, 0, 0},
    {0, 0, 1, 0, 0, 0, 1, 0, 0},
    {0, 0, 1, 0, 1, 0, 1, 0, 0},
    {1, 0, 1, 0, 0, 0, 1, 0, 1},
    {1, 0, 1, 0, 1, 0, 1, 0, 1},
    {1, 0, 1, 1, 0, 1, 1, 0, 1}
  };
  GraphicsGrid[Map[Graphics, Partition[grid[[n]], 3] /. rules, {2}],
    Background -> color, Spacings -> 10, ImageSize -> 40]
]
```

Using Array, we create a list of the six dice.

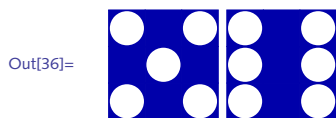
```
In[34]:= Array[Dice, {6}]
```



Rolling a pair is randomly choosing (with replacement).

```
In[35]:= RollDice[] := GraphicsRow[RandomChoice[Array[Dice, {6}], {2}]]
```

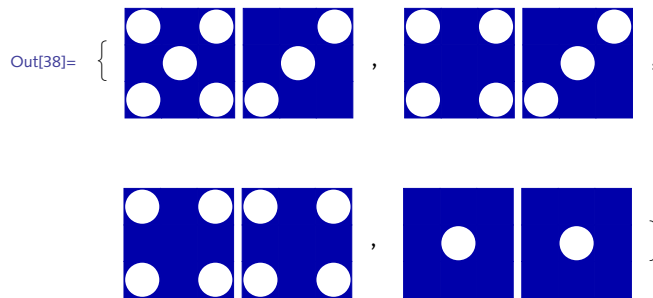
```
In[36]:= RollDice[]
```



And here is the rule for rolling the pair of dice  $n$  times.

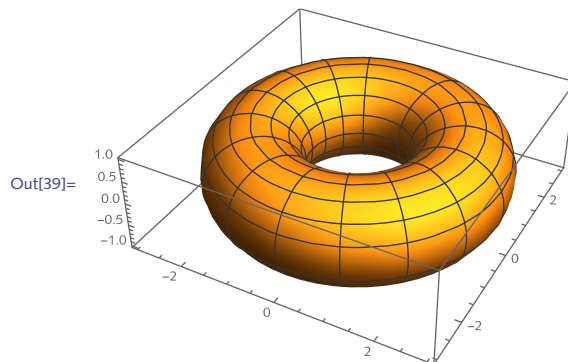
```
In[37]:= RollDice[n_] := Table[RollDice[], {n}]
```

```
In[38]:= RollDice[4]
```



9. Here is the torus:

```
In[39]:= torus = ParametricPlot3D[{(2 + Cos[v]) Sin[u], (2 + Cos[v]) Cos[u], Sin[v]},  
  {u, 0, 2 π}, {v, 0, 2 π}]
```



The internal form of the graphic shows the form we are looking for.

```
In[40]:= Short[InputForm[torus], 5]
```

```
Out[40]//Short= Graphics3D[GraphicsComplex[  
  {{1.3464 × 10-6, 2.999999999999597, 4.48799 × 10-7}, {<< 3 >>}, << 2296 >>},  
  {-2.0674956344646818, 2.067494242630337, -0.38268379517170614}},  
  << 2 >>], {<< 9 >>}]
```

Mirroring what was done in the plot example in this section, here is the expression to extract only coordinate triples.

```
In[41]:= points = Cases[torus, GraphicsComplex[pts : {{_, _, _} ..}, ___] :> pts, Infinity]
```

Out[41]=

```
{ { { 1.3464 × 10-6, 3., 4.48799 × 10-7 },  
  ... 2297 ... , { -2.0675, 2.06749, -0.382684 } } }
```

large output

show less

show more

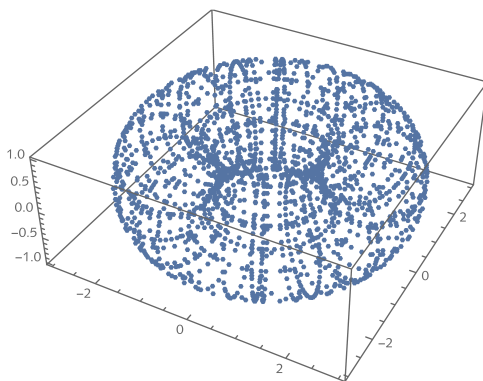
show all

set size limit...

And here is the scatter plot of these points.

```
In[42]:= ListPointPlot3D[points, PlotStyle -> PointSize[Small]]
```

Out[42]=



### 4.3 Examples: exercises

1. Create a predicate function `compositeQ` that tests whether a positive integer is composite. Check it against the built-in `CompositeQ`.
2. Plot the function `Sinc[x]` over the interval  $[-2\pi, 2\pi]$  and then use a transformation rule to display a reflection in the  $y$ -axis. Use `Show[plot /. expr :> rule, PlotRange -> All]` to display the transformed plot in such a way that a new plot range is computed.
3. Occasionally, when collecting data from an instrument, the collector fails or returns a bad value. In analyzing the data, the analyst has to make a decision about what to use to replace these bad values. One approach is to replace them with a column mean. Given an array of numbers such as the following, create a function to replace each "NAN" with the mean of the numbers that appear in that column:

```
In[1]:= data = { { 0.9034, "NAN", 0.7163, 0.8588 },  
  { 0.3031, 0.5827, 0.2699, 0.8063 },  
  { 0.0418, 0.8426, "NAN", 0.8634 },  
  { "NAN", 0.8913, 0.0662, 0.8432 } };
```

4. Given a two-column array of data

```
In[2]:= data = RandomInteger[{0, 9}, {5, 2}];
```

```
In[3]:= MatrixForm[data, TableAlignments → "."]
```

```
Out[3]//MatrixForm=
```

$$\begin{pmatrix} 7 & 0 \\ 8 & 2 \\ 1 & 5 \\ 8 & 0 \\ 6 & 7 \end{pmatrix}$$

create a new array that consists of three columns where the first two columns are identical to the original, but the third column consists of the mean of the two numbers from the first two columns.

$$\begin{pmatrix} 7 & 0 & \frac{7}{2} \\ 8 & 2 & 5 \\ 1 & 5 & 3 \\ 8 & 0 & 4 \\ 6 & 7 & \frac{13}{2} \end{pmatrix}$$

- Given a graphic produced by Plot, use a transformation rule to halve the y-coordinate of each point used to construct the plot, then display the result.
- Extend the counting coins example to take images of coins as the argument to the function.

```
In[4]:= CountChange[{, , , , }]
```

```
Out[4]= 0.42
```

- Create a function FindSubsequence[*digits*, *subseq*] to find the positions of a subsequence *subseq* within a sequence of numbers given by *digits*. Assume both *digits* and *subseq* are lists of numbers. Your function should return a list of the starting and ending positions where the subsequence occurs in the sequence, similar to what Position returns. For example, here are the first 50 digits of  $\pi$ :

```
In[5]:= pidigs = First[RealDigits[ $\pi$ , 10, 100]]
```

```
Out[5]= {3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7, 9, 3, 2, 3, 8, 4, 6, 2, 6, 4, 3,
3, 8, 3, 2, 7, 9, 5, 0, 2, 8, 8, 4, 1, 9, 7, 1, 6, 9, 3, 9, 9, 3, 7, 5, 1,
0, 5, 8, 2, 0, 9, 7, 4, 9, 4, 4, 5, 9, 2, 3, 0, 7, 8, 1, 6, 4, 0, 6, 2, 8,
6, 2, 0, 8, 9, 9, 8, 6, 2, 8, 0, 3, 4, 8, 2, 5, 3, 4, 2, 1, 1, 7, 0, 6, 7}
```

The subsequence 38 appears in two locations in pidigs.

```
In[6]:= FindSubsequence[pidigs, {3, 8}]
```

```
Out[6]= {{18, 19}, {26, 27}}
```

- Write another definition of FindSubsequence that takes two integers as arguments. So, for example, the following should work:

```

In[7]:= SeedRandom[6];
n = RandomInteger[10200]

Out[8]:= 38 962 167 906 640 602 500 170 931 211 955 779 575 023 497 774 170 227 858 878 429 522 794 529 :
744 062 342 783 143 699 902 237 900 976 316 871 609 846 545 097 431 390 396 795 087 845 924 :
977 005 230 435 025 177 652 637 766 538 981 421 277 296 525 589 205 107 229 653

In[9]:= FindSubsequence[n, 965]

Out[9]:= {{181, 183}, {197, 199}}

```

9. Compute the area of a triangle using the following formula for three two-dimensional coordinates  $(x_i, y_i, z_i)$  embedded in three-dimensional space:

$$A_{\Delta} = \frac{1}{2} \sqrt{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix}^2}$$

10. Using historical global surface temperatures, make a plot showing the difference in °C from the 1950–1980 average for each year. Data is available from numerous sources, including NASA's Goddard Institute for Space Studies ([NASA 2015](#)). After importing the data you will need to remove header and footer information before pouring the pairs  $\{year, yearly\_temp\}$  into `TimeSeries`. Make the plot using `DateListPlot` and include a smoothed five-year moving average together with the plot of the raw data.
11. Sunspots are caused by magnetic fields which in turn are caused by the current generated by the motion of hot plasma inside the sun. In regions where the induced magnetic field is most intense, the increased pressure causes the region to rise to the surface causing darker regions – sunspots – where the temperature is lower. The mean magnetic field of the sun has been recorded since 1975 and is available from the Wilcox Solar Observatory at Stanford University:

```

In[10]:= data = Import["http://wso.stanford.edu/meanfld/MF_timeseries.txt", "Table"];
Take[data, 10]

Out[11]:= {{Date, Daily, MF, (uT)}, {1975:05:16_20h, 29},
{1975:05:17_20h, 22}, {1975:05:18_20h, 24}, {1975:05:19_20h, 24},
{1975:05:20_20h, 13}, {1975:05:21_20h, 4}, {1975:05:22_20h, 4},
{1975:05:23_20h, XXXX}, {1975:05:24_20h, -3}}

```

Import the solar magnetic field data and create rules to convert the timestamps to a form that the time series functions can work with. You will need to convert the missing measurements (denoted "XXXX" in the data) of the magnetic field strength to `Missing[]` which the time series functions will handle more gracefully. Finally, make a plot of the data over the time period using `DateListPlot`.

To convert the date to a usable format, use `DateList` and specify the explicit delimiters.

```
In[12]:= DateList[
  {"1975:05:18_20h", {"Year", ":", "Month", ":", "Day", "_", "Hour", "h"}}]
Out[12]:= {1975, 5, 18, 20, 0, 0.}
```

### 4.3 Solutions

1. First, the argument is checked to see if it has head Integer and if it is greater than one.

```
In[1]:= compositeQ[n_Integer /; n > 1] := Not[PrimeQ[n]]
```

Check a few numbers for compositeness.

```
In[2]:= compositeQ[16]
```

```
Out[2]:= True
```

```
In[3]:= compositeQ[231 - 1]
```

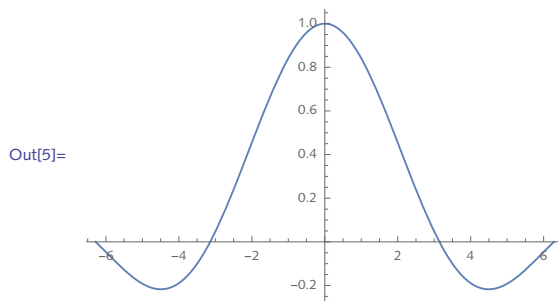
```
Out[3]:= False
```

```
In[4]:= compositeQ[263 - 1] === CompositeQ[263 - 1]
```

```
Out[4]:= True
```

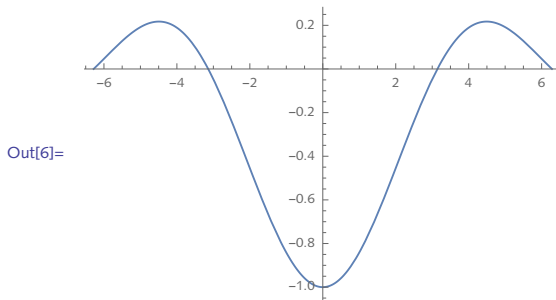
2. Here is the plot of the sinc function.

```
In[5]:= splot = Plot[Sinc[x], {x, -2  $\pi$ , 2  $\pi$ }]
```



This replacement rule replaces each pair of numbers  $\{x, y\}$  with the pair  $\{x, -y\}$ , giving a reflection in the  $y$ -axis. Note the need to modify the plot range here.

```
In[6]:= Show[plot /. {x_?NumberQ, y_?NumberQ} -> {x, -y}, PlotRange -> All]
```



The argument checking (`_?NumberQ`) is necessary here so that pairs of arbitrary expressions embedded somewhere in the graphics expression are not pattern matched. We only want to interchange pairs of numbers, not pairs of options or other expressions that might be present in the underlying expression representing the graphic.

- First, here is the data with which we will work.

```
In[7]:= array = {
  {0.9034, "NAN", 0.7163, 0.8588, 0.1228},
  {0.3031, 0.5827, 0.2699, 0.8063, "NAN"},
  {0.0418, 0.8426, "NAN", 0.8634, 0.9682},
  {0.9163, 0.8913, 0.0662, 0.8432, 0.0547},
  {0.7937, 0.6905, 0.9105, 0.5589, 0.8993}
};
```

Get only the numeric values from the second column.

```
In[8]:= col2 = array[[All, 2]];
Cases[col2, _?NumberQ]
```

```
Out[9]:= {0.5827, 0.8426, 0.8913, 0.6905}
```

Compute the mean of the second column.

```
In[10]:= Mean[Cases[col2, _?NumberQ]]
```

```
Out[10]:= 0.751775
```

Replace the string with the column mean.

```
In[11]:= col2 /. "NAN" -> Mean[Cases[col2, _?NumberQ]] // MatrixForm
```

```
Out[11]//MatrixForm=
```

$$\begin{pmatrix} 0.751775 \\ 0.5827 \\ 0.8426 \\ 0.8913 \\ 0.6905 \end{pmatrix}$$

Turn it into a function.

```
In[12]:= fixcolumn[col_] :=
  array[[All, col]] /. "NAN" -> Mean[Cases[array[[All, col]], _?NumberQ]]
```



Try this function out on column 1 of our matrix.

```
In[13]:= fixcolumn[1]
```

```
Out[13]= {0.9034, 0.3031, 0.0418, 0.9163, 0.7937}
```

Map this function across all the columns.

```
In[14]:= Map[fixcolumn, Range[Length[First[array]]]] // MatrixForm
```

```
Out[14]//MatrixForm=
```

$$\begin{pmatrix} 0.9034 & 0.3031 & 0.0418 & 0.9163 & 0.7937 \\ 0.751775 & 0.5827 & 0.8426 & 0.8913 & 0.6905 \\ 0.7163 & 0.2699 & 0.490725 & 0.0662 & 0.9105 \\ 0.8588 & 0.8063 & 0.8634 & 0.8432 & 0.5589 \\ 0.1228 & 0.51125 & 0.9682 & 0.0547 & 0.8993 \end{pmatrix}$$

This operated on the columns, so the array is a list of the transformed column vectors. Transpose it back to put things right.

```
In[15]:= MatrixForm[Transpose[%]]
```

```
Out[15]//MatrixForm=
```

$$\begin{pmatrix} 0.9034 & 0.751775 & 0.7163 & 0.8588 & 0.1228 \\ 0.3031 & 0.5827 & 0.2699 & 0.8063 & 0.51125 \\ 0.0418 & 0.8426 & 0.490725 & 0.8634 & 0.9682 \\ 0.9163 & 0.8913 & 0.0662 & 0.8432 & 0.0547 \\ 0.7937 & 0.6905 & 0.9105 & 0.5589 & 0.8993 \end{pmatrix}$$

Next, turn this into a reusable function, FixArray.

```
In[16]:= FixArray[mat_] := Module[{fixcolumn},
  fixcolumn[col_] :=
    mat[[All, col]] /. "NaN" -> Mean[Cases[mat[[All, col]], _?NumberQ]];
  Transpose[Map[fixcolumn, Range[Length[First[mat]]]]]
]
```

```
In[17]:= FixArray[array] // MatrixForm
```

```
Out[17]//MatrixForm=
```

$$\begin{pmatrix} 0.9034 & 0.751775 & 0.7163 & 0.8588 & 0.1228 \\ 0.3031 & 0.5827 & 0.2699 & 0.8063 & 0.51125 \\ 0.0418 & 0.8426 & 0.490725 & 0.8634 & 0.9682 \\ 0.9163 & 0.8913 & 0.0662 & 0.8432 & 0.0547 \\ 0.7937 & 0.6905 & 0.9105 & 0.5589 & 0.8993 \end{pmatrix}$$

4. We are embedding the two-dimensional data into a three-dimensional array. The embedding function is written directly as a transformation rule.

```
In[18]:= data = RandomReal[{0, 1}, {8, 2}]
```

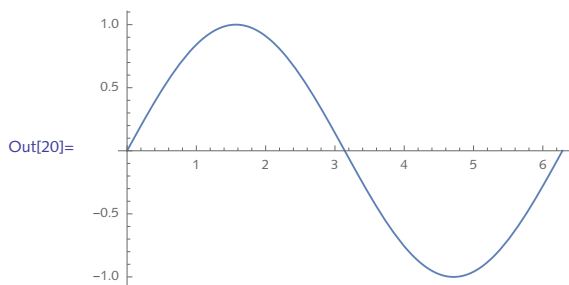
```
Out[18]= {{0.0130644, 0.16409}, {0.639823, 0.731402},
  {0.997011, 0.621673}, {0.0157075, 0.409658}, {0.705118, 0.514003},
  {0.0233511, 0.539084}, {0.487938, 0.503521}, {0.848247, 0.874698}}
```

```
In[19]:= data /. {x_, y_} -> {x, y, Mean[{x, y}]} // MatrixForm
Out[19]//MatrixForm=
```

```
( 0.0130644  0.16409  0.088577 )
( 0.639823  0.731402  0.685612 )
( 0.997011  0.621673  0.809342 )
( 0.0157075 0.409658  0.212683 )
( 0.705118  0.514003  0.60956 )
( 0.0233511 0.539084  0.281217 )
( 0.487938  0.503521  0.495729 )
( 0.848247  0.874698  0.861472 )
```

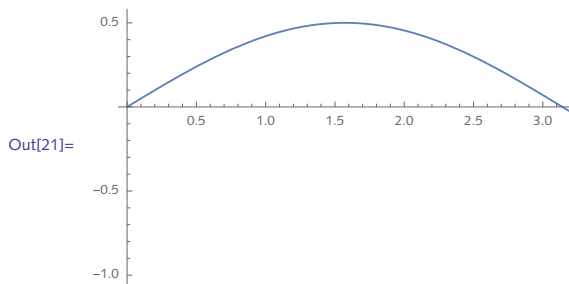
5. Here is a plot of the sin function.

```
In[20]:= plot = Plot[Sin[x], {x, 0, 2  $\pi$ }]
```



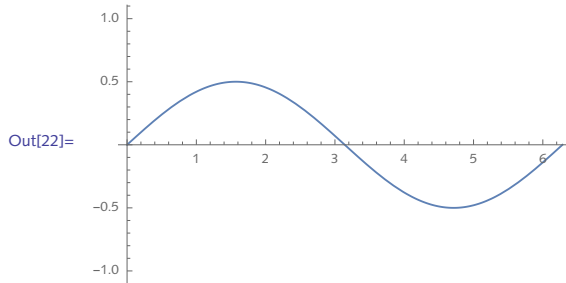
This rule replaces each pair  $\{x, y\}$  with the pair  $\{x, y/2\}$ .

```
In[21]:= plot /. {x_?NumericQ, y_?NumericQ} -> {x, y/2}
```



Because the plot range is simply inherited from the original plot we lose some information after the transformation. To adjust the plot range, wrap the expression in Show and give an explicit plot range.

```
In[22]:= Show[
  plot /. {x_?NumericQ, y_?NumericQ} :> {x, y/2},
  PlotRange -> {{0, 2  $\pi$ }, {-1, 1}}]
```



6. For U.S. coins, you can import images from the U.S. Mint ([www.usmint.gov](http://www.usmint.gov)); adjust accordingly for different currencies.

```
In[23]:= {q, d, n, p} = {, , , };
```

```
In[24]:= coins = {p, p, q, n, d, d, p, q, q, p}
```



Here are the values, given by a list of rules.

```
In[25]:= values = {p -> .01, n -> .05, d -> .10, q -> .25};
```

This replaces each coin by its value.

```
In[26]:= coins /. values
```

```
Out[26]:= {0.01, 0.01, 0.25, 0.05, 0.1, 0.1, 0.01, 0.25, 0.25, 0.01}
```





And here is the value of the set of coins.

```
In[27]:= Total[coins /. values]
```

```
Out[27]:= 1.04
```

Finally, here is a function that wraps up all these steps.

```
In[28]:= CountChange[coins_List] :=
```

Total[coins /. { → .01,  → .05,  → .10,  
  
 → .25}]

```
In[29]:= CountChange[{ ,  ,  ,  ,  ,  
  
 ,  ,  ,  ,  }]
```

```
Out[29]= 1.04
```

7. To prototype, we will only work with a small number of digits so we can easily check on our progress. Here are the first 50 digits of  $\pi$ , starting from the right of the decimal point.

```
In[30]:= pidigs = First[RealDigits[ $\pi$ , 10, 50]]
```

```
Out[30]= {3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7, 9, 3, 2, 3, 8, 4, 6, 2, 6, 4, 3,  
3, 8, 3, 2, 7, 9, 5, 0, 2, 8, 8, 4, 1, 9, 7, 1, 6, 9, 3, 9, 9, 3, 7, 5, 1}
```

The subsequence we are searching for is also given as a list of digits.

```
In[31]:= subseq = {3, 2, 3, 8};
```

One approach to this problem is to partition the list of digits in `pidigs` into lists of the same length as the list `subseq`, with overlapping sublists of offset one. This means that we will examine all sublists of length four from `pidigs`.

```

In[32]:= p = Partition[pidigs, Length[subseq], 1]
Out[32]:= {{3, 1, 4, 1}, {1, 4, 1, 5}, {4, 1, 5, 9}, {1, 5, 9, 2}, {5, 9, 2, 6}, {9, 2, 6, 5},
           {2, 6, 5, 3}, {6, 5, 3, 5}, {5, 3, 5, 8}, {3, 5, 8, 9}, {5, 8, 9, 7}, {8, 9, 7, 9},
           {9, 7, 9, 3}, {7, 9, 3, 2}, {9, 3, 2, 3}, {3, 2, 3, 8}, {2, 3, 8, 4},
           {3, 8, 4, 6}, {8, 4, 6, 2}, {4, 6, 2, 6}, {6, 2, 6, 4}, {2, 6, 4, 3},
           {6, 4, 3, 3}, {4, 3, 3, 8}, {3, 3, 8, 3}, {3, 8, 3, 2}, {8, 3, 2, 7},
           {3, 2, 7, 9}, {2, 7, 9, 5}, {7, 9, 5, 0}, {9, 5, 0, 2}, {5, 0, 2, 8},
           {0, 2, 8, 8}, {2, 8, 8, 4}, {8, 8, 4, 1}, {8, 4, 1, 9}, {4, 1, 9, 7},
           {1, 9, 7, 1}, {9, 7, 1, 6}, {7, 1, 6, 9}, {1, 6, 9, 3}, {6, 9, 3, 9},
           {9, 3, 9, 9}, {3, 9, 9, 3}, {9, 9, 3, 7}, {9, 3, 7, 5}, {3, 7, 5, 1}}

```

Now we are ready for the pattern match. From the list `p` above, we are looking for the positions of any sublist that matches `{3, 2, 3, 8}`. The subsequence 3238 occurs starting at the sixteenth digit in `pidigs` (fifteen digits to the right of the decimal point).

```

In[33]:= pos = Position[p, subseq]
Out[33]:= {{16}}

```

To mirror the default output of `Position`, we give the starting and ending positions of this match.

```

In[34]:= pos /. {num_?IntegerQ} :> {num, num + Length[subseq] - 1}
Out[34]:= {{16, 19}}

```

Finally, let us turn this into a function and test it on a much larger example. Note that we use the pattern `_List` on both arguments, `digits` and `subseq`, so that `FindSubsequence` will only match arguments that have head `List`.

```

In[35]:= FindSubsequence[digits_List, subseq_List] :=
  Position[Partition[digits, Length[subseq], 1], subseq] /.
    {num_?IntegerQ} :> {num, num + Length[subseq] - 1}

```

Store the first 10 000 000 digits of  $\pi$  in the symbol `pidigs`.

```

In[36]:= pidigs = First[RealDigits[ $\pi$ , 10, 107, -1]];

```

The subsequence 314159 occurs seven times in the first 10 000 000 digits of  $\pi$ , starting with the 176 451st digit.

```

In[37]:= FindSubsequence[pidigs, {3, 1, 4, 1, 5, 9}] // Timing
Out[37]:= {10.2297, {{176 451, 176 456},
                    {1 259 351, 1 259 356}, {1 761 051, 1 761 056}, {6 467 324, 6 467 329},
                    {6 518 294, 6 518 299}, {9 753 731, 9 753 736}, {9 973 760, 9 973 765}}}

```

In Section 7.2 we will see a different approach to this problem, one using string-processing functions that gives a substantial speedup compared to the computation above.

8. This creates another rule associated with `FindSubsequence` that simply takes each integer argument, converts it to a list of integer digits, and then passes that off to the rule above.

```
In[38]:= FindSubsequence[n_Integer, subseq_Integer] :=  
        FindSubsequence[IntegerDigits[n], IntegerDigits[subseq]]
```

Create the list of the first 100 000 digits of  $\pi$ .

```
In[39]:= pi = FromDigits[First@RealDigits[N[Pi, 10^5] - 3]];
```

The subsequence 1415 occurs seven times at the following locations in this digit expansion of  $\pi$ .

```
In[40]:= FindSubsequence[pi, 1415]  
Out[40]= {{1, 4}, {6955, 6958}, {29 136, 29 139}, {45 234, 45 237},  
          {79 687, 79 690}, {85 880, 85 883}, {88 009, 88 012}}
```

9. This is a direct translation of the formula given in the exercise using a transformation rule to embed the two-dimensional vectors in three-space. It is important to get the pattern on the left-hand side of this definition correct so it matches a list consisting of three expressions (the three points).

```
In[41]:= area[Triangle[pts : {_, _, _}]] :=  
        1  
        2  
         $\sqrt{(\text{Det}[pts[[All, \{2, 3\}]] /. \{a_, b_\} \Rightarrow \{a, b, 1\}]^2 +$   
           $\text{Det}[pts[[All, \{3, 1\}]] /. \{a_, b_\} \Rightarrow \{a, b, 1\}]^2 +$   
           $\text{Det}[pts[[All, \{1, 2\}]] /. \{a_, b_\} \Rightarrow \{a, b, 1\}]^2)}$ 
```

And here is a check against the built-in function.

```
In[42]:= pts = {{0, 0, 0}, {3, 4, 0}, {5, 5, 5}};
```

```
In[43]:= Area[Triangle[pts]]
```

```
Out[43]= 5  $\sqrt{\frac{13}{2}}$ 
```

```
In[44]:= area[Triangle[pts]]
```

```
Out[44]= 5  $\sqrt{\frac{13}{2}}$ 
```

10. First, import data from NASA's Goddard Institute for Space Studies.

```
In[45]:= data = Import["http://data.giss.nasa.gov/gistemp/tabledata_v3/GLB.Ts+dSST.txt",  
                      {"Table", "Data"}];
```

We need to strip out the header and footer information and just deal with the raw data: inspection of the data (not shown here) indicates the header info is in the first eight rows and footer info is in the last twelve rows.



```
In[46]:= data[[9 ;; -13]];
```

Some filtering is also necessary as the web page repeats header row (column names) frequently.

```
In[47]:= DeleteCases[data[[9 ;; -13]], {"Year", __} | {}];
```

Finally, we want row fourteen, the yearly temp difference from the norm. Also, divide by 100 to get actual °C.

```
In[48]:= tempData = TimeSeries[%[[All, 14]]/100., {"1880", "2014"}]
```

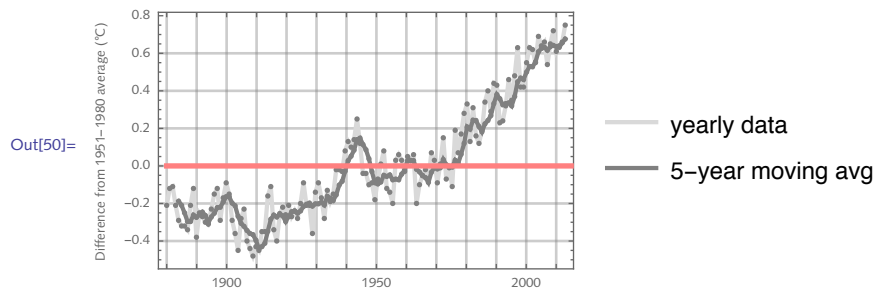
```
Out[48]= TimeSeries[  Time: 01 Jan 1880 to 01 Jan 2014  
Data points: 136]
```

Smooth with a five-year moving average.

```
In[49]:= avg = MovingAverage[tempData, 5];
```

Plot original data (light gray) together with smoothed data (gray). Put a thick red line on the mean.

```
In[50]:= DateListPlot[{tempData, avg}, PlotStyle -> {LightGray, Gray}, Mesh -> All,
  MeshStyle -> Directive[PointSize[Small], Gray],
  GridLines -> {ToString/@Range[1880, 2010, 10], Automatic}, ImageSize -> Small,
  Epilog -> {Pink, Thick, Line[{{"1880", 0}, {"2015", 0}}]},
  FrameLabel -> {None, "Difference from 1951-1980 average (°C)"},
  PlotLegends -> {"yearly data", "5-year moving avg"}]
```



```
In[51]:= Clear[x, y, z, f, g, p, q, d, n, a, b, c, d, e]
```

II. This imports the data from the Wilcox Observatory:

```
In[52]:= data = Import["http://wso.stanford.edu/meanfld/MF_timeseries.txt", "Table"]
```

```
Out[52]= {{Date, Daily, MF, (uT)}, {1975:05:16_20h, 29},
          {1975:05:17_20h, 22}, ... 14 320 ..., {2014:08:01_20h, 69},
          {2014:08:02_20h, 34}, {2014:08:03_20h, 6}}
```

large output

show less

show more

show all

set size limit...

The data is of the form  $\{date, mf\}$  where *date* is a string and the magnetic field measurement an integer giving the mean magnetic field measurement in  $\mu T$  (micro-Teslas).

```
In[53]:= data[[2]] // InputForm
```

```
Out[53]//InputForm= {"1975:05:16_20h", 29}
```

Using the suggestion in the exercise, create a function to convert the string to a date object.

```
In[54]:= dateConvert[str_String] :=
         DateList[{str, {"Year", ":", "Month", ":", "Day", "_", "Hour", "h"}}]
```

Then, using a rule, convert all the dates:

```
In[55]:= data2 = Rest@data /. {date_String, val_} -> {dateConvert[date], val}
```

```
Out[55]= {{{{1975, 5, 16, 20, 0, 0.}, 29}, {{1975, 5, 17, 20, 0, 0.}, 22},
           {{1975, 5, 18, 20, 0, 0.}, 24}, ... 14 319 ..., {{2014, 8, 1, 20, 0, 0.}, 69},
           {{2014, 8, 2, 20, 0, 0.}, 34}, {{2014, 8, 3, 20, 0, 0.}, 6}}}
```

large output

show less

show more

show all

set size limit...

Next, we need to deal with missing data. In the data set, they are indicated by the string "XXXX".

```
In[56]:= data2[[8]] // InputForm
```

```
Out[56]//InputForm= {{1975, 5, 23, 20, 0, 0.}, "XXXX"}
```

The next rule converts this string to `Missing[]` which the time series functions can deal with better.



```
In[57]:= data3 = data2 /. {date_List, _String} -> {date, Missing[]}
```

```
Out[57]= {{ {1975, 5, 16, 20, 0, 0.}, 29}, { {1975, 5, 17, 20, 0, 0.}, 22},
          { {1975, 5, 18, 20, 0, 0.}, 24}, ... 14 319 ..., { {2014, 8, 1, 20, 0, 0.}, 69},
          { {2014, 8, 2, 20, 0, 0.}, 34}, { {2014, 8, 3, 20, 0, 0.}, 6} }
```

large output

show less

show more

show all

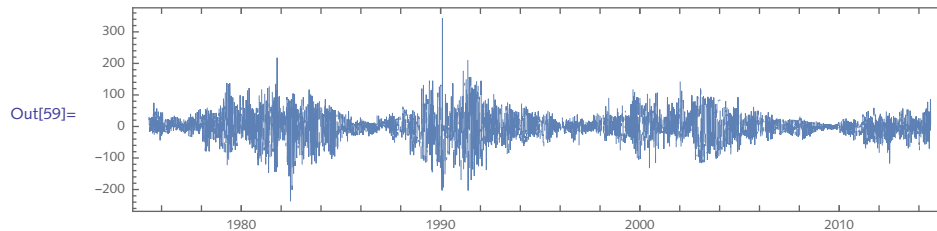
set size limit...

Now, convert data3 to a time series object and plot.

```
In[58]:= tsData = TimeSeries[data3]
```

```
Out[58]= TimeSeries[   Time: 16 May 1975 to 03 Aug 2014  
Data points: 14 325 ]
```

```
In[59]:= DateListPlot[tsData, AspectRatio -> 1 / 4, PlotStyle -> Thin, PlotRange -> All]
```



An alternative approach could use the `DateStringFormat` option to `Import` to put the dates in the proper form for the time series, but it is a bit slower to do the processing during import.

```
In[60]:= data = Rest@Import["http://wso.stanford.edu/meanfld/MF_timeseries.txt",
    "Table", "DateStringFormat" ->
    {"Year", ":", "Month", ":", "Day", "_", "Hour", "h"}]
```

```
Out[60]= {{  Fri 16 May 1975 , 29}, {  Sat 17 May 1975 , 22}, {  Sun 18 May 1975 , 24}, {  Mon 19 May 1975 , 24},
          ... 14 317 ..., {  Thu 31 Jul 2014 , 87}, {  Fri 1 Aug 2014 , 69}, {  Sat 2 Aug 2014 , 34}, {  Sun 3 Aug 2014 , 6} }
```

large output

show less

show more

show all

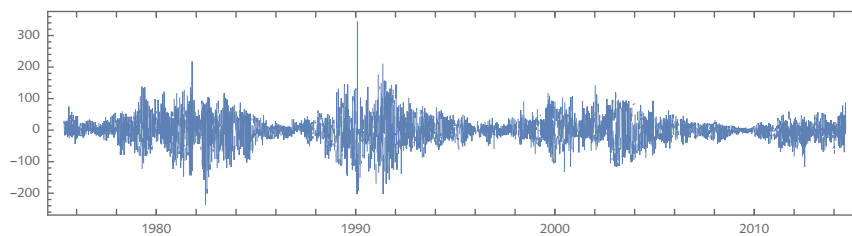
set size limit...

```
In[61]:= tsData = TimeSeries[data]
```

```
Out[61]= TimeSeries[   Time: 16 May 1975 to 03 Aug 2014  
Data points: 14 325 ]
```

```
In[62]:= DateListPlot[tsData, AspectRatio → 1 / 4, PlotStyle → Thin, PlotRange → All]
```

Out[62]=



---

## 5 Functions

### 5.1 Functions for manipulating expressions: exercises

1. Use `Partition` and `Mean` to create a two-term moving average. Then repeat for a three-term moving average. Check your results against the built-in `MovingAverage` function.

```
In[1]:= MovingAverage[{a, b, c, d, e}, 2]
```

```
Out[1]= {  $\frac{a+b}{2}$ ,  $\frac{b+c}{2}$ ,  $\frac{c+d}{2}$ ,  $\frac{d+e}{2}$  }
```

2. Use `Apply` to rewrite the definition of `squareMatrixQ` given in Section 4.1.
3. Use `Inner` to replicate the result in the text that used `Thread` to create a list of equations.

```
In[2]:= Thread[Equal[{x, y, z}, {a, b, c}]]
```

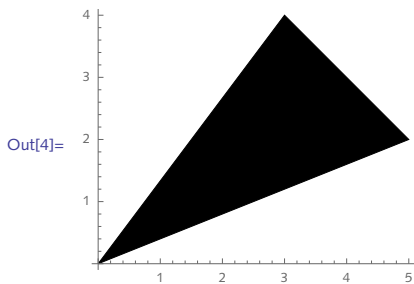
```
Out[2]= {x == a, y == b, z == c}
```

4. Heron's formula for the area of a triangle is given as follows:

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where  $a$ ,  $b$ , and  $c$  are the lengths of the three sides of the triangle and  $s$  is the semiperimeter of the triangle, defined by  $s = (a + b + c)/2$ . Compute the area of any triangle using Heron's formula. You can check your result against the built-in `Area` function.

```
In[3]:= pt1 = {0, 0}; pt2 = {5, 2}; pt3 = {3, 4};
Graphics[Triangle[{pt1, pt2, pt3}], Axes → Automatic]
```



```
In[5]:= Area[Triangle[{pt1, pt2, pt3}]]
```

Out[5]= 7

5. Find all square numbers that contain the digits one through nine exactly once (see [Madachy 1979](#)). Use SquareNumberQ from Exercise 4 in Section 2.4.
6. In Section 2.4 we defined a predicate PerfectQ[n] that returns True if n is a perfect number (n is a perfect number if it is equal to the sum of its proper divisors). Create a function PerfectSearch[n] that finds all perfect numbers less than n. Then find all perfect numbers less than  $10^6$ .
7. Turn the computation from Exercise 6 of Section 3.3 into a reusable function PrimesLessThan[n] that returns all prime numbers less than n.
8. One of the tasks in analyzing DNA sequences is determining the frequency with which the individual nucleotides G, C, A, and T occur. In addition, the frequency of longer “words” is often also of interest. For example, words of length two (or longer) such as AG, AT, CC are sometimes used to find locations in the sequence where an evolutionary change may have occurred.

Nucleotides are generally represented as strings, so starting with the list {"A", "T", "G", "C"}, create a list of all possible two-letter words from this alphabet. You can use StringJoin to return a string consisting of the concatenation of two strings.

```
In[6]:= StringJoin[{"A", "T"}] // InputForm
```

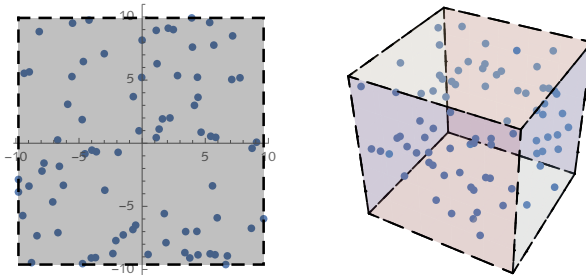
Out[6]//InputForm= AT

Finally, generalize this process to create words of length n for any such alphabet. For example, you could use a list of the amino acids as your alphabet.

```
In[7]:= aa = {"A", "R", "N", "D", "C", "E", "Q", "G", "H", "I", "L", "K", "M", "F",
             "P", "O", "U", "S", "T", "W", "Y", "V"};
```

9. Create a function `LeadingDigit[n]` that takes an integer  $n$  as an argument and returns the leading digit of  $n$  (see Exercise 7, Section 3.3). Set up your function so that it returns the leading digits of a list of numbers such as the first 10 000 Fibonacci numbers.
10. Given a set of points in the plane, find the bounding rectangle that fully encloses the points. For three-dimensional sets of points, find the bounding rectangular box. (See Figure 5.1.)

FIGURE 5.1. Bounding boxes (dashed lines) for points in two and three dimensions.



11. Given a set of points in the plane (or 3-space), find the maximum distance between any pair of these points. This is often called the *diameter* of the point set. If your definition is general enough it should be able to handle points in any dimension.
12. Create a graphic that consists of  $n$  randomly colored circles in the plane with random centers and random radii. Consider using `Thread` or `MapThread` to thread `Circle[...]` across the lists of centers and radii.
13. While matrices can easily be added using `Plus`, matrix multiplication is a bit more involved. The `Dot` function, written as a single period, is used.

```
In[8]:= {{1, 2}, {3, 4}} . {x, y}
```

```
Out[8]= {x + 2 y, 3 x + 4 y}
```

Perform matrix multiplication on  $\{\{1, 2\}, \{3, 4\}\}$  and  $\{x, y\}$  without using `Dot`.

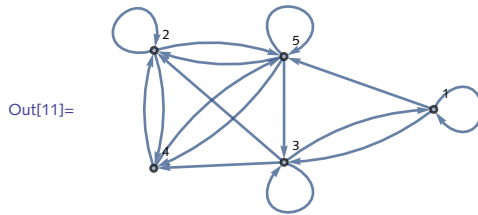
14. An adjacency matrix can be thought of as representing a graph of vertices and edges where a value of one in position  $a_{ij}$  indicates an edge between vertex  $i$  and vertex  $j$ , whereas a value of zero indicates no such edge between vertices  $i$  and  $j$ .

```
In[9]:= mat = RandomInteger[1, {5, 5}];
MatrixForm[mat]
```

```
Out[10]//MatrixForm=
```

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

```
In[11]:= AdjacencyGraph[mat, VertexLabels -> "Name"]
```



Compute the total number of edges for each vertex in both the adjacency matrix and graph representations. For example, you should get the following edge counts for the five vertices represented in the above graph. Note: self-loops count as two edges each.

```
{5, 7, 7, 5, 8}
```

15. Create a function `ToEdges[lis]` that takes a list of pairs of elements and transforms it into a list of directed edges suitable for a graph. For example:

```
In[12]:= lis = RandomInteger[9, {12, 2}]
```

```
Out[12]= {{1, 4}, {1, 5}, {1, 7}, {1, 9}, {8, 0},
          {3, 1}, {9, 2}, {6, 2}, {3, 4}, {4, 4}, {7, 6}, {2, 8}}
```

```
In[13]:= ToEdges[lis]
```

```
Out[13]= {1 -> 4, 1 -> 5, 1 -> 7, 1 -> 9, 8 -> 0, 3 -> 1, 9 -> 2, 6 -> 2, 3 -> 4, 4 -> 4, 7 -> 6, 2 -> 8}
```

Make sure that your function also works in the case where its argument is a single list of a pair of elements.

```
In[14]:= ToEdges[{3, 6}]
```

```
Out[14]= 3 -> 6
```

16. `FactorInteger[n]` returns a nested list of prime factors and their exponents for the number  $n$ .

```
In[15]:= FactorInteger[3628800]
```

```
Out[15]= {{2, 8}, {3, 4}, {5, 2}, {7, 1}}
```

Use `Apply` to reconstruct the original number from this nested list.

17. Repeat the above exercise but instead use `Inner` to reconstruct the original number  $n$  from the factorization given by `FactorInteger[n]`.
18. Create a function `PrimeFactorForm[n]` that formats its argument  $n$  in prime factorization form. You will need to use `Superscript` and `CenterDot` to format the factored integer.

```
In[16]:= PrimeFactorForm[12]
```

```
Out[16]= 22 · 31
```

19. The Vandermonde matrix arises in Lagrange interpolation and in reconstructing statistical distributions from their moments. Construct the Vandermonde matrix of order  $n$ :

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}$$

20. Using `Inner`, write a function `div[vecs, vars]` that computes the divergence of an  $n$ -dimensional vector field,  $\text{vecs} = \{e_1, e_2, \dots, e_n\}$ , dependent upon  $n$  variables,  $\text{vars} = \{v_1, v_2, \dots, v_n\}$ . The divergence is given by the sum of the pairwise partial derivatives.

$$\frac{\partial e_1}{\partial v_1} + \frac{\partial e_2}{\partial v_2} + \cdots + \frac{\partial e_n}{\partial v_n}$$

21. Using `Outer`, create a function `JacobianMatrix[vec, vars]` that returns the Jacobian of the vector `vec` in the variables given by the list `vars`. Then use `JacobianMatrix` to compute the volume of the hypersphere in two (circle) and three (sphere) dimensions by integrating the absolute value of the determinant of the Jacobian.
22. The example in this section on `Select` and `Pick` found Mersenne numbers  $2^n - 1$  that are prime for exponents  $n$  from 1 to 100. Modify that example to only use prime exponents – a basic theorem in number theory states that a Mersenne number with composite exponent must be composite (Crandall and Pomerance 2005).

## 5.1 Solutions

1. If we are doing a 2-term moving average, then partition into 2-element lists.

```
In[1]:= Clear[a, b, c, d, e, f, g, h, i]
In[2]:= Partition[{a, b, c, d, e}, 2, 1]
Out[2]:= {{a, b}, {b, c}, {c, d}, {d, e}}
```

Then take the mean of each pair.

```
In[3]:= Map[Mean, %]
Out[3]:= {a + b / 2, b + c / 2, c + d / 2, d + e / 2}
```

Check against the built-in function.

```
In[4]:= MovingAverage[{a, b, c, d, e}, 2]
Out[4]:= {a + b / 2, b + c / 2, c + d / 2, d + e / 2}
```

Similarly for the 3-term moving average.

```
In[5]:= Map[Mean, Partition[{a, b, c, d, e}, 3, 1]]
Out[5]= {1/3 (a + b + c), 1/3 (b + c + d), 1/3 (c + d + e)}
```

2. First, here is the definition given in Section 4.1.

```
In[6]:= squareMatrixQ[mat_?MatrixQ] := Dimensions[mat][[1]] == Dimensions[mat][[2]]
```

For a matrix, `Dimensions` returns a list of two integers. Applying `Equal` to the list will return `True` if the two dimensions are identical, that is, if the matrix is square.

```
In[7]:= squareMatrixQ[mat_?MatrixQ] := Apply[Equal, Dimensions[mat]]
```

```
In[8]:= squareMatrixQ[{a, b}, {c, d}, {e, f}]
```

```
Out[8]= False
```

```
In[9]:= squareMatrixQ[{a, b, c}, {d, e, f}, {g, h, i}]
```

```
Out[9]= True
```

3. Here is the example from the text.

```
In[10]:= Clear[x, y, z]
```

```
In[11]:= Thread[Equal[{x, y, z}, {a, b, c}]]
```

```
Out[11]= {x == a, y == b, z == c}
```

The first argument to `Inner` is a function that will be threaded over the following lists. Afterwards, the fourth argument to `Inner` will be applied to the result. So we use `Equal` as that first function.

```
In[12]:= Inner[Equal, {x, y, z}, {a, b, c}, List]
```

```
Out[12]= {x == a, y == b, z == c}
```

4. Given three points that define a triangle, we need the distances between every pair of points, that is, the length of the sides of the triangles.

```
In[13]:= pt1 = {0, 0};
          pt2 = {5, 2};
          pt3 = {3, 5};
```

First, make a list of the possible pairs of points.

```
In[16]:= pairs = Subsets[{pt1, pt2, pt3}, {2}]
```

```
Out[16]= {{0, 0}, {5, 2}}, {{0, 0}, {3, 5}}, {{5, 2}, {3, 5}}
```

Then apply `EuclideanDistance` at level one to get the distance between each pair.

```
In[17]:= {a, b, c} = Apply[EuclideanDistance, pairs, {1}]
```

```
Out[17]= {sqrt(29), sqrt(34), sqrt(13)}
```

Here is the semiperimeter:



```
In[18]:= s = (a + b + c) / 2
```

```
Out[18]=  $\frac{1}{2} \left( \sqrt{13} + \sqrt{29} + \sqrt{34} \right)$ 
```

And finally, the area computation given by Heron's formula:

```
In[19]:=  $\sqrt{s (s - a) (s - b) (s - c)}$  // Simplify
```

```
Out[19]=  $\frac{19}{2}$ 
```

Check:

```
In[20]:= Area[Triangle[{pt1, pt2, pt3}]]
```

```
Out[20]=  $\frac{19}{2}$ 
```

```
In[21]:= Clear[a, b, c, s]
```

5. Here is the definition of SquareNumberQ from Exercise 4 in Section 2.4.

```
In[22]:= SquareNumberQ[n_Integer] := IntegerQ[ $\sqrt{n}$ ]
```

First, create the numbers that contain each of the digits using FromDigits.

```
In[23]:= FromDigits[{1, 2, 3, 4, 5, 6, 7, 8, 9}]
```

```
Out[23]= 123456789
```

This then creates all such permutations.

```
In[24]:= nums = Map[FromDigits, Permutations[Range[9]]]
```

```
Out[24]= {123456789, 123456798, 123456879, 123456897,
  123456978, 123456987, 123457689, 123457698, 123457869,
  ... 362862 ..., 987653241, 987653412, 987653421, 987654123,
  987654132, 987654213, 987654231, 987654312, 987654321}
```

large output

show less

show more

show all

set size limit...

And here are those numbers from nums that are square.

```
In[25]:= Select[nums, SquareNumberQ]
```

```
Out[25]= {139854276, 152843769, 157326849, 215384976, 245893761, 254817369,
  326597184, 361874529, 375468129, 382945761, 385297641, 412739856,
  523814769, 529874361, 537219684, 549386721, 587432169, 589324176,
  597362481, 615387249, 627953481, 653927184, 672935481, 697435281,
  714653289, 735982641, 743816529, 842973156, 847159236, 923187456}
```

A quick check that these are all square numbers.

```
In[26]:=  $\sqrt{\%}$ 
Out[26]:= {11 826, 12 363, 12 543, 14 676, 15 681, 15 963, 18 072, 19 023, 19 377, 19 569,
          19 629, 20 316, 22 887, 23 019, 23 178, 23 439, 24 237, 24 276, 24 441, 24 807,
          25 059, 25 572, 25 941, 26 409, 26 733, 27 129, 27 273, 29 034, 29 106, 30 384}
```

What would happen if you first found all 9-digit square numbers and then determined which of those contained only the digits one through nine? A bit of thought should convince you that that approach would be quite time and resource intensive as the first step would require checking every number below  $10^9$  to see if it was a square.

6. Here is the definition of PerfectQ.

```
In[27]:= PerfectQ[n_] := Total[Divisors[n]] == 2 n
```

To find all perfect numbers less than  $n$ , use Select on the list given by Range  $[n]$ , using PerfectQ as the test.

```
In[28]:= PerfectSearch[n_] := Select[Range[n], PerfectQ]
```

This finds the four perfect numbers less than one million.

```
In[29]:= PerfectSearch[106] // Timing
```

```
Out[29]:= {9.25325, {6, 28, 496, 8128}}
```

This is quite compute-intensive. You can speed things up by using a built-in function that is designed specifically for this task, DivisorSigma.

```
In[30]:= PerfectQ[n_] := DivisorSigma[1, n] == 2 n
```

```
In[31]:= PerfectSearch[106] // Timing
```

```
Out[31]:= {4.1752, {6, 28, 496, 8128}}
```

7. Based on Exercise 6 from Section 3.3, here is the definition:

```
In[32]:= PrimesLessThan[n_] := Table[Prime[p], {p, PrimePi[n]}]
```

```
In[33]:= PrimesLessThan[100]
```

```
Out[33]:= {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31,
          37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97}
```

8. Since we are interested in all possible two-letter combinations of the list of nucleotides, the function that should come to mind is Outer. The only question is what function to thread across the lists? A moment's thought should convince you that StringJoin is what is needed.

```
In[34]:= lis = Outer[StringJoin, {"A", "C", "T", "G"}, {"A", "C", "T", "G"}]
```

```
Out[34]:= {{AA, AC, AT, AG}, {CA, CC, CT, CG}, {TA, TC, TT, TG}, {GA, GC, GT, GG}}
```

You can see what is happening by using a symbolic function sj instead of StringJoin:

```
In[35]:= Outer[sj, {"A", "C", "T", "G"}, {"A", "C", "T", "G"}]
Out[35]= {{sj[A, A], sj[A, C], sj[A, T], sj[A, G]},
          {sj[C, A], sj[C, C], sj[C, T], sj[C, G]},
          {sj[T, A], sj[T, C], sj[T, T], sj[T, G]},
          {sj[G, A], sj[G, C], sj[G, T], sj[G, G]}}
```

Finally, we need to flatten the list `lis`.

```
In[36]:= Flatten[lis]
Out[36]= {AA, AC, AT, AG, CA, CC, CT, CG, TA, TC, TT, TG, GA, GC, GT, GG}
```

Generalizing is a bit tricky. Here is what we would need to do for words of length three.

```
In[37]:= Outer[StringJoin, {"A", "C", "T", "G"}, {"A", "C", "T", "G"},
               {"A", "C", "T", "G"}] // Flatten
Out[37]= {AAA, AAC, AAT, AAG, ACA, ACC, ACT, ACG, ATA, ATC, ATT, ATG, AGA, AGC, AGT, AGG, CAA,
          CAC, CAT, CAG, CCA, CCC, CCT, CCG, CTA, CTC, CTT, CTG, CGA, CGC, CGT, CGG, TAA,
          TAC, TAT, TAG, TCA, TCC, TCT, TCG, TTA, TTC, TTT, TTG, TGA, TGC, TGT, TGG, GAA,
          GAC, GAT, GAG, GCA, GCC, GCT, GCG, GTA, GTC, GTT, GTG, GGA, GGC, GGT, GGG}
```

But what about words of length four or fourteen? Surely we can't manually keep adding the list of nucleotides inside `Outer`. Well, for words of length four say, we want a *sequence* of four copies of the list `{"A", "C", "T", "G"}`. We can use `Table` to generate that but then we have too many nested lists to pass to `Outer`.

```
In[38]:= Table[{"A", "C", "T", "G"}, {4}]
Out[38]= {{A, C, T, G}, {A, C, T, G}, {A, C, T, G}, {A, C, T, G}}
In[39]:= Outer[StringJoin, %]
Out[39]= {{A, C, T, G}, {A, C, T, G}, {A, C, T, G}, {A, C, T, G}}
```

The key here is to pass a *sequence* of these four lists. There is a function designed specifically for this situation: `Sequence`. We will apply it to the result of `Table` above and then give that sequence as the second argument to `Outer`.

```
In[40]:= Apply[Sequence, Table[{"A", "C", "T", "G"}, {4}]]
Out[40]= Sequence[{A, C, T, G}, {A, C, T, G}, {A, C, T, G}, {A, C, T, G}]
```

```
In[41]:= Outer[StringJoin, %] // Flatten
Out[41]= {AAAA, AAAC, AAAT, AAAG, AACA, AACC, AACT, AACG, AATA, AATC, AATT, AATG, AAGA,
AAGC, AAGT, AAGG, ACAA, ACAC, ACAT, ACAG, ACCA, ACCC, ACCT, ACCG, ACTA, ACTC,
ACTT, ACTG, ACGA, ACGC, ACGT, ACGG, ATAA, ATAC, ATAT, ATAG, ATCA, ATCC, ATCT,
ATCG, ATTA, ATTC, ATTT, ATTG, ATGA, ATGC, ATGT, ATGG, AGAA, AGAC, AGAT, AGAG,
AGCA, AGCC, AGCT, AGCG, AGTA, AGTC, AGTT, AGTG, AGGA, AGGC, AGGT, AGGG, CAAA,
CAAC, CAAT, CAAG, CACA, CACC, CACT, CACG, CATA, CATC, CATT, CATG, CAGA, CAGC,
CAGT, CAGG, CCAA, CCAC, CCAT, CCAG, CCCA, CCCC, CCCT, CCCG, CCTA, CCTC, CCTT,
CCTG, CCGA, CCGC, CCGT, CCGG, CTAA, CTAC, CTAT, CTAG, CTCA, CTCC, CTCT, CTCG,
CTTA, CTTC, CTTT, CTTG, CTGA, CTGC, CTGT, CTGG, CGAA, CGAC, CGAT, CGAG, CGCA,
CGCC, CGCT, CGCG, CGTA, CGTC, CGTT, CGTG, CGGA, CGGC, CGGT, CGGG, TAAA, TAAC,
TAAT, TAAG, TACA, TACC, TACT, TACG, TATA, TATC, TATT, TATG, TAGA, TAGC, TAGT,
TAGG, TCAA, TCAC, TCAT, TCAG, TCCA, TCCC, TCCT, TCCG, TCTA, TCTC, TCTT, TCTG,
TCGA, TCGC, TCGT, TCGG, TTAA, TTAC, TTAT, TTAG, TTCA, TTCC, TTCT, TTTC, TTGA,
TTGC, TTGT, TTGG, TGAA, TGAC, TGAT, TGAG, TGCA, TGCC,
TGCT, TGCG, TGTA, TGTC, TGTT, TGTG, TGGA, TGGC, TGGT, TGGG, GAAA, GAAC, GAAT,
GAAG, GACA, GACC, GACT, GACG, GATA, GATC, GATT, GATG, GAGA, GAGC, GAGT, GAGG,
GCAA, GCAC, GCAT, GCAG, GCCA, GCCC, GCCT, GCCG, GCTA, GCTC, GCTT, GCTG,
GCGA, GCGC, GCGT, GCGG, GTAA, GTAC, GTAT, GTAG, GTCA, GTCC, GTCT, GTCG,
GTTA, GTTC, GTTT, GTTG, GTGA, GTGC, GTGT, GTGG, GGAA, GGAC, GGAT, GGAG,
GGCA, GGCC, GGCT, GGCG, GGTA, GGTC, GGTT, GG TG, GGG A, GGGC, GGGT, GGGG}
```

Here then is a function that generalizes this process. We have included some pattern matching of the arguments to insure that the alphabet is a list of one or more strings and that the word length  $n$  is a positive integer.

```
In[42]:= Clear[NGrams]
In[43]:= NGrams[alphabet : {__String}, n_Integer?Positive] :=
  Flatten[Outer[StringJoin, Apply[Sequence, Table[alphabet, {n}]]]]
In[44]:= NGrams[{"A", "C", "T", "G"}, 3]
Out[44]= {AAA, AAC, AAT, AAG, ACA, ACC, ACT, ACG, ATA, ATC, ATT, ATG, AGA, AGC, AGT, AGG, CAA,
CAC, CAT, CAG, CCA, CCC, CCT, CCG, CTA, CTC, CTT, CTG, CGA, CGC, CGT, CGG, TAA,
TAC, TAT, TAG, TCA, TCC, TCT, TCG, TTA, TTC, TTT, TTG, TGA, TGC, TGT, TGG, GAA,
GAC, GAT, GAG, GCA, GCC, GCT, GCG, GTA, GTC, GTT, GTG, GGA, GGC, GGT, GGG}
```

Here is a list of all possible words of length two from the alphabet of amino acids.

```
In[45]:= alphabet = {"K", "P", "W", "G", "E", "V", "Y", "L", "M", "Q", "R", "S",
  "F", "D", "H", "I", "C", "T", "N", "A"};
```

```
In[46]:= NGrams[alphabet, 2]
```

```
Out[46]= {KK, KP, KW, KG, KE, KV, KY, KL, KM, KQ, KR, KS, KF, KD, KH, KI, KC, KT, KN, KA, PK,
PP, PW, PG, PE, PV, PY, PL, PM, PQ, PR, PS, PF, PD, PH, PI, PC, PT, PN, PA, WK,
WP, WW, WG, WE, WV, WY, WL, WM, WQ, WR, WS, WF, WD, WH, WI, WC, WT, WN, WA, GK,
GP, GW, GG, GE, GV, GY, GL, GM, GQ, GR, GS, GF, GD, GH, GI, GC, GT, GN, GA, EK,
EP, EW, EG, EE, EV, EY, EL, EM, EQ, ER, ES, EF, ED, EH, EI, EC, ET, EN, EA, VK,
VP, VW, VG, VE, VV, VY, VL, VM, VQ, VR, VS, VF, VD, VH, VI, VC, VT, VN, VA, YK,
YP, YW, YG, YE, YV, YY, YL, YM, YQ, YR, YS, YF, YD, YH, YI, YC, YT, YN, YA, LK,
LP, LW, LG, LE, LV, LY, LL, LM, LQ, LR, LS, LF, LD, LH, LI, LC, LT, LN, LA, MK,
MP, MW, MG, ME, MV, MY, ML, MM, MQ, MR, MS, MF, MD, MH, MI, MC, MT, MN, MA, QK,
QP, QW, QG, QE, QV, QY, QL, QM, QQ, QR, QS, QF, QD, QH, QI, QC, QT, QN, QA, RK,
RP, RW, RG, RE, RV, RY, RL, RM, RQ, RR, RS, RF, RD, RH, RI, RC, RT, RN, RA, SK,
SP, SW, SG, SE, SV, SY, SL, SM, SQ, SR, SS, SF, SD, SH, SI, SC, ST, SN, SA, FK,
FP, FW, FG, FE, FV, FY, FL, FM, FQ, FR, FS, FF, FD, FH, FI, FC, FT, FN, FA, DK,
DP, DW, DG, DE, DV, DY, DL, DM, DQ, DR, DS, DF, DD, DH, DI, DC, DT, DN, DA, HK,
HP, HW, HG, HE, HV, HY, HL, HM, HQ, HR, HS, HF, HD, HH, HI, HC, HT, HN, HA, IK,
IP, IW, IG, IE, IV, IY, IL, IM, IQ, IR, IS, IF, ID, IH, II, IC, IT, IN, IA, CK,
CP, CW, CG, CE, CV, CY, CL, CM, CQ, CR, CS, CF, CD, CH, CI, CC, CT, CN, CA, TK,
TP, TW, TG, TE, TV, TY, TL, TM, TQ, TR, TS, TF, TD, TH, TI, TC, TT, TN, TA, NK,
NP, NW, NG, NE, NV, NY, NL, NM, NQ, NR, NS, NF, ND, NH, NI, NC, NT, NN, NA, AK,
AP, AW, AG, AE, AV, AY, AL, AM, AQ, AR, AS, AF, AD, AH, AI, AC, AT, AN, AA}
```

9. `IntegerDigits[n]` gives a list of the digits in  $n$ . First returns the first element in that list. Here then is the function definition.

```
In[47]:= LeadingDigit[n_Integer] := First[IntegerDigits[n]]
```

```
In[48]:= LeadingDigit[2495874985797]
```

```
Out[48]= 2
```

By default, this function has no attributes, in particular it is not listable.

```
In[49]:= LeadingDigit[{434, 826, 5632}]
```

```
Out[49]= LeadingDigit[{434, 826, 5632}]
```

We could map the function across lists, but instead we will give it the attribute that causes it to automatically map across lists. Set the function to have the `Listable` attribute.

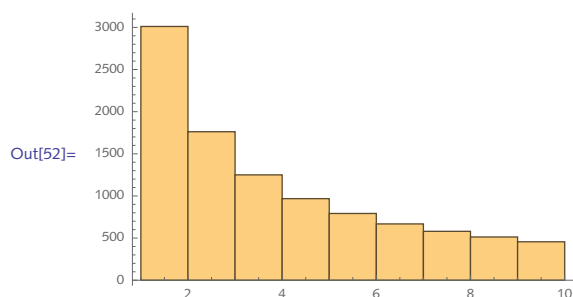
```
In[50]:= SetAttributes[LeadingDigit, Listable]
```

Now, create a table of the first 10 000 Fibonacci numbers.

```
In[51]:= fibs = Table[Fibonacci[i], {i, 10000}];
```

Because `LeadingDigit` is now a listable function, it automatically maps across the list `fibs`. Here then is a histogram of the leading digits of the first 10 000 Fibonacci digits.

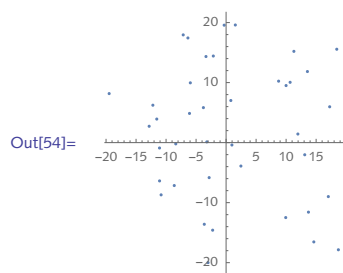
```
In[52]:= Histogram[LeadingDigit[fibs]]
```



10. Start with a set of points in the plane.

```
In[53]:= data = RandomReal[{-20, 20}, {40, 2}];
```

```
In[54]:= ListPlot[data, AspectRatio -> Automatic]
```



The data are of the form  $\{\{x_1, y_1\}, \{x_2, y_2\}, \dots\}$ .

```
In[55]:= Take[data, 2]
```

```
Out[55]= {{9.93547, -12.4866}, {-2.23917, -14.5885}}
```

Transposing the data gives a list all the  $x$ -coordinates, followed by a list of all the  $y$ -coordinates.

```
In[56]:= Transpose[{{x1, y1}, {x2, y2}, {x3, y3}}]
```

```
Out[56]= {{x1, x2, x3}, {y1, y2, y3}}
```

Then map `Min` and `Max` over this list.

```
In[57]:= Map[Min, %]
```

```
Out[57]= {Min[x1, x2, x3], Min[y1, y2, y3]}
```

Here then are the computations for our data. Notice the parallel assignments saving some typing.

```
In[58]:= {xmin, ymin} = Map[Min, Transpose[data]]
```

```
Out[58]= {-19.4808, -19.9974}
```

```
In[59]:= {xmax, ymax} = Map[Max, Transpose[data]]
```

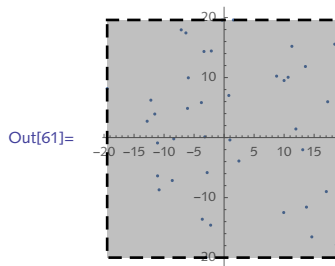
```
Out[59]:= {18.7094, 19.5983}
```

As it turns out, a built-in function is available for this task. `MinMax` operates on vectors of numbers. So we will map it across the transposed data. Note the form of the data that is returned: first a list of the min and max x-coordinates and then likewise for the y-coordinates. A bit of rearranging is needed for `Rectangle`.

```
In[60]:= {{xmin, xmax}, {ymin, ymax}} = Map[MinMax, Transpose[data]]
```

```
Out[60]:= {{-19.4808, 18.7094}, {-19.9974, 19.5983}}
```

```
In[61]:= ListPlot[data, AspectRatio -> Automatic, Epilog -> {
    Opacity[.25], EdgeForm[Dashed], Rectangle[{xmin, ymin}, {xmax, ymax}]]]
```



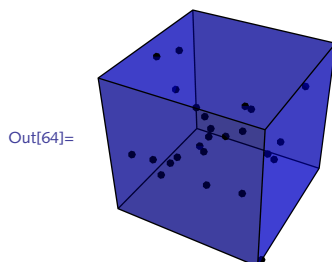
And here is the three-dimensional case:

```
In[62]:= data3d = RandomReal[{-10, 10}, {25, 3}];
```

```
In[63]:= {{xmin, xmax}, {ymin, ymax}, {zmin, zmax}} = Map[MinMax, Transpose[data3d]]
```

```
Out[63]:= {{-9.06054, 8.82294}, {-9.13641, 8.77756}, {-9.47419, 8.14716}}
```

```
In[64]:= Graphics3D[{
    Point[data3d],
    Blue, Opacity[.5], Cuboid[{xmin, ymin, zmin}, {xmax, ymax, zmax}]
}, Boxed -> False]
```



II. First create a set of points with which to work.

```
In[65]:= pts = RandomReal[1, {100, 2}];
```

The set of all two-element subsets is given by:

```
In[66]:= pairs = Subsets[pts, {2}];
```

Apply the distance function to pairs. Note the need to apply EuclideanDistance at level one.

```
In[67]:= Apply[EuclideanDistance, pairs, {1}];
```

The maximum distance (diameter) is given by Max.

```
In[68]:= Max[%]
```

```
Out[68]= 1.27562
```

Here is a function that puts it all together.

```
In[69]:= PointsetDiameter[pts_List] :=  
      Max[Apply[EuclideanDistance, Subsets[pts, {2}], {1}]]
```

```
In[70]:= PointsetDiameter[pts]
```

```
Out[70]= 1.27562
```

To get the coordinate points that give this maximum distance, use MaximalBy:

```
In[71]:= MaximalBy[pairs, Apply[EuclideanDistance, #] &]
```

```
Out[71]= {{0.94701, 0.0541054}, {0.0448236, 0.95592}}
```

A quick check:

```
In[72]:= Apply[EuclideanDistance, %, {1}]
```

```
Out[72]= {1.27562}
```

In fact, this function works on  $n$ -dimensional point sets.

```
In[73]:= pts3D = RandomReal[1, {5, 3}]
```

```
Out[73]= {{0.00512186, 0.739216, 0.594529},  
          {0.88659, 0.837747, 0.72644}, {0.274863, 0.754823, 0.71913},  
          {0.291825, 0.990644, 0.528335}, {0.88648, 0.339045, 0.871665}}
```

```
In[74]:= PointsetDiameter[pts3D]
```

```
Out[74]= 1.00684
```

Because of the use of Subsets, this computation will not scale well. Computing subsets has computational complexity  $O(n^2)$  and so the time to compute subsets is quadratic in the size of the set.

```
In[75]:= Timing[Subsets[Range[1000], {2}];]
```

```
Out[75]= {0.07042, Null}
```

Twice as large of a set should cause the time to quadruple.



```
In[76]:= Timing[Subsets[Range[2000], {2}];]
```

```
Out[76]:= {0.315402, Null}
```

Exercise 5, Section 9.1 explores another approach – one that involves the convex hull – to speed up the computation.

12. First, create the random centers and radii.

```
In[77]:= n = 12;
centers = RandomReal[{-1, 1}, {n, 2}]
```

```
Out[78]:= {{-0.391653, 0.830624}, {-0.747273, -0.460966}, {0.656809, -0.910538},
{-0.187957, -0.421397}, {0.574283, 0.718045}, {-0.620718, -0.831547},
{-0.548517, -0.791468}, {-0.578859, -0.448743}, {0.330763, 0.848178},
{-0.948835, -0.268074}, {0.911755, 0.460441}, {0.146787, 0.898121}}
```

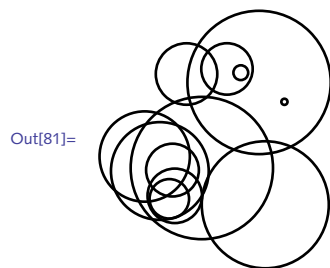
```
In[79]:= radii = RandomReal[1, {n}]
```

```
Out[79]:= {0.409292, 0.650746, 0.8457, 0.947663, 0.954289, 0.262659,
0.362816, 0.351422, 0.0984283, 0.601188, 0.0410166, 0.342364}
```

MapThread is perfect for the task of grabbing one center, one radii, and wrapping Circle around them.

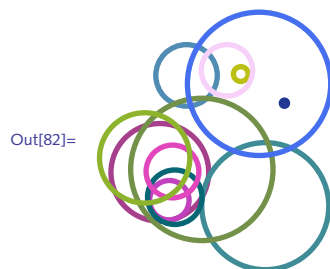
```
In[80]:= circles = MapThread[Circle, {centers, radii}];
```

```
In[81]:= Graphics[circles]
```



And here is a rule to transform each circle into a scoped list that includes Thick and RandomColor. Note the need for the delayed rule ( $\rightarrow$ ); try it with an immediate rule to understand why.

```
In[82]:= Graphics[circles /. Circle[x__] -> {Thick, RandomColor[], Circle[x]}]
```



13. This can be done either in two steps, or by using the Inner function.

```
In[83]:= Transpose[{ {1, 2}, {3, 4} } {x, y}
Out[83]= {{x, 3 x}, {2 y, 4 y}}

In[84]:= Total[%]
Out[84]= {x + 2 y, 3 x + 4 y}

In[85]:= Inner[Times, {{1, 2}, {3, 4}}, {x, y}, Plus]
Out[85]= {x + 2 y, 3 x + 4 y}
```

14. Here is a test matrix.

```
In[86]:= mat = RandomInteger[1, {5, 5}];
MatrixForm[mat]
```

Out[87]//MatrixForm=

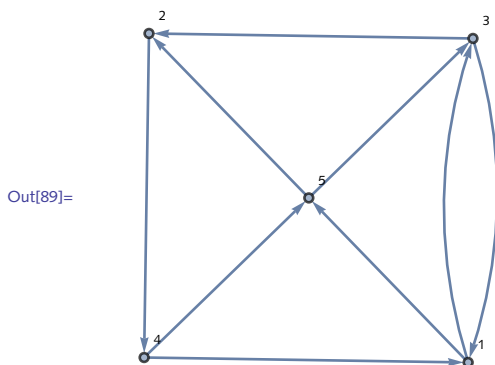
$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

A bit of thought should convince you that adding the matrix to its transpose and then totaling all the ones in each row will give the correct count.

```
In[88]:= Map[Total, mat + Transpose[mat]]
Out[88]= {4, 3, 4, 3, 4}
```

Using graphs you can accomplish the same thing directly using VertexDegree.

```
In[89]:= gr = AdjacencyGraph[mat, VertexLabels -> "Name"]
```



```
In[90]:= VertexDegree[gr]
Out[90]= {4, 3, 4, 3, 4}
```

15. Applying DirectedEdge at level one will do the trick.

```

In[91]:= ToEdges[lis : { {_, _} .. } ] := Apply[DirectedEdge, lis, {1}]
In[92]:= lis = RandomInteger[9, {12, 2}];
          ToEdges[lis]
Out[93]= {3 ↔ 6, 2 ↔ 4, 0 ↔ 5, 1 ↔ 2, 3 ↔ 6, 7 ↔ 2, 3 ↔ 6, 0 ↔ 0, 2 ↔ 8, 4 ↔ 9, 3 ↔ 7, 6 ↔ 6}

```

This rule fails for the case when the argument is a single flat list of a pair of elements.

```

In[94]:= ToEdges[{3, 6}]
Out[94]= ToEdges[{3, 6}]

```

A second rule is needed for this case.

```

In[95]:= ToEdges[lis : {_, _}] := Apply[DirectedEdge, lis]
In[96]:= ToEdges[{3, 6}]
Out[96]= 3 ↔ 6

```

16. To get down to the level of the nested lists, you have to use a second argument to Apply.

```

In[97]:= facs = FactorInteger[3 628 800]
Out[97]= {{2, 8}, {3, 4}, {5, 2}, {7, 1}}
In[98]:= Apply[Power, facs, {1}]
Out[98]= {256, 81, 25, 7}

```

One more use of Apply is needed to multiply these terms.

```

In[99]:= Apply[Times, %]
Out[99]= 3 628 800

```

Here is a function that puts this all together.

```

In[100]:= ExpandFactors[lis_] := Apply[Times, Apply[Power, lis, {1}]]
In[101]:= FactorInteger[295 232 799 039 604 140 847 618 609 643 520 000 000]
Out[101]= {{2, 32}, {3, 15}, {5, 7}, {7, 4}, {11, 3},
           {13, 2}, {17, 2}, {19, 1}, {23, 1}, {29, 1}, {31, 1}}
In[102]:= ExpandFactors[%]
Out[102]= 295 232 799 039 604 140 847 618 609 643 520 000 000

```

17. Here is a factorization we can use to work through this problem.

```

In[103]:= facs = FactorInteger[3 628 800]
Out[103]= {{2, 8}, {3, 4}, {5, 2}, {7, 1}}

```

Another approach uses Transpose to separate the bases from their exponents, then uses Inner to put things back together.

```
In[104]:= {base, exponents} = Transpose[facs]
Out[104]= {{2, 3, 5, 7}, {8, 4, 2, 1}}

In[105]:= Inner[Power, base, exponents, Times]
Out[105]= 3 628 800
```

Since Transpose returns a list of two lists in this example, we need to strip the outer list. This is done by applying Sequence.

```
In[106]:= ExpandFactors2[lis_] := Inner[Power, Sequence @@ Transpose[lis], Times]
In[107]:= ExpandFactors2[facs]
Out[107]= 3 628 800
```

18. First, here is the prime factorization of a test integer:

```
In[108]:= lis = FactorInteger[10! ]
Out[108]= {{2, 8}, {3, 4}, {5, 2}, {7, 1}}
```

Apply Superscript at level one to each of the sublists:

```
In[109]:= Apply[Superscript, lis, {1}]
Out[109]= {28, 34, 52, 71}
```

Finally, apply CenterDot to this list.

```
In[110]:= Apply[CenterDot, %]
Out[110]= 28 · 34 · 52 · 71
```

Put it all together (using shorthand notation for Apply) and Apply at level one.

```
In[111]:= PrimeFactorForm[p_] := CenterDot @@ (Superscript @@@ FactorInteger[p])
In[112]:= PrimeFactorForm[20! ]
Out[112]= 218 · 38 · 54 · 72 · 111 · 131 · 171 · 191
```

Unfortunately, this rule fails for numbers that have only one prime factor.

```
In[113]:= PrimeFactorForm[9]
Out[113]= CenterDot[32]
```

A second rule is needed for this special case.

```
In[114]:= PrimeFactorForm[p_?PrimePowerQ] := First[Superscript @@@ FactorInteger[p]]
In[115]:= PrimeFactorForm[9]
Out[115]= 32
```

A subtle point is that *Mathematica* has automatically ordered these two rules, putting the one involving prime powers first.

```
In[116]:= ?PrimeFactorForm
```

```
Global PrimeFactorForm
```

```
PrimeFactorForm[p_?PrimePowerQ] :=
  First[Apply[Superscript, FactorInteger[p], {1}]]

PrimeFactorForm[p_] :=
  CenterDot @@ Apply[Superscript, FactorInteger[p], {1}]
```

This reordering (we evaluated the rules in a different order) is essential for this function to work properly. If the general rule was checked first, it would apply to arguments that happen to be prime powers and it would give wrong answers.

One final point: the expressions returned by PrimeFactorForm will not evaluate like ordinary expressions due to the use of CenterDot which has no evaluation rules associated with it. You could add an “interpretation” to such expressions by using Interpretation[*disp*, *expr*] as follows.

```
In[117]:= PrimeFactorForm[p_Integer] := With[{fp = FactorInteger[p]},
  Interpretation[
    CenterDot @@ (Superscript @@@ fp),
    Times @@ (Power @@@ fp) ]]
```

Now the output of the following expression can be evaluated directly to get an interpreted result.

```
In[118]:= PrimeFactorForm[12!]
```

```
Out[118]= 210 · 35 · 52 · 71 · 111
```

19. This is a straightforward application of the Outer function.

```
In[119]:= VandermondeMatrix[n_, x_] :=
  Outer[Power, Table[xi, {i, 1, n}], Range[0, n - 1]]
```

```
In[120]:= VandermondeMatrix[4, x] // MatrixForm
```

```
Out[120]//MatrixForm=
```

$$\begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix}$$

20. If we first look at a symbolic result, we should be able to see how to construct our function. For three vectors and three variables, here is the divergence (think of *d* as the derivative operator).

```
In[121]:= Inner[d, {e1, e2, e3}, {v1, v2, v3}, Plus]
```

```
Out[121]:= d[e1, v1] + d[e2, v2] + d[e3, v3]
```

So for arbitrary-length vectors and variables, we have:

```
In[122]:= div[vecs_, vars_] := Inner[D, vecs, vars, Plus]
```

As a check, we can compute the divergence of the standard gravitational or electric force field, which should be zero.

```
In[123]:= div[{x, y, z}/(x^2 + y^2 + z^2)^(3/2), {x, y, z}]
```

```
Out[123]:= - 3 x^2 / (x^2 + y^2 + z^2)^(5/2) - 3 y^2 / (x^2 + y^2 + z^2)^(5/2) - 3 z^2 / (x^2 + y^2 + z^2)^(5/2) + 3 / (x^2 + y^2 + z^2)^(3/2)
```

```
In[124]:= Simplify[%]
```

```
Out[124]:= 0
```

Finally, we should note that this definition of divergence is a bit delicate as we are doing no argument checking at this point. For example, it would be sensible to insure that the length of the vector list is the same as the length of the variable list before starting the computation. Refer to Chapter 4 for a discussion of how to use pattern matching to deal with this issue.

21. The Jacobian is given by the following outer product:

```
In[125]:= JacobianMatrix[vec_List, vars_List] := Outer[D, vec, vars]
```

Add a condition that the dimensions of vec and vars are the same.

```
In[126]:= JacobianMatrix[vec_List, vars_List] :=  
Outer[D, vec, vars] /; Dimensions[vec] == Dimensions[vars]
```

To compute the volume of the hypersphere in dimension two, start with a point in  $\mathbb{R}^2$  expressed in polar coordinates:

```
In[127]:= x = ρ Cos[θ];  
y = ρ Sin[θ];
```

Then compute the determinant of the Jacobian.

```
In[129]:= JacobianMatrix[{x, y}, {ρ, θ}] // MatrixForm
```

```
Out[129]//MatrixForm=
```

$$\begin{pmatrix} \cos[\theta] & -\rho \sin[\theta] \\ \sin[\theta] & \rho \cos[\theta] \end{pmatrix}$$

```
In[130]:= Det[%] // Simplify
```

```
Out[130]= ρ
```

The volume of the hypersphere is obtained by integrating the determinant of the Jacobian over  $\rho$  and  $\theta$ :

```
In[131]:= Integrate[Abs@Det[JacobianMatrix[{x, y}, {ρ, θ}], {ρ, 0, r}, {θ, 0, 2 π},
  Assumptions → r > 0]
```

```
Out[131]= π r2
```

And here is the computation for dimension three:

```
In[132]:= x = ρ Cos[θ] Sin[φ];
  y = ρ Sin[θ] Sin[φ];
  z = ρ Cos[φ];
```

```
In[135]:= Integrate[Abs@Det@JacobianMatrix[{x, y, z}, {ρ, θ, φ}], {ρ, 0, r},
  {θ, -π/2, π/2}, {φ, 0, 2 π}]
```

```
Out[135]=  $\frac{4 \pi r^3}{3}$ 
```

22. First create a table of primes and then use that list for values of p in the second table.

```
In[136]:= primes = Table[Prime[n], {n, 1, 50}]
```

```
Out[136]= {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67,
  71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149,
  151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229}
```

```
In[137]:= Select[Table[2p - 1, {p, primes}], PrimeQ]
```

```
Out[137]= {3, 7, 31, 127, 8191, 131 071, 524 287, 2 147 483 647, 2 305 843 009 213 693 951,
  618 970 019 642 690 137 449 562 111, 162 259 276 829 213 363 391 578 010 288 127,
  170 141 183 460 469 231 731 687 303 715 884 105 727}
```

Or you could do the same thing more directly.

```
In[138]:= Select[Table[2Prime[n] - 1, {n, 1, 50}], PrimeQ]
```

```
Out[138]= {3, 7, 31, 127, 8191, 131 071, 524 287, 2 147 483 647, 2 305 843 009 213 693 951,
  618 970 019 642 690 137 449 562 111, 162 259 276 829 213 363 391 578 010 288 127,
  170 141 183 460 469 231 731 687 303 715 884 105 727}
```

```
In[139]:= Clear[a, b, c, d, e, x, y, z, pt1, pt2, pt3]
```

## 5.2 Iterating functions: exercises

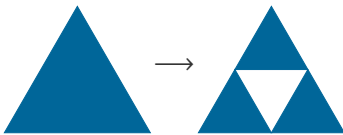
1. Use `NestWhileList` to iterate the `julia` function similarly to how it was iterated with `FixedPointList` in the text.
2. Use `NestList` to iterate the process of summing cubes of digits, that is, starting with an initial integer generate a list of the successive sums of cubes of its digits. For example, starting with 4, the list should look like {4, 64, 280, 520, 133, ...}, since  $4^3 = 64$ ,  $6^3 + 4^3 = 280$ , etc. Extend the list to at least fifteen values, experiment with other starting values, and look for patterns in the sequences.

3. Following on the example in this section iterating rotations of a triangle, use `Translate` to iterate the translation of a square or other polygon.
4. Using `Fold`, create a function `fac[n]` that takes an integer  $n$  as argument and returns the factorial of  $n$ , that is,  $n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$ .
5. The naive way to multiply  $x^{22}$  would be to repeatedly multiply  $x$  by itself, performing 21 multiplications. But going as far back as about 200 BC in the Hindu classic *Chandah-sutra*, another method has been known that significantly reduces the total number of multiplications in performing such exponentiation. The idea is to first express the exponent in base 2.

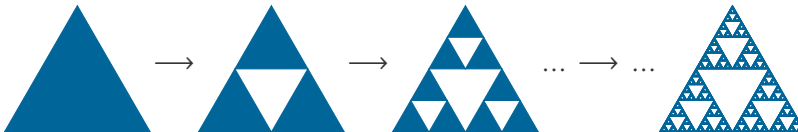
```
In[1]:= IntegerDigits[22, 2]
Out[1]= {1, 0, 1, 1, 0}
```

Then, starting with the second bit from the left, interpret a 1 to mean square the existing expression and multiply by  $x$ , and a 0 to mean multiply just by  $x$ . Implement this algorithm using `FoldList`.

6. The Sierpiński triangle is a classic iteration example. It can be constructed by starting with an equilateral triangle and removing the inner triangle formed by connecting the midpoints of each side of the original triangle.



The process is iterated by repeating the same computation on each of the resulting smaller triangles (other types of iteration can be used).



One approach is to take the starting equilateral triangle and, at each iteration, perform the appropriate transformations using `Scale` and `Translate`, then iterate. Implement this algorithm, but be careful about nesting large symbolic graphics structures too deeply.

## 5.2 Solutions

1. Here is the code with `FixedPointList`. The iteration stops when the distance of the iterates to the origin exceeds 4.0.

```
In[1]:= Clear[f, z, julia]
In[2]:= f[z_] := z^2 + -0.8 - 0.156 i
julia[z_] := FixedPointList[f, z, SameTest -> (Abs[#2] > 4.0 &)]
```



And here it is using `NestWhileList`. Iteration continues so long as the distance of the iterates to the origin is less than 4.0.

```
In[4]:= NestWhileList[julia, -0.5 + 1.5 I, Abs[#] < 4.0 &]
Out[4]= {-0.5 + 1.5 i, {-0.5 + 1.5 i, -2.8 - 1.656 i, 4.29766 + 9.1176 i}}
```

- For a given number, first we need to get its digits and then add their cubes.

```
In[5]:= IntegerDigits[64]
Out[5]= {6, 4}

In[6]:= IntegerDigits[64]^3
Out[6]= {216, 64}

In[7]:= Total[IntegerDigits[64]^3]
Out[7]= 280

In[8]:= sumsOfCubes[n_] := Total[IntegerDigits[n]^3]
```

Then iterate:

```
In[9]:= NestList[sumsOfCubes, 4, 12]
Out[9]= {4, 64, 280, 520, 133, 55, 250, 133, 55, 250, 133, 55, 250}
```

In fact, it appears as if many initial values enter into cycles.

```
In[10]:= NestList[sumsOfCubes, 32, 12]
Out[10]= {32, 35, 152, 134, 92, 737, 713, 371, 371, 371, 371, 371}

In[11]:= NestList[sumsOfCubes, 123, 12]
Out[11]= {123, 36, 243, 99, 1458, 702, 351, 153, 153, 153, 153, 153}
```

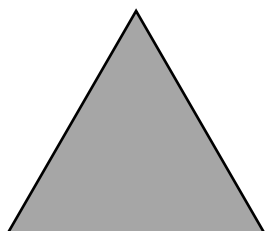
- To start, here is a triangle we will translate.

```
In[12]:= tri = Triangle[CirclePoints[3]]
Out[12]= Triangle[{ {  $\frac{\sqrt{3}}{2}$ ,  $-\frac{1}{2}$  }, {0, 1}, {  $-\frac{\sqrt{3}}{2}$ ,  $-\frac{1}{2}$  } }]
```

We will display with opacity turned on so that overlapping triangles will show through.

```
In[13]:= Graphics[{Opacity[.35], EdgeForm[Black], tri}]
```

Out[13]=



`Translate[gr, vecs]` takes a graphics object *gr*, and translates it according to the vectors *vecs*. So for example, here are some translation vectors; the first vector translates by  $1/2$  unit to the right, the second vector translates up and to the right.

```
In[14]:= vecs = {{1/2, 0}, {1/4, sqrt(3)/4}};
```

This translates the triangle using the first translation vector.

```
In[15]:= Translate[tri, {1/2, 0}]
```

```
Out[15]= Translate[Triangle[{{sqrt(3)/2, -1/2}, {0, 1}, {-sqrt(3)/2, -1/2}}, {1/2, 0}]]
```

The following translation function creates two objects translated by the vectors *vecs*.

```
In[16]:= translation[gr_] := Translate[gr, vecs]
```

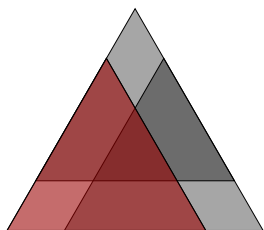
```
In[17]:= tranTri = NestList[translation, tri, 1]
```

```
Out[17]= {Triangle[{{sqrt(3)/2, -1/2}, {0, 1}, {-sqrt(3)/2, -1/2}}], Translate[
  Triangle[{{sqrt(3)/2, -1/2}, {0, 1}, {-sqrt(3)/2, -1/2}}, {{1/2, 0}, {1/4, sqrt(3)/4}}]]}
```

Here are the translated objects along with a red version of the original triangle.

```
In[18]:= Graphics[{Opacity[.35], EdgeForm[{Black, Thin}],
  tranTri, {Red, tri}}]
```

```
Out[18]=
```



- Starting with 1, fold the `Times` function across the first  $n$  integers.

```
In[19]:= fac[n_] := Fold[Times, 1, Range[n]]
```

```
In[20]:= fac[10]
```

```
Out[20]= 3 628 800
```

- To compute  $x^{25}$  say, start by expressing the exponent in base 2.

```
In[21]:= IntegerDigits[25, 2]
```

```
Out[21]= {1, 1, 0, 0, 1}
```

Now starting with the second bit from the left (use `Rest`), interpret a one to mean square the

existing expression and multiply by  $x$ , and a zero to mean multiply the existing expression by  $x$ .

```
In[22]:= Clear[f, a, b, x]
In[23]:= f[a_, b_] := a^2 x^b
In[24]:= FoldList[f, x, Rest[{1, 1, 0, 0, 1}]]
Out[24]= {x, x^3, x^6, x^12, x^25}
```

Once you are familiar with pure functions (Section 5.5), this is done directly (without the need to first define an auxiliary function  $f$ ) as follows:

```
In[25]:= FoldList[#1^2 x^#2 &, x, Rest[{1, 1, 0, 0, 1}]]
Out[25]= {x, x^3, x^6, x^12, x^25}
```

Or you can start with the first bit but you will have to adjust the initial value accordingly.

```
In[26]:= FoldList[#1^2 x^#2 &, 1, {1, 1, 0, 0, 1}]
Out[26]= {1, x, x^3, x^6, x^12, x^25}
```

Here is a larger computation.

```
In[27]:= exp = 4523;
         digs = IntegerDigits[exp, 2];
         FoldList[#1^2 x^#2 &, 1, digs]
Out[29]= {1, x, x^2, x^4, x^8, x^17, x^35, x^70, x^141, x^282, x^565, x^1130, x^2261, x^4523}
```

6. First create the vertices of the triangle. Wrapping them in  $N[\dots]$  helps to keep the graphical structures small (see Section 8.3 for more on numeric vs. symbolic expressions in graphics).

```
In[30]:= vertices = N[{{0, 0}, {1, 0}, {1/2, 1}}];
```

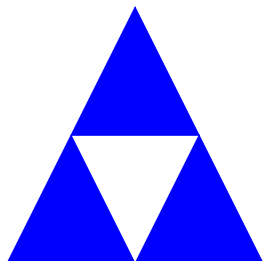
This gives the three different translation vectors.

```
In[31]:= translateVecs = 0.5 vertices
Out[31]= {{0., 0.}, {0.5, 0.}, {0.25, 0.5}}
```

Here is the set of transformations of the triangle described by `vertices`, scaled by 0.5, and translated according to the translation vectors.

```
In[32]:= tri = Polygon[vertices];
Graphics[{
  Blue, Translate[Scale[tri, 0.5, {0., 0.}], translateVecs]
}]
```

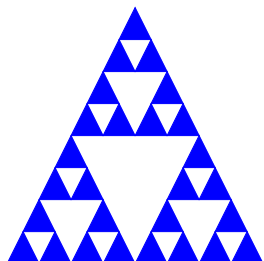
Out[33]=



Finally, iterate the transformations by wrapping them in Nest.

```
In[34]:= Graphics[
  {Blue, Nest[{Blue, Translate[Scale[#, 0.5, {0., 0.}], translateVecs]} &,
    Polygon[vertices], 3]}]
```

Out[34]=

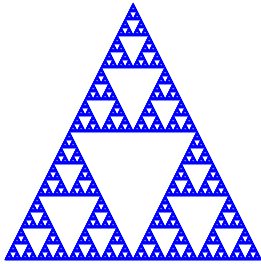


Once you have been through the rest of this chapter, you should be able to turn this into a reusable function, scoping local variables, using pure functions, and adding options.

```
In[37]:= SierpinskiTriangle[iter_, opts : OptionsPattern[Graphics]] :=
Module[{vertices, vecs},
  vertices = N[{{0, 0}, {1, 0}, {1/2, 1}}];
  vecs = 0.5 vertices;
  Graphics[{Blue, Nest[{Blue, Translate[Scale[#, 0.5, {0., 0.}], vecs]} &,
    Polygon[vertices], iter]}, opts]]
```

```
In[36]:= SierpinskiTriangle[9]
```

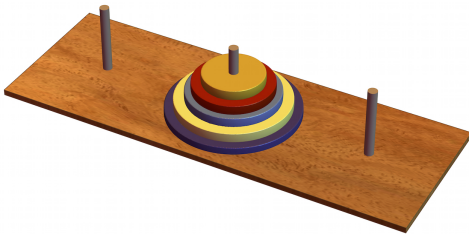
```
Out[36]=
```



### 5.3 Recursive functions: exercises

1. Create a recursive function that computes the  $n$ th power of two.
2. Create a recursive function that returns the factorial of  $n$ , for  $n$  a nonnegative integer.
3. The Tower of Hanoi puzzle is a popular game invented by the French mathematician Édouard Lucas in 1883. Given three pegs, one of which contains  $n$  disks of increasing radii, the object is to move the stack of disks to another peg in the fewest number of moves without putting a larger disk on a smaller one (Figure 5.2).

FIGURE 5.2. *Tower of Hanoi.*



Create a recursive function `TowerOfHanoi[n]` that computes the minimal number of moves for a stack of  $n$  disks over three pegs. Results for  $n = 1, 2, \dots, 10$  are as follows:

{1, 3, 7, 15, 31, 63, 127, 255, 511, 1023}

Legend has it that the priests in the Indian temple Kashi Vishwanath have a room with a stack of 64 disks and three pegs and that when they complete moving the stack to a new peg, the world will end. If they move one disk per second, compute how long until the “end of the world.”

4. For each of the following sequences of numbers, see if you can deduce the pattern and write a function to compute the general term:

a. 

2,	3,	6,	18,	108,	1944,	209 952,	...
$A_1$	$A_2$	$A_3$	$A_4$	$A_5$	$A_6$	$A_7$	...

- b. 

0,	1,	-1,	2,	-3,	5,	-8,	13,	-21,	...
$B_1$	$B_2$	$B_3$	$B_4$	$B_5$	$B_6$	$B_7$	$B_8$	$B_9$	...
- c. 

0,	1,	2,	3,	6,	11,	20,	37,	68,	...
$C_1$	$C_2$	$C_3$	$C_4$	$C_5$	$C_6$	$C_7$	$C_8$	$C_9$	...

5. The Fibonacci sequence can also be defined for negative integers using the following formula (Graham, Knuth, and Patashnik 1994):

$$F_{-n} = (-1)^{n-1} F_n$$

The first few terms are

$$\begin{array}{cccccccccccc} 0 & 1 & -1 & 2 & -3 & 5 & -8 & 13 & -21 & \dots \\ F_0 & F_{-1} & F_{-2} & F_{-3} & F_{-4} & F_{-5} & F_{-6} & F_{-7} & F_{-8} & \dots \end{array}$$

Write the definitions for Fibonacci numbers with negative integer arguments.

6. Create a recursive function to reverse the elements in a flat list.
7. Create a recursive function to transpose the elements of two lists. Write an additional rule to transpose the elements of three lists.
8. Using dynamic programming is one way to speed up the computation of Fibonacci numbers, but another is to use different algorithms. A more efficient algorithm is based on the following identities:

$$\begin{aligned} F_1 &= 1 \\ F_2 &= 1 \\ F_{2n} &= 2F_{n-1}F_n + F_n^2, \quad \text{for } n \geq 1 \\ F_{2n+1} &= F_{n+1}^2 + F_n^2, \quad \text{for } n \geq 1 \end{aligned}$$

Program a function to generate Fibonacci numbers using these identities.

9. You can still speed up the code for generating Fibonacci numbers in the previous exercise by using dynamic programming. Do so, and construct tables like those in this section, giving the number of additions performed by the two programs.
10. An Eulerian number, denoted  $\left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle$ , gives the number of permutations of  $n$ -element sets with  $k$  increasing runs of elements. For example, for  $n = 3$  the permutations of  $\{1, 2, 3\}$  contain four increasing runs of length one, namely  $\{1, 3, 2\}$ ,  $\{2, 1, 3\}$ ,  $\{2, 3, 1\}$ , and  $\{3, 1, 2\}$ . Hence,  $\left\langle \begin{smallmatrix} 3 \\ 1 \end{smallmatrix} \right\rangle = 4$ .

```
In[1]:= Permutations[{1, 2, 3}]
```

```
Out[1]= {{1, 2, 3}, {1, 3, 2}, {2, 1, 3}, {2, 3, 1}, {3, 1, 2}, {3, 2, 1}}
```

This can be programmed using the following recursive definition, where  $n$  and  $k$  are assumed to be integers:

$$\left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle = (k+1) \left\langle \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\rangle + (n-k) \left\langle \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\rangle, \text{ for } n > 0,$$

$$\left\langle \begin{smallmatrix} 0 \\ k \end{smallmatrix} \right\rangle = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0. \end{cases}$$

Create a function `EulerianNumber [n, k]`. You can check your work against Table 5.1, which displays the first few Eulerian numbers.

TABLE 5.1. *Eulerian number triangle*

$n$	$\left\langle \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\rangle$	$\left\langle \begin{smallmatrix} n \\ 1 \end{smallmatrix} \right\rangle$	$\left\langle \begin{smallmatrix} n \\ 2 \end{smallmatrix} \right\rangle$	$\left\langle \begin{smallmatrix} n \\ 3 \end{smallmatrix} \right\rangle$	$\left\langle \begin{smallmatrix} n \\ 4 \end{smallmatrix} \right\rangle$	$\left\langle \begin{smallmatrix} n \\ 5 \end{smallmatrix} \right\rangle$	$\left\langle \begin{smallmatrix} n \\ 6 \end{smallmatrix} \right\rangle$	$\left\langle \begin{smallmatrix} n \\ 7 \end{smallmatrix} \right\rangle$	$\left\langle \begin{smallmatrix} n \\ 8 \end{smallmatrix} \right\rangle$
0	1								
1	1	0							
2	1	1	0						
3	1	4	1	0					
4	1	11	11	1	0				
5	1	26	66	26	1	0			
6	1	57	302	302	57	1	0		
7	1	120	1191	2416	1191	120	1	0	
8	1	247	4293	15619	15619	4293	247	1	0

Because of the triple recursion, you will find it necessary to use a dynamic programming implementation to compute any Eulerian numbers of even modest size.

Hint: Although the above formulas will compute it, you can add the following rule to simplify some of the computation (Graham, Knuth, and Patashnik 1994):

$$\left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle = 0, \text{ for } k \geq n$$

11. The Collatz sequence is generated as follows: starting with a number  $n$ , if it is even, then output  $n/2$ ; if it is odd, then output  $3n+1$ . Iterate this process while the value of the iterate is not equal to one. Using recursion and dynamic programming, create the function `collatz [n, i]`, which computes the  $i$ th iterate of the Collatz sequence starting with integer  $n$ . Compare its speed with that of the procedural approach in Exercise 10 of Section 5.4.

## 5.3 Solutions

1. Thinking about the powers of two recursively, we have  $2^n = 2 \times 2^{n-1}$ . The base case is  $2^0 = 1$ .

```
ln[1]:= powersOf2 [0] = 1;
```

```
ln[2]:= powersOf2 [n_] := 2 powersOf2 [n - 1]
```

```
In[3]:= Table[powersOf2[j], {j, 0, 10}]
Out[3]= {1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024}
```

2. Here are the two rules for the factorial function, defined recursively.

```
In[4]:= fact[0] = 1;
In[5]:= fact[n_Integer?NonNegative] := n fact[n - 1]
```

And here are the first ten factorials.

```
In[6]:= Table[fact[j], {j, 1, 10}]
Out[6]= {1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800}
```

Check that the rule is not called for nonintegers or negative numbers.

```
In[7]:= fact[3.6]
Out[7]= fact[3.6]

In[8]:= fact[-4]
Out[8]= fact[-4]
```

3. For the base case, it only takes one move to move a disk from one peg to another.

```
In[9]:= hanoi[1] = 1;
```

To move a stack of  $n$  disks, move the top  $n - 1$  disks, then move the bottom disk, then move the  $n - 1$  disks again. Here is the recursion.

```
In[10]:= hanoi[n_] := 2 hanoi[n - 1] + 1
```

And here are the number of moves for puzzles with one through ten disks.

```
In[11]:= Table[hanoi[i], {i, 10}]
Out[11]= {1, 3, 7, 15, 31, 63, 127, 255, 511, 1023}
```

In fact, there is a closed form formula for this.

```
In[12]:= Table[2^n - 1, {n, 10}]
Out[12]= {1, 3, 7, 15, 31, 63, 127, 255, 511, 1023}
```

4. The key here is to get the stopping conditions right in each case.

- a. This is a straightforward recursion, multiplying the previous two values to get the next.

```
In[13]:= Clear[a, b, c]
In[14]:= a[1] := 2
          a[2] := 3
          a[i_] := a[i - 1] a[i - 2]
```



```
In[17]:= Table[a[i], {i, 1, 8}]
Out[17]= {2, 3, 6, 18, 108, 1944, 209 952, 408 146 688}
```

b. The sequence is obtained by taking the difference of the previous two values.

```
In[18]:= b[1] := 0
          b[2] := 1
          b[i_] := b[i - 2] - b[i - 1]
In[21]:= Table[b[i], {i, 1, 9}]
Out[21]= {0, 1, -1, 2, -3, 5, -8, 13, -21}
```

c. Here we add the previous three values.

```
In[22]:= c[1] := 0
          c[2] := 1
          c[3] := 2
          c[i_] := c[i - 3] + c[i - 2] + c[i - 1]
In[26]:= Table[c[i], {i, 1, 9}]
Out[26]= {0, 1, 2, 3, 6, 11, 20, 37, 68}
```

5. You can use your earlier definition of the Fibonacci numbers, or use the built-in Fibonacci.

```
In[27]:= f[n_Integer?NonPositive] := (-1)n-1 Fibonacci[-n]
In[28]:= f[0] = 0;
          f[-1] = 1;
In[30]:= Table[f[i], {i, 0, -8, -1}]
Out[30]= {0, 1, -1, 2, -3, 5, -8, 13, -21}
```

6. This is similar to the length function in the text – recursion is on the tail. The base case is a list consisting of a single element.

```
In[31]:= reverse[{x_, y_}] := Join[reverse[{y}], {x}]
In[32]:= reverse[{x_}] := {x}
In[33]:= reverse[{1, β, 3/4, "practice makes perfect"}]
Out[33]= {practice makes perfect,  $\frac{3}{4}$ , β, 1}
```

7. Recursion is on the tails of the two lists, here denoted r1 and r2.

```
In[34]:= transpose[{x1_, r1_}, {x2_, r2_}] :=
          Join[{x1, x2}, transpose[{r1}, {r2}]]
In[35]:= transpose[{x_}, {y_}] := {{x, y}}
In[36]:= transpose[{x1, x2}, {y1, y2}]
Out[36]= {{x1, y1}, {x2, y2}}
```

```
In[37]:= transpose[%]
```

```
Out[37]= {{x1, x2}, {y1, y2}}
```

8. This implementation uses the identities given in the exercise together with some pattern matching for the even and odd cases.

```
In[38]:= F[1] := 1
          F[2] := 1
```

```
In[40]:= F[n_?EvenQ] := 2 F[n/2 - 1] F[n/2] + F[n/2]^2
          F[n_?OddQ] := F[n/2 + 1]^2 + F[n/2]^2
```

```
In[42]:= Map[F, Range[10]]
```

```
Out[42]= {1, 1, 2, 3, 5, 8, 13, 21, 34, 55}
```

```
In[43]:= Timing[F[10^4];]
```

```
Out[43]= {0.405646, Null}
```

9. The use of dynamic programming speeds up the computation by several orders of magnitude.

```
In[44]:= FF[1] := 1
          FF[2] := 1
```

```
In[46]:= FF[n_?EvenQ] := FF[n] = 2 FF[n/2 - 1] FF[n/2] + FF[n/2]^2
          FF[n_?OddQ] := FF[n] = FF[n/2 + 1]^2 + FF[n/2]^2
```

```
In[48]:= Map[FF, Range[10]]
```

```
Out[48]= {1, 1, 2, 3, 5, 8, 13, 21, 34, 55}
```

```
In[49]:= Timing[FF[10^5];]
```

```
Out[49]= {0.000664, Null}
```

This is fairly fast, even compared with the built-in `Fibonacci` which uses a method based on the binary digits of  $n$ .

```
In[50]:= Timing[Fibonacci[10^5];]
```

```
Out[50]= {0.000171, Null}
```

10. Here are the rules translated directly from the formulas given in the exercise.

```
In[51]:= EulerianNumber[0, k_] = 0;
          EulerianNumber[n_Integer, 0] = 1;
          EulerianNumber[n_Integer, k_Integer] /; k ≥ n = 0;
```

```
In[54]:= EulerianNumber[n_Integer, k_Integer] :=
          (k + 1) EulerianNumber[n - 1, k] + (n - k) EulerianNumber[n - 1, k - 1]
```

```

In[55]:= Table[EulerianNumber[n, k], {n, 0, 7}, {k, 0, 7}] // TableForm
Out[55]//TableForm=


|   |     |      |      |      |     |   |   |
|---|-----|------|------|------|-----|---|---|
| 0 | 0   | 0    | 0    | 0    | 0   | 0 | 0 |
| 1 | 0   | 0    | 0    | 0    | 0   | 0 | 0 |
| 1 | 1   | 0    | 0    | 0    | 0   | 0 | 0 |
| 1 | 4   | 1    | 0    | 0    | 0   | 0 | 0 |
| 1 | 11  | 11   | 1    | 0    | 0   | 0 | 0 |
| 1 | 26  | 66   | 26   | 1    | 0   | 0 | 0 |
| 1 | 57  | 302  | 302  | 57   | 1   | 0 | 0 |
| 1 | 120 | 1191 | 2416 | 1191 | 120 | 1 | 0 |


```

Because of the triple recursion, computing larger values is not only time and memory intensive but also bumps up against the built-in recursion limit.

```

In[56]:= EulerianNumber[25, 15] // Timing
Out[56]= {16.0102, 531 714 261 368 950 897 339 996}

```

This is a good candidate for dynamic programming. In the following implementation we have temporarily reset the value of `$RecursionLimit` using `Block`.

```

In[57]:= Clear[EulerianNumber];
In[58]:= EulerianNumber[0, k_] = 0;
EulerianNumber[n_Integer, 0] = 1;
EulerianNumber[n_Integer, k_Integer] /; k ≥ n = 0;
In[61]:= EulerianNumber[n_Integer, k_Integer] := Block[{ $RecursionLimit = Infinity },
EulerianNumber[n, k] = (k + 1) EulerianNumber[n - 1, k] +
(n - k) EulerianNumber[n - 1, k - 1] ]
In[62]:= EulerianNumber[25, 15] // Timing
Out[62]= {0.001165, 531 714 261 368 950 897 339 996}
In[63]:= EulerianNumber[600, 65]; // Timing
Out[63]= {0.296621, Null}
In[64]:= N[EulerianNumber[600, 65]]
Out[64]= 4.998147102049161 × 101091

```

II. Recursion is on the tail of the iterator  $i$ .

```

In[65]:= collatz[n_, 0] := n
In[66]:= collatz[n_, i_] := (collatz[n, i] =  $\frac{\text{collatz}[n, i - 1]}{2}$ ) /; EvenQ[collatz[n, i - 1]]
In[67]:= collatz[n_, i_] := (collatz[n, i] = 3 collatz[n, i - 1] + 1) /;
OddQ[collatz[n, i - 1]]

```

Here is the fifth iterate of the Collatz sequence for 27.

```
In[68]:= collatz[27, 5]
```

```
Out[68]= 31
```

Here is the Collatz sequence for 27. This sequence takes a while to settle down to the cycle 4, 2, 1.

```
In[69]:= Table[collatz[27, i], {i, 0, 114}]
```

```
Out[69]= {27, 82, 41, 124, 62, 31, 94, 47, 142, 71, 214, 107, 322, 161, 484, 242, 121, 364,
182, 91, 274, 137, 412, 206, 103, 310, 155, 466, 233, 700, 350, 175, 526, 263, 790,
395, 1186, 593, 1780, 890, 445, 1336, 668, 334, 167, 502, 251, 754, 377, 1132,
566, 283, 850, 425, 1276, 638, 319, 958, 479, 1438, 719, 2158, 1079, 3238, 1619,
4858, 2429, 7288, 3644, 1822, 911, 2734, 1367, 4102, 2051, 6154, 3077, 9232,
4616, 2308, 1154, 577, 1732, 866, 433, 1300, 650, 325, 976, 488, 244, 122, 61,
184, 92, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1}
```

## 5.4 Loops and flow control: exercises

1. Create a function `UpperTriangularMatrix[n]` that generates an  $n \times n$  matrix with ones on and above the diagonal and zeros below the diagonal. Create an alternative rule that defaults to the value one for the upper values, but allows the user to specify a nondefault upper value.

```
In[1]:= UpperTriangularMatrix[3] // MatrixForm
```

```
Out[1]//MatrixForm= 
$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

```

```
In[2]:= UpperTriangularMatrix[3, 5] // MatrixForm
```

```
Out[2]//MatrixForm= 
$$\begin{pmatrix} 0 & 5 & 5 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

```

2. Write a function `sign[x]` which, when applied to an integer  $x$ , returns  $-1$ ,  $0$ , or  $1$  if  $x$  is less than, equal to, or greater than zero, respectively. Write it in four different ways: using three clauses, using a single clause with `If`, using a single clause with `Which`, and using `Piecewise`.
3. Use `If` to define a function that, given a list of numbers, doubles all the positive numbers but leaves the negative numbers unchanged.
4. The definition of the absolute value function in this section does not handle complex numbers well:

```
In[3]:= abs[3 + 4 I]
```

```
GreaterEqual::nord : Invalid comparison with 3 + 4 I attempted. >>
```

```
Less::nord : Invalid comparison with 3 + 4 I attempted. >>
```

```
Out[3]= abs[3 + 4 i]
```

Rewrite `abs` to include a specific rule for the case where its argument is complex.

5. One of the fastest methods for computing Fibonacci numbers (Section 5.3) involves iterating multiplication of the matrix  $\begin{Bmatrix} 0 & 1 \\ 1 & 1 \end{Bmatrix}$  and pulling off the appropriate part. For example, the last element in the output of `mat9` is the tenth Fibonacci number.

```
In[4]:= mat = {{0, 1}, {1, 1}};
        MatrixPower[mat, 9]

Out[5]= {{21, 34}, {34, 55}}

In[6]:= Fibonacci[10]

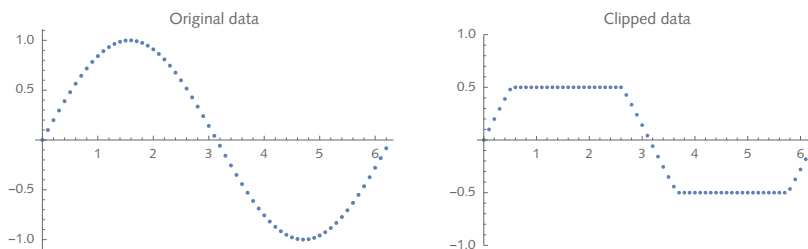
Out[6]= 55
```

Without using `MatrixPower`, create a function `FibMat[n]` that iterates the matrix multiplication and then pulls off the correct element from the resulting matrix to give the  $n$ th Fibonacci number. Check the speed of your implementation against both `MatrixPower` and the built-in `Fibonacci`.

6. Using an `If` control structure, create a function `median[lis]` that computes the median (the middle value) of a one-dimensional list. You will need to consider the case when the length of the list is odd and the case when it is even. In the latter case, the median is given by the average of the middle two elements of the sorted list.
7. Given a set of data representing a sine wave, perform a clipping operation where values greater than 0.5 are clipped to 0.5, values less than -0.5 are clipped to -0.5, and all other values are left unchanged (Figure 5.3).

```
In[7]:= data = Table[{x, Sin[x]}, {x, 0, 2  $\pi$ , 0.1}];
```

FIGURE 5.3. Discrete data of sine wave together with data clipped at amplitude 0.5.



8. Rewrite the `WhatAmI` function from this section so that it properly deals with expressions such as  $\pi$  and  $e$  that are numerical but not explicit numbers.
9. The bibliography example in Section 3.4 is unable to properly handle a key that has a missing value. For example, the following association has no value for both the `"Issue"` key and the `"Pages"` key:

```

In[8]:= art2 = Association[{ "Author" → "Hathaway, David H.", "Title" → "The solar cycle",
    "Journal" → "Living Reviews in Solar Physics", "Year" → 2010,
    "Volume" → 7, "Issue" → "", "Pages" → "",
    "Url" → "http://dx.doi.org/10.12942/lrsp-2010-1" }];

In[9]:= art2["Issue"]

Out[9]=

```

Suppose you were interested in creating a formatted bibliographic reference that displayed volume 7, issue 4 as 7(4). But if the issue value is missing, it should display the volume value only. Create a function that takes the volume and issue values and displays the correct information regardless of whether or not the issue number is present.

```

In[10]:= volIss[art2]

Out[10]= 7

```

10. In Exercise 11, Section 5.3 we introduced Collatz numbers using recursion. Write a procedural implementation, `CollatzSequence[n]`, that produces the Collatz sequence for any positive integer  $n$ . Consider using `NestWhileList`. Here is the Collatz sequence for initial value 22:

```

In[11]:= CollatzSequence[22]

Out[11]= {22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1}

```

11. Write a version of the `Manipulate` example visualizing interpolation order that used `Which` to instead use `Switch`.
12. Compute the square root of two using `Nest` (see Section 5.2) and compare with the versions in this section using a `Do` loop.
13. `Do` is closely related to `Table`, the main difference being that `Do` does not return any value, whereas `Table` does. Use `Table` instead of `Do` to rewrite one of the `findRoot` functions given in this section. Compare the efficiency of the two approaches.
14. Compute Fibonacci numbers iteratively. The first few values in the Fibonacci sequence are 1, 1, 2, 3, 5, 8, 13, ..., where, after the first two 1s, each number is the sum of the previous two numbers in the sequence. You will need two variables, say `this` and `prev`, giving the two most recent Fibonacci numbers, so that after the  $i$ th iteration, `this` and `prev` have the values  $F_i$  and  $F_{i-1}$ , respectively.
15. As mentioned in the discussion of Newton's method for root finding, one type of difficulty that can arise occurs when the derivative of the function in question is either difficult or impossible to compute. As a very simple example, consider the function  $|x + 3|$ , which has a root at  $x = -3$ . Both the built-in function `FindRoot` and our user-defined `findRoot` will fail with this function, since a symbolic derivative cannot be computed.

```

In[12]:= D[Abs[x + 3], x]

Out[12]= Abs'[3 + x]

```

One way around such problems is to use a numerical derivative (as opposed to an analytic derivative). The secant method approximates  $f'(x_k)$  using the difference quotient:

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

Implement a version of `findRoot` using the secant method by creating a rule that takes two initial values: `findRoot [f, {var, a, b}]`.

16. Using a `While` loop, write a function `gcd [m, n]` that computes the greatest common divisor (gcd) of  $m$  and  $n$ . The Euclidean algorithm for computing the gcd of two positive integers  $m$  and  $n$ , sets  $m = n$  and  $n = m \bmod n$ . It iterates this process until  $n = 0$ , at which point the gcd of  $m$  and  $n$  is left in the value of  $m$ .

17. Write the function `gcd [m, n]` implementing the Euclidean algorithm using an `If` function.

18. A permutation of the elements of a list is a reordering of the elements such that the original list and the reordered list are in one-to-one correspondence. For example, the permutations of the list  $\{a, b, c\}$  are  $\{\{a, b, c\}, \{a, c, b\}, \{b, a, c\}, \{b, c, a\}, \{c, a, b\}, \{c, b, a\}\}$ .

One way to create a permutation of the elements of a list `lis` is to start by randomly selecting one element from `lis` and putting it in a temporary list, `lis2` say. Then select one element from the complement of `lis` and `lis2` and repeat the process. Using a `Do` loop, create a function `randomPermutation` that implements this procedure.

19. Create a program `InversePermutation [p]` that takes a list  $p$ , that is a permutation of the numbers one through  $n$ , and returns the inverse permutation  $ip$ . The inverse permutation  $ip$  is such that  $p[ip[i]] = ip[p[i]] = i$ . Check you answer against the built-in `Ordering` function. See [Sedgewick and Wayne \(2007\)](#) for a discussion of inverse permutations.

20. Create a procedural definition for each of the following functions. For each function, create a definition using a `Do` loop and another using `Table`.

For example, the following function first creates an array consisting of zeros of the same dimension as `mat`. Then inside the `Do` loop it assigns the element in position  $\{j, k\}$  in `mat` to position  $\{k, j\}$  in `matA`, effectively performing a transpose operation. Finally, it returns `matA`, since the `Do` loop itself does not return a value.

```
In[13]:= transposeDo [mat_] :=
Module[{matA, rows = Length[mat], cols = Length[mat[[1]]], j, k},
  matA = ConstantArray[0, {rows, cols}];
  Do[matA[[j, k]] = mat[[k, j],
    {j, 1, rows},
    {k, 1, cols}];
  matA]
```

```
In[14]:= mat1 = {{a, b, c}, {d, e, f}, {g, h, i}};
          MatrixForm[mat1]
```

```
Out[15]//MatrixForm=
```

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

```
In[16]:= MatrixForm[transposeDo[mat1]]
```

```
Out[16]//MatrixForm=
```

$$\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

This same computation could be performed with a *structured iteration* using Table.

```
In[17]:= transposeTable[mat_?MatrixQ] := Module[{matA, rows, cols},
          {rows, cols} = Dimensions[mat];
          matA = ConstantArray[0, {rows, cols}];
          Table[matA[[j, k]] = mat[[k, j]], {j, rows}, {k, cols}]
        ]
```

```
In[18]:= transposeTable[mat1] // MatrixForm
```

```
Out[18]//MatrixForm=
```

$$\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

- a. Create the function `reverse[vec]` that reverses the elements in the list `vec`.
  - b. Create a function `rotateRight[lis, n]` that rotates the elements in the list `lis` `n` places to the right.
21. The *digit sum* of a number is given by adding the digits of that number. For example, the digit sum of 7763 is  $7 + 7 + 6 + 3 = 23$ . If you iterate the digit sum until the resulting number has only one digit, this is called the *digit root* of the original number. So the digit root of 7763 is  $7763 \rightarrow 7 + 7 + 6 + 3 = 23 \rightarrow 2 + 3 = 5$ . Create a function to compute the digit root of any positive integer.
  22. Quadrants in the Euclidean plane are traditionally numbered counterclockwise from quadrant I ( $x$  and  $y$  positive) to quadrant IV ( $x$  positive,  $y$  negative) with some convention adopted for points that lie on either of the axes. Use `Piecewise` to define a quadrant function that returns the quadrant value for any point in the plane.
  23. Using a `Do` loop create a function to tally the elements in a list, returning a list of the form  $\{\{elem_1, cnt_1\}, \dots, \{elem_n, cnt_n\}\}$ . Check your result against the built-in `Tally` function.

```
In[19]:= Tally[{a, c, a, a, c, b, a, a, b, c, a, b}]
```

```
Out[19]= {{a, 6}, {c, 3}, {b, 3}}
```



## 5.4 Solutions

1. If, for element  $a_{ij}$ ,  $i$  is bigger than  $j$ , then we are below the diagonal and should insert 0, otherwise insert a 1.

```
In[1]:= UpperTriangularMatrix[{m_, n_}] := Table[If[i ≥ j, 0, 1], {i, m}, {j, n}]
```

A default value can be given for an optional argument that specifies the elements above the diagonal.

```
In[2]:= UpperTriangularMatrix[{m_, n_}, val_: 1] :=  
Table[If[i ≥ j, 0, val], {i, m}, {j, n}]
```

```
In[3]:= UpperTriangularMatrix[{5, 5}, α] // MatrixForm
```

```
Out[3]//MatrixForm=
```

$$\begin{pmatrix} 0 & \alpha & \alpha & \alpha & \alpha \\ 0 & 0 & \alpha & \alpha & \alpha \\ 0 & 0 & 0 & \alpha & \alpha \\ 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

2. Here are the conditional definitions.

```
In[4]:= sign[x_ /; x < 0] := -1  
sign[x_ /; x > 0] := 1  
sign[0] = 0;  
sign[0.0] = 0;
```

Actually the last two rules can be combined using alternatives.

```
In[8]:= sign[0 | 0.0] = 0;  
In[9]:= Map[sign, {-2, 0, 1}]  
Out[9]= {-1, 0, 1}
```

Here is the signum function defined using If.

```
In[10]:= signIf[x_] := If[x < 0, -1, If[x == 0, 0, 1]]  
In[11]:= Map[signIf, {-2, 0, 1}]  
Out[11]= {-1, 0, 1}
```

Here is the signum function defined using Which.

```
In[12]:= signWhich[x_] := Which[x < 0, -1, x > 0, 1, True, 0]  
In[13]:= Map[signWhich, {-2, 0, 1}]  
Out[13]= {-1, 0, 1}
```

Finally, here is the signum function defined using Piecewise.

```
In[14]:= Piecewise[{ {-1, x < 0}, {1, x > 0}, {0, x == 0} }]
```

```
Out[14]= 
$$\begin{cases} -1 & x < 0 \\ 1 & x > 0 \\ 0 & \text{True} \end{cases}$$

```

3. The pure function doubles its argument if it is greater than zero.

```
In[15]:= doublePos[lis_] := Map[If[# > 0, 2 #, #] &, lis]
```

```
In[16]:= doublePos[{4, 0, -3, x}]
```

```
Out[16]= {8, 0, -3, If[x > 0, 2 x, x]}
```

4. The test as the first argument of If on the right-hand side checks to see if  $x$  is an element of the domain of complex numbers and, if it is, then  $\sqrt{\text{re}(x)^2 + \text{im}(x)^2}$  is computed. If  $x$  is not complex, nothing is done, but then the other definitions for abs will be checked.

```
In[17]:= Clear[abs];
```

```
abs[x_] := Sqrt[Re[x]^2 + Im[x]^2] /; x ∈ Complexes;
```

```
abs[x_] := x /; x ≥ 0
```

```
abs[x_] := -x /; x < 0
```

```
In[21]:= abs[3 + 4 I]
```

```
Out[21]= 5
```

```
In[22]:= abs[-3]
```

```
Out[22]= 3
```

The condition itself can appear on the left-hand side of the function definition, as part of the pattern match. Here is a slight variation on the abs definition.

```
In[23]:= Clear[abs]
```

```
abs[x_] := If[x ≥ 0, x, -x]
```

```
abs[x_ /; x ∈ Complexes] := Sqrt[Re[x]^2 + Im[x]^2]
```

```
In[26]:= abs[3 + 4 I]
```

```
Out[26]= 5
```

```
In[27]:= abs[-3]
```

```
Out[27]= 3
```

5. The iteration can be done with a Do loop. First, initialize the matrix tempmat to {{0, 1}, {1, 1}}. Then multiply tempmat by the original matrix and reset tempmat to this new value. Repeat  $n - 2$  times and then pull off the second element in the second row.

```
In[28]:= FibMat[n_] := Module[{tempmat = {{0, 1}, {1, 1}}},
  Do[tempmat = tempmat.{{0, 1}, {1, 1}}, {n - 2}];
  Part[tempmat, 2, 2]
]
```

```
In[29]:= Timing[FibMat[201]]
Out[29]:= {0.004827, 453 973 694 165 307 953 197 296 969 697 410 619 233 826}

In[30]:= Timing[Fibonacci[201]]
Out[30]:= {0.000019, 453 973 694 165 307 953 197 296 969 697 410 619 233 826}
```

6. If the number of elements in the list is odd, then the median is the middle element of the *sorted* list. Divide the length in two and take the next greater integer using *Ceiling* to get the position of the middle element.

```
In[31]:= lis = {5, 7, 2, 13, 1};
          Ceiling[Length[lis]/2]
Out[32]:= 3
```

We want the element in the third position of the sorted list.

```
In[33]:= Part[Sort[lis], Ceiling[Length[lis]/2]]
Out[33]:= 5
```

If the length of the list is even, we take the average of the middle two elements of the sorted list.

```
In[34]:= lis = {65, 2, 78, 5};
In[35]:= len = Length[lis];
          Part[Sort[lis], len/2 ;; len/2 + 1]
Out[36]:= {5, 65}

In[37]:= Mean[%]
Out[37]:= 35
```

Here then is the function using *If* to branch when the length of the list is odd or even. The pattern `lis : {__}` is matched by a list with one or more elements. We name the pattern `lis` so that we can refer to it on the right-hand side of the definition.

```
In[38]:= medianP[lis : {__}] := Module[{len = Length[lis]},
    If[OddQ[len],
      Part[Sort[lis], Ceiling[len/2]],
      Mean@Part[Sort[lis], len/2 ;; len/2 + 1]
    ]]
```

Here are some test data.

```
In[39]:= data0 = RandomInteger[10 000, 100 001];
          dataE = RandomInteger[10 000, 100 000];
```

This compares our function with the built-in *Median* function.

```
In[41]:= medianP[data0] // Timing
```

```
Out[41]:= {0.014395, 4981}
```

```
In[42]:= Median[data0] // Timing
```

```
Out[42]:= {0.01061, 4981}
```

```
In[43]:= medianP[dataE] // Timing
```

```
Out[43]:= {0.010609, 4987}
```

```
In[44]:= Median[dataE] // Timing
```

```
Out[44]:= {0.011003, 4987}
```

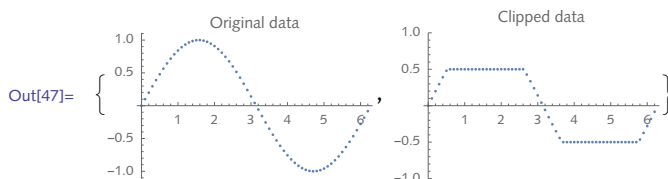
7. First, here is the data we will work with.

```
In[45]:= data = Table[{x, Sin[x]}, {x, 0, 2  $\pi$ , 0.1}];
```

There are several approaches we could use. We will use Piecewise for the conditions as stated in the exercise.

```
In[46]:= clipped = data /. {a_, b_}  $\Rightarrow$  {a, Piecewise[{{0.5, b > 0.5}, {-0.5, b < -0.5}}, b]};
```

```
In[47]:= {
  ListPlot[data, PlotLabel  $\rightarrow$  "Original data"],
  ListPlot[clipped, PlotLabel  $\rightarrow$  "Clipped data", PlotRange  $\rightarrow$  {-1, 1}]
}
```



8. An additional clause is need in the Which expression, one that handles expressions that are numeric but not explicit numbers.

```
In[48]:= WhatAmI[expr_] := Switch[expr,
  _Integer, "I am an integer",
  _Rational, "I am rational",
  _Real, "I am real",
  _Complex, "I am complex",
  _?NumericQ, "I am numeric",
  _, "I am not a number"]
```

```
In[49]:= WhatAmI[ $\pi$ ]
```

```
Out[49]:= I am numeric
```

```
In[50]:= WhatAmI["a string"]
```

```
Out[50]:= I am not a number
```

9. Here is the function to extract and format the volume information. Here we make the volume number format in bold.

```
In[51]:= getVolume[ref_] := Style[ref["Volume"], "TR", FontWeight → "Bold"]
In[52]:= getIssue[ref_] := If[ref["Issue"] == "", "",
    Style[Row[{(" ", ref["Issue"], " ") }], "TR"]]
In[53]:= volIss[ref_] := Row[{getVolume[ref], getIssue[ref]}]
In[54]:= art2 = Association[{"Author" → "Hathaway, David H.", "Title" → "The solar cycle",
    "Journal" → "Living Reviews in Solar Physics", "Year" → 2010,
    "Volume" → 7, "Issue" → "", "Pages" → "",
    "Url" → "http://dx.doi.org/10.12942/lrsp-2010-1"}];
In[55]:= volIss[art2]
Out[55]= 7
```

10. First, define the auxiliary function using conditional statements.

```
In[56]:= collatz[n_] :=  $\frac{n}{2}$  ; EvenQ[n]
In[57]:= collatz[n_] := 3 n + 1 ; OddQ[n]
```

Alternatively, use If.

```
In[58]:= collatz[n_Integer?Positive] := If[EvenQ[n], n / 2, 3 n + 1]
```

Then iterate Collatz, starting with  $n$ , and continue while  $n$  is not equal to 1.

```
In[59]:= CollatzSequence[n_] := NestWhileList[collatz, n, # ≠ 1 &]
```

```
In[60]:= CollatzSequence[17]
```

```
Out[60]= {17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1}
```

11. Here is the version of the Manipulate example using Switch instead of Which.

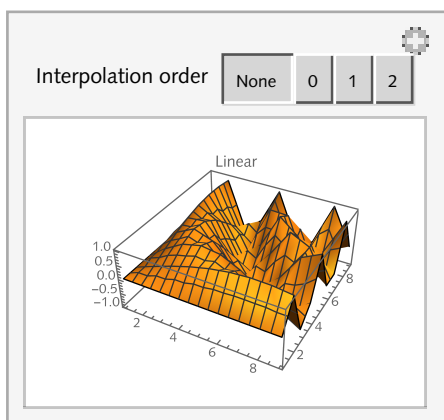
```
In[61]:= data3D = Table[Sin[x y], {x, 0, 4, 0.5}, {y, 0, 4, 0.5}];
```

```

In[62]:= Manipulate[
  ListPlot3D[data3D, InterpolationOrder → order,
    PlotLabel →
      Switch[order,
        "None", "Linear",
        0, "Voronoi cells",
        1, "Baricentric",
        2, "Natural neighbor"]],
  {{order, "None", "Interpolation order"}, {"None", 0, 1, 2}},
  SaveDefinitions → True]

```

Out[62]=



12. To compute the square root of a number  $r$ , iterate the following expression.

```

In[63]:= fun[x_] := x^2 - r;
Simplify[x - fun[x] / fun'[x]]

```

Out[64]=  $\frac{r + x^2}{2x}$

This can be written as a pure function, with a second argument giving the initial guess. Here we iterate ten times, starting with a high-precision initial value, 2.0 to 30-digit precision.

```

In[65]:= nestSqrt[r_, init_] := Nest[
  (r + #^2) / (2 #) &, init, 10]

```

```

In[66]:= nestSqrt[2, N[2, 30]]

```

Out[66]= 1.41421356237309504880168872

13. Here is a first basic attempt to replace the Do loop with Table.

```

In[67]:= f[x_] := x^2 - 2

```

```
In[68]:= a = 2;
```

```
Table[a = N[a -  $\frac{f[a]}{f'[a]}$ ], {10}]
```

```
Out[69]= {1.5, 1.41667, 1.41422, 1.41421, 1.41421,
          1.41421, 1.41421, 1.41421, 1.41421, 1.41421}
```

```
In[70]:= findRoot[fun_Symbol, {var_, init_}, iter_ : 10] := Module[{xi = init},
```

```
Table[xi = N[xi -  $\frac{fun[xi]}{fun'[xi]}$ ], {iter}];
```

```
{var -> xi}]
```

```
In[71]:= findRoot[f, {x, 2}]
```

```
Out[71]= {x -> 1.41421}
```

This runs the iteration only three times.

```
In[72]:= findRoot[f, {x, 2}, 3]
```

```
Out[72]= {x -> 1.41422}
```

14. Note that this version of the Fibonacci function is much more efficient than the simple recursive version given in Section 5.3, and is closer to the version there that uses dynamic programming.

```
In[73]:= Clear[fib]
```

```
In[74]:= fib[n_] := Module[{prev = 0, this = 1, next},
```

```
Do[next = prev + this;
```

```
prev = this;
```

```
this = next,
```

```
{n - 1}];
```

```
this]
```

```
In[75]:= Table[fib[i], {i, 1, 10}]
```

```
Out[75]= {1, 1, 2, 3, 5, 8, 13, 21, 34, 55}
```

Actually, this code can be simplified a bit by using parallel assignments.

```
In[76]:= fib2[n_] := Module[{f1 = 0, f2 = 1},
```

```
Do[{f1, f2} = {f2, f1 + f2},
```

```
{n - 1}];
```

```
f2]
```

```
In[77]:= Table[fib2[i], {i, 1, 10}]
```

```
Out[77]= {1, 1, 2, 3, 5, 8, 13, 21, 34, 55}
```

Both of these implementations are quite fast and avoid the deep recursion of the classical definition.

```
In[78]:= {Timing[fib[100000];], Timing[fib2[100000];]}
Out[78]:= {{0.188238, Null}, {0.132801, Null}}
```

15. Here is an implementation of the secant method, essentially modifying our earlier Newton iteration to use a difference quotient to approximate the derivative. Two values are needed to compute the quotient which is reflected in the updated argument structure.

```
In[79]:= findRoot[f_, {var_, a_, b_}] := Module[{x1 = a, x2 = b, df},
  While[Abs[f[x2]] >  $\frac{1}{10^{10}}$ ,
    df =  $\frac{f[x2] - f[x1]}{x2 - x1}$ ;
    {x1, x2} = {x2, x2 -  $\frac{f[x2]}{df}$ };
    {var → x2}]
In[80]:= f[x_] := Abs[x + 3]
In[81]:= findRoot[f, {x, -3.1, -1.8}]
Out[81]:= {x → -3.}
```

16. This is a direct implementation of the Euclidean algorithm.

```
In[82]:= gcd[m_, n_] := Module[{a = m, b = n, tmpa},
  While[b > 0,
    tmpa = a;
    a = b;
    b = Mod[tmpa, b];
  a]
In[83]:= With[{m = 12782, n = 5531207},
  gcd[m, n]]
Out[83]:= 11
```

You can avoid the need for the temporary variable `tmpa` by performing a parallel assignment as in the following function. In addition, some argument checking insures that  $m$  and  $n$  are integers.

```
In[84]:= gcd[m_Integer, n_Integer] := Module[{a = m, b = n},
  While[b > 0,
    {a, b} = {b, Mod[a, b]};
  a]
In[85]:= With[{m = 12782, n = 5531207},
  gcd[m, n]]
Out[85]:= 11
```



17. Here is the gcd function implemented using an If structure.

```
In[86]:= Clear[gcd]
In[87]:= gcd[m_Integer, n_Integer] := If[m > 0, gcd[Mod[n, m], m], gcd[m, n] = n]
In[88]:= With[{m = 12782, n = 5531207},
  gcd[m, n]]
Out[88]= 11
```

18. To build this function up step-by-step, start with a small list of ten elements.

```
In[89]:= lis = Range[10]
Out[89]= {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
```

The idea is to choose a position within the list at random and remove the element in that position and put it into a new list lis2.

```
In[90]:= x = RandomChoice[lis]
Out[90]= 4
In[91]:= lis2 = {};
  lis2 = Append[lis2, x]
Out[92]= {4}
```

We then repeat the above process on the remaining elements of the list. Note that lis is assigned the value of this new list, thus overwriting the previous value.

```
In[93]:= lis = Complement[lis, {x}]
Out[93]= {1, 2, 3, 5, 6, 7, 8, 9, 10}
In[94]:= x = RandomChoice[lis]
  lis2 = Append[lis2, x]
  lis = Complement[lis, lis2]
Out[94]= 8
Out[95]= {4, 8}
Out[96]= {1, 2, 3, 5, 6, 7, 9, 10}
```

In this example we know explicitly how many iterations to perform in our Do loop:  $n$  iterations, where  $n$  is the length of the list, lis.

```
In[97]:= Clear[lis, lis2, x];
```

Now we just put the pieces of the previous computations together in one input.

```
In[98]:= lis = Range[10];
lis2 = {};
Do[
  x = RandomChoice[lis];
  lis2 = Append[lis2, x];
  lis = Complement[lis, lis2],
  {i, 1, Length[lis]}]
```

When we are done, the result is left in the new list `lis2`.

```
In[101]:= lis2
Out[101]= {2, 4, 7, 6, 9, 1, 5, 3, 8, 10}
```

```
In[102]:= lis = Range[10];
lis2 = {};
Do[
  x = RandomChoice[lis];
  lis2 = Append[lis2, x];
  lis = Complement[lis, lis2],
  {i, 1, Length[lis]}]
```

Here then is our function `randomPermutation` that takes a list as an argument and generates a random permutation of that list's elements.

```
In[105]:= randomPermutation[arg_List] := Module[{lis = arg, x, lis2 = {}},
  Do[
    x = RandomChoice[lis];
    lis2 = Append[lis2, x];
    lis = Complement[lis, lis2],
    {i, 1, Length[arg]}];
  lis2]
```

Here is a permutation of the list consisting of the first 20 integers.

```
In[106]:= randomPermutation[Range[20]]
Out[106]= {11, 17, 16, 8, 2, 9, 14, 5, 19, 7, 1, 12, 4, 10, 20, 13, 6, 18, 15, 3}
```

And here is a random permutation of the lowercase letters of the English alphabet.

```
In[107]:= alphabet = CharacterRange["a", "z"]
Out[107]= {a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z}

In[108]:= randomPermutation[alphabet]
Out[108]= {b, j, s, k, q, v, u, n, h, m, e, d, c, p, x, z, f, y, r, o, i, t, a, g, w, l}
```

This functionality is built into *Mathematica* via the `RandomSample` function.

```

In[109]:= RandomSample[CharacterRange["a", "z"]]
Out[109]= {w, z, t, y, q, g, f, h, e, k, u, p, r, v, i, n, c, x, l, b, a, o, d, s, j, m}

In[110]:= Clear[x, lis, lis2, alphabet]

```

19. Start with a random permutation on the first ten integers.

```

In[111]:= Clear[p, i, ip];
In[112]:= p = RandomSample[Range[10]]
Out[112]= {1, 8, 3, 2, 6, 10, 5, 7, 9, 4}

```

Initialize the inverse permutation to a list of the same length as `p` but filled with zeros to start.

```

In[113]:= ip = Table[0, {i, Length[p]}]
Out[113]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

In[114]:= For[i = 1, i ≤ Length[p], i++, ip[[p[[i]]]] = i];
ip
Out[115]= {1, 4, 3, 10, 7, 5, 8, 2, 9, 6}

In[116]:= p[[ip]] == ip[[p]]
Out[116]= True

```

The built-in `Ordering` function gives essentially the same result.

```

In[117]:= Ordering[p]
Out[117]= {1, 4, 3, 10, 7, 5, 8, 2, 9, 6}

```

20. Each solution mirrors that of the transpose example in the exercise.

a. Create a list `vecA` of zeros, then use a `Do` loop to set `vecA[[i]]` to `vec[[n - i]]`, where `n` is the length of `vec`.

```

In[118]:= Clear[reverse, a, b, c, d, e]
In[119]:= reverse[vec_] := Module[{vecA, n = Length[vec]},
  vecA = ConstantArray[0, {n}];
  Do[vecA[[i]] = vec[[n - i + 1]],
    {i, 1, n}];
  vecA]
In[120]:= reverse[{a, b, c, d, e}]
Out[120]= {e, d, c, b, a}

In[121]:= reverseStruc[vec_] := Module[{vecA, n = Length[vec]},
  vecA = ConstantArray[0, {n}];
  Table[vecA[[i]] = vec[[n - i + 1]], {i, n}]
]

```

```
In[122]:= reverseStruc[{a, b, c, d, e}]
```

```
Out[122]= {e, d, c, b, a}
```

- b. The key to this problem is to use the Mod operator to compute the target address for any item from `vec`. That is, the element `vec[i]` must move to, roughly speaking, position  $n + i \bmod \text{Length}[\text{vec}]$ . The “roughly speaking” is due to the fact that the Mod operator returns values in the range 0 to  $\text{Length}[\text{vec}] - 1$ , whereas vectors are indexed by values  $i$  up to  $\text{Length}[\text{vec}]$ . This causes a little trickiness in this problem.

```
In[123]:= rotateRight[vec_, n_] := Module[{vecA, len = Length[vec]},
    vecA = ConstantArray[0, {len}];
    Do[vecA[[1 + Mod[n + i - 1, len]]] = vec[[i]], {i, 1, len}];
    vecA]
```

```
In[124]:= rotateRight[{a, b, c, d, e}, 2]
```

```
Out[124]= {d, e, a, b, c}
```

```
In[125]:= rotateRightStruc[vec_, n_] := Module[{vecA, len = Length[vec]},
    vecA = ConstantArray[0, {len}];
    Table[vecA[[1 + Mod[n + i - 1, len]]] = vec[[i]], {i, len}];
    vecA
]
```

```
In[126]:= rotateRightStruc[{a, b, c, d, e}, 3]
```

```
Out[126]= {c, d, e, a, b}
```

21. Given an integer, this totals the list of its digits.

```
In[127]:= Total[IntegerDigits[7763]]
```

```
Out[127]= 23
```

This can be accomplished without iteration as follows:

```
In[128]:= digitRoot[n_Integer?Positive] := If[Mod[n, 9] == 0, 9, Mod[n, 9]]
```

```
In[129]:= digitRoot[7763]
```

```
Out[129]= 5
```

Looking ahead to Chapter 6 where localization is discussed, you could also accomplish this with an iteration using a While loop.

```
In[130]:= digitRoot2[n_Integer?Positive] := Module[{locn = n, lis},
    While[
        Length[lis = IntegerDigits@locn] > 1,
        locn = Total[lis];
    locn]
```

```
In[131]:= digitRoot2[7763]
```

```
Out[131]= 5
```

22. This is a direct implementation using Piecewise.

```
In[132]:= Piecewise[{{0, x == 0 && y == 0}, {-1, y == 0}, {-2, x == 0}, {1, x > 0 && y > 0},
  {2, x < 0 && y > 0}, {3, x < 0 && y < 0}}, 4]
```

```
Out[132]= {
  0    x == 0 && y == 0
 -1    y == 0
 -2    x == 0
  1    x > 0 && y > 0
  2    x < 0 && y > 0
  3    x < 0 && y < 0
  4    True
}
```

```
In[133]:= quadrantPw[{x_, y_}] :=
  Piecewise[{{0, x == 0 && y == 0}, {-1, y == 0}, {-2, x == 0}, {1, x > 0 && y > 0},
    {2, x < 0 && y > 0}, {3, x < 0 && y < 0}}, 4]
```

```
In[134]:= Map[quadrantPw, {{0, 0}, {4, 0}, {0, 1.3}, {2, 4}, {-2, 4}, {-2, -4},
  {2, -4}, {2, 0}, {3, -4}}]
```

```
Out[134]= {0, -1, -2, 1, 2, 3, 4, -1, 4}
```

23. Union[lis] will give a list of the unique elements in lis (this is ulist below). Also, we need a temporary list of the counts initialized to zero (cnts). Then, starting with the first element in the input list (i = 1), check to see if it equals the first element in ulist, then the second, and so on, incrementing by one each time they are equal. When the loops are done, transpose the list of unique elements with the corresponding counts. Here is the function:

```
In[135]:= Clear[tally]
```

```
In[136]:= tally[lis_List] := Module[{ulist = Union[lis], cnts},
  cnts = Table[0, {Length[ulist]}];
  Do[
    Do[If[lis[[i]] == ulist[[j]], ++cnts[[j]]], {j, 1, Length[ulist]}],
    {i, 1, Length[lis]}];
  Transpose[{ulist, cnts}]
]
```

```
In[137]:= lis = RandomChoice[{a, b, c}, {20}]
```

```
Out[137]= {c, c, a, c, a, b, c, a, a, a, a, b, b, b, a, a, a, a, b, b}
```

```
In[138]:= tally[lis]
```

```
Out[138]= {{a, 10}, {b, 6}, {c, 4}}
```

Check against the built-in function.

```
In[139]:= Tally[lis] === tally[lis]
```

```
Out[139]= False
```

Here is another implementation. Here we are creating rules for each unique element (counter[ $\#$ ]) and then incrementing the values for those rules each time the counter comes across an element.

```
In[140]:= tally2[lis_] := Module[{ulist = Union[lis], counter},
    counter[_] = 0;
    Map[counter[ $\#$ ] ++ &, lis] ;
    Map[{ $\#$ , counter[ $\#$ ]} &, ulist]
]
```

```
In[141]:= tally2[lis]
```

```
Out[141]:= {{a, 10}, {b, 6}, {c, 4}}
```

Check the speed of these two functions.

```
In[142]:= data = RandomInteger[{0, 9}, {50000}];
```

```
In[143]:= Timing[tally[data]]
```

```
Out[143]:= {0.975904, {{0, 5078}, {1, 5041}, {2, 4928}, {3, 4973},
    {4, 4969}, {5, 5037}, {6, 4875}, {7, 5040}, {8, 5014}, {9, 5045}}}
```

```
In[144]:= Timing[tally2[data]]
```

```
Out[144]:= {0.180405, {{0, 5078}, {1, 5041}, {2, 4928}, {3, 4973},
    {4, 4969}, {5, 5037}, {6, 4875}, {7, 5040}, {8, 5014}, {9, 5045}}}
```

```
In[145]:= Timing[Tally[data]]
```

```
Out[145]:= {0.000122, {{2, 4928}, {8, 5014}, {5, 5037}, {4, 4969},
    {3, 4973}, {9, 5045}, {6, 4875}, {7, 5040}, {0, 5078}, {1, 5041}}}
```

## 5.5 Pure functions: exercises

1. Using pure functions, compute  $\sin(x)/x$  for each  $x$  in the list  $\{0, \pi/6, \pi/4, \pi/3, \pi/2\}$ . Check your answer against the built-in `Sinc` function.
2. Create a pure function to select all integers between one and one thousand that are square numbers; that is, numbers that are some integer squared.
3. In Exercise II, Section 5.1 you were asked to create a function to compute the diameter of a set of points in  $n$ -dimensional space. Modify that solution by instead using the `Norm` function and pure functions to find the diameter.
4. Rewrite the code from Section 5.2 for finding the next prime after a given integer so that it uses pure functions instead of relying upon the auxiliary definition `addOne` and the built-in function `CompositeQ`.
5. Create a function `RepUnit[n]` that generates integers of length  $n$  consisting entirely of ones. For example, `RepUnit[7]` should produce `1111111`.

6. Rewrite the example from Section 5.2 in which `NestList` was used to perform several rotations of a triangle to instead use pure functions and dispense with the auxiliary function `rotation` used to specify the rotation angles.
7. Given a set of numerical data, extract all those data points that are within one standard deviation of the mean of the data.

```
In[1]:= data = RandomVariate[NormalDistribution[0, 1], {2500}];
```

8. Using the built-in `Fold` function, write a function `fromDigits[lis, b]` that accepts a list of digits in any base  $b$  (less than 20) and converts it to a base 10 number. For example,  $1101_2$  is 13 in base 10, so your function should handle this as follows:

```
In[2]:= fromDigits[{1, 1, 0, 1}, 2]
```

```
Out[2]= 13
```

Check your solution against the built-in `FromDigits` function.

9. Write a pure function that moves a random walker from one location on a square lattice to one of the four adjoining locations with equal probability (a two-dimensional lattice walk). For example, starting at  $\{0, 0\}$ , the function should return  $\{0, 1\}$ ,  $\{0, -1\}$ ,  $\{1, 0\}$ , or  $\{-1, 0\}$  with equal likelihood. Use this pure function with `NestList` to generate the list of step locations for an  $n$ -step random walk starting at the origin.
10. Modify the `findRoot` function from this section to allow for equations of the form  $x^2 == 2$  or  $\text{Cos}[x] == 0$  as the first argument.
11. Create a function `findRootList` that is based on `findRoot` and returns all the intermediate values that are computed by the Newton iteration.
12. A naive approach to polynomial arithmetic would require three additions and six multiplications to carry out the arithmetic in the expression  $ax^3 + bx^2 + cx + d$ . Using Horner's method for fast polynomial multiplication, this expression can be represented as  $d + x(c + x(b + ax))$ , where there are now half as many multiplications. You can see this using the `MultiplyCount` function developed in Exercise 7 of Section 4.2.

```
In[3]:= MultiplyCount[a x^3 + b x^2 + c x + d]
```

```
Out[3]= 6
```

```
In[4]:= MultiplyCount[d + x (c + x (b + a x))]
```

```
Out[4]= 3
```

In general, the number of multiplications in an  $n$ -degree polynomial in traditional form is given by:

```
In[5]:= Binomial[n + 1, 2]
```

```
Out[5]=  $\frac{1}{2} n (1 + n)$ 
```

This, of course, grows quadratically with  $n$ , whereas Horner's method grows linearly. Create a function `HornerPolynomial [lis, var]` that gives a representation of a polynomial in Horner form. Here is some sample output that your function should generate:

```
In[6]:= HornerPolynomial[{a, b, c, d}, x]
Out[6]= d + x (c + x (b + a x))

In[7]:= Expand[%]
Out[7]= d + c x + b x^2 + a x^3
```

13. Find all words in the dictionary that start with the letter *q* and are of length five. The following gets all the words in the dictionary that comes with *Mathematica* and displays a random sample:

```
In[8]:= words = DictionaryLookup[];
RandomSample[words, 18]

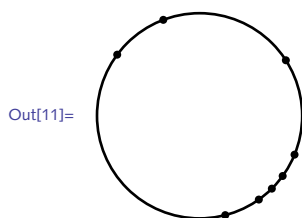
Out[9]= {dare, condole, forums, attributive, impassible,
minefields, skittish, thereabout, jugful, fishiest, canter,
choral, Nanjing, toupee, solemnner, Marlboro, Whigs, healthier}
```

14. Given a list of angles between zero and  $2\pi$ , map them to points of the form  $\{x, y\}$  on the unit circle.

```
In[10]:= angles = RandomReal[{0, 2 Pi}, {8}]
Out[10]= {5.8948, 2.50319, 1.93239, 5.65609, 0.571937, 5.32786, 4.96526, 5.49373}
```

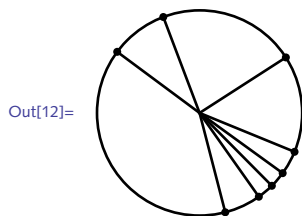
Once created, display the points graphically using code similar to the following:

```
In[11]:= Graphics[{Circle[], Point[pts]}]
```



Finally, create a set of lines from the origin to each of the points on the circle and display using code similar to the following:

```
In[12]:= Graphics[{Circle[], Point[pts], Line[lines]}]
```





15. Use `FoldList` to compute an exponential moving average of a list  $\{x_1, x_2, x_3\}$ . You can check your result against the built-in `ExponentialMovingAverage`.

```
In[13]:= ExponentialMovingAverage[{x1, x2, x3}, α]
```

```
Out[13]= {x1, -(-1 + α) x1 + α x2, -(-1 + α) (-(-1 + α) x1 + α x2) + α x3}
```

16. A well-known programming exercise in many languages is to generate Hamming numbers, sometimes referred to as *regular numbers*. These are numbers that divide powers of 60 (the choice of that number goes back to the Babylonians, who used 60 as a number base). Generate a sorted sequence of all Hamming numbers less than 1000. The key observation is that these numbers have only 2, 3, and 5 as prime factors.
17. In the text, we developed `FunctionsWithAttributes` to find all built-in functions with a particular attribute. Create a new function to find all built-in functions with a given option. For example:

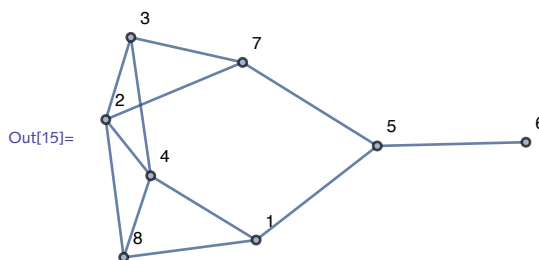
```
In[14]:= FunctionsWithOption[StepMonitor]
```

```
Out[14]= {FindArgMax, FindArgMin, FindFit, FindMaximum, FindMaxValue,
  FindMinimum, FindMinValue, FindRoot, NArgMax, NArgMin, NDSolve,
  NDSolveValue, NMaximize, NMaxValue, NMinimize, NMinValue,
  NonlinearModelFit, NRroots, ParametricNDSolve, ParametricNDSolveValue}
```

Several changes from `FunctionsWithAttributes` will be necessary: `Options` does not take a string as an argument; and options are often given as nested lists of rules, requiring mapping at the appropriate level. You might also want to delete all functions beginning with `$`, such as `$Context`, to help speed the computation.

18. Graphs that are not too dense are often represented using adjacency structures which consist of a list for each vertex  $v_i$  that includes those other vertices that  $v_i$  is connected to. Create an adjacency structure for any graph, directed or undirected. For example, consider the graph `gr` below.

```
In[15]:= gr = RandomGraph[{8, 12}, VertexLabels -> "Name"]
```



Start by creating an adjacency list for any given vertex; that is, a list of those vertices to which the given vertex is connected. The adjacency list for vertex 4 in the above graph would be  $\{1, 8, 2, 3\}$ .

The adjacency structure is then the list of adjacency lists for every vertex in that graph. It is common to prepend each adjacency list with its vertex; typically the adjacency structure takes

the following form where this syntax indicates that vertex 1 is connected to vertices 4, 5, and 8; vertex 2 is connected to vertices 3, 4, 7, and 8; and so on.

$$\{\{1, \{4, 5, 8\}\}, \{2, \{3, 4, 7, 8\}\}, \{3, \{2, 4, 7\}\}, \{4, \{1, 2, 3, 8\}\}, \\ \{5, \{1, 6, 7\}\}, \{6, \{5\}\}, \{7, \{2, 3, 5\}\}, \{8, \{1, 2, 4\}\}\}$$

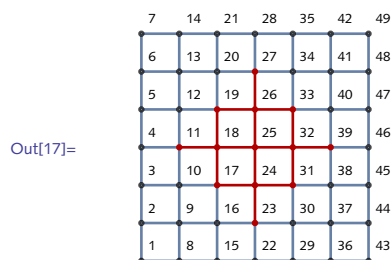
19. One common use of graphs is to analyze the relationships of the objects under study. For example, if you were interested in finding all objects a certain distance from a given object, you could use `NeighborhoodGraph`.

Here is a grid graph on which we can start prototyping:

```
In[16]:= gr = GridGraph[ {7, 7}, VertexLabels -> "Name" ];
```

`NeighborhoodGraph` gives all those vertices up to distance two from vertex 25.

```
In[17]:= HighlightGraph[gr, NeighborhoodGraph[gr, 25, 2]]
```



But suppose you were interested in all those vertices *precisely* distance two from vertex 25.

Implement this using two different approaches: one in which you use `NeighborhoodGraph` and another using a two-argument form of `VertexList` in which the second argument is a pattern.

20. Create a “composition” using the digits of  $\pi$  to represent notes on the C scale, where a digit  $n$  is interpreted as a note  $n$  semitones from middle C. For example, the first few digits 1, 4, 1, 5 would give the notes one, four, one, and five semitones from middle C.

You can generate tones using `Sound` and `SoundNote`. For example, this emits a tone five semitones above middle C of duration one second:

```
In[18]:= Sound[SoundNote[5, 1]] // EmitSound
```

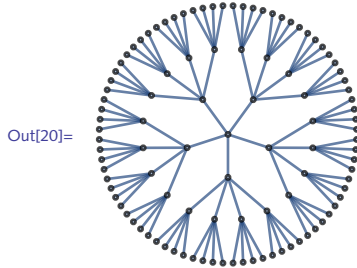
And this emits the same tone but using a midi instrument instead of the default:

```
In[19]:= Sound[SoundNote[5, 1, "Vibraphone"]] // EmitSound
```

21. A matrix is *nilpotent* if it is a square matrix all of whose eigenvalues are zero. Alternatively, a square matrix  $A$  is nilpotent if  $A^n$  is the zero matrix for some positive integer  $n$ . Create a predicate `NilpotentMatrixQ[mat]` that returns `True` if `mat` is nilpotent. Check your function by verifying that any directed acyclic graph (DAG) has a nilpotent adjacency matrix. A list of

acyclic graphs is available via `GraphData["Acyclic"]`. See McKay et al. (2004) for information about acyclic graphs and the eigenvalues of their adjacency matrices.

```
In[20]:= gr = GraphData[{"CayleyTree", {5, 3}}]
```



```
In[21]:= AcyclicGraphQ[gr]
```

```
Out[21]= True
```

## 5.5 Solutions

1. The pure function is `Sin[#] / # &`.

```
In[1]:= Map[Sin[#] / # &, {0, π/6, π/4, π/3, π/2}]
```

Power::infy : Infinite expression  $\frac{1}{0}$  encountered. >>

Infinity::indet : Indeterminate expression 0 ComplexInfinity encountered. >>

```
Out[1]= {Indeterminate,  $\frac{3}{\pi}$ ,  $\frac{2\sqrt{2}}{\pi}$ ,  $\frac{3\sqrt{3}}{2\pi}$ ,  $\frac{2}{\pi}$ }
```

Since `Sinc` is listable, it automatically maps across lists.

```
In[2]:= Sinc[{0, π/6, π/4, π/3, π/2}]
```

```
Out[2]= {1,  $\frac{3}{\pi}$ ,  $\frac{2\sqrt{2}}{\pi}$ ,  $\frac{3\sqrt{3}}{2\pi}$ ,  $\frac{2}{\pi}$ }
```

Use a piecewise function to properly deal with the point at zero.

```
In[3]:= Map[Piecewise[{{1, # == 0}, {Sin[#] / #, # != 0}}] &, {0, π/6, π/4, π/3, π/2}]
```

```
Out[3]= {1,  $\frac{3}{\pi}$ ,  $\frac{2\sqrt{2}}{\pi}$ ,  $\frac{3\sqrt{3}}{2\pi}$ ,  $\frac{2}{\pi}$ }
```

2. One test for a number to be a square number is that its square root is an integer.

```
In[4]:= Select[Range[1000], IntegerQ[Sqrt[#]] &]
```

```
Out[4]= {1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289,
324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900, 961}
```

3. To compute the distance between two points, use either `EuclideanDistance` or `Norm`.

```
In[5]:= pts = RandomReal[1, {4, 2}]
Out[5]= {{0.578624, 0.852785}, {0.917576, 0.509619},
          {0.187069, 0.755342}, {0.879409, 0.277697}}

In[6]:= Norm[pts[[1]] - pts[[2]]]
Out[6]= 0.482339

In[7]:= EuclideanDistance[pts[[1]], pts[[2]]]
Out[7]= 0.482339
```

Now we need the distance between every pair of points. So we first create the set of pairs.

```
In[8]:= pairs = Subsets[pts, {2}]
Out[8]= {{ {0.578624, 0.852785}, {0.917576, 0.509619} },
          { {0.578624, 0.852785}, {0.187069, 0.755342} },
          { {0.578624, 0.852785}, {0.879409, 0.277697} },
          { {0.917576, 0.509619}, {0.187069, 0.755342} },
          { {0.917576, 0.509619}, {0.879409, 0.277697} },
          { {0.187069, 0.755342}, {0.879409, 0.277697} } }
```

Then we compute the distance between each pair and take the `Max`.

```
In[9]:= Apply[Norm[#1 - #2] &, pairs, {1}]
Out[9]= {0.482339, 0.403498, 0.648998, 0.770727, 0.235042, 0.841118}

In[10]:= Max[%]
Out[10]= 0.841118
```

Or, use `Outer` on the set of points directly, but note the need to get the level correct.

```
In[11]:= Max@Outer[Norm[#1 - #2] &, pts, pts, 1]
Out[11]= 0.841118
```

Now put it all together using a pure function in place of the distance function. The `diameter` function operates on lists of pairs of numbers, so we need to specify them in our pure function as `#1` and `#2`.

```
In[12]:= diameter[lis_] := Max[Apply[Norm[#1 - #2] &, Subsets[lis, {2}], {1}]]
In[13]:= diameter[pts]
Out[13]= 0.841118
```

`EuclideanDistance` is a bit faster here, but for large data sets, the difference is more pronounced.

```

In[14]:= pts = RandomReal[1, {1500, 2}];
         diameter[pts] // Timing
Out[15]= {3.29299, 1.40019}

In[16]:= Max[Apply[EuclideanDistance, Subsets[pts, {2}], {1}]] // Timing
Out[16]= {1.1718, 1.40019}

```

4. Pure functions are needed to replace both addOne and CompositeQ:

```

In[17]:= nextPrime[n_Integer /; n > 1] := NestWhile[# + 1 &, n, Not[PrimeQ[#]] &]

```

Here is a quick check for correctness.

```

In[18]:= nextPrime[2123] == NextPrime[2123]
Out[18]= True

```

Compare timing with the built-in function.

```

In[19]:= Timing[nextPrime[22500]];
Out[19]= {0.217708, Null}

In[20]:= Timing[NextPrime[22500]];
Out[20]= {0.207308, Null}

```

5. This function is ideally written as an iteration.

```

In[21]:= RepUnit[n_] := Nest[(10 # + 1) &, 1, n - 1]
In[22]:= RepUnit[7]
Out[22]= 1 111 111

In[23]:= Map[RepUnit[#] &, Range[12]]
Out[23]= {1, 11, 111, 1111, 11 111, 111 111, 1111 111, 1111 111, 11 111 111,
          111 111 111, 1111 111 111, 1111 111 111, 111 111 111 111}

```

This can also be done in a functional style by first creating a list of the appropriate number of ones using ConstantArray, then converting that to an integer with FromDigits.

```

In[24]:= RepUnit[n_] := FromDigits[ConstantArray[1, {n}]]
In[25]:= RepUnit[11]
Out[25]= 11 111 111 111

```

6. First, here is the triangle.

```

In[26]:= vertices = {{0, 0}, {1, 0}, {1/2,  $\sqrt{3}/2$ }};

```

```
In[27]:= tri = Triangle[vertices];
Graphics[{LightBlue, EdgeForm[Gray], tri}]
```

Out[28]=



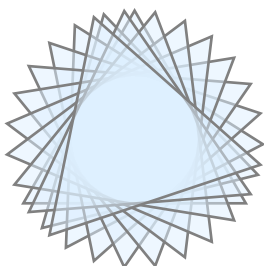
Here is the rotation function as defined in Section 5.2:

```
In[29]:= rotation[gr_] := Rotate[gr,  $\pi/13$ , RegionCentroid[tri]]
```

To turn that into a pure function, replace the argument gr with the slot into which each successive triangle will go.

```
In[30]:= Graphics[{
  LightBlue, Opacity[.5], EdgeForm[Gray],
  NestList[Rotate[#,  $\pi/13$ , RegionCentroid[tri]] &, tri, 10]
}]
```

Out[30]=



7. Here are some sample data taken from a normal distribution.

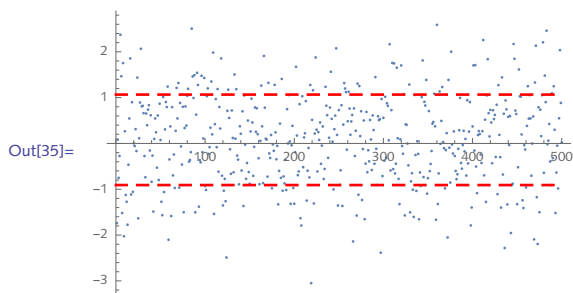
```
In[31]:= data = RandomVariate[NormalDistribution[0, 1], {500}];
```

Quickly visualize the data together with dashed lines drawn one standard deviation from the mean.

```

In[32]:=  $\mu$  = Mean[data];
 $\sigma$  = StandardDeviation[data];
len = Length[data];
ListPlot[data,
  Epilog → {Dashed, Red,
    Line[{0,  $\mu + \sigma$ }, {len,  $\mu + \sigma$ }]},
    Line[{0,  $\mu - \sigma$ }, {len,  $\mu - \sigma$ }]}}
]

```



Select those data elements whose distance to the mean is less than one standard deviation.

```

In[36]:= filtered = Select[data, (Abs[# -  $\mu$ ] <  $\sigma$  &)] ;

```

Here is a quick check that we get about the value we might expect (we would expect about 68% for normally distributed data).

```

In[37]:= N[Length[filtered] / Length[data]]

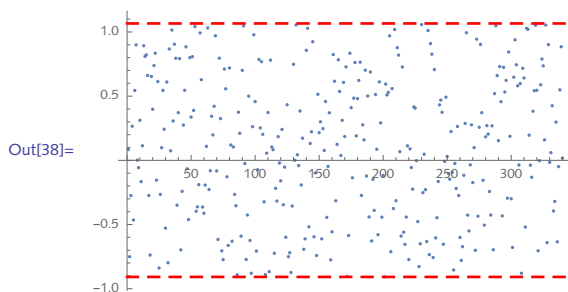
```

Out[37]= 0.68

```

In[38]:= ListPlot[filtered, PlotRange → All,
  Epilog → {Dashed, Red,
    Line[{0,  $\mu + \sigma$ }, {len,  $\mu + \sigma$ }]},
    Line[{0,  $\mu - \sigma$ }, {len,  $\mu - \sigma$ }]}}
]

```



8. This function uses a default value of 2 for the base. (Try replacing `Fold` with `FoldList` to see more clearly what this function is doing.)

```
In[39]:= convert[digits_List, base_ : 2] := Fold[(base #1 + #2) &, 0, digits]
```

Here are the digits for 9 in base 2:

```
In[40]:= IntegerDigits[9, 2]
```

```
Out[40]= {1, 0, 0, 1}
```

This converts them back to the base 10 representation.

```
In[41]:= convert[%]
```

```
Out[41]= 9
```

Note, this functionality is built into the function `FromDigits[lis, base]`.

```
In[42]:= FromDigits[{1, 0, 0, 1}, 2]
```

```
Out[42]= 9
```

This function is essentially an implementation of Horner's method for fast polynomial multiplication.

```
In[43]:= convert[{a, b, c, d, e}, x]
```

```
Out[43]= e + x (d + x (c + x (b + a x)))
```

```
In[44]:= Expand[%]
```

```
Out[44]= e + d x + c x^2 + b x^3 + a x^4
```

9. Using the list of step increments in the north, south, east, and west directions, this ten-step walk starts at the origin.

```
In[45]:= SeedRandom[0];
```

```
NSEW = {{1, 0}, {-1, 0}, {0, 1}, {0, -1}};
```

```
NestList[#1 + RandomChoice[NSEW] &, {0, 0}, 10]
```

```
Out[47]= {{0, 0}, {0, 1}, {0, 2}, {-1, 2}, {-1, 1},
          {0, 1}, {1, 1}, {1, 2}, {2, 2}, {3, 2}, {3, 3}}
```

Except for the initial value, you can get the same result with `Accumulate` which generates cumulative sums.

```
In[48]:= SeedRandom[0];
```

```
Accumulate[RandomChoice[NSEW, 10]]
```

```
Out[49]= {{0, 1}, {0, 2}, {-1, 2}, {-1, 1}, {0, 1}, {1, 1}, {1, 2}, {2, 2}, {3, 2}, {3, 3}}
```

10. The key here is to take any expression of the form  $lhs == rhs$  and extract the appropriate function whose root you will compute using the previous implementations of `findRoot`. Looking at the internal form of an equation, some thought should convince you that  $lhs - rhs$  is the expression whose root we want.



```
In[50]:= expr =  $x^2 == 2$ ;
FullForm[expr]
```

```
Out[51]//FullForm= Equal[Power[x, 2], 2]
```

And here is the pure function that we will use with NestWhile.

```
In[52]:= var = x;
expr /. Equal[a_, b_]  $\Rightarrow$  Function[Evaluate@var, a - b]
```

```
Out[53]= Function[x,  $x^2 - 2$ ]
```

```
In[54]:= Clear[findRoot];
findRoot[expr_Equal, {var_, init_},  $\epsilon$  : 0.0001] := Module[{result, fun},
  fun = expr /. Equal[a_, b_]  $\Rightarrow$  Function[Evaluate@var, a - b];
  result = NestWhile[# -  $\frac{\text{fun}[\text{\#}]}{\text{fun}'[\text{\#}]}$  &, init, Abs[fun[#]] >  $\epsilon$  &];
  {var  $\rightarrow$  result}]
```

```
In[56]:= findRoot[ $x^2 == 2$ , {x, 1.0}]
```

```
Out[56]= {x  $\rightarrow$  1.41422}
```

```
In[57]:= findRoot[Cos[x] == 0, {x, N[1, 20]},  $10^{-18}$ ]
```

```
Out[57]= {x  $\rightarrow$  1.570796326794896619}
```

II. Perhaps the easiest way to see the intermediate values is to simply use NestWhileList.

```
In[58]:= findRootList[expr_, {var_, init_},  $\epsilon$  :  $10^{-15}$ ] :=
  Module[{fun = Function[Evaluate[var], expr]},
    NestWhileList[# -  $\frac{\text{fun}[\text{\#}]}{\text{fun}'[\text{\#}]}$  &, N[init], Abs[fun[#]] >  $\epsilon$  &]]
```

```
In[59]:= findRoot[expr_Equal, {var_, init_},  $\epsilon$  : 0.0001] := Module[{fun},
  fun = expr /. Equal[a_, b_]  $\Rightarrow$  Function[Evaluate@var, a - b];
  NestWhileList[# -  $\frac{\text{fun}[\text{\#}]}{\text{fun}'[\text{\#}]}$  &, N[init], Abs[fun[#]] >  $\epsilon$  &]]
```

```
In[60]:= findRootList[ $x^2 - 2$ , {x, 1.0}]
```

```
Out[60]= {1., 1.5, 1.41667, 1.41422, 1.41421, 1.41421}
```

```
In[61]:= findRootList[Cos[x], {x, 1.0}]
```

```
Out[61]= {1., 1.64209, 1.57068, 1.5708, 1.5708}
```

Alternatively, you could use Reap and Sow. Be careful where you place the closing bracket for Sow.

```

In[62]:= Clear[findRootList];
findRootList[expr_, {var_, init_},  $\epsilon$ _: 0.0001] :=
Module[{result, fun = Function[Evaluate[var], expr]},
  Reap@NestWhile[Sow[# -  $\frac{\text{fun}[\#]}{\text{fun}'[\#]}$ ] &, N[init], Abs[fun[#]] >  $\epsilon$  &]]

In[64]:= findRootList[Cos[x], {x, .5}, 10-15]
Out[64]:= {1.5708, {{2.33049, 1.38062, 1.57312, 1.5708, 1.5708}}}
```

12. Using Fold, this pure function requires two arguments. The key is to start with an initial value of zero.

```

In[65]:= HornerPolynomial[list_List, var_] := Fold[var #1 + #2 &, 0, list]

In[66]:= HornerPolynomial[{a, b, c, d, e}, x]
Out[66]:= e + x (d + x (c + x (b + a x)))

In[67]:= Expand[%]
Out[67]:= e + d x + c x2 + b x3 + a x4
```

13. Here are the words from the built-in *Mathematica* dictionary.

```

In[68]:= words = DictionaryLookup[];

Here are those words that start with the letter q.

In[69]:= DictionaryLookup["q" ~~ __];
RandomSample[%, 20]

Out[70]:= {quadruples, quaffs, quark, quarantining, quahogs, quaff, quickie,
  quadrupled, quires, quit, quell, quints, quizzes, quadriplegia,
  quotation, quacking, quilter, queered, quintuple, quarterfinals}
```

And here are those words that start with the letter *q* and are of length five. Note the need for StringLength, not Length.

```

In[71]:= Select[DictionaryLookup["q" ~~ __], StringLength[#] == 5 &]

Out[71]:= {quack, quads, quaff, quail, quake, quaky, qualm, quark, quart,
  quash, quasi, quays, queen, queer, quell, quern, query, quest, queue,
  quick, quids, quiet, quiff, quill, quilt, quins, quint, quips, quire,
  quirk, quirt, quite, quits, quoin, quoit, quota, quote, quoth}
```

14. First, create a set of angles between zero and  $2\pi$ .

```

In[72]:= angles = RandomReal[{0, 2  $\pi$ }, 6];
```

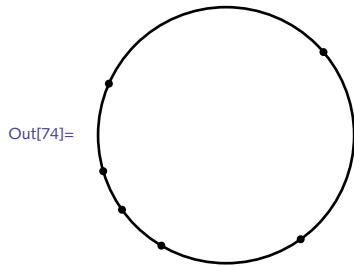
To turn these angles into points on the unit circle, note that the cosine of the angle gives the *x*-coordinate, and sine of the angle gives the *y*-coordinate. So the following creates a list of the form {Cos[ $\theta$ ], Sin[ $\theta$ ]} for each angle  $\theta$ .

```
In[73]:= pts = Map[{Cos[#], Sin[#]} &, angles]
```

```
Out[73]= {{0.760612, 0.649207}, {-0.505615, -0.862759}, {-0.915506, 0.402304},
          {-0.813205, -0.581977}, {0.5826, -0.812759}, {-0.959663, -0.281152}}
```

Here is the graphic.

```
In[74]:= Graphics[{Circle[], Point[pts]}]
```

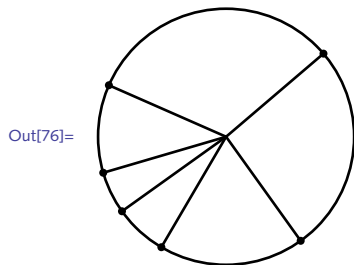


Line takes a list of two points. One of those points will be the origin for each of our coordinate pairs. Here is the code to create the pairs of points that will be passed to Line.

```
In[75]:= lines = Map[{#, {0, 0}} &, pts]
```

```
Out[75]= {{{0.760612, 0.649207}, {0, 0}}, {{-0.505615, -0.862759}, {0, 0}},
          {{-0.915506, 0.402304}, {0, 0}}, {{-0.813205, -0.581977}, {0, 0}},
          {{0.5826, -0.812759}, {0, 0}}, {{-0.959663, -0.281152}, {0, 0}}}
```

```
In[76]:= Graphics[{Circle[], Point[pts], Line[lines]}]
```



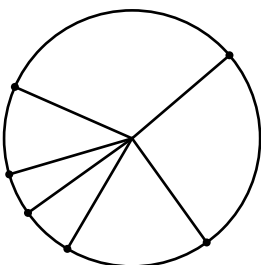
This can also be done more directly with AngleVectors which takes a polar angle as argument and returns a point on the unit circle with that polar angle.

```
In[77]:= pts = Map[AngleVector, angles]
```

```
Out[77]= {{0.760612, 0.649207}, {-0.505615, -0.862759}, {-0.915506, 0.402304},
          {-0.813205, -0.581977}, {0.5826, -0.812759}, {-0.959663, -0.281152}}
```

```
In[78]:= Graphics[{
    Circle[],
    Point[pts],
    Map[Line[{{0, 0}, #}] &, pts]
}]
```

Out[78]=



15. The key to solving this problem is thinking carefully about the initial value for `FoldList`.

```
In[79]:= FoldList[#1 +  $\alpha$  (#2 - #1) &, x1, {x2, x3}]
```

Out[79]= {x1, x1 +  $\alpha$  (-x1 + x2), x1 +  $\alpha$  (-x1 + x2) +  $\alpha$  (-x1 -  $\alpha$  (-x1 + x2) + x3)}

If you were defining your own function, you would need to extract the first element of the (data) list as the initial value of `FoldList`.

```
In[80]:= expMovingAverage[lis_,  $\alpha$ _] := FoldList[#1 +  $\alpha$  (#2 - #1) &, First[lis], Rest[lis]]
```

```
In[81]:= expMovingAverage[{a, b, c},  $\alpha$ ]
```

Out[81]= {a, a + (-a + b)  $\alpha$ , a + (-a + b)  $\alpha$  +  $\alpha$  (-a + c - (-a + b)  $\alpha$ )}

```
In[82]:= ExponentialMovingAverage[{a, b, c},  $\alpha$ ]
```

Out[82]= {a, -a (-1 +  $\alpha$ ) + b  $\alpha$ , c  $\alpha$  - (-1 +  $\alpha$ ) (-a (-1 +  $\alpha$ ) + b  $\alpha$ )}

```
In[83]:= Simplify[% == %]
```

Out[83]= True

16. A first, naive implementation will use the fact that the prime factors are all less than 6. Here are the factors for a single integer.

```
In[84]:= facs = FactorInteger[126]
```

Out[84]= {{2, 1}, {3, 2}, {7, 1}}

This extracts only the prime factors.

```
In[85]:= Map[First, facs]
```

Out[85]= {2, 3, 7}

In this case, they are not all less than 6.



```
In[94]:= Options["Integrate"]
```

```
Out[94]= {}
```

To work around this, convert strings to symbols.

```
In[95]:= Options[Symbol["Integrate"]]
```

```
Out[95]= {Assumptions -> $Assumptions,
          GenerateConditions -> Automatic, PrincipalValue -> False}
```

The second issue is that options are given as a list of rules which is a more deeply nested expression structure than the list of attributes.

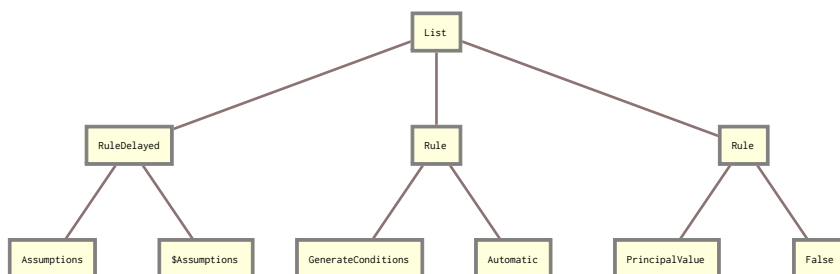
```
In[96]:= MemberQ[Options[Integrate], Assumptions]
```

```
Out[96]= False
```

The tree structure shows that the option names occur down at level two.

```
In[97]:= TreeForm[Options[Integrate]]
```

```
Out[97]//TreeForm=
```



The optional third argument to MemberQ can be used to specify the level at which the pattern match should take place. In this case, at level two.

```
In[98]:= MemberQ[Options[Integrate], Assumptions, 2]
```

```
Out[98]= True
```

Next, note that many built-in symbols have no options, as indicated by the empty list returned.

```
In[99]:= Options[Round]
```

```
Out[99]= {}
```

Rather than search through these functions, let's remove them from the search as well as any function that starts with "\$" such as \$RecursionLimit.

Here then is the function:

```
In[100]:= FunctionsWithOption[opt_Symbol] := Module[{names, lis},
  names = Complement[Names["System`*"], Names["System`$*"]];
  lis = DeleteCases[names, f_ /; Options[Symbol[f]] == {}];
  Select[lis, MemberQ[Options[Symbol[#]], opt, 2] &]]
```

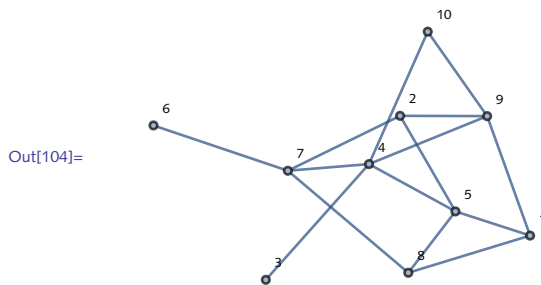
```
In[101]:= FunctionsWithOption[Compiled] // Timing
Out[101]:= {17.3194, {FindArgMax, FindArgMin, FindFit, FindMaximum,
  FindMaxValue, FindMinimum, FindMinValue, FindRoot, LiftingFilterData,
  NDSolve, NDSolveValue, NExpectation, NIntegrate, NProbability,
  NProduct, NSum, ParametricNDSolve, ParametricNDSolveValue, Play}}
```

With a little analysis you should see that we reduced the large list of functions to search from over 5000 to a little more than 1200 (check the length of `lis`). The function would be quite a bit slower otherwise.

```
In[102]:= Length[Names["System`*"]]
Out[102]:= 5491
```

18. First, create a prototype graph to work with.

```
In[103]:= SeedRandom[16];
gr = RandomGraph[{10, 15}, VertexLabels -> "Name"]
```



And here are its edges and its vertices:

```
In[105]:= EdgeList[gr]
Out[105]:= {1 ↔ 5, 1 ↔ 8, 1 ↔ 9, 2 ↔ 5, 2 ↔ 7, 2 ↔ 9, 3 ↔ 4,
  4 ↔ 5, 4 ↔ 7, 4 ↔ 9, 4 ↔ 10, 5 ↔ 8, 6 ↔ 7, 7 ↔ 8, 9 ↔ 10}

In[106]:= VertexList[gr]
Out[106]:= {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
```

Below are those edges from vertex 3 to any other vertex. In other words, this gives the adjacency list for vertex 3.

```
In[107]:= With[{u = 3},
  Select[VertexList[gr], (EdgeQ[gr, UndirectedEdge[u, #]] &)]
]
Out[107]:= {4}
```

The case for directed graphs is similar. Here then is a function that returns the adjacency list for

a given vertex  $u$  in graph  $gr$ .

```
In[108]:= adjacencyList[gr_, u_] :=
  If[DirectedGraphQ[gr],
    Select[VertexList[gr], EdgeQ[gr, DirectedEdge[u, #]] &],
    Select[VertexList[gr], EdgeQ[gr, UndirectedEdge[u, #]] &]
  ]
```

The adjacency structure is then given by mapping the above function across the vertex list.

```
In[109]:= AdjacencyStructure[gr_Graph] := Map[{#, adjacencyList[gr, #]} &, VertexList[gr]]
```

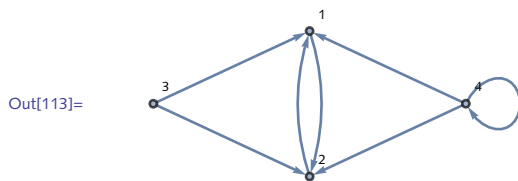
A built-in function can also be used to get the adjacency lists.

```
In[110]:= Map[{#, AdjacencyList[gr, #]} &, VertexList[gr]]
Out[110]:= {{1, {5, 8, 9}}, {2, {5, 7, 9}}, {3, {4}},
  {4, {3, 5, 7, 9, 10}}, {5, {1, 2, 4, 8}}, {6, {7}},
  {7, {2, 4, 6, 8}}, {8, {1, 5, 7}}, {9, {1, 2, 4, 10}}, {10, {4, 9}}}

In[111]:= AdjacencyStructure[gr_Graph] := Map[{#, AdjacencyList[gr, #]} &, VertexList[gr]]
In[112]:= AdjacencyStructure[gr]
Out[112]:= {{1, {5, 8, 9}}, {2, {5, 7, 9}}, {3, {4}},
  {4, {3, 5, 7, 9, 10}}, {5, {1, 2, 4, 8}}, {6, {7}},
  {7, {2, 4, 6, 8}}, {8, {1, 5, 7}}, {9, {1, 2, 4, 10}}, {10, {4, 9}}}
```

Check that it works for a directed graph also.

```
In[113]:= gr2 = Graph[{1 → 2, 2 → 1, 3 → 1, 3 → 2, 4 → 1, 4 → 2, 4 → 4},
  VertexLabels → "Name"]
```

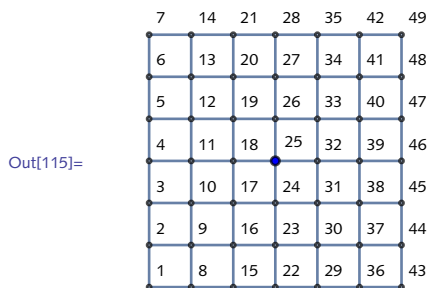


```
In[114]:= AdjacencyStructure[gr2]
Out[114]:= {{1, {2, 3, 4}}, {2, {1, 3, 4}}, {3, {1, 2}}, {4, {1, 2}}}
```

19. To start, here is a  $7 \times 7$  grid graph.



```
In[115]:= gr = GridGraph[ {7, 7}, VertexLabels -> "Name", VertexStyle -> {25 -> Blue},
    VertexSize -> {25 -> Medium}]
```



A somewhat brute force way to solve this problem is to find all vertices within distance three of vertex 25 and subtract out the vertices within distance two of vertex 25.

```
In[116]:= Complement[VertexList@NeighborhoodGraph[gr, 25, 3],
    VertexList@NeighborhoodGraph[gr, 25, 2]]
```

Out[116]= {4, 10, 12, 16, 20, 22, 28, 30, 34, 38, 40, 46}

```
In[117]:= HighlightGraph[gr, %]
```

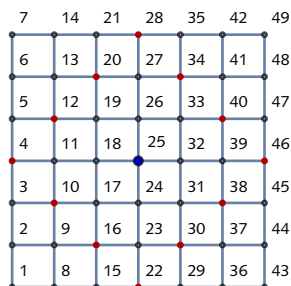


Alternatively, create a function `VertexNeighbors` that takes a graph `gr`, a vertex `v`, and a distance `dist`, and, using `GraphDistance`, finds all those vertices in `gr` that are distance `dist` from `v`.

```
In[118]:= VertexNeighbors[gr_, v_, dist_] :=
    VertexList[gr, _? (GraphDistance[gr, v, #] == dist &)]
```

```
In[119]:= HighlightGraph[gr, VertexNeighbors[gr, 25, 3]]
```

Out[119]=



20. First generate 100 digits for a 100-note “composition”.

```
In[120]:= digs = First[RealDigits[N[ $\pi$ , 50]]];
```

Fix note duration at 0.5 seconds.

```
In[121]:= Sound[SoundNote[#, 0.5] & /@ digs] // EmitSound
```

Change the duration to be dependent upon the digit. Also change the MIDI instrument.

```
In[122]:= Sound[SoundNote[#, 1 / (#+ 1), "Vibraphone"] & /@ digs] // EmitSound
```

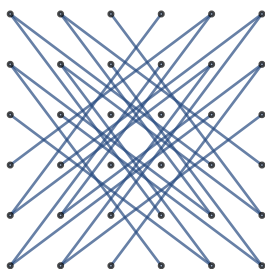
Go a bit further, expanding the range of notes that will be played.

```
In[123]:= Sound[SoundNote[1 + 2 #, 1 / (#+ 1), "Vibraphone"] & /@ digs] // EmitSound
```

21. As a test graph, we create a directed graph and check that it is acyclic.

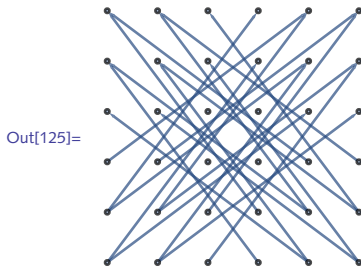
```
In[124]:= gr = GraphData[{"Antelope", {6, 6}}]
```

Out[124]=



This converts the undirected graph gr to a directed graph.

```
In[125]:= dirgr = DirectedGraph[gr, "Acyclic"]
```



Check that it is indeed acyclic.

```
In[126]:= AcyclicGraphQ[dirgr]
```

```
Out[126]= True
```

Here is its adjacency matrix.

```
In[127]:= mat = AdjacencyMatrix[dirgr];
```

This gives the eigenvalues of mat.

```
In[128]:= Eigenvalues[mat]
```

```
Out[128]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
           0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Here then is a function that checks if the eigenvalues of a matrix are all zero. Note the use of logical or (||) to check for both exact and approximate zero.

```
In[129]:= NilpotentMatrixQ[mat_?SquareMatrixQ] :=
           AllTrue[Eigenvalues[mat], (# == 0 || # == 0.0) &]
```

```
In[130]:= NilpotentMatrixQ[mat]
```

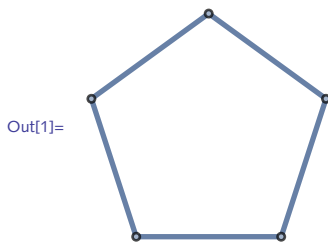
```
Out[130]= True
```

## 5.6 Examples: exercises

- I. Write a version of the function that computes Hamming distance by using Count with an appropriate pattern to find the number of nonidentical pairs of corresponding numbers in two binary signals.
2. Write an implementation of Hamming distance using the Total function and modular arithmetic base 2 to compare two bits. Compare running times with the other versions discussed in this chapter.

3. Rewrite the median function from Exercise 6, Section 5.4 so that instead of using an If control structure, you create two separate rules: one rule for the case when the list has an odd number of elements and another rule for the case when the length of the list is even.
4. Extend the survivor function developed in this section to a function of two arguments, so that `survivor[n, m]` returns the survivor starting from a list of  $n$  people and executing every  $m$ th person.
5. In Section 4.3 we created a function `CountChange[lis]` that took a list of coins and, using transformation rules, returned the monetary value of that list of coins. Rewrite `CountChange` to use a purely functional approach. Consider using `Dot`, or `Inner`, or `Tally`.
6. Create a function `Regular2Graph[n]` that outputs an  $n$ -sided regular graph where every vertex has degree two.

```
In[1]:= Regular2Graph[5]
```



7. Extend the visualization of PPI networks from this section by coloring vertices according to the biological process in which they are involved. The built-in `ProteinData` contains this information, for example:

```
In[2]:= ProteinData["KLKB1", "BiologicalProcesses"]
Out[2]= {BloodCoagulation, Fibrinolysis, InflammatoryResponse, Proteolysis}
```

8. Extend the range of `ReplaceElement` developed in this section to accept a list of strings considered as nonnumeric matrix entries, each of which should be replaced by a column mean.
9. Imagine a random walk on a graph: starting at vertex  $v_i$ , the probability that you next move to vertex  $v_j$  is given by

$$P(v_i, v_j) = \begin{cases} \frac{1}{d_{v_i}}, & \text{if } v_i \leftrightarrow v_j \\ 0, & \text{otherwise} \end{cases}$$

where  $d_{v_i}$  is the degree of vertex  $v_i$ . This defines what is known as a *transition probability matrix* and can be used to create a Markov model to simulate random walks on graphs.

Given a graph, create its transition probability matrix where matrix element  $a_{ij} = P(v_i, v_j)$  as defined above. You will need the built-in functions `VertexCount` and `VertexDegree` as well as `EdgeQ`. For more on random walks on graphs, see [Lovász \(1993\)](#) or [Aldous and Fill \(2014\)](#).

10. Random graphs have a rich and deep history in spite of the fact that they were only first defined in the mid-twentieth century in a paper by Erdős and Rényi (1959). They have since been used to study areas as diverse as percolation, telecommunications, social networks, and many more.

Perhaps the simplest random graph model is one in which both the number of vertices and the number of edges are fixed. In this model,  $G(n, m)$ , the  $m$  edges are placed at random among all  $\binom{n}{2}$  possible edges. This is the model that the built-in function `RandomGraph` is based upon. In an alternate model, commonly referred to as  $G(n, p)$ , the probability of an edge between any two of the  $n$  vertices is fixed. As a result, the number of edges can vary from none, for example when  $p = 0$ , to  $\binom{n}{2}$  when  $p = 1$ . In this exercise you are asked to create a simplified model of the  $G(n, p)$  random graph.

Starting with a set of edges, assign a probability to each. The edges can be taken from a complete graph, a graph in which there is an edge between every pair of vertices.

```
In[3]:= n = 7;
        cg = CompleteGraph[n];
        EdgeRules[cg]

Out[5]= {1 → 2, 1 → 3, 1 → 4, 1 → 5, 1 → 6, 1 → 7, 2 → 3, 2 → 4, 2 → 5, 2 → 6,
        2 → 7, 3 → 4, 3 → 5, 3 → 6, 3 → 7, 4 → 5, 4 → 6, 4 → 7, 5 → 6, 5 → 7, 6 → 7}
```

Next choose an edge with probability  $p$ . An edge being chosen is a binary choice: either it is chosen or it isn't. This is essentially a coin toss distribution which, in *Mathematica*, is represented by `BernoulliDistribution`. We create a list of probabilities the same length as the number of edges in the complete graph.

```
In[6]:= p = 0.45;
        probs = RandomVariate[BernoulliDistribution[p], EdgeCount[cg]]

Out[7]= {1, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 1, 1, 1, 0, 1, 0, 0, 0, 1}
```

Finally, use `Pick` to include only edges whose corresponding probability is less than the threshold value  $p$  and display the resulting graph with `Graph`.

11. Extend the grid example from Exercise 5, Section 3.1 to color the prime grid elements (Figure 5.4).

FIGURE 5.4. Grid of integers 1 through  $n^2$  with prime elements highlighted.

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	32	33	34	35
36	37	38	39	40	41	42
43	44	45	46	47	48	49

12. Using `Grid`, create a truth table for a logical expression such as  $(A \vee B) \Rightarrow C$  (Figure 5.5).

FIGURE 5.5. Truth table for  $A \vee B \Rightarrow C$ .

A	B	C	$A \vee B \Rightarrow C$
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	T

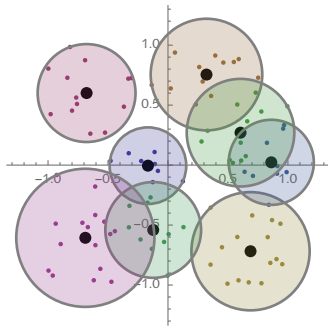
13. Create a function `RandomNote[n]` that plays a random sequence of  $n$  notes taken from the twelve-tone scale. The twelve tones in C major can be generated using `Sound` and `SoundNote` to create the sound objects; use `EmitSound` to play them through the speakers of your computer.

```
In[8]:= Sound[Map[SoundNote, Range[0, 11]]] // EmitSound
```

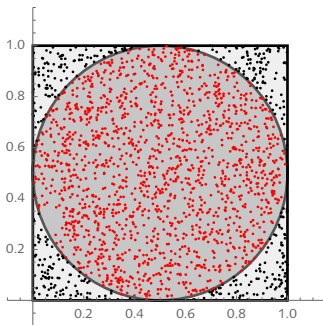
A second argument to `SoundNote` can be used to alter the duration of each note, which, by default, is for one second. Set up `RandomNote` to have random durations as well as random notes. Finally, set up `RandomNote` to accept a second argument to set the midi instrument through which the sound will be played.

14. Because of the completely random nature of how the notes are chosen in the previous exercise, the resulting “tunes” will have no autocorrelation and the result is quite uninteresting. Create a new function to generate sequences of notes where the randomness is applied to the distance between notes, essentially performing a “random walk” through the C major scale. Music generated in such a way is called Brownian because it behaves much like the movement of particles suspended in liquid – Brownian motion.
15. If you read musical notation, take a musical composition such as one of Bach’s *Brandenburg Concertos* and write down a list of the frequency intervals  $x$  between successive notes. Then find a function that interpolates the power spectrum of these frequency intervals and determine if this function is of the form  $f(x) = c/x$  for some constant  $c$ . (Hint: To get the power spectrum, you will need to square the magnitude of the Fourier transform: take `Abs[Fourier[...]]2` of your data.) Compute the power spectra of different types of music using this procedure.
16. Modify the clustering example in this section so that the cluster disks enclose every point in their respective clusters (Figure 5.6).

FIGURE 5.6. Clustering with each disk enclosing entire cluster.



17. Given a set of  $n$  points in the plane, find a tour that visits each of them once, returning to the first point visited: choose a point  $v_1$  from the set at random; then find the point that is closest to  $v_1$ , call it  $v_2$ ; of the points not chosen so far, find the point that is closest to  $v_2$ ; call it  $v_3$ . In this way create a running list  $\{v_1, v_2, \dots, v_n, v_1\}$  that gives the points to visit in order, returning to the first point visited. This is essentially a nearest-neighbor algorithm for solving the traveling salesman problem. It is known that this solution is sub-optimal but it will give you a good feeling for these kinds of problems. See Section 8.4 for some variations on traveling salesman-type problems.
18. Starting with a set of  $n$  random points in the unit square, find the proportion that are also in the square's inscribed disk (Figure 5.7). As  $n$  increases, this proportion (slowly!) approaches the value  $\pi/4$ .

FIGURE 5.7. Monte Carlo simulation to approximate  $\pi$ .

In Exercise 5, Section 9.2 this computation is run numerous times as a Monte Carlo simulation to get better and better approximations to the value of  $\pi$ .

## 5.6 Solutions

1. Here are two small sample lists.

```
In[1]:= l1 = {1, 0, 0, 1, 1};
        l2 = {0, 1, 0, 1, 0};
```

First, pair them.

```
In[3]:= l1 = Transpose[{l1, l2}]
Out[3]= {{1, 0}, {0, 1}, {0, 0}, {1, 1}, {1, 0}}
```

Here is the conditional pattern that matches any pair where the two elements are *not* identical. The Hamming distance is the number of such nonidentical pairs.

```
In[4]:= Count[l1, {p_, q_} /; p ≠ q]
Out[4]= 3
```

Finally, here is a function that puts this all together.

```
In[5]:= HammingDistance3[lis1_List, lis2_List] :=
        Count[Transpose[{lis1, lis2}], {p_, q_} /; p ≠ q]
In[6]:= HammingDistance3[l1, l2]
Out[6]= 3
```

The running times of this version of HammingDistance are quite a bit slower than those where we used bit operators. This is due to additional computation (Transpose) and the use of pattern matching and comparisons at every step.

```
In[7]:= HammingDistance2[lis1_, lis2_] := Total[BitXor[lis1, lis2]]
In[8]:= data1 = RandomInteger[1, {10^6}];
In[9]:= data2 = RandomInteger[1, {10^6}];
In[10]:= Timing[HammingDistance2[data1, data2]]
Out[10]= {0.020373, 501 050}
In[11]:= Timing[HammingDistance3[data1, data2]]
Out[11]= {0.798615, 501 050}
```

- Using Total on the binary sum, Hamming distance can be computed as follows:

```
In[12]:= HammingDistance4[lis1_, lis2_] := Total[Mod[lis1 + lis2, 2]]
```

Timing tests show that the implementation with Total is quite a bit more efficient than the previous versions, although still slower than the version that uses bit operators.

```
In[13]:= sig1 = RandomInteger[1, {10^6}];
In[14]:= sig2 = RandomInteger[1, {10^6}];
In[15]:= HammingDistance1[lis1_, lis2_] :=
        Count[MapThread[SameQ, {lis1, lis2}], False]
```



```

In[16]:= Map[{#, Timing[#[{sig1, sig2}]]] &,
            {HammingDistance1, HammingDistance2, HammingDistance3, HammingDistance4}] //
            Grid
Out[16]:= HammingDistance1 {0.352712, 499.983}
          HammingDistance2 {0.011396, 499.983}
          HammingDistance3 {0.796302, 499.983}
          HammingDistance4 {0.038783, 499.983}

```

3. The median of a list containing an odd number of elements is the middle element of the sorted list.

```

In[17]:= median[lis_List /; OddQ[Length[lis]]] :=
          Part[Sort[lis], Ceiling[Length[lis]/2]]

```

When the list has an even number of elements, take the mean of the middle two.

```

In[18]:= median[lis_List /; EvenQ[Length[lis]]] := Module[{len = Length[lis]/2},
          Mean[Part[Sort[lis], len ;; len + 1]]
        ]

```

Check the two cases – an even number of elements, and an odd number of elements. Then compare with the built-in Median.

```

In[19]:= dataE = RandomInteger[10000, 100000];
In[20]:= dataO = RandomInteger[10000, 100001];
In[21]:= median[dataE] // Timing
Out[21]:= {0.010258, 4990}

In[22]:= Median[dataE] // Timing
Out[22]:= {0.013946, 4990}

In[23]:= median[dataO] // Timing
Out[23]:= {0.011299, 5009}

In[24]:= Median[dataO] // Timing
Out[24]:= {0.01085, 5009}

```

The two rules given here should be more careful about the input, using pattern matching to insure that these rules only apply to one-dimensional lists. The following modifications handle that more robustly.

```

In[25]:= Clear[median]
In[26]:= median[lis : {__} /; OddQ[Length[lis]]] :=
          Part[Sort[lis], Ceiling[Length[lis]/2]]
In[27]:= median[lis : {__} /; EvenQ[Length[lis]]] := Module[{len = Length[lis]/2},
          Mean[Part[Sort[lis], len ;; len + 1]]
        ]

```

4. Just one change is needed here: add a second argument to `RotateLeft` that specifies the number of positions to rotate. We have used `NestList` to display the intermediate steps.

```
In[28]:= survivor[n_, m_] := NestList[Rest[RotateLeft[#, m - 1]] &, Range[n], n - 1]
```

```
In[29]:= survivor[11, 3]
```

```
Out[29]:= {{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}, {4, 5, 6, 7, 8, 9, 10, 11, 1, 2},
           {7, 8, 9, 10, 11, 1, 2, 4, 5}, {10, 11, 1, 2, 4, 5, 7, 8}, {2, 4, 5, 7, 8, 10, 11},
           {7, 8, 10, 11, 2, 4}, {11, 2, 4, 7, 8}, {7, 8, 11, 2}, {2, 7, 8}, {2, 7}, {7}}
```

5. Here is a list of coins (modify for other currencies).

```
In[30]:= coins = {p, p, q, n, d, d, p, q, q, p};
```

First count the occurrences of each.

```
In[31]:= Map[Count[coins, #] &, {p, n, d, q}]
```

```
Out[31]:= {4, 1, 2, 3}
```

Then a dot product of this count vector with a value vector does the trick.

```
In[32]:= %.{.01, .05, .10, .25}
```

```
Out[32]:= 1.04
```

```
In[33]:= CountChange[lis_] :=
           Dot[Map[Count[lis, #] &, {p, n, d, q}], {.01, .05, .10, .25}]
```

```
In[34]:= CountChange[coins]
```

```
Out[34]:= {16 + 2 d + 3 q, 44 + 2 d + 3 q, 24 + 2 d + 3 q, 20 + 2 d + 3 q, 36 + 2 d + 3 q,
           52 + 2 d + 3 q, 32 + 2 d + 3 q, 40 + 2 d + 3 q, 48 + 2 d + 3 q, 28 + 2 d + 3 q}
```

```
In[35]:= CountChange2[lis_] :=
           Inner[Times, Map[Count[lis, #] &, {p, n, d, q}], {.01, .05, .10, .25}, Plus]
```

```
In[36]:= CountChange2[coins]
```

```
Out[36]:= 1.04
```

And here is a rule-based approach.

```
In[37]:= Tally[coins] /. {d -> .10, n -> .05, p -> .01, q -> .25}
```

```
Out[37]:= {{0.01, 4}, {0.25, 3}, {0.05, 1}, {0.1, 2}}
```

```
In[38]:= Total[Apply[Times, %, {1}]]
```

```
Out[38]:= 1.04
```

```
In[39]:= CountChange3[lis_] := Module[{freq},
           freq = Tally[lis] /. {p -> .01, n -> .05, d -> .10, q -> .25};
           Total[Apply[Times, freq, {1}]]]
```

```
In[40]:= CountChange3[coins]
```

```
Out[40]= 1.04
```

6. If you think of the vertex indices as one through  $n$ , then each vertex needs to be connected to the vertex with index one less and also one more.

Create the pairs using `Partition` with `overlap` one and `set` to cycle from the last to the first element.

```
In[41]:= Partition[Range[5], 2, 1, 1]
```

```
Out[41]= {{1, 2}, {2, 3}, {3, 4}, {4, 5}, {5, 1}}
```

Then apply `UndirectedEdge` to these pairs at level one.

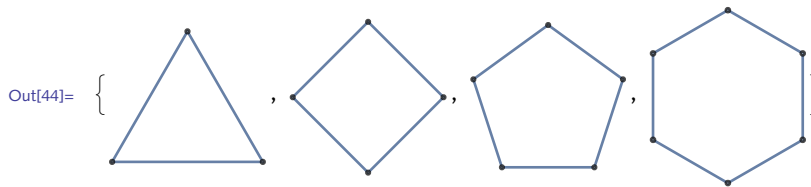
```
In[42]:= Apply[UndirectedEdge, Partition[Range[5], 2, 1, 1], {1}]
```

```
Out[42]= {1 ↔ 2, 2 ↔ 3, 3 ↔ 4, 4 ↔ 5, 5 ↔ 1}
```

This bundles up the code, using the shorthand notation for `Apply` at level one, `@@@`.

```
In[43]:= Regular2Graph[n_] := Graph[UndirectedEdge @@@ Partition[Range[n], 2, 1, 1]]
```

```
In[44]:= Table[Regular2Graph[n], {n, 3, 6, 1}]
```



7. (\* solution to appear \*)

8. Column 4 of this matrix contains several different nonnumeric values.

```
In[45]:= mat3 = {{0.796495, "N/A", 0.070125, "nan", 0.806554},
  {"nn", -0.100365, 0.992736, -0.320560, -0.0805351},
  {0.473571, 0.460741, 0.030060, -0.412400, 0.788522},
  {0.614974, -0.503201, 0.615744, 0.966053, -0.011776},
  {-0.828415, 0.035514, 0.8911617, "N/A", -0.453926}};
MatrixForm[col4 = mat3[[All, 4]]]
```

```
Out[46]//MatrixForm=
```

$$\begin{pmatrix} \text{nan} \\ -0.32056 \\ -0.4124 \\ 0.966053 \\ \text{N/A} \end{pmatrix}$$

To pattern match on either "N/A" or "nan", use `Alternatives[]`.

```

In[47]:= col4 /. "N/A" | "nan" → Mean[Cases[mat3[[All, 4]], _?NumberQ]] //
MatrixForm
Out[47]//MatrixForm=

$$\begin{pmatrix} 0.0776977 \\ -0.32056 \\ -0.4124 \\ 0.966053 \\ 0.0776977 \end{pmatrix}$$


```

Convert the list of strings to a set of alternatives.

```

In[48]:= Apply[Alternatives, {"N/A", "nan", "nn"}]
Out[48]= N/A | nan | nn

```

Here is a third set of definitions, including a new rule for ReplaceElement where the second argument is a list of strings. And another rule for ReplaceElement accommodates the new argument structure of colMean.

```

In[49]:= colMean[col_, {strings___String}] :=
col /. Apply[Alternatives, {strings}] → Mean[Cases[col, _?NumberQ]]

In[50]:= ReplaceElement[mat_, {strings___}] :=
Transpose[Map[colMean[#, {strings}] &, Transpose[mat]]]

In[51]:= ReplaceElement[mat3, {"N/A", "nan", "nn"}] // MatrixForm
Out[51]//MatrixForm=

$$\begin{pmatrix} 0.796495 & -0.0268277 & 0.070125 & 0.0776977 & 0.806554 \\ 0.264156 & -0.100365 & 0.992736 & -0.32056 & -0.0805351 \\ 0.473571 & 0.460741 & 0.03006 & -0.4124 & 0.788522 \\ 0.614974 & -0.503201 & 0.615744 & 0.966053 & -0.011776 \\ -0.828415 & 0.035514 & 0.891162 & 0.0776977 & -0.453926 \end{pmatrix}$$

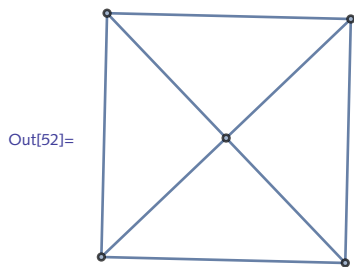

```

9. Let's start with a random graph.

```

In[52]:= gr = RandomGraph[{5, 8}]

```



Let us create a list of all possible edges in gr.

```
In[53]:= With[{n = VertexCount[gr]},
  Array[UndirectedEdge, {n, n}]]
```

```
Out[53]= {{1 ↔ 1, 1 ↔ 2, 1 ↔ 3, 1 ↔ 4, 1 ↔ 5},
  {2 ↔ 1, 2 ↔ 2, 2 ↔ 3, 2 ↔ 4, 2 ↔ 5}, {3 ↔ 1, 3 ↔ 2, 3 ↔ 3, 3 ↔ 4, 3 ↔ 5},
  {4 ↔ 1, 4 ↔ 2, 4 ↔ 3, 4 ↔ 4, 4 ↔ 5}, {5 ↔ 1, 5 ↔ 2, 5 ↔ 3, 5 ↔ 4, 5 ↔ 5}}
```

The transition probability matrix assigns the value  $1 / \text{VertexDegree}$  in position  $\{i, j\}$  if there is an edge between vertex  $i$  and vertex  $j$  and zero otherwise. Here is the pure function that does this.

```
In[54]:= If[EdgeQ[gr, #], 1/VertexDegree[gr, First@#], 0] &;
```

This needs to be mapped at level two to account for the nested lists in the array.

```
In[55]:= With[{n = VertexCount[gr]},
  Map[EdgeQ[gr, #] &, Array[UndirectedEdge, {n, n}], {2}]]
```

```
Out[55]= {{False, True, True, False, True},
  {True, False, False, True, True}, {True, False, False, True, True},
  {False, True, True, False, True}, {True, True, True, True, False}}
```

Finally, here then is the transition matrix.

```
In[56]:= With[{n = VertexCount[gr]},
  Map[If[EdgeQ[gr, #], 1/VertexDegree[gr, First@#], 0] &,
  Array[UndirectedEdge, {n, n}], {2}]]
```

```
Out[56]= {{0, 1/3, 1/3, 0, 1/3}, {1/3, 0, 0, 1/3, 1/3},
  {1/3, 0, 0, 1/3, 1/3}, {0, 1/3, 1/3, 0, 1/3}, {1/4, 1/4, 1/4, 1/4, 0}}
```

```
In[57]:= TransitionMatrix[gr_Graph] := With[{n = VertexCount[gr]},
  Map[If[EdgeQ[gr, #], 1/VertexDegree[gr, First@#], 0] &,
  Array[UndirectedEdge, {n, n}], {2}]
]
```

```
In[58]:= T = TransitionMatrix[gr]
```

```
Out[58]= {{0, 1/3, 1/3, 0, 1/3}, {1/3, 0, 0, 1/3, 1/3},
  {1/3, 0, 0, 1/3, 1/3}, {0, 1/3, 1/3, 0, 1/3}, {1/4, 1/4, 1/4, 1/4, 0}}
```

To see how to use  $\mathcal{T}$  to create a random walk using Markov processes, first create some starting probabilities for each vertex.

```
In[59]:= Π = ConstantArray[1/VertexCount[gr], {VertexCount[gr]}]
```


```
Out[59]= {1/5, 1/5, 1/5, 1/5, 1/5}
```

Here is the Markov process using the initial probabilities and the transition matrix.

```
In[60]:=  $\mathcal{P}$  = DiscreteMarkovProcess[ $\Pi$ ,  $\mathcal{T}$ ];
```

Create ten steps in the process.

```
In[61]:= data = RandomFunction[ $\mathcal{P}$ , {0, 10}]
```

```
Out[61]= TemporalData[ Time: 0 to 10  
Data points: 11 Paths: 1]
```

These are the vertices that were visited on the walk.

```
In[62]:= data["Values"]
```

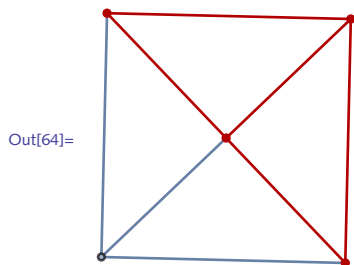
```
Out[62]= {1, 2, 4, 5, 1, 2, 5, 1, 5, 2, 4}
```

Turn the successive vertices into edges and highlight them with the original graph.

```
In[63]:= walk = UndirectedEdge @@@ Partition[data["Values"], 2, 1]
```

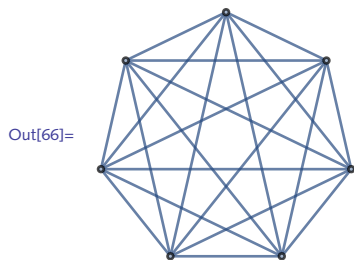
```
Out[63]= {1 ↔ 2, 2 ↔ 4, 4 ↔ 5, 5 ↔ 1, 1 ↔ 2, 2 ↔ 5, 5 ↔ 1, 1 ↔ 5, 5 ↔ 2, 2 ↔ 4}
```

```
In[64]:= HighlightGraph[gr, Graph[walk]]
```



10. Starting with a set of edges, we assign a probability to each. The edges are taken from a complete graph, a graph in which there is an edge between every pair of vertices. Then, using Pick, we include only edges whose corresponding probability is less than some threshold value,  $p$ .

```
In[65]:= n = 7;  
cg = CompleteGraph[n]
```



Here are the edges.

```
In[67]:= EdgeRules[cg]
```

```
Out[67]:= {1 → 2, 1 → 3, 1 → 4, 1 → 5, 1 → 6, 1 → 7, 2 → 3, 2 → 4, 2 → 5, 2 → 6,
           2 → 7, 3 → 4, 3 → 5, 3 → 6, 3 → 7, 4 → 5, 4 → 6, 4 → 7, 5 → 6, 5 → 7, 6 → 7}
```

We want to choose an edge with probability  $p$ . An edge being chosen is a binary choice: either it is chosen or it isn't. This is essentially a coin toss distribution which, in *Mathematica* is represented by `BernoulliDistribution`.

```
In[68]:= p = 0.45;
```

```
probs = RandomVariate[BernoulliDistribution[p], EdgeCount[cg]]
```

```
Out[69]:= {0, 1, 1, 1, 1, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 0}
```

We will use a three-argument form of `Pick` to choose the edges that meet our criteria.

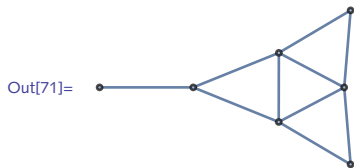
`Pick[lis, sel, patt]` returns those elements in *lis* for which the corresponding element of *sel* matches the pattern *patt*. In our case, we choose edges if the corresponding Bernoulli distribution returns a value of one.

```
In[70]:= includedEdges = Pick[EdgeRules[cg], probs, 1]
```

```
Out[70]:= {1 → 3, 1 → 4, 1 → 5, 1 → 6, 2 → 3, 2 → 6, 3 → 5, 3 → 6, 4 → 6, 4 → 7}
```

Finally, turn this list of included (undirected) edges into a graph.

```
In[71]:= Graph[includedEdges, DirectedEdges → False]
```



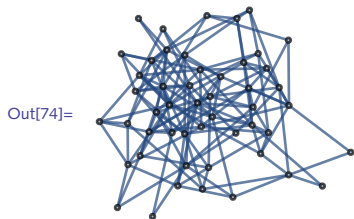
Gather all these pieces and scale up the size of the graph.

```
In[72]:= n = 50; p = .12;
```

```
cg = CompleteGraph[n];
```

```
probs = RandomVariate[BernoulliDistribution[p], EdgeCount[cg]];
```

```
gr = Graph[Pick[EdgeRules[cg], probs, 1], DirectedEdges → False]
```



A quick check that 50 vertices are returned and that the ratio of edges to *possible* edges is approximately equal to the probability we specified.

```
In[75]:= {VertexCount[gr], N@GraphDensity[gr]}
```

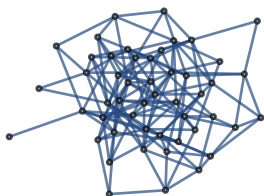
```
Out[75]= {50, 0.114286}
```

Several exercises in Chapter 6 ask you to bundle up the code here and turn it into a reusable function inheriting options from `Graph`.

As an aside, the above functionality is built into `BernoulliGraphDistribution[n, pr]` which constructs an  $n$ -vertex graph, starting with an edge connecting every pair of vertices and then selects edges independently via a Bernoulli trial with probability  $pr$ .

```
In[76]:= RandomGraph[BernoulliGraphDistribution[50, 0.12]]
```

```
Out[76]=
```



II. First construct the grid for the  $n^2$  elements, partitioning on  $n$ .

```
In[77]:= n = 7;
lis = Partition[Range[n^2], n];
Grid[lis, Frame -> All]
```

```
Out[79]=
```

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	32	33	34	35
36	37	38	39	40	41	42
43	44	45	46	47	48	49

Extract the positions of the prime elements.

```
In[80]:= Position[lis, p_ /; PrimeQ[p]]
```

```
Out[80]= {{1, 2}, {1, 3}, {1, 5}, {1, 7}, {2, 4}, {2, 6}, {3, 3},
          {3, 5}, {4, 2}, {5, 1}, {5, 3}, {6, 2}, {6, 6}, {7, 1}, {7, 5}}
```

```
In[81]:= Position[lis, p_?PrimeQ]
```

```
Out[81]= {{1, 2}, {1, 3}, {1, 5}, {1, 7}, {2, 4}, {2, 6}, {3, 3},
          {3, 5}, {4, 2}, {5, 1}, {5, 3}, {6, 2}, {6, 6}, {7, 1}, {7, 5}}
```

Map a rule of the form  $position \rightarrow \text{Pink}$  across the positions of the prime numbers.

```
In[82]:= Map[ (# -> Pink) &, Position[lis, p_?PrimeQ]]
```

```
Out[82]= {{1, 2} -> Pink, {1, 3} -> Pink, {1, 5} -> Pink, {1, 7} -> Pink, {2, 4} -> Pink,
          {2, 6} -> Pink, {3, 3} -> Pink, {3, 5} -> Pink, {4, 2} -> Pink, {5, 1} -> Pink,
          {5, 3} -> Pink, {6, 2} -> Pink, {6, 6} -> Pink, {7, 1} -> Pink, {7, 5} -> Pink}
```



Use the rules with the Background option to Grid.

```
In[83]:= n = 7;
lis = Partition[Range[n^2], n];
Grid[lis,
  Frame → All,
  Background → Join[{None, None}, {Map[(<#> → Pink) &, Position[lis, p_?PrimeQ]]}]
]
```

Out[85]=

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	32	33	34	35
36	37	38	39	40	41	42
43	44	45	46	47	48	49

```
In[86]:= Clear[n, lis]
```

12. Start with a prototype logical expression.

```
In[87]:= Clear[A, B]
```

```
In[88]:= expr = (A || B) ⇒ C;
```

```
In[89]:= vars = {A, B, C};
```

List all the possible truth value assignments for the variables.

```
In[90]:= tuples = Tuples[{True, False}, Length[vars]]
```

```
Out[90]= {{True, True, True}, {True, True, False},
  {True, False, True}, {True, False, False}, {False, True, True},
  {False, True, False}, {False, False, True}, {False, False, False}}
```

Next, create a list of rules, associating each of the triples of truth values with a triple of variables.

```
In[91]:= rules = Map[Thread[vars → #] &, tuples]
```

```
Out[91]= {{A → True, B → True, C → True}, {A → True, B → True, C → False},
  {A → True, B → False, C → True}, {A → True, B → False, C → False},
  {A → False, B → True, C → True}, {A → False, B → True, C → False},
  {A → False, B → False, C → True}, {A → False, B → False, C → False}}
```

Replace the logical expression with each set of rules.

```
In[92]:= expr /. rules
```

```
Out[92]= {True, False, True, False, True, False, True, True}
```

Put these last values at the end of each “row” of the tuples.

```
In[93]:= table = Transpose@Join[Transpose[tuples], {expr /. rules}]
```

```
Out[93]= {{True, True, True, True}, {True, True, False, False},
           {True, False, True, True}, {True, False, False, False},
           {False, True, True, True}, {False, True, False, False},
           {False, False, True, True}, {False, False, False, True}}
```

Create a header for table.

```
In[94]:= head = Append[vars, TraditionalForm[expr]]
```

```
Out[94]= {A, B, C,  $A \vee B \Rightarrow C$ }
```

Prepend head to table.

```
In[95]:= Prepend[table, head]
```

```
Out[95]= {{A, B, C,  $A \vee B \Rightarrow C$ }, {True, True, True, True}, {True, True, False, False},
           {True, False, True, True}, {True, False, False, False},
           {False, True, True, True}, {False, True, False, False},
           {False, False, True, True}, {False, False, False, True}}
```

Pour into a grid.

```
In[96]:= Grid[Prepend[table, head]]
```

```
Out[96]=
      A      B      C       $A \vee B \Rightarrow C$ 
      True   True   True   True
      True   True   False  False
      True   False  True   True
      True   False  False  False
      False  True   True   True
      False  True   False  False
      False  False  True   True
      False  False  False  True
```

Replace True with "T" and False with "F".

```
In[97]:= Grid[Prepend[table /. {True -> "T", False -> "F"}, head]]
```

```
Out[97]=
      A B C  $A \vee B \Rightarrow C$ 
      T T T      T
      T T F      F
      T F T      T
      T F F      F
      F T T      T
      F T F      F
      F F T      T
      F F F      T
```

Add formatting via options to Grid.

```
In[98]:= Grid[Prepend[table /. {True → "T", False → "F"}, head],
  Frame → True, Dividers → {{-2 → LightGray}, {2 → LightGray}},
  ItemStyle → {"Menu", 7}]
```

Out[98]=

A	B	C	$A \vee B \Rightarrow C$
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	T

13. A simple “melody” with no correlation can be generated by randomly selecting notes from a scale. First we generate the frequencies of the 12 semitones from a C major scale. This is just a chromatic scale beginning with middle C.

```
In[99]:= cmajor = Table[SoundNote[i], {i, 0, 11}]
```

```
Out[99]= {SoundNote[0], SoundNote[1], SoundNote[2], SoundNote[3],
  SoundNote[4], SoundNote[5], SoundNote[6], SoundNote[7],
  SoundNote[8], SoundNote[9], SoundNote[10], SoundNote[11]}
```

Here is a list of ten notes randomly selected from the C major scale.

```
In[100]:= Sound[RandomChoice[cmajor, 10]]
```



Add some rests and randomize the durations of each note. The symbol None is interpreted by Sound as a rest.

```
In[101]:= notes = Join[{None}, Range[0, 11]]
```

```
Out[101]= {None, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}
```

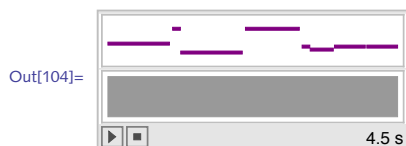
```
In[102]:= durations = Range[1/8, 1, 1/8]
```

```
Out[102]= {1/8, 1/4, 3/8, 1/2, 5/8, 3/4, 7/8, 1}
```

```
In[103]:= MapThread[SoundNote[#1, #2] &,
  {RandomChoice[notes, 8], RandomChoice[durations, 8]}]
```

```
Out[103]= {SoundNote[4, 1], SoundNote[9, 1/8], SoundNote[1, 1], SoundNote[9, 7/8],
  SoundNote[3, 1/8], SoundNote[2, 3/8], SoundNote[3, 1/2], SoundNote[3, 1/2]}
```

```
In[104]:= Sound[%]
```



Here then is a function that takes the number of notes and the instrument as arguments.

```
In[105]:= RandomNotes[n_Integer, instrument_ : "Piano"] :=  
  With[{notes = Join[{None}, Range[0, 11]], durations = Range[1/8, 1, 1/8]},  
    Sound[{instrument, MapThread[SoundNote[#1, #2] &,  
      {RandomChoice[notes, n], RandomChoice[durations, n]}]}]]
```

```
In[106]:= RandomNotes[20, "Vibraphone"]
```



14. First set up the options structure.

```
In[107]:= Options[BrownianSoundList] = {Weights -> Automatic};  
In[108]:= BrownianSoundList[steps_Integer, instr_ : "Vibraphone", OptionsPattern[]] :=  
  Module[{walk, durs, weights},  
    weights = If[OptionValue[Weights] === Automatic, Table[1/9, {9}],  
      OptionValue[Weights]];  
    walk[n_] := Accumulate[RandomChoice[weights -> Range[-4, 4], n]];  
    durs = RandomChoice[Range[1/16, 1, 1/16], {steps}];  
    Sound@MapThread[SoundNote[#1, #2, instr] &, {walk[steps], durs}]]  
In[109]:= BrownianSoundList[18, "Vibraphone"] // EmitSound  
In[110]:= BrownianSoundList[18, "Marimba",  
  Weights -> Abs@RandomVariate[NormalDistribution[0, 4], 9]] // EmitSound
```

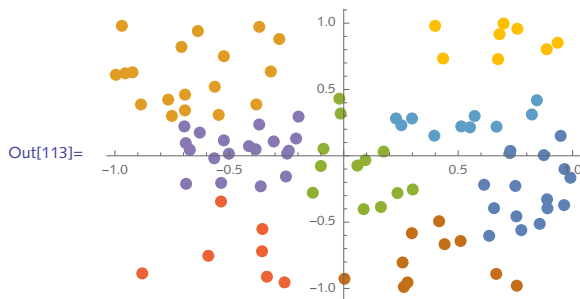
15. (\* solution to appear \*)

16. Start with one hundred points.

```
In[111]:= data = RandomReal[{-1, 1}, {100, 2}];
```

Partition into eight clusters.

```
In[112]:= clusters = FindClusters[data, 8];
ListPlot[clusters, PlotStyle -> PointSize@Medium]
```



The centroid and color computations are as before.

```
In[114]:= centroids = Map[Mean, clusters]
Out[114]:= {{0.813483, -0.257487}, {-0.664783, 0.609614},
{0.0631635, -0.0791516}, {-0.473583, -0.731627}, {-0.446728, 0.04197},
{0.389006, -0.792897}, {0.514577, 0.263179}, {0.683372, 0.871155}}

In[115]:= colors = Take[ColorData[1, "ColorList"], Length[clusters]]
Out[115]:= {Blue, Purple, Green, Yellow, Orange, Red, Brown, Pink}
```

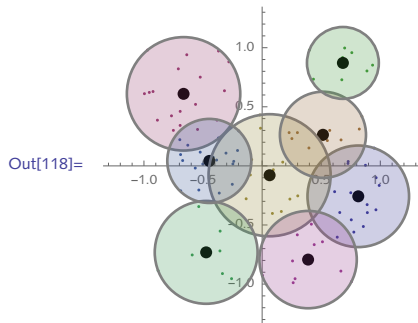
To find the distance from each centroid to the point in its cluster farthest away, we want the max of the set of distances between the cluster and the points in that cluster.

```
In[116]:= distances = Table[
  Max@Map[EuclideanDistance[centroids[[i]], #] &, clusters[[i]]],
  {i, 1, Length[clusters]}]
Out[116]:= {0.429688, 0.479782, 0.516458, 0.435077, 0.355637, 0.411775, 0.364234, 0.30396}
```

The code to create the disks is identical to that in this section.

```
In[117]:= disks = MapThread[{#1, Disk[#2, #3]} &, {colors, centroids, distances}];
```

```
In[118]:= ListPlot[clusters, PlotStyle -> colors, AspectRatio -> Automatic, Epilog -> {
    PointSize[Medium], Point[centroids],
    Opacity[0.25], EdgeForm[Gray], disks}, PlotRangePadding -> .35]
```



17. First create a set of points with which to prototype. Choose a small set to make it easier to inspect the progress.

```
In[119]:= SeedRandom[6];
pts = RandomInteger[40, {10, 2}]
```

```
Out[120]= {{16, 18}, {28, 12}, {25, 35}, {10, 13},
{26, 12}, {24, 15}, {30, 34}, {5, 0}, {16, 38}, {1, 6}}
```

Next, choose a point at random from this set of points.

```
In[121]:= base = RandomChoice[pts]
```

```
Out[121]= {16, 38}
```

Add it to a list path that we will maintain as the algorithm progresses.

```
In[122]:= path = {base}
```

```
Out[122]= {{16, 38}}
```

Next, create a list of all those points with this base points deleted, resetting pts to this new value.

```
In[123]:= pts = Complement[pts, path]
```

```
Out[123]= {{1, 6}, {5, 0}, {10, 13}, {16, 18},
{24, 15}, {25, 35}, {26, 12}, {28, 12}, {30, 34}}
```

Find the point nearest this list of points. Call it the new base point.

```
In[124]:= base = First@Nearest[pts, path]
```

```
Out[124]= {{25, 35}}
```

Add this new base point to existing list path and reset the value of path:

```
In[125]:= path = Join[path, base]
Out[125]= {{16, 38}, {25, 35}}
```

Iterate:

```
In[126]:= SeedRandom[6];
pts = RandomInteger[40, {10, 2}];

base = RandomChoice[pts];
path = {base};
Do[
  pts = Complement[pts, path];
  base = First@Nearest[pts, path];
  path = Join[path, base],

  {Length[pts] - 1}];
path
Out[131]= {{16, 38}, {25, 35}, {30, 34}, {16, 18},
  {24, 15}, {10, 13}, {26, 12}, {28, 12}, {1, 6}, {5, 0}}
```

Find the length of the path, connecting the last point to the first to make a closed loop.

```
In[132]:= ArcLength[Line[path /. {a_, b_} -> {a, b, a}]] // N
Out[132]= 150.993
```

Compare the built-in function for finding shortest tours. The first element in the returned list is the length of the tour and the second element is the ordering on the positions of the points in the tour.

```
In[133]:= SeedRandom[6];
pts = RandomInteger[40, {10, 2}];
FindShortestTour[pts]
Out[135]= {27 + 3  $\sqrt{10}$  + 2  $\sqrt{13}$  +  $\sqrt{26}$  +  $\sqrt{61}$  + 3  $\sqrt{65}$  +  $\sqrt{130}$  +  $\sqrt{397}$ ,
  {1, 4, 10, 8, 5, 2, 6, 7, 3, 9, 1}}
```

```
In[136]:= N[%[[1]]]
Out[136]= 112.121
```

18. First, create 1000 points in the unit square.

```
In[137]:= pts = RandomReal[{-1, 1}, {1000, 2}];
```

Next create regions that represent the unit disk and unit square.

```
In[138]:= D = Disk[{0, 0}, 1];
P = Polygon[{{-1, -1}, {1, -1}, {1, 1}, {-1, 1}, {-1, -1}}];
```

Now select those point that are members of the disk region. You could use `Select` or `Count`.

```
In[140]:= Length@Select[pts, RegionMember[d, #] &]
```

```
Out[140]= 782
```

```
In[141]:= Count[pts, p_ /; RegionMember[d, p]]
```

```
Out[141]= 782
```

The following ratio is a crude approximation to  $\pi/4$ .

```
In[142]:= Length@Select[pts, RegionMember[d, #] &] / Length[pts]
```

```
Out[142]=  $\frac{391}{500}$ 
```

```
In[143]:= 4 N[%]
```

```
Out[143]= 3.128
```



---

## 6

# Programs

### 6.1 Scoping constructs: exercises

1. The condition number of a square matrix gives a measure of how close the matrix is to being singular. First proposed by Alan Turing, it is often defined as the ratio of the largest singular value of the matrix to the smallest singular value. The condition number can give a sense of how much the solution of the matrix equation  $Ax = b$  can change for small perturbations of the vector  $b$ .

Create a function `ConditionNumber [mat]` that takes a square matrix *mat* and returns its condition number. Then test it on several Hilbert matrices (`HilbertMatrix`), which are known to have very large condition numbers, thus rendering numerical computations with them problematic from a precision point of view.

2. Turn the random graph example in Exercise 10, Section 5.6 into a function `RandomGraphGnp [n, p]` that returns a random graph on  $n$  vertices with edges chosen randomly (`BernoulliDistribution`) using probability  $p$ .
3. Create a function `EquilateralTriangleQ` that takes a `Triangle` object as an argument and returns a value of `True` if that triangle is equilateral and returns a value of `False` if it is not. A triangle is equilateral if the lengths of its three sides are identical. Create two implementations, one using `EuclideanDistance` to measure the side lengths and another using `ArcLength` with `MeshRegion` and `MeshPrimitives`.
4. Based on the row-switching example in this section, create a function to swap matrix columns.
5. Rewrite the function `FindSubsequence` from Exercise 7, Section 4.3 using `Module` to localize the length of the subsequence `subseq`.
6. The `PerfectSearch` function defined in Exercise 6, Section 5.1 is impractical for checking large numbers because it has to check all numbers from 1 through  $n$ . It is inefficient to check all numbers from 1 to 1000 if you are only looking for perfect numbers in the range 500 to 1000. Modify `PerfectSearch` so that it is self-contained, defining the perfect predicate inside the body of the function. Also, set up the argument structure to accept two numbers as input and

have the function output all perfect numbers between those inputs. For example, `PerfectSearch[a, b]` will return a list of all perfect numbers in the range from  $a$  to  $b$ .

7. A number  $n$  is  $k$ -perfect if the sum of its proper divisors equals  $kn$ . Redefine `PerfectSearch` from the previous exercise so that it accepts as input two numbers  $a$  and  $b$ , a positive integer  $k$ , and computes all  $k$ -perfect numbers in the range from  $a$  to  $b$ . Use your rule to find the only three 4-perfect numbers less than 2 200 000.
8. Often in processing files you are presented with expressions that need to be converted into a format that can be more easily manipulated. For example, a file may contain dates in the form 20160702 to represent July 2, 2016. *Mathematica* represents its dates as a list in the following form:

`{year, month, day, hour, minutes, seconds}`

Write a function `convertToDate[n]` to convert a number consisting of eight digits such as 20160702 into a list of the form `{2016, 7, 2}`.

```
In[1]:= convertToDate[20160702]
```

```
Out[1]= {2016, 7, 2}
```

9. Create a function `zeroColumns[mat, m ; ; n]` that zeros out columns  $m$  through  $n$  in matrix  $mat$ . Include rules to handle the cases of zeroing out one column or a list of nonconsecutive columns.
10. Following on the code for the one-dimensional random walk, create a function `walk2D[n]` that generates  $n$  steps of a two-dimensional random walk on the integer lattice (see Exercise 8, Section 2.1). For such a walk, the directions can be thought of as the two-dimensional vectors `{0, 1}`, `{0, -1}`, `{1, 0}`, and `{-1, 0}` pointing in the direction of the compass directions north, south, east, and west. Then create a function `walk3D[n]` that returns an  $n$ -step random walk on the three-dimensional integer lattice.

## 6.1 Solutions

1. Let us start with the  $3 \times 3$  Hilbert matrix.

```
In[1]:= mat = HilbertMatrix[3]
```

```
Out[1]= {{1, 1/2, 1/3}, {1/2, 1/3, 1/4}, {1/3, 1/4, 1/5}}
```

Here are its singular values:

```
In[2]:= sv = SingularValueList[mat]
```

```
Out[2]= {Sqrt[Root[-1 + 138 537 #1 - 9 323 424 #1^2 + 4 665 600 #1^3 &, 3]],
          Sqrt[Root[-1 + 138 537 #1 - 9 323 424 #1^2 + 4 665 600 #1^3 &, 2]],
          Sqrt[Root[-1 + 138 537 #1 - 9 323 424 #1^2 + 4 665 600 #1^3 &, 1]]}
```

We are interested in the ratio of the largest to the smallest.

```
In[3]:= N[Max[sv]/Min[sv]]
```

```
Out[3]= 524.057
```

Actually, `SingularValueList` returns the singular values ordered from greatest to smallest. So we can speed up the computation by only pulling off those positions rather than computing maximum and minimum values.

```
In[4]:= N[First[sv]/Last[sv]]
```

```
Out[4]= 524.057
```

Here is the function definition. We use `Module` to localize a variable `sv` that is used in the body of the function. Also, the definition given in the exercise is good for square matrices so we do some argument checking on the left-hand side of the definition.

```
In[5]:= ConditionNumber[mat_?SquareMatrixQ] := Module[{sv = SingularValueList[mat]},
  First[sv]/Last[sv]]
```

Here are the condition numbers for the first five Hilbert matrices

```
In[6]:= Table[ConditionNumber[HilbertMatrix[i]], {i, 1, 5}] // N
```

```
Out[6]= {1., 19.2815, 524.057, 15513.7, 476607.}
```

And this tests with a known pathological example of a  $4 \times 4$  matrix (Rump 1991).

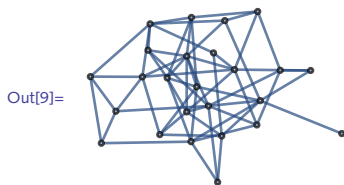
```
In[7]:= mat = {{-5046135670319638, -3871391041510136, -5206336348183639,
  -6745986988231149}, {-640032173419322, 8694411469684959,
  -564323984386760, -2807912511823001},
  {-16935782447203334, -18752427538303772, -8188807358110413,
  -14820968618548534}, {-1069537498856711, -14079150289610606,
  7074216604373039, 7257960283978710}};
```

```
In[8]:= ConditionNumber[mat] // N
```

```
Out[8]= 6.41354 × 1064
```

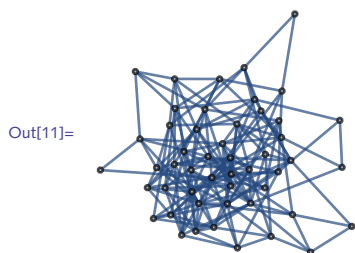
- Here are the code fragments from Exercise 10 in Section 5.6.

```
In[9]:= With[{n = 25, p = .2},
  cg = CompleteGraph[n];
  probs = RandomVariate[BernoulliDistribution[p], EdgeCount[cg]];
  Graph[Pick[EdgeRules[cg], probs, 1], DirectedEdges → False]]
```



And here is the function, localizing `cg` and `probs`.

```
In[10]:= RandomGraphGnp[n_, p_] := Module[{cg = CompleteGraph[n], probs},
  probs = RandomVariate[BernoulliDistribution[p], EdgeCount[cg]];
  Graph[Pick[EdgeRules[cg], probs, 1], DirectedEdges → False]
In[11]:= RandomGraphGnp[50, .15]
```



3. Start by creating a prototypical triangle to work with.

```
In[12]:= pts = {{0, 0}, {1, 0}, {1/2, Sqrt[3]/2}};
In[13]:= tri = Triangle[pts]
```

```
Out[13]= Triangle[{{0, 0}, {1, 0}, {1/2, Sqrt[3]/2}}]
```

To measure the lengths of the sides, create all possible pairs of the points and apply `EuclideanDistance` (at level one).

```
In[14]:= Apply[EuclideanDistance, Subsets[pts, {2}], {1}]
Out[14]= {1, 1, 1}
```

Finally, check if all distances are equal.

```
In[15]:= Apply[Equal, %]
Out[15]= True
```

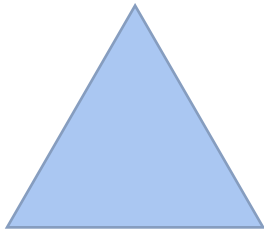
Here then is the first implementation:

```
In[16]:= EquilateralTriangleQ[Triangle[pts : {__}]] := Module[{dists},
  dists = Apply[EuclideanDistance, Subsets[pts, {2}], {1}];
  Apply[Equal, dists]]
```

To use the region framework, first generate a mesh region from the coordinate points. `MeshRegion` takes two arguments: a list of coordinates and a region (in our case, `Triangle`) specified by the coordinate indices.

```
In[17]:= T = MeshRegion[pts, Triangle[{1, 2, 3}]]
```

Out[17]=



The lines are the one-dimensional primitives in this region.

In[18]:= `MeshPrimitives[ $\mathcal{T}$ , 1]`Out[18]= `{Line[{{0., 0.}, {1., 0.}}],  
Line[{{1., 0.}, {0.5, 0.866025}}], Line[{{0.5, 0.866025}, {0., 0.}}]}`

Here are their lengths:

In[19]:= `Map[ArcLength, %]`Out[19]= `{1., 1., 1.}`

Here then is the implementation using regions.

```
In[20]:= EquilateralTriangleQ[Triangle[pts : {__}]] := Module[{dists},
  dists = Map[ArcLength, MeshPrimitives[MeshRegion[pts, Triangle[{1, 2, 3}]],
    1]];
  Apply[Equal, dists]]
```

```
In[21]:= tri = Triangle[{{-1, 0}, {1, 0}, {2, 1}}];
Graphics[tri]
```

Out[22]=

In[23]:= `EquilateralTriangleQ[tri]`Out[23]= `False`

```
In[24]:= pts = {{0, 0}, {1, 0}, {1/2,  $\sqrt{3}/2$ }};
EquilateralTriangleQ[Triangle[pts]]
```

Out[25]= `True`

4. Switching columns is basically switching rows of the transposed matrix and then transposing back.

```
In[26]:= switchRows[mat_, {r1_, r2_}] := Module[{lmat = mat},
  lmat[[{r1, r2}]] = lmat[[{r2, r1}]];
  lmat]
```

```
In[27]:= switchColumns[mat_, {c1_, c2_}] :=
  Transpose@switchRows[Transpose[mat], {c1, c2}]
```

```
In[28]:= mat = {{a, b, c}, {d, e, f}, {g, h, i}};
switchColumns[mat, {2, 3}] // MatrixForm
```

```
Out[29]//MatrixForm=
```

$$\begin{pmatrix} a & c & b \\ d & f & e \\ g & i & h \end{pmatrix}$$

5. The FindSubsequence function from Section 4.3 computed Length[*subseq*] twice on the right-hand side of the definition. Setting up a local variable len that computes this value as part of the initialization shortens the code and makes it a bit more efficient.

```
In[30]:= FindSubsequence[digits_List, subseq_List] := Module[{len = Length[subseq]},
  Position[Partition[digits, len, 1], subseq] /.
  {num_?IntegerQ} :> {num, num + len - 1}
]
```

```
In[31]:= pidigs = First[RealDigits[ $\pi$ , 10, 107, -1]];
```

```
In[32]:= FindSubsequence[pidigs, {3, 1, 4, 1, 5, 9}] // Timing
```

```
Out[32]= {10.2822, {{176451, 176456},
  {1259351, 1259356}, {1761051, 1761056}, {6467324, 6467329},
  {6518294, 6518299}, {9753731, 9753736}, {9973760, 9973765}}}
```

6. The following function creates a local function perfectQ using the Module construct. It then checks every other number between *n* and *m* by using a third argument to the Range function.

```
In[33]:= PerfectSearch[n_, m_] := Module[{perfectQ},
  perfectQ[j_] := DivisorSigma[1, j] == 2 j;
  Select[Range[n, m, 2], perfectQ]]
```

```
In[34]:= PerfectSearch[2, 107] // Timing
```

```
Out[34]= {26.3649, {6, 28, 496, 8128}}
```

This function does not guard against the user supplying “bad” inputs. For example, if the user starts with an odd number, then this version of PerfectSearch will check every other odd number, and, since it is known that there are no odd perfect numbers below at least  $10^{300}$ , none is reported.

```
In[35]:= PerfectSearch[1, 10000]
```

```
Out[35]= {}
```

You can fix this situation by using the (as yet unproved) assumption that there are no odd perfect numbers. This next version first checks that the first argument is an even number.

```
In[36]:= Clear[PerfectSearch]
```

```
In[37]:= PerfectSearch[n_?EvenQ, m_] := Module[{perfectQ},
  perfectQ[j_] := Total[Divisors[j]] == 2 j;
  Select[Range[n, m, 2], perfectQ]]
```

Now, the function only works if the first argument is even.

```
In[38]:= PerfectSearch[2, 10 000]
```

```
Out[38]= {6, 28, 496, 8128}
```

```
In[39]:= PerfectSearch[1, 1000]
```

```
Out[39]= PerfectSearch[1, 1000]
```

7. This function requires a third argument.

```
In[40]:= Clear[PerfectSearch];
PerfectSearch[n_, m_, k_] := Module[{perfectQ},
  perfectQ[j_] := Total[Divisors[j]] == k j;
  Select[Range[n, m], perfectQ]]
```

The following computation can be quite time consuming and requires a fair amount of memory to run to completion. If your computer's resources are limited, you should split up the search intervals into smaller units or try running this in parallel. See Section 9.2 for a discussion on how to set up parallel computation.

```
In[42]:= PerfectSearch[1, 2 200 000, 4] // AbsoluteTiming
```

```
Out[42]= {21.925, {30 240, 32 760, 2 178 540}}
```

8. Many implementations are possible for `convertToDate`. The task is made easier by observing that `DateList` handles this task directly if its argument is a string.

```
In[43]:= DateList["20151015"]
```

```
Out[43]= {2015, 10, 15, 0, 0, 0.}
```

The string is necessary otherwise `DateList` will interpret the integer as an absolute time (from Jan 1 1900).

```
In[44]:= DateList[20151015]
```

```
Out[44]= {1900, 8, 22, 5, 30, 15.}
```

So we need to convert the integer to a string first,

```
In[45]:= DateList[ToString[20151015]]
```

```
Out[45]= {2015, 10, 15, 0, 0, 0.}
```

and then take the first three elements.

```
In[46]:= Take[%, 3]
```

```
Out[46]= {2015, 10, 15}
```

Here is the function that puts these steps together.

```
In[47]:= convertToDate[n_Integer] := Take[DateList[ToString[n]], 3]
```

```
In[48]:= convertToDate[20151015]
```

```
Out[48]= {2015, 10, 15}
```

With a bit more manual work, you could also do this with `StringTake`.

```
In[49]:= convertToDate2[n_Integer /; Length[IntegerDigits[n]] == 8] :=  
Module[{str = ToString[n]},  
  {StringTake[str, 4], StringTake[str, {5, 6}], StringTake[str, -2]}]
```

```
In[50]:= convertToDate2[20151015]
```

```
Out[50]= {2015, 10, 15}
```

You could avoid working with strings by making use of `FromDigits`. This uses `With` to create a local constant `d`, as this expression never changes throughout the body of the function.

```
In[51]:= convertToDate3[num_] := With[{d = IntegerDigits[num]},  
  {FromDigits[Take[d, 4]],  
   FromDigits[Take[d, {5, 6}]],  
   FromDigits[Take[d, {7, 8}]]}]
```

```
In[52]:= convertToDate3[20151015]
```

```
Out[52]= {2015, 10, 15}
```

9. The computation of zeroing out one or more columns of a matrix can be handled with list component assignment. We need to use a local variable here to avoid changing the original matrix.

```
In[53]:= mat = RandomReal[1, {5, 5}];  
MatrixForm[mat]
```

```
Out[54]//MatrixForm=
```

$$\begin{pmatrix} 0.560325 & 0.895832 & 0.087212 & 0.889873 & 0.405831 \\ 0.626251 & 0.787507 & 0.45469 & 0.384032 & 0.892384 \\ 0.296374 & 0.505962 & 0.319666 & 0.327262 & 0.825282 \\ 0.121677 & 0.0413138 & 0.994354 & 0.0078785 & 0.77109 \\ 0.205495 & 0.959287 & 0.337399 & 0.940264 & 0.900914 \end{pmatrix}$$

Here is a rule for zeroing out one column:

```
In[55]:= zeroColumns[mat_, n_Integer] := Module[{lmat = mat},  
  lmat[[All, n]] = 0;  
  lmat]
```

This next rule is for zeroing out a range of columns:

```
In[56]:= zeroColumns[mat_, Span[m_, n_]] := Module[{lmat = mat},  
  lmat[[All, m ;; n]] = 0;  
  lmat]
```

We also need a final rule for zeroing out a discrete set of columns whose positions are given by a list.



```
In[57]:= zeroColumns[mat_, lis : {__}] := Module[{lmat = mat},
  lmat[[All, lis]] = 0;
  lmat]
```

```
In[58]:= zeroColumns[mat, 3] // MatrixForm
```

```
Out[58]//MatrixForm=
```

$$\begin{pmatrix} 0.560325 & 0.895832 & 0 & 0.889873 & 0.405831 \\ 0.626251 & 0.787507 & 0 & 0.384032 & 0.892384 \\ 0.296374 & 0.505962 & 0 & 0.327262 & 0.825282 \\ 0.121677 & 0.0413138 & 0 & 0.0078785 & 0.77109 \\ 0.205495 & 0.959287 & 0 & 0.940264 & 0.900914 \end{pmatrix}$$

```
In[59]:= zeroColumns[mat, 1 ;; 2] // MatrixForm
```

```
Out[59]//MatrixForm=
```

$$\begin{pmatrix} 0 & 0 & 0.087212 & 0.889873 & 0.405831 \\ 0 & 0 & 0.45469 & 0.384032 & 0.892384 \\ 0 & 0 & 0.319666 & 0.327262 & 0.825282 \\ 0 & 0 & 0.994354 & 0.0078785 & 0.77109 \\ 0 & 0 & 0.337399 & 0.940264 & 0.900914 \end{pmatrix}$$

```
In[60]:= zeroColumns[mat, {1, 3, 5}] // MatrixForm
```

```
Out[60]//MatrixForm=
```

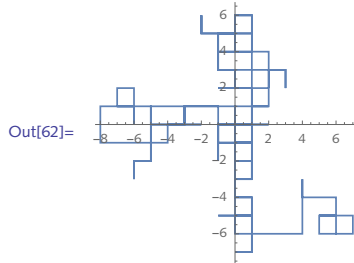
$$\begin{pmatrix} 0 & 0.895832 & 0 & 0.889873 & 0 \\ 0 & 0.787507 & 0 & 0.384032 & 0 \\ 0 & 0.505962 & 0 & 0.327262 & 0 \\ 0 & 0.0413138 & 0 & 0.0078785 & 0 \\ 0 & 0.959287 & 0 & 0.940264 & 0 \end{pmatrix}$$

10. Here is the code for the two-dimensional walk. Except for the direction vectors, it is identical to the code for the one-dimensional walk.

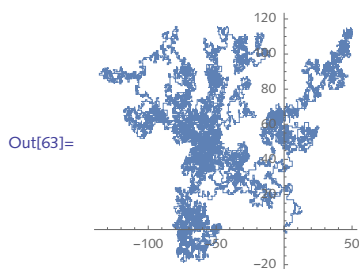
```
In[61]:= walk2D[t_] := Module[{dirs, steps},
  dirs = {{0, 1}, {0, -1}, {1, 0}, {-1, 0}};
  steps = RandomChoice[dirs, {t}];
  Accumulate[steps]
]
```

Try it out for a small number of steps and a large number of steps.

```
In[62]:= ListLinePlot[walk2D[250], AspectRatio -> 1]
```



```
In[63]:= ListLinePlot[walk2D[25000], AspectRatio → 1, PlotStyle → Thin]
```



A bit of thought is needed to come up with the eight directions in the three-dimensional integer lattice.

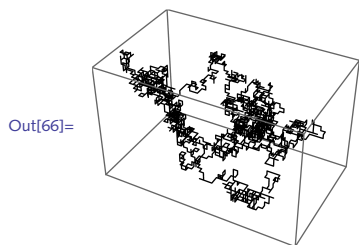
```
In[64]:= dirs3 = {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}, {-1, 0, 0}, {0, -1, 0}, {0, 0, -1}};
```

Here then is the three-dimensional walk function.

```
In[65]:= walk3D[t_] := Accumulate[RandomChoice[dirs3, t]]
```

And here is a 2500-step walk on the three-dimensional integer lattice.

```
In[66]:= Graphics3D[Line[walk3D[2500]]]
```



## 6.2 Options and messages: exercises

1. Modify the random graph example in Exercise 2, Section 6.1 to inherit options from Graph.
2. In Section 6.1 we developed a function `switchRows` that interchanged two rows in a matrix. Create a message for this function that is issued whenever a row index greater than the size of the matrix is used as an argument. For example,

```
In[1]:= mat = RandomInteger[{0, 9}, {4, 4}];
MatrixForm[mat]
```

Out[2]//MatrixForm=

$$\begin{pmatrix} 8 & 0 & 9 & 2 \\ 7 & 2 & 1 & 9 \\ 4 & 5 & 0 & 4 \\ 2 & 8 & 6 & 4 \end{pmatrix}$$

```
In[3]:= switchRows[mat, {5, 2}]
```

switchRows::badargs : The absolute value of the row indices 5 and 2 in switchRows[mat,{5,2}] must be between 1 and 4, the size of the matrix.

```
Out[3]:= {{8, 0, 9, 2}, {7, 2, 1, 9}, {4, 5, 0, 4}, {2, 8, 6, 4}}
```

You should also trap for a row index of zero.

```
In[4]:= switchRows[mat, {0, 2}]
```

switchRows::badargs : The absolute value of the row indices 0 and 2 in switchRows[mat,{0,2}] must be between 1 and 4, the size of the matrix.

```
Out[4]:= {{8, 0, 9, 2}, {7, 2, 1, 9}, {4, 5, 0, 4}, {2, 8, 6, 4}}
```

3. Create an error message for StemPlot, developed in this section, so that an appropriate message is issued if the argument is not a list of numbers.
4. Extend the definitions for ToEdges from Exercise 15, Section 5.1 to include an option to output directed or undirected edges.
5. The function MultiplyCount given in the solution to Exercise 7, Section 4.2 does not check that the expression passed to it is in fact a polynomial. Rewrite this function so that MultiplyCount[poly, var] checks that poly is a polynomial in the independent variable var and issues an appropriate message if the polynomial test fails.

## 6.2 Solutions

1. First, set up the options inherited from Graph.

```
In[1]:= Options[RandomGraphGnp] = Options[Graph];
```

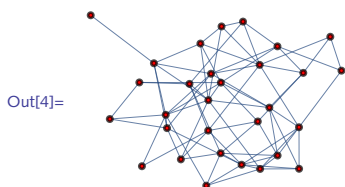
```
In[2]:= RandomGraphGnp[n_, p_, opts : OptionsPattern[]] := Module[{cg, probs},
  cg = CompleteGraph[n];
  probs = RandomVariate[BernoulliDistribution[p], EdgeCount[cg]];
  Graph[Pick[EdgeRules[cg], probs, 1], opts, DirectedEdges → False]]
```

Alternatively, you could use RandomChoice to pick the True/False values.

```
In[3]:= RandomGraphGnp[n_, p_, opts : OptionsPattern[]] := Module[{cg, probs},
  cg = CompleteGraph[n];
  probs = RandomChoice[{p, 1 - p} → {True, False}, EdgeCount[cg]];
  Graph[Pick[EdgeRules[cg], probs], opts, DirectedEdges → False]]
```

Exercise the options.

```
In[4]:= RandomGraphGnp[30, .2, VertexSize → Large, VertexStyle → Red, EdgeStyle → Thin]
```



- The message will slot in the values of the row indices being passed to the function `switchRows`, as well as the length of the matrix, that is, the number of matrix rows.

```
In[5]:= switchRows::badargs =
  "The absolute value of the row indices '1' and '2' in switchRows[mat,'1','2']
  must be between 1 and '3', the size of the matrix.";
```

The message is issued if either of the row indices have absolute value greater than the length of the matrix or if either of these indices is equal to zero.

```
In[6]:= switchRows[mat_, {r1_Integer, r2_Integer}] :=
  Module[{lmat = mat, len = Length[mat]},
    If[Abs[r1] > len || Abs[r2] > len || r1 r2 == 0,
      Message[switchRows::badargs, r1, r2, len],
      lmat[[{r1, r2}]] = lmat[[{r2, r1}]]];
  lmat]
```

```
In[7]:= mat = RandomInteger[9, {4, 4}];
MatrixForm[mat]
```

Out[8]//MatrixForm=

$$\begin{pmatrix} 4 & 8 & 5 & 4 \\ 4 & 4 & 3 & 7 \\ 0 & 5 & 7 & 3 \\ 8 & 4 & 1 & 7 \end{pmatrix}$$

```
In[9]:= switchRows[mat, {0, 4}]
```

switchRows::badargs :

The absolute value of the row indices 0 and 4 in switchRows[mat,0,4] must be between 1 and 4, the size of the matrix.

```
Out[9]= {{4, 8, 5, 4}, {4, 4, 3, 7}, {0, 5, 7, 3}, {8, 4, 1, 7}}
```

```
In[10]:= switchRows[mat, {2, 8}]
```

switchRows::badargs :

The absolute value of the row indices 2 and 8 in switchRows[mat,2,8] must be between 1 and 4, the size of the matrix.

```
Out[10]= {{4, 8, 5, 4}, {4, 4, 3, 7}, {0, 5, 7, 3}, {8, 4, 1, 7}}
```

- If the first argument is not a list containing numbers, then issue a message.

```
In[11]:= MatchQ[{1, 2, a}, {__?NumericQ}]
```

```
Out[11]= False
```

Here is the message:

```
In[12]:= StemPlot::badarg = "The first argument to StemPlot must be a list of numbers.";
```

```
In[13]:= Options[StemPlot] = Options[ListPlot];
```

```
In[14]:= StemPlot[lis_, opts : OptionsPattern[]] :=
  If[MatchQ[lis, {__?NumericQ}],
    ListPlot[lis, opts, Filling -> Axis],
    Message[StemPlot::badarg]
  ]
```

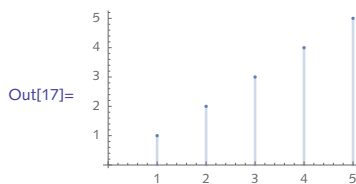
```
In[15]:= StemPlot[4]
```

StemPlot::badarg : The first argument to StemPlot must be a list of numbers.

```
In[16]:= StemPlot[{1, 2, c}]
```

StemPlot::badarg : The first argument to StemPlot must be a list of numbers.

```
In[17]:= StemPlot[{1, 2, 3, 4, 5}]
```



4. First set up the option for ToEdges.

```
In[18]:= Options[ToEdges] = {DirectedEdges -> True};
```

```
In[19]:= ToEdges[lis : {_, _} .., OptionsPattern[]] :=
  If[OptionValue[DirectedEdges], DirectedEdge @@@ lis, UndirectedEdge @@@ lis]
```

```
In[20]:= ToEdges[lis : {_, _}, OptionsPattern[]] :=
  If[OptionValue[DirectedEdges], DirectedEdge @@ lis, UndirectedEdge @@ lis]
```

```
In[21]:= pairs = Partition[Range[5], 2, 1]
```

```
Out[21]= {{1, 2}, {2, 3}, {3, 4}, {4, 5}}
```

```
In[22]:= ToEdges[pairs]
```

```
Out[22]= {1 -> 2, 2 -> 3, 3 -> 4, 4 -> 5}
```

```
In[23]:= ToEdges[pairs, DirectedEdges -> False]
```

```
Out[23]= {1 ↔ 2, 2 ↔ 3, 3 ↔ 4, 4 ↔ 5}
```

5. We use If to check if the expression *expr* is a polynomial in the independent variable *var*. If it is, use the same code from Exercise 7, Section 4.2 to count the number of multiplies. If the test fails, then issue a message.

```

In[24]:= MultiplyCount[expr_, var_] :=
  If[PolynomialQ[expr, var],
    Total@Cases[{expr}, Power[_ , exp_] :> exp - 1, Infinity] +
    Total@Cases[{expr}, fac_Times :> Length[fac] - 1, Infinity],
    Message[MultiplyCount::poly, expr]]

```

Note that we have done something a bit different here. We have used an internal usage message `poly` with our `MultiplyCount` function. Here is the message issued by the built-in function `PolynomialRemainder`.

```

In[25]:= PolynomialRemainder[{3 x}, x + a, x]
PolynomialRemainder::poly : {3 x} is not a polynomial. >>

Out[25]= PolynomialRemainder[{3 x}, a + x, x]

In[26]:= MultiplyCount[{a x^4 + b x + 1}, x]
MultiplyCount::poly : {1 + b x + a x^4} is not a polynomial. >>

In[27]:= MultiplyCount[a x^4 + b x + 1, x]
Out[27]= 5

```

### 6.3 Examples: exercises

1. Add a message to the `RandomWalk` function developed in this chapter so that a warning is issued when a nonpositive integer is given as the first argument to `RandomWalk`.
2. Consider a sequence of numbers generated by the following iterative process: starting with the list of odd integers 1, 3, 5, 7, ..., the first odd number greater than 1 is 3, so delete every third number from the list; from the list of remaining numbers, the next number is 7, so delete every seventh number; and so on. The numbers that remain are referred to as *lucky numbers*. Use a sieving method to find all lucky numbers less than 1000. See [Weisstein \(Lucky Number\)](#) for more information.
3. Create a function `TruthTable[expr, vars]` that takes a logical expression such as  $A \wedge B$  and outputs a truth table similar to those in Section 2.4. You can create a list of truth values using `Tuples`:

```

In[1]:= Tuples[{True, False}, 2]
Out[1]= {{True, True}, {True, False}, {False, True}, {False, False}}

```

You will also find it helpful to consider threading rules over the tuples using `MapThread` or `Thread`. Alternatively, consider using the built-in function `BooleanTable`.

4. Given a list of expressions, *lis*, create a function `NearTo [lis, elem, {n}]` that returns all elements of *lis* that are exactly *n* positions away from *elem*, which is assumed to be a member of *lis*. For example:

```
In[2]:= chars = CharacterRange["a", "z"]
Out[2]= {a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z}

In[3]:= NearTo[chars, "q", {3}]
Out[3]= {{n}, {t}}
```

Write a second rule, `NearTo [lis, elem, n]`, that returns all elements in *lis* that are within *n* positions of *elem*.

```
In[4]:= NearTo[chars, "q", 4]
Out[4]= {{m, n, o, p, q, r, s, t, u}}
```

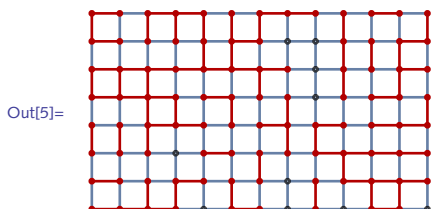
Finally, create your own distance function (`DistanceFunction`) and use it with the built-in `Nearest` to do the same computation. Two useful functions for these tasks are `Position` and `Extract`. `Extract [expr, pos]` returns elements from *expr* whose positions *pos* are given by `Position`.

5. A *Smith number* is a composite number such that the sum of its digits is equal to the sum of the digits of its prime factors. For example, the prime factorization of 852 is  $2^2 \cdot 3^1 \cdot 71^1$ , and so the sum of the digits of its prime factors is  $2 + 2 + 3 + 7 + 1 = 15$ , which is equal to the sum of its digits,  $8 + 5 + 2 = 15$ . Write a program to find all Smith numbers less than 10 000.
6. Develop auxiliary functions for one-, two-, and three-dimensional off-lattice random walks. Then create a function `offLatticeWalk [steps, dim]` that uses these auxiliary functions to return an off-lattice walk of length *steps*, in dimension *dim*. Finally, modify `RandomWalk` to include an option `LatticeWalk` that, when set to `True`, calls the `latticeWalk` auxiliary function and when set to `False`, calls this new `offLatticeWalk` auxiliary function.
7. Here is some code to run a bond percolation simulation on an  $m \times n$  lattice grid. For each edge in the grid graph *gg*, it creates a probability from a Bernoulli distribution (think coin flip – only two possible outcomes) and then picks those edges that are below the threshold probability *prob*.

```

In[5]:= With[{m = 8, n = 13, prob = 0.47},
  gg = GridGraph[{m, n}];
  probs = RandomVariate[BernoulliDistribution[prob], EdgeCount[gg]];
  perc = Graph[Pick[EdgeList[gg], probs, 1]];
  HighlightGraph[gg, perc]

```



Turn the above code into a function `BondPercolation[{m, n}, prob, opts]` that outputs a graph like that above. Include a check on the arguments *m* and *n*, returning an appropriate message if they are not positive integers. Finally, have your function inherit options from `Graph` and pass them into the `GridGraph` function.

- Create a function `ColorResidues[seq]` that takes an amino acid sequence *seq* and returns a visualization like that below where the amino acids are colored according to a scheme such as *Amino*, where more polar residues are brighter and more nonpolar residues are colored darker. In addition to a `ColorScheme` option, add options to control the frame around each residue.

```

In[6]:= Amino = {{ "P", #1 }, { "W", #2 }, { "L", #3 }, { "V", #4 }, { "I", #5 }, { "N", #6 },
  { "Q", #7 }, { "S", #8 }, { "T", #9 }, { "C", #10 }, { "M", #11 }, { "H", #12 },
  { "A", #13 }, { "G", #14 }, { "F", #15 }, { "Y", #16 }, { "K", #17 }, { "R", #18 },
  { "D", #19 }, { "E", #20 } };

In[7]:= muc6 = StringTake[ProteinData["MUC6", "Sequence"], 75]
Out[7]= MVQRWLLSCCGALLSAGLANTSYPGLQRLKDSPTAPDKGCSTWGAGHFSTFDHHVYDFSGTCNYIFAATC

In[8]:= ColorResidues[muc6, ColorScheme -> Amino, FrameMargins -> 2]
Out[8]= 

```

## 6.3 Solutions

- Here is the message text that will be issued when the first argument to `RandomWalk` is not a positive integer.

```

In[1]:= RandomWalk::rwn = "Argument '1' is not a positive integer.";

```

And here is the message for the `Dimensions` option.



```
In[2]:= RandomWalk::baddim =
  "The value `1` of the option Dimension is not a positive integer.";
```

Because we now have several conditions to trap for (non positive integer first argument, bad value for Dimension option), it is best to use Which to catch the various conditions and set appropriate actions.

```
In[3]:= latticeWalk[steps_, dim_] := Module[{w, mat},
  mat = Join[IdentityMatrix[dim], -IdentityMatrix[dim]];
  w = Accumulate[RandomChoice[mat, steps]];
  If[dim == 1, Flatten[w], w]
]
```

To simplify the code it is useful to have an auxiliary predicate that tests if its argument is a positive integer.

```
In[4]:= PositiveIntegerQ[n_] := IntegerQ[n] && Positive[n]
In[5]:= Options[RandomWalk] = {Dimension -> 2};
In[6]:= RandomWalk[t_, OptionsPattern[]] := Module[{dim, latticeQ},
  dim = OptionValue[Dimension];
  Which[
    ! PositiveIntegerQ[t], Message[RandomWalk::rwn, t],
    ! PositiveIntegerQ[dim], Message[RandomWalk::baddim, dim],
    True, latticeWalk[t, dim]
  ]
]
```

First check that correct results are returned for correct values of the arguments and options.

```
In[7]:= RandomWalk[3]
Out[7]= {{1, 0}, {1, 1}, {1, 0}}

In[8]:= RandomWalk[4, Dimension -> 5]
Out[8]= {{0, 0, -1, 0, 0}, {0, 0, -1, 0, -1}, {0, 0, -1, 0, -2}, {0, 0, -2, 0, -2}}
```

Then check for bad input.

```
In[9]:= RandomWalk[4, Dimension -> -5]
RandomWalk::baddim : The value -5 of the option Dimension is not a positive integer.

In[10]:= RandomWalk[4.3, Dimension -> -5]
RandomWalk::rwn : Argument 4.3` is not a positive integer.
```

2. Start with a small list of odd numbers.

```
In[11]:= ints = Range[1, 100, 2]
```

```
Out[11]= {1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31,
          33, 35, 37, 39, 41, 43, 45, 47, 49, 51, 53, 55, 57, 59, 61, 63, 65,
          67, 69, 71, 73, 75, 77, 79, 81, 83, 85, 87, 89, 91, 93, 95, 97, 99}
```

On the first iteration, drop every third number, that is, drop 5, 11, 17, and so on.

```
In[12]:= p = ints[[2]];
          ints = Drop[ints, p ;; -1 ;; p]
```

```
Out[13]= {1, 3, 7, 9, 13, 15, 19, 21, 25, 27, 31, 33, 37, 39, 43, 45, 49,
          51, 55, 57, 61, 63, 67, 69, 73, 75, 79, 81, 85, 87, 91, 93, 97, 99}
```

Get the next number, 7, in the list `ints`; then drop every seventh number.

```
In[14]:= p = ints[[3]];
          ints = Drop[ints, p ;; -1 ;; p]
```

```
Out[15]= {1, 3, 7, 9, 13, 15, 21, 25, 27, 31, 33, 37, 43, 45,
          49, 51, 55, 57, 63, 67, 69, 73, 75, 79, 85, 87, 91, 93, 97, 99}
```

Iterate. You will need to be careful about the upper limit of the iterator `i`.

```
In[16]:= ints = Range[1, 1000, 2];
          Do[
            p = ints[[i]];
            ints = Drop[ints, p ;; -1 ;; p],
            {i, 2, 32}]
          ints
```

```
Out[18]= {1, 3, 7, 9, 13, 15, 21, 25, 31, 33, 37, 43, 49, 51, 63, 67, 69, 73, 75, 79,
          87, 93, 99, 105, 111, 115, 127, 129, 133, 135, 141, 151, 159, 163, 169,
          171, 189, 193, 195, 201, 205, 211, 219, 223, 231, 235, 237, 241, 259,
          261, 267, 273, 283, 285, 289, 297, 303, 307, 319, 321, 327, 331, 339, 349,
          357, 361, 367, 385, 391, 393, 399, 409, 415, 421, 427, 429, 433, 451, 463,
          475, 477, 483, 487, 489, 495, 511, 517, 519, 529, 535, 537, 541, 553, 559,
          577, 579, 583, 591, 601, 613, 615, 619, 621, 631, 639, 643, 645, 651, 655,
          673, 679, 685, 693, 699, 717, 723, 727, 729, 735, 739, 741, 745, 769, 777,
          781, 787, 801, 805, 819, 823, 831, 841, 855, 867, 873, 883, 885, 895, 897,
          903, 925, 927, 931, 933, 937, 957, 961, 975, 979, 981, 991, 993, 997}
```

It would be more efficient if you did not need to manually determine the upper limit of the iteration. A `While` loop is better for this task. The test checks that the value of the iterator has not gone past the length of the successively shortened lists.

```
In[19]:= LuckyNumbers[n_Integer?Positive] := Module[{p, i = 2, ints = Range[1, n, 2]},
  While[ints[[i]] < Length[ints],
    p = ints[[i]];
    ints = Drop[ints, p ;; -1 ;; p];
    i++];
  ints]
```

```
In[20]:= LuckyNumbers[1000]
```

```
Out[20]:= {1, 3, 7, 9, 13, 15, 21, 25, 31, 33, 37, 43, 49, 51, 63, 67, 69, 73, 75, 79,
  87, 93, 99, 105, 111, 115, 127, 129, 133, 135, 141, 151, 159, 163, 169,
  171, 189, 193, 195, 201, 205, 211, 219, 223, 231, 235, 237, 241, 259,
  261, 267, 273, 283, 285, 289, 297, 303, 307, 319, 321, 327, 331, 339, 349,
  357, 361, 367, 385, 391, 393, 399, 409, 415, 421, 427, 429, 433, 451, 463,
  475, 477, 483, 487, 489, 495, 511, 517, 519, 529, 535, 537, 541, 553, 559,
  577, 579, 583, 591, 601, 613, 615, 619, 621, 631, 639, 643, 645, 651, 655,
  673, 679, 685, 693, 699, 717, 723, 727, 729, 735, 739, 741, 745, 769, 777,
  781, 787, 801, 805, 819, 823, 831, 841, 855, 867, 873, 883, 885, 895, 897,
  903, 925, 927, 931, 933, 937, 957, 961, 975, 979, 981, 991, 993, 997}
```

This latter approach is also reasonably fast. Here is the time it takes to compute all lucky numbers less than one million; there are 71 918 of them.

```
In[21]:= Length[LuckyNumbers[106]] // Timing
```

```
Out[21]:= {0.302607, 71 918}
```

### 3. Start with a prototype logical expression.

```
In[22]:= Clear[A, B]
```

```
In[23]:= expr = (A || B) ==> C;
```

```
In[24]:= vars = {A, B, C};
```

List all the possible truth value assignments for the variables.

```
In[25]:= tuples = Tuples[{True, False}, Length[vars]]
```

```
Out[25]:= {{True, True, True}, {True, True, False},
  {True, False, True}, {True, False, False}, {False, True, True},
  {False, True, False}, {False, False, True}, {False, False, False}}
```

Next, create a list of rules, associating each of the triples of truth values with a triple of variables.

```
In[26]:= rules = Map[Thread[vars ==> #] &, tuples]
```

```
Out[26]:= {{A ==> True, B ==> True, C ==> True}, {A ==> True, B ==> True, C ==> False},
  {A ==> True, B ==> False, C ==> True}, {A ==> True, B ==> False, C ==> False},
  {A ==> False, B ==> True, C ==> True}, {A ==> False, B ==> True, C ==> False},
  {A ==> False, B ==> False, C ==> True}, {A ==> False, B ==> False, C ==> False}}
```

Replace the logical expression with each set of rules.

```
In[27]:= expr /. rules
```

```
Out[27]:= {True, False, True, False, True, False, True, True}
```

Put these last values at the end of each “row” of the tuples.

```
In[28]:= table = Transpose@Join[Transpose[tuples], {expr /. rules}]
```

```
Out[28]:= {{True, True, True, True}, {True, True, False, False},
           {True, False, True, True}, {True, False, False, False},
           {False, True, True, True}, {False, True, False, False},
           {False, False, True, True}, {False, False, False, True}}
```

Create a header for table.

```
In[29]:= head = Append[vars, TraditionalForm[expr]]
```

```
Out[29]:= {A, B, C,  $A \vee B \Rightarrow C$ }
```

Prepend head to table.

```
In[30]:= Prepend[table, head]
```

```
Out[30]:= {{A, B, C,  $A \vee B \Rightarrow C$ }, {True, True, True, True}, {True, True, False, False},
           {True, False, True, True}, {True, False, False, False},
           {False, True, True, True}, {False, True, False, False},
           {False, False, True, True}, {False, False, False, True}}
```

Pour into a grid.

```
In[31]:= Grid[Prepend[table, head]]
```

```
Out[31]=
      A      B      C       $A \vee B \Rightarrow C$ 
      True   True   True   True
      True   True   False  False
      True   False  True   True
      True   False  False  False
      False  True   True   True
      False  True   False  False
      False  False  True   True
      False  False  False  True
```

Replace True with "T" and False with "F".

```
In[32]:= Grid[Prepend[table /. {True -> "T", False -> "F"}, head]]
```

```
Out[32]=
      A  B  C   $A \vee B \Rightarrow C$ 
      T  T  T      T
      T  T  F      F
      T  F  T      T
      T  F  F      F
      F  T  T      T
      F  T  F      F
      F  F  T      T
      F  F  F      T
```

Add formatting via options to Grid.

```
In[33]:= Grid[Prepend[table /. {True -> "T", False -> "F"}, head],
  Frame -> True, Dividers -> {{-2 -> LightGray}, {2 -> LightGray}},
  ItemStyle -> {"Menu", 8}]
```

Out[33]=

A	B	C	$A \vee B \Rightarrow C$
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	T

Put the pieces together.

```
In[34]:= TruthTable[expr_, vars_] :=
  Module[{len = Length[vars], tuples, rules, table, head},
    tuples = Tuples[{True, False}, len];
    rules = Thread[vars -> #1] & /@ tuples;
    table = Transpose@Join[Transpose[tuples], {expr /. rules}];
    head = Append[vars, TraditionalForm[expr]];
    Grid[Prepend[table /. {True -> "T", False -> "F"}, head],
      Frame -> True, FrameStyle -> Thin,
      Dividers -> {{-2 -> LightGray}, {2 -> LightGray}},
      BaseStyle -> {"Menu", 8}]]
```

```
In[35]:= TruthTable[A ∧ B ⇒ ¬ C, {A, B, C}]
```

Out[35]=

A	B	C	$A \wedge B \Rightarrow \neg C$
T	T	T	F
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

Alternatively, the table of values can be generated directly with BooleanTable.

```
In[36]:= TruthTable[expr_, vars_] := Module[{table, head},
  table = BooleanTable[Flatten[{vars, expr}]];
  head = Append[vars, TraditionalForm[expr]];
  Grid[Prepend[table /. {True -> "T", False -> "F"}, head],
    Frame -> True, FrameStyle -> Thin,
    Dividers -> {{-2 -> LightGray}, {2 -> LightGray}},
    BaseStyle -> {"Menu", 8}]]
```

```
In[37]:= TruthTable[A ∧ B ⇒ ¬ C, {A, B, C}]
```

Out[37]=

A	B	C	$A \wedge B \Rightarrow \neg C$
T	T	T	F
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

4. `Position[lis, elem]` returns a list of positions at which *elem* occurs in *lis*. `Extract[lis, pos]` returns those elements whose positions are specified by `Position`.

```
In[38]:= NearTo[lis_List, elem_, {n_}] :=  
Module[{pos = Position[lis, elem]}, Extract[lis, {pos - n, pos + n}]]
```

```
In[39]:= NearTo[lis_List, elem_, n_] :=  
Module[{pos = Position[lis, elem]}, Extract[lis, Range[pos - n, pos + n]]]
```

```
In[40]:= chars = CharacterRange["a", "z"];
```

```
In[41]:= NearTo[chars, "q", {3}]
```

```
Out[41]= {{n}, {t}}
```

```
In[42]:= NearTo[chars, "q", 4]
```

```
Out[42]= {{m, n, o, p, q, r, s, t, u}}
```

The key to writing the distance function is to observe that it must be a function of two variables and return a numeric value (the distance metric). We are finding the difference of the positions of a target element in the list with the element in question, *y* and *x*, respectively in the pure function. The use of `[[1, 1]]` is to strip off extra braces returned by `Position`.

```
In[43]:= NearToN[lis_, elem_, n_] :=  
Nearest[lis, elem, {2 n + 1, n},  
DistanceFunction →  
Function[{x, y}, Abs[(Position[lis, y] - Position[lis, x])[[1, 1]]]]
```

```
In[44]:= NearToN[chars, "q", 4]
```

```
Out[44]= {q, p, r, o, s, n, t, m, u}
```

5. Rather than try to incorporate all the conditions into one rule, it is cleaner and more efficient to write separate rules for the cases where the input is prime or less than or equal to one.

```
In[45]:= SmithNumberQ[n_ /; n ≤ 1] := False
```

```
In[46]:= SmithNumberQ[n_?PrimeQ] := False
```

Given the factorization of a number, separate the prime bases from their exponents using `Transpose`.

```
In[47]:= lis = FactorInteger[852]
Out[47]= {{2, 2}, {3, 1}, {71, 1}}

In[48]:= Transpose[lis]
Out[48]= {{2, 3, 71}, {2, 1, 1}}
```

This multiplies the integer digits of each base by its multiplicity.

```
In[49]:= MapThread[IntegerDigits[#1] #2 &, %]
Out[49]= {{4}, {3}, {7, 1}}
```

Here is the sum.

```
In[50]:= Total[Flatten[%]]
Out[50]= 15
```

Check that it equals the sum of the digits of the original number:

```
In[51]:= Total[IntegerDigits[852]]
Out[51]= 15
```

This puts the pieces together for the general rule.

```
In[52]:= SmithNumberQ[n_] := With[{lis = FactorInteger[n]},
  Total[Flatten[MapThread[IntegerDigits[#1] #2 &, Transpose[lis]]]] ==
  Total[IntegerDigits[n]]]
```

Here are the Smith numbers less than 100.

```
In[53]:= Select[Range[100], SmithNumberQ]
Out[53]= {4, 22, 27, 58, 85, 94}
```

There are 376 Smith numbers less than 10 000.

```
In[54]:= Select[Range[104], SmithNumberQ] // Length
Out[54]= 376
```

As an interesting aside, you can also generate Smith numbers using rep units (see Exercise 5 in Section 5.5). For example, multiply any prime repunit by a suitable factor, e.g., 1540. For details of the relationship between repunits and Smith numbers, see [Hoffman \(1999\)](#).

```
In[55]:= RepUnit[n_] := Nest[(10 # + 1) &, 1, n - 1]
In[56]:= PrimeQ[RepUnit[19]]
Out[56]= True

In[57]:= SmithNumberQ[1540 RepUnit[23]]
Out[57]= True
```

6. In the one-dimensional case, steps of unit length give the lattice walk described above. For our off-lattice walk, we will take step directions chosen to be any real number between  $-1$  and  $1$ . Of course, this means that for this case, steps are not of length one.

```
In[58]:= walk1DOffLattice[t_] := Accumulate[RandomReal[{-1, 1}, t]]
```

In the two-dimensional case, we essentially compute polar points and so the directions are polar angles between  $0$  and  $2\pi$ ; the coordinates of the points are given by the pair  $(\cos \theta, \sin \theta)$ , which gives steps of unit length.

```
In[59]:= walk2DOffLattice[t_] :=
  Accumulate[Map[{Cos[#], Sin[#]} &, RandomReal[{0, 2 \pi}, t]]]
```

Let us quickly check that each step is of length one.

```
In[60]:= walk2D = walk2DOffLattice[4]
```

```
Out[60]= {{-0.856498, -0.51615}, {-0.330181, -1.36644},
  {-1.32972, -1.33606}, {-0.945803, -2.25943}}
```

```
In[61]:= Partition[walk2D, 2, 1]
```

```
Out[61]= {{{-0.856498, -0.51615}, {-0.330181, -1.36644}},
  {{-0.330181, -1.36644}, {-1.32972, -1.33606}},
  {{-1.32972, -1.33606}, {-0.945803, -2.25943}}}
```

```
In[62]:= Apply[EuclideanDistance, %, 1]
```

```
Out[62]= {1., 1., 1.}
```

There are several different ways to approach the three-dimensional off-lattice walk. Using a spherical coordinate system, a point uniformly distributed on the sphere can be obtained from the following equations ([Marsaglia 1972](#)):

$$\begin{aligned}x &= \cos(\theta) \sqrt{1 - u^2}, \\y &= \sin(\theta) \sqrt{1 - u^2}, \\z &= u.\end{aligned}$$

We need to produce *pairs* of random numbers  $\theta$  and  $u$  with  $\theta$  in the interval  $[0, 2\pi)$  and  $u$  in the interval  $[-1, 1]$ . Here then is the function to generate  $t$  steps of an off-lattice random walk in three dimensions.

```
In[63]:= walk3DOffLattice[t_] :=
  Accumulate[
    Table[Function[{ \theta, u}, {Cos[\theta] \sqrt{1 - u^2}, Sin[\theta] \sqrt{1 - u^2}, u}] @@
      {RandomReal[{0, 2 \pi}], RandomReal[{-1, 1}]}, {t}]
  ]
```



Again, check that each step is of unit length.

```
In[64]:= walk3D = walk3DOffLattice[5]
Out[64]= {{-0.0516908, 0.919087, -0.390649},
          {-0.190662, 1.07929, -1.3679}, {0.404071, 1.86198, -1.55146},
          {1.37636, 1.74985, -1.75659}, {1.82819, 2.53074, -1.32523}}

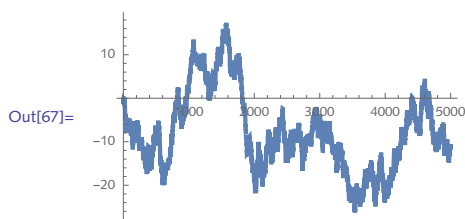
In[65]:= Apply[EuclideanDistance, Partition[walk3D, 2, 1], 1]
Out[65]= {1., 1., 1., 1.}
```

We now use the common elements to simplify our code, similarly to what we did earlier with the lattice walk code. The only difference amongst these three cases is the function that we are accumulating. We will use `Which` to slot in the appropriate function to `Accumulate`, based on the value of the dimension argument, `dim`.

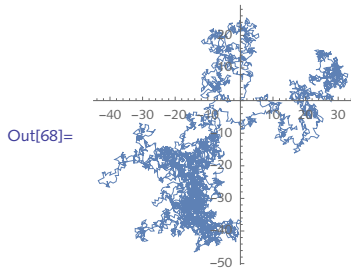
```
In[66]:= offLatticeWalk[t_, dim_] := Module[{f1, f2, f3},
  f1 := RandomReal[{-1, 1}, t];
  f2 := Map[{Cos[#], Sin[#]} &, RandomReal[{0, 2 π}, t]];
  f3 := Table[Function[{θ, u}, {Cos[θ] √(1 - u²), Sin[θ] √(1 - u²), u}] @@
    {RandomReal[{0, 2 π}], RandomReal[{-1, 1}]}, {t}];
  Which[
    dim == 1, Accumulate[f1],
    dim == 2, Accumulate[f2],
    dim == 3, Accumulate[f3]
  ]
]
```

Try out the code for dimensions one through three.

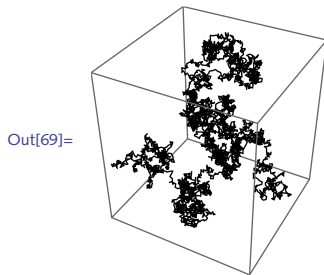
```
In[67]:= ListLinePlot[offLatticeWalk[5000, 1]]
```



```
In[68]:= ListLinePlot[offLatticeWalk[5000, 2], AspectRatio → Automatic,
  PlotStyle → Thin]
```

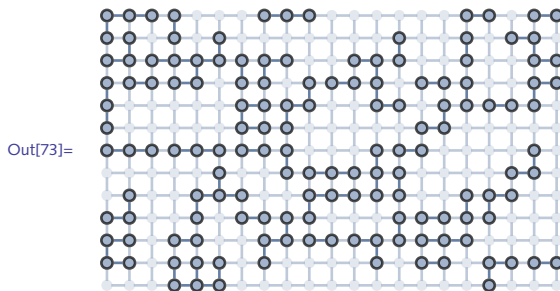


```
In[69]:= Graphics3D[Line@offLatticeWalk[5000, 3]]
```



7. Here is the function for running bond percolation simulations taken from the exercise. Options are inherited from GridGraph.

```
In[70]:= Clear[BondPercolation];
In[71]:= Options[BondPercolation] = Options[GridGraph];
In[72]:= BondPercolation[{m_, n_, prob_}, opts : OptionsPattern[]] := Module[{gg, gr},
  gg = GridGraph[{m, n}, opts];
  gr =
    Graph[Pick[EdgeList[gg], RandomVariate[BernoulliDistribution[prob],
      EdgeCount[gg], 1]],
      HighlightGraph[gg, gr, GraphHighlightStyle → "DehighlightFade"]]
In[73]:= BondPercolation[{13, 21, .2}, VertexSize → .5]
```



Add a check on the arguments  $m$  and  $n$  and issue a message if they are not positive integers.

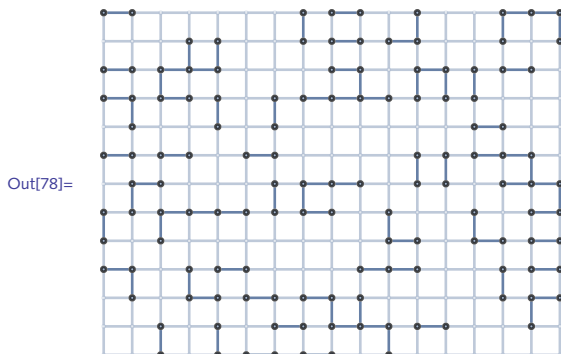
```
In[74]:= PositiveIntegerQ[n_] := IntegerQ[n] && Positive[n]

In[75]:= BondPercolation::badargs =
  "The parameters `1` and `2` must be positive integers.";

In[76]:= Clear[BondPercolation];
BondPercolation[{m_, n_, prob_}, opts : OptionsPattern[]] := Module[{gg, gr},
  If[AllTrue[{m, n}, PositiveIntegerQ],
    gg = GridGraph[{m, n}, opts];
    gr =
      Graph[Pick[EdgeList[gg], RandomVariate[BernoulliDistribution[prob],
        EdgeCount[gg], 1]], 1];
    HighlightGraph[gg, gr, GraphHighlightStyle -> "DehighlightFade"],

    Message[BondPercolation::badargs, m, n]
  ]
]

In[78]:= BondPercolation[{13, 17, .2}]
```























```
In[79]:= BondPercolation[{-13, 17, .2}]

BondPercolation::badargs: The parameters -13 and 17 must be positive integers.
```

8. Start with the color scheme. This is the *Amino* scheme used in *RasMol*. The color swatches can be entered as RGB values and then the swatch can be copied and pasted.

```
In[80]:= RGBColor[{220/255, 150/255, 130/255}]
```

Out[80]=

```
In[81]:= Amino = {{ "P", , { "W", , { "L", , { "V", , { "I", , { "N", ,
  { "Q", , { "S", , { "T", , { "C", , { "M", , { "H", ,
  { "A", , { "G", , { "F", , { "Y", , { "K", , { "R", ,
  { "D", , { "E", 
```

Set up the options framework. ColorResidues will inherit options from Framed and also have

a unique option to select different color schemes. Amino will be the default scheme.

```
In[82]:= Options[ColorResidues] = Join[{ColorScheme → Amino}, Options[Framed]];
```

Write a usage message for ColorResidues.

```
In[83]:= ColorResidues::usage =
  "ColorResidues[seq] displays a sequence of amino acids from the
  protein sequence seq.";
```

This creates an auxiliary function that turns the color scheme into a list of rules that will be applied in the ColorResidues function below.

```
In[84]:= makeAabox[scheme_, opts : OptionsPattern[]] := Module[{aa, col},
  {aa, col} = Transpose[scheme];
  MapThread[#1 → Framed[#1, opts, Background → #2, FrameStyle → #2] &, {aa, col}]]
```

```
In[85]:= makeAabox[Amino]
```

```
Out[85]= {P → P, W → W, L → L, V → V, I → I, N → N,
  Q → Q, S → S, T → T, C → C, M → M, H → H, A → A,
  G → G, F → F, Y → Y, K → K, R → R, D → D, E → E}
```

Finally, here is the function with the first 100 residues of the protein MUC6 as argument.

```
In[86]:= ColorResidues[str_String, opts : OptionsPattern[]] :=
  Module[{cs = OptionValue[ColorScheme]},
    Row[Characters[str] /. makeAabox[cs, FilterRules[{opts}, Options[Framed]]]]]
```

```
In[87]:= Map[{#, ProteinData["MUC6", #]} &,
  {"BiologicalProcesses", "Memberships", "MolecularFunctions"}] //
  TableForm
```

```
Out[87]//TableForm= BiologicalProcesses MaintenanceOfGastrointestinalEpithelium
  ExtracellularMatrixStructuralConstituent
  ExtracellularRegion
  Memberships MaintenanceOfGastrointestinalEpithelium
  Proteins
  MolecularFunctions ExtracellularMatrixStructuralConstituent
```

```
In[88]:= muc6 = StringTake[ProteinData["MUC6", "Sequence"], 100];
```

```
In[89]:= ColorResidues[muc6, ColorScheme → Amino, FrameMargins → 1]
```

```
Out[89]= M V Q R W L L L S C C G A L L S A G L A N T S Y T S P G L Q R L K D
  S P Q T A P D K G Q C S T W G A G H F S T F D H H V Y D F S G T C
  N Y I F A A T C K D A F P T F S V Q L R R G P D G S I S R I I V E
```

---

## 7 Strings

### 7.1 Structure and syntax: exercises

1. Convert the first character in a string (which you may assume to be a lowercase letter) to uppercase.
2. Given a string of digits, for example, "10495", convert it to its integer value.
3. Create a function `UniqueCharacters[str]` that takes a string as its argument and returns a list of the unique characters in that string. For example, `UniqueCharacters["Mississippi"]` should return `{M, i, s, p}`.
4. Using a first-name database, find all names consisting entirely of combinations of the first letters on the keys of a typical phone (Figure 7.1), that is, names that only contain the letters *a, d, g, j, m, p, t, and w*; for example, *PAT*. As a database source, the US Census Bureau curates lists of common names.

FIGURE 7.1. *Telephone number pad.*



5. A somewhat simplistic cipher, known as the XOR cipher, uses binary eight-bit keys to encode strings. The idea is to first convert each letter in a plaintext string to their character code in eight-bit binary. So the letter A, whose character code is 65, is converted into 1000001. A key string, say the letter K, is similarly converted to an eight-bit binary representation. Then each letter in the plaintext is encoded by performing a bit XOR operation on the plaintext letter and the key, both using their eight-bit binary representation. The resulting ciphertext could remain in binary or it could be converted back from character codes to encoded text, the ciphertext.

Create an XOR cipher and encode a plaintext string to produce a ciphertext. See Exercise 6, Section 2.2 for information on converting a string to its binary representation. As an aside, the XOR cipher is a terribly insecure cipher as simply reversing the encoding operations makes it easy to recover the key ([Churchhouse 2001](#)).

## 7.1 Solutions

- I. Here is a test string we will use for this exercise.

```
In[1]:= str = "this is a test string"
```

```
Out[1]= this is a test string
```

This extracts the first character from `str`.

```
In[2]:= StringTake[str, 1]
```

```
Out[2]= t
```

Here is its character code.

```
In[3]:= ToCharacterCode[%]
```

```
Out[3]= {116}
```

For each lowercase letter of the English alphabet, subtracting 32 gives the corresponding uppercase character.

```
In[4]:= % - 32
```

```
Out[4]= {84}
```

Convert back to a character.

```
In[5]:= FromCharacterCode[%]
```

```
Out[5]= T
```

Take the original string minus its first character.

```
In[6]:= StringDrop[str, 1]
```

```
Out[6]= his is a test string
```

Finally, join the previous string with the capital `T`.

```
In[7]:= StringJoin[%%, %]
```

```
Out[7]= This is a test string
```

You can do this more efficiently using `ToUpperCase` and `StringTake`. This approach is more general in that it does not assume that the first character in your string is lower case.

```

In[8]:= ToUpperCase[StringTake[str, 1]]
Out[8]= T

In[9]:= StringTake[str, 2 ;; -1]
Out[9]= his is a test string

In[10]:= ToUpperCase[StringTake[str, 1]] <> StringTake[str, 2 ;; -1]
Out[10]= This is a test string

```

This can be done most directly with `Capitalize` which automatically handles many of the issues discussed above.

```

In[11]:= Capitalize[str]
Out[11]= This is a test string

```

2. One approach converts the string to character codes.

```

In[12]:= ToCharacterCode["10495"]
Out[12]= {49, 48, 52, 57, 53}

In[13]:= % - 48
Out[13]= {1, 0, 4, 9, 5}

In[14]:= Table[10j, {j, 4, 0, -1}]
Out[14]= {10 000, 1000, 100, 10, 1}

In[15]:= %.%%
Out[15]= 10 495

```

This is a good place to use `Fold`. Using `FoldList`, you can see how the expression is built up.

```

In[16]:= FoldList[#2 + 10 #1 &, 0, ToCharacterCode["10495"] - 48]
Out[16]= {0, 1, 10, 104, 1049, 10 495}

```

Much more directly, use `ToExpression`.

```

In[17]:= ToExpression["10495"]
Out[17]= 10 495

```

3. Start by extracting the individual characters in a string.

```

In[18]:= str = "Mississippi";
          Characters[str]
Out[19]= {M, i, s, s, i, s, s, i, p, p, i}

```

This gives the set of unique characters in this string.

```
In[20]:= Union[Characters[str]]
```

```
Out[20]= {i, M, p, s}
```

Union sorts the list whereas DeleteDuplicates does not.

```
In[21]:= DeleteDuplicates[Characters[str]]
```

```
Out[21]= {M, i, s, p}
```

Here then is the function.

```
In[22]:= UniqueCharacters[str_String] := DeleteDuplicates[Characters[str]]
```

Try it out on a more interesting example.

```
In[23]:= protein = ProteinData["PP2672"]
```

```
Out[23]= MKSSEELQCLKQMEEEELFLKAGQGSQRARLTPLPRALQGNFGAPALCGIWF AEHLHPAVGMPPNYNSSMLSLSPERT :
          ILSGWWSGKQTQQPVPLRLTLLRSPFSLHKSSQPGSPKASQRIHPLFHSIPRSQLHSVLLGLPLLFITRPSPPA :
          QYGAQMPLRYICFPNIFWGSKKPQKE
```

```
In[24]:= UniqueCharacters[protein]
```

```
Out[24]= {M, K, S, E, L, Q, C, F, A, G, R, T, P, N, I, W, H, V, Y}
```

It even works in the degenerate case.

```
In[25]:= UniqueCharacters[""]
```

```
Out[25]= {}
```

4. The U.S. Census Bureau curates lists of common first and last names. This imports a list of common female first names.

```
In[26]:= data =
```

```
  Import[
    "http://www2.census.gov/topics/genealogy/1990surnames/dist.female.first",
    {"Table"}];
```

The names themselves are in the first column.

```
In[27]:= names = data[[All, 1]];
```

```
RandomChoice[names, 12]
```

```
Out[28]= {CARISA, EXIE, MICHELIN, RENEE, BRETT,
          INA, KATHE, ELYSE, ANDREE, LELIA, THOMASENA, KARRI}
```

The first letters on the keys on the phone pad are the following. We have input them here all uppercase as the names from our database are in that format. You may have to make adjustments to the case (ToUpperCase, ToLowerCase) depending upon your sources.

```
In[29]:= firstLetters = {"A", "D", "G", "J", "M", "P", "T", "W"};
```

Here are the letters in one of the names in the database.



```
In[30]:= nChars = Characters["PAT"]
```

```
Out[30]= {P, A, T}
```

This list of letters is a subset of the first letters from the phone pad.

```
In[31]:= SubsetQ[firstLetters, nChars]
```

```
Out[31]= True
```

Here then is a predicate that checks if a name consists of letters that are a subset of the phone pad first letters.

```
In[32]:= PhoneNameQ[name_String] :=
Module[{nChars, firstLetters = {"A", "D", "G", "J", "M", "P", "T", "W"}},
  nChars = Characters[name];
  SubsetQ[firstLetters, nChars]
]
```

```
In[33]:= PhoneNameQ["PAT"]
```

```
Out[33]= True
```

```
In[34]:= Length[names]
```

```
Out[34]= 4275
```

```
In[35]:= lis = Select[names, PhoneNameQ]
```

```
Out[35]= {ADA, PAM, PAT, MA, MAGDA, JADA, AMADA, AJA, TAM, PA, JA, TAMA, ADAM, TA, JAMA}
```

5. First convert a string to binary 8-bit.

```
In[36]:= text = IntegerDigits[ToCharacterCode["Orange"], 2, 8]
```

```
Out[36]= {{0, 1, 0, 0, 1, 1, 1, 1}, {0, 1, 1, 1, 0, 0, 1, 0}, {0, 1, 1, 0, 0, 0, 0, 1},
  {0, 1, 1, 0, 1, 1, 1, 0}, {0, 1, 1, 0, 0, 1, 1, 1}, {0, 1, 1, 0, 0, 1, 0, 1}}
```

Now we need an 8-bit key to do the encoding. It could be a random sequence of bits or it might be a letter in binary Ascii. For this example, our key will be the binary Ascii code for the letter K.

```
In[37]:= key = First@IntegerDigits[ToCharacterCode["K"], 2, 8]
```

```
Out[37]= {0, 1, 0, 0, 1, 0, 1, 1}
```

Map the BitXor operator with this key across each of the binary representations of the letters in text.

```
In[38]:= lis = Map[BitXor[#, key] &, text]
```

```
Out[38]= {{0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 1, 1, 1, 0, 0, 1}, {0, 0, 1, 0, 1, 0, 1, 0},
  {0, 0, 1, 0, 0, 1, 0, 1}, {0, 0, 1, 0, 1, 1, 0, 0}, {0, 0, 1, 0, 1, 1, 1, 0}}
```

Convert from binary to the decimal character codes.

```
In[39]:= Map[FromDigits[#, 2] &, lis]
```

```
Out[39]= {4, 57, 42, 37, 44, 46}
```

Convert from character codes to Ascii letters. Although only five characters appear to be in the ciphertext below, there are in fact six characters; the first character is a nonprintable Ascii character with character code 4.

```
In[40]:= ciphertext = FromCharCode[%]
```

```
Out[40]= \.049*%, .
```

This puts the pieces together in a function.

```
In[41]:= XorEncode[text_String, key_String] := Module[{lis, bitText, bitKey},
  bitText = IntegerDigits[ToCharacterCode[text], 2, 8];
  bitKey = First@IntegerDigits[ToCharacterCode[key], 2, 8];
  lis = Map[BitXor[#, bitKey] &, bitText];
  FromCharCode[Map[FromDigits[#, 2] &, lis]]
]
```

```
In[42]:= XorEncode["Orange", "K"]
```

```
Out[42]= \.049*%, .
```

## 7.2 Operations on strings: exercises

1. Create a function `PalindromeQ[str]` that returns a value of `True` if its argument `str` is a palindrome, that is, if the string `str` is the same forward and backward. For example, the word *refer* is a palindrome.
2. Several dozen words in an English dictionary contain two consecutive double letters: *balloon*, *coffee*, *succeed*, for example. Find all words in the dictionary that contain three consecutive double letters. This puzzle appeared on the *Car Talk* radio show ([CarTalk 2007](#)).
3. Given two strings of equal length, create a function `StringTranspose` that transposes the characters in the two strings and then joins them into a single string.

```
In[1]:= StringTranspose["abc", "def"]
```

```
Out[1]= adbecf
```

4. The built-in function `StringRotateLeft` rotates the characters in a string by a specified amount.

```
In[2]:= StringRotateLeft["a quark for Muster Mark ", 8]
```

```
Out[2]= for Muster Mark a quark
```

Perform the same operation without using `StringRotateLeft`.

5. Create a function `StringPermutations [str]` that returns all permutations of the string `str`. For example:

```
In[3]:= StringPermutations["ABC"] // InputForm
```

```
Out[3]//InputForm= {ABC, ACB, BAC, BCA, CAB, CBA}
```

6. Rewrite the function `SmarandacheWellin` from Exercise 10 of Section 2.3 to instead use strings to construct these numbers. Test your implementation for speed against the function from Section 2.3 that uses numerical functions only.
7. When developing algorithms that operate on large structures (for example, large systems of equations), it is often helpful to be able to create a set of unique symbols with which to work. Create a function `MakeVarList` that creates unique symbols. For example:

```
In[4]:= MakeVarList[x, 8]
```

```
Out[4]= {x1, x2, x3, x4, x5, x6, x7, x8}
```

```
In[5]:= MakeVarList[var, {10, 15}]
```

```
Out[5]= {var10, var11, var12, var13, var14, var15}
```

8. Create a function `StringTally` that counts each unique character in a string and returns a list similar to that returned by the built-in `Tally` function. Include the option `IgnoreCase` with default value `False`. When set to `True`, your function should convert all characters to lower-case before doing the tally.

```
In[6]:= StringTally["One fish, two fish, red fish, blue fish"]
```

```
Out[6]= {{0, 1}, {n, 1}, {e, 3}, { , 7}, {f, 4}, {i, 4}, {s, 4}, {h, 4},
{., 3}, {t, 1}, {w, 1}, {o, 1}, {r, 1}, {d, 1}, {b, 1}, {l, 1}, {u, 1}}
```

```
In[7]:= StringTally["One fish, two fish, red fish, blue fish", IgnoreCase -> True]
```

```
Out[7]= {{o, 2}, {n, 1}, {e, 3}, { , 7}, {f, 4}, {i, 4}, {s, 4}, {h, 4},
{., 3}, {t, 1}, {w, 1}, {r, 1}, {d, 1}, {b, 1}, {l, 1}, {u, 1}}
```

When done, import a sample text and do a frequency analysis on the letters in that text. Letter frequency analysis can be used to spot transposition ciphers in encoded messages as the frequency of the letters is unchanged in such simple encoding schemes.

Exercise 9 in Section 7.4 asks you to extend this function to include an option to specify if punctuation and digits should be included.

9. Generalize the Caesar cipher so that it encodes by shifting  $n$  places to the right. Include the space character in the alphabet.
10. A mixed-alphabet cipher is created by first writing a keyword followed by the remaining letters of the alphabet and then using this key to encode the text. For example, if the keyword is *django*, the encoding alphabet would be

djangobcefhiklmpqrstuvwxyz

So, *a* is replaced with *d*, *b* is replaced with *j*, *c* is replaced with *a*, and so on. As an example, the text

*the sheik of araby*

would be encoded as

*tcg scgeh mo drdijy*

Implement this cipher and go one step further to output the ciphertext in blocks of length five, omitting spaces and punctuation.

11. Modify the alphabet permutation cipher so that, instead of being based on single letters, it is based on adjacent pairs of letters. The single letter cipher has  $26!$  permutations:

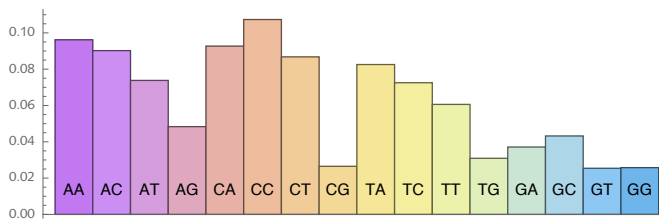
$$26! = 403\,291\,461\,126\,605\,635\,584\,000\,000$$

The adjacent pairs cipher will have  $26^2! = 1.883707684133810 \times 10^{1621}$  permutations.

12. Create a function `StringPad[str, {n}]` that pads the end of a string with *n* whitespace characters. Then create a second rule `StringPad[str, n]` that pads the string out to length *n*. If the input string has length greater than *n*, issue a warning message. Finally, mirroring the argument structure for the built-in `PadLeft`, create a third rule `StringPad[str, n, m]` that pads with *n* whitespaces at the front and *m* whitespaces at the end of the string.
13. Fibonacci words are formed in a similar manner as Fibonacci numbers except, instead of adding the previous two elements, Fibonacci words *concatenate* the previous two elements (Knuth 1997). Starting with the two strings “o” and “oi,” create a function `FibonacciWord[n]` to generate the *n*th Fibonacci word. This can be generalized to start with any two strings, say “a” and “b.” Fibonacci words are examples of a well-known object from combinatorics, Sturmian words.
14. In Exercise 8, Section 5.1 a function was created to generate *n*-grams from a given alphabet. For example, that function could be used to create all bigrams (words of length two) from the nucleotide alphabet { “G”, “C”, “A”, “T” }.

Import a nucleotide sequence such as the human mitochondrial genome `hsmi` to below and then create a histogram (as in Figure 7.2) showing the frequency of each of the sixteen possible bigrams AA, AC, AT, etc.

```
In[8]:= hsmi = First@Import["ExampleData/mitochondrion.fa.gz"] // Short
Out[8]//Short= GATCACAGGTCTATACCCT ... CTAAATAAGACATCACGATG
```

FIGURE 7.2. Distribution of nucleotide words of length two in *Homo sapiens* mitochondria genome.

## 7.2 Solutions

- Here is the function that checks if a string is a palindrome.

```
In[1]:= PalindromeQ[str_String] := StringReverse[str] == str
```

```
In[2]:= PalindromeQ["mood"]
```

```
Out[2]= False
```

```
In[3]:= PalindromeQ["PoP"]
```

```
Out[3]= True
```

Get all words in the dictionary that comes with *Mathematica*.

```
In[4]:= words = DictionaryLookup[];
```

Select those that pass the PalindromeQ test.

```
In[5]:= Select[words, PalindromeQ]
```

```
Out[5]= {a, aha, aka, bib, bob, boob, bub, CFC, civic, dad, deed, deified, did, dud, DVD,
eke, ere, eve, ewe, eye, gag, gig, huh, I, kayak, kook, level, ma'am, madam, mam,
MGM, minim, mom, mum, nan, non, noon, nun, oho, pap, peep, pep, pip, poop, pop,
pup, radar, redder, refer, repaper, revival, rotor, sagas, sees, seres, sexes,
shahs, sis, solos, SOS, stats, stets, tat, tenet, TNT, toot, tot, tut, wow, WWW}
```

An argument that is a number can be converted to a string and then the previous rule is called.

```
In[6]:= PalindromeQ[num_Integer] := PalindromeQ[ToString[num]]
```

```
In[7]:= PalindromeQ[12522521]
```

```
Out[7]= True
```

- First, we get the words in the dictionary.

```
In[8]:= words = DictionaryLookup[];
```

Here is the string pattern we are looking for: a letter (pattern named *x*) repeated once, followed by a letter (pattern named *y*) repeated once, followed by another letter repeated (pattern named *z*). The pattern can be passed directly to `DictionaryLookup`.

```
In[9]:= DictionaryLookup[___ ~~ x_ ~~ x_ ~~ y_ ~~ y_ ~~ z_ ~~ z_ ~~ ___]
Out[9]= {bookkeeper, bookkeepers, bookkeeping}
```

3. First, get the characters from the two strings.

```
In[10]:= With[{str1 = "abc", str2 = "def"},
  Characters[{str1, str2}]]
Out[10]= {{a, b, c}, {d, e, f}}
```

Then perform the transpose on these two lists.

```
In[11]:= Transpose[%]
Out[11]= {{a, d}, {b, e}, {c, f}}
```

StringJoin automatically flattens the sublists.

```
In[12]:= StringJoin[%]
Out[12]= adbecf

In[13]:= Clear[StringTranspose];
StringTranspose[str1_, str2_] :=
  StringJoin[Transpose[Characters[{str1, str2}]]]

In[15]:= StringTranspose["abc", "def"]
Out[15]= adbecf
```

Transpose will, helpfully, return an error message if the two strings are of unequal length.

```
In[16]:= StringTranspose["abcd", "efg"]
Transpose::nmtx: The first two levels of {{a, b, c, d}, {e, f, g}} cannot be transposed. >>
StringJoin::string:
String expected at position 1 in StringJoin[Transpose[{{a, b, c, d}, {e, f, g}}]]. >>

Out[16]= StringJoin[Transpose[{{a, b, c, d}, {e, f, g}}]]
```

4. Use the argument structure of RotateLeft.

```
In[17]:= stringRotateLeft[str_, n_: 1] := StringJoin[RotateLeft[Characters[str], n]]

In[18]:= stringRotateLeft["squeamish ossifrage"]
Out[18]= queamish ossifrages

In[19]:= stringRotateLeft["squeamish ossifrage", 5]
Out[19]= mish ossifragesquea
```

5. Here is a list of the characters in a sample string.

```
In[20]:= chars = Characters["cold"]
Out[20]= {c, o, l, d}
```

And here are the permutations of this list of characters.

```
In[21]:= Permutations[%]
Out[21]:= {{c, o, l, d}, {c, o, d, l}, {c, l, o, d}, {c, l, d, o}, {c, d, o, l}, {c, d, l, o},
           {o, c, l, d}, {o, c, d, l}, {o, l, c, d}, {o, l, d, c}, {o, d, c, l}, {o, d, l, c},
           {l, c, o, d}, {l, c, d, o}, {l, o, c, d}, {l, o, d, c}, {l, d, c, o}, {l, d, o, c},
           {d, c, o, l}, {d, c, l, o}, {d, o, c, l}, {d, o, l, c}, {d, l, c, o}, {d, l, o, c}}
```

Finally, join each set of characters with StringJoin.

```
In[22]:= Map[StringJoin, %]
Out[22]:= {cold, codl, clod, cldo, cdol, cdlo, ocld, ocdl, olcd, oldc, odcl, odlc,
           lcod, lcdo, locd, lodc, ldco, ldoc, dcol, dclo, docl, dolc, dlco, dloc}
```

And here is the function putting these pieces together.

```
In[23]:= StringPermutations[str_String] := Map[StringJoin, Permutations[Characters[str]]]
In[24]:= StringPermutations["ACGT"]
Out[24]:= {ACGT, ACTG, AGCT, AGTC, ATCG, ATGC, CAGT, CATG, CGAT, CGTA, CTAG, CTGA,
           GACT, GATC, GCAT, GCTA, GTAC, GTCA, TACG, TAGC, TCAG, TCGA, TGAC, TGCA}
```

- The approach here, in comparison with that in Exercise 10 in Section 2.3, is to convert the integers to strings and operate on the strings. When done, we convert the string back to an integer. To start, here are the first few primes.

```
In[25]:= Prime[Range[10]]
Out[25]:= {2, 3, 5, 7, 11, 13, 17, 19, 23, 29}
```

Convert to strings and then concatenate.

```
In[26]:= Map[ToString, %] // InputForm
Out[26]//InputForm= {"2", "3", "5", "7", "11", "13", "17", "19", "23",
                    "29"}

In[27]:= StringJoin[%] // InputForm
Out[27]//InputForm= "2357111317192329"
```

Finally, convert the above string to a number.

```
In[28]:= ToExpression[%] // InputForm
Out[28]//InputForm= 2357111317192329

In[29]:= Prime[Range[12]]
Out[29]:= {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37}

In[30]:= NumberJoin[{x__}] := ToExpression[StringJoin[Map[ToString, {x}]]]
In[31]:= SmarandacheWellin2[n_Integer?Positive] := NumberJoin[Prime[Range[n]]]
```

```
In[32]:= SmarandacheWellin2[10]
```

```
Out[32]= 2 357 111 317 192 329
```

Compared with the solution from Section 2.3, the string-based solution is a bit faster, even with all the conversion from numbers to strings and back.

```
In[33]:= SmarandacheWellin[n_Integer?Positive] :=  
          FromDigits[Flatten[IntegerDigits@Table[Prime[i], {i, n}]]]
```

```
In[34]:= Timing[SmarandacheWellin[50000];]
```

```
Out[34]= {0.228937, Null}
```

```
In[35]:= Timing[SmarandacheWellin2[50000];]
```

```
Out[35]= {0.136362, Null}
```

```
In[36]:= SmarandacheWellin[50000] == SmarandacheWellin2[50000]
```

```
Out[36]= True
```

7. Although there is a built-in function, `Unique`, that does this, it has some limitations for this particular task.

```
In[37]:= Table[Unique["x"], {8}]
```

```
Out[37]= {x55, x56, x57, x58, x59, x60, x61, x62}
```

One potential limitation of `Unique` is that it uses the first unused symbol of a particular form. It does this to avoid overwriting existing symbols.

```
In[38]:= Table[Unique["x"], {8}]
```

```
Out[38]= {x63, x64, x65, x66, x67, x68, x69, x70}
```

However, if you want to explicitly create a list of indexed symbols with a set of specific indices, it is useful to create a different function. First, note that a string can be converted to a symbol using `ToExpression` or by wrapping the string in `Symbol`.

```
In[39]:= Head["x1"]
```

```
Out[39]= String
```

```
In[40]:= ToExpression["x1"] // Head
```

```
Out[40]= Symbol
```

```
In[41]:= Symbol["x1"] // Head
```

```
Out[41]= Symbol
```

`StringJoin` is used to concatenate strings. So, let us concatenate the variable with the index, first with one number and then with a range of numbers.



```
In[42]:= StringJoin["x", "8"] // FullForm
```

```
Out[42]//FullForm= "x8"
```

```
In[43]:= ToExpression[Map["x" <> ToString[#] &, Range[12]]]
```

```
Out[43]= {x1, x2, x3, x4, x5, x6, x7, x8, x9, x10, x11, x12}
```

Put all the pieces of code together.

```
In[44]:= MakeVarList[x_Symbol, n_Integer] :=
  ToExpression[Map[ToString[x] <> ToString[#] &, Range[n]]]
```

Let us create an additional rule for this function that takes a range specification as its second argument.

```
In[45]:= MakeVarList[x_Symbol, {n_Integer, m_Integer}] :=
  ToExpression[Map[ToString[x] <> ToString[#] &, Range[n, m]]]
```

```
In[46]:= MakeVarList[tmp, {20, 30}]
```

```
Out[46]= {tmp20, tmp21, tmp22, tmp23, tmp24, tmp25, tmp26, tmp27, tmp28, tmp29, tmp30}
```

Note that we have not been too careful about argument checking.

```
In[47]:= MakeVarList[tmp, {-2, 2}]
```

```
Out[47]= {-2 + tmp, -1 + tmp, tmp0, tmp1, tmp2}
```

Using a negative index is a problem when the string is converted using ToExpression.

```
In[48]:= Clear[x]
```

```
In[49]:= ToString[x] <> ToString[-2] // FullForm
```

```
Out[49]//FullForm= "x-2"
```

```
In[50]:= ToExpression[%] // FullForm
```

```
Out[50]//FullForm= Plus[-2, x]
```

Argument checking with a different pattern corrects this problem.

```
In[51]:= Clear[MakeVarList]
```

```
In[52]:= MakeVarList[x_Symbol, {n_Integer?Positive, m_Integer?Positive}] :=
  ToExpression[Map[ToString[x] <> ToString[#] &, Range[n, m]]]
```

```
In[53]:= MakeVarList[tmp, {2, 4}]
```

```
Out[53]= {tmp2, tmp3, tmp4}
```

8. Starting with one word, first get a list of the characters in the lowercase version of that word:

```
In[54]:= str = "Mississippi";
  Characters[ToLowerCase@str]
```

```
Out[55]= {m, i, s, s, i, s, s, i, p, p, i}
```

```
In[56]:= Tally[%]
```

```
Out[56]:= {{m, 1}, {i, 4}, {s, 4}, {p, 2}}
```

```
In[57]:= StringTally[str_] := Tally[Characters[ToLowerCase@str]]
```

To try this out on a large text, first import Lewis Carroll's *Alice's Adventures in Wonderland* from [gutenberg.org](http://www.gutenberg.org).

```
In[58]:= alice = Import["http://www.gutenberg.org/ebooks/28885.txt.utf-8", "Text"];
```

```
In[59]:= StringTally[alice] // Sort
```

```
Out[59]:= {{!, 457}, {@, 2}, {#, 1}, {%, 1}, {&, 2}, {*, 48}, {(, 89},
  {), 89}, {_, 514}, {-, 884}, {[, 41}, {], 41}, {., 1240}, {,, 2591},
  {;, 198}, {", 2307}, {?, 212}, {' , 802}, {/, 31}, {:, 277}, {
  , 4045}, { , 31914}, {0, 27}, {1, 77}, {2, 23}, {3, 22}, {4, 18}, {5, 21},
  {6, 16}, {7, 9}, {8, 31}, {9, 16}, {a, 10191}, {b, 1806}, {c, 3114},
  {d, 5678}, {e, 15877}, {f, 2446}, {g, 3029}, {h, 8102}, {i, 8949},
  {j, 238}, {k, 1314}, {l, 5456}, {m, 2546}, {n, 8358}, {o, 9775},
  {p, 2051}, {q, 237}, {r, 6881}, {s, 7498}, {t, 12637}, {u, 4134}, {ù, 1},
  {v, 1000}, {w, 3049}, {x, 185}, {y, 2672}, {z, 79}, {$, 2}, {., 13}}
```

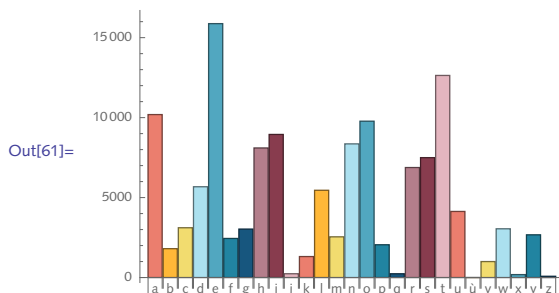
Let's sort on the characters, the first element in each sublist; then take the twenty-six pairs that give the Ascii letter tallies.

```
In[60]:= counts = Take[Sort[StringTally[alice]], -29 ;; -3]
```

```
Out[60]:= {{a, 10191}, {b, 1806}, {c, 3114}, {d, 5678}, {e, 15877}, {f, 2446},
  {g, 3029}, {h, 8102}, {i, 8949}, {j, 238}, {k, 1314}, {l, 5456}, {m, 2546},
  {n, 8358}, {o, 9775}, {p, 2051}, {q, 237}, {r, 6881}, {s, 7498}, {t, 12637},
  {u, 4134}, {ù, 1}, {v, 1000}, {w, 3049}, {x, 185}, {y, 2672}, {z, 79}}
```

Here is a bar chart of the tallies for each of these letters.

```
In[61]:= BarChart[counts[[All, 2]], ChartLabels -> counts[[All, 1]], ChartStyle -> 24]
```



- This is a simple modification of the code given in the text. But first we add the space character to the alphabet.

```

In[62]:= ToCharacterCode[" "]
Out[62]= {32}

In[63]:= alphabet = Join[{FromCharacterCode[32]}, CharacterRange["a", "z"]]
Out[63]= { , a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z}

```

Giving coderules an argument  $n$  will allow you to shift an arbitrary number of places. We give  $n$  a default value of 1 so that omitting the argument will default to the value  $n = 1$ .

```

In[64]:= coderules[n_ : 1] := Thread[alphabet → RotateRight[alphabet, n]]
In[65]:= decoderules[n_] := Map[Reverse, coderules[n]]
In[66]:= code[str_String, n_] := Apply[StringJoin, Characters[str] /. coderules[n]]
In[67]:= decode[str_String, n_] := Apply[StringJoin, Characters[str] /. decoderules[n]]
In[68]:= code["squeamish ossifrage", 5]
Out[68]= nlp whdncvjnnndamwb

In[69]:= decode[%, 5]
Out[69]= squeamish ossifrage

```

10. First, here is the list of characters from the plaintext alphabet.

```

In[70]:= PlainAlphabet = CharacterRange["a", "z"]
Out[70]= {a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z}

```

Here is our key, *django*:

```

In[71]:= key = "django"
Out[71]= django

```

And here is the cipher text alphabet, prepending the key:

```

In[72]:= StringJoin[Characters@key, Complement[PlainAlphabet, Characters@key]]
Out[72]= djangobcefhiklmpqrstuvwxyz

```

Make a reusable function.

```

In[73]:= CipherAlphabet[key_String] := With[{k = Characters[key]},
  StringJoin[k, Complement[CharacterRange["a", "z"], k]]]

```

Generate the coding rules:

```

In[74]:= codeRules = Thread[PlainAlphabet → Characters@CipherAlphabet["django"]]
Out[74]= {a → d, b → j, c → a, d → n, e → g, f → o, g → b, h → c, i → e, j → f, k → h, l → i, m → k,
  n → l, o → m, p → p, q → q, r → r, s → s, t → t, u → u, v → v, w → w, x → x, y → y, z → z}

```

The encoding function follows that in the text of this section.

```
In[75]:= encode[str_String] := StringJoin[Characters[str] /. codeRules]
In[76]:= encode["the sheik of araby"]
Out[76]= tcg scgeh mo drdjy
```

Omit spaces and punctuation and output in blocks of length 5 (using stringPartition from Section 7.5).

```
In[77]:= stringPartition[str_String, seq_]:=
  Map[StringJoin, Partition[Characters[str], seq]]
In[78]:= StringSplit[encode["the sheik of araby"], RegularExpression["\\W+"]]
Out[78]= {tcg, scgeh, mo, drdjy}
In[79]:= StringJoin[Riffle[stringPartition[StringJoin[%], 5, 5, 1, ""], " "]]
Out[79]= tcgsc gehmo drdjy
```

Finally, this puts all these pieces together.

```
In[80]:= Clear[encode];
encode[str_String, key_String, blocksize_: 5] :=
  Module[{CipherAlphabet, codeRules, s1, s2, s3},
    CipherAlphabet[k_] :=
      StringJoin[Characters[k], Complement[CharacterRange["a", "z"],
        Characters[k]]];
    codeRules =
      Thread[CharacterRange["a", "z"] → Characters@CipherAlphabet[key]];
    s1 = StringJoin[Characters[str] /. codeRules];
    s2 = StringSplit[s1, RegularExpression["\\W+"]];
    s3 = stringPartition[StringJoin[s2], blocksize, blocksize, 1, ""];
    StringJoin[Riffle[s3, " "]]]
In[82]:= encode["the sheik of araby", "django", 3]
Out[82]= tcg scg ehm odr djy
```

II. (\* solution to appear \*)

12. First, using StringJoin, put  $n$  spaces at the end of the string.

```
In[83]:= StringPad[str_String, {n_}] := StringJoin[str, Table[" ", {n}]]
In[84]:= StringPad["ciao", {5}] // FullForm
Out[84]//FullForm= "ciao "
```

For the second rule, first create a message that will be issued if the string is longer than the pad length,  $n$ .

```
In[85]:= StringPad::badlen =  
    "Pad length '1' must be greater than the length of string '2'.";
```

```
In[86]:= StringPad[str_String, n_] :=  
    With[{len = StringLength[str]},  
        If[len > n, Message[StringPad::badlen, n, str], StringPad[str, {n - len}]]]
```

```
In[87]:= StringPad["ciao", 8] // FullForm
```

```
Out[87]//FullForm= "ciao      "
```

```
In[88]:= StringPad["ciao", 3]
```

StringPad::badlen : Pad length 3 must be greater than the length of string ciao.

Finally, here is a rule for padding at the beginning and end of the string.

```
In[89]:= StringPad[str_String, n_, m_] :=  
StringJoin[Table[" ", {n}], str, Table[" ", {m}] ]  
  
In[90]:= StringPad["ciao", 3, 8] // FullForm  
  
Out[90]//FullForm= "      ciao      "
```

Note, `StringInsert` could also be used.

```
In[91]:= StringInsert["ciao", " ", {1, -1}] // FullForm
Out[91]//FullForm= " ciao "

In[92]:= StringPad2[str_String, n_, m_] :=
StringInsert[str, " ", Join[Table[1, {n}], T
In[93]:= StringPad2["ciao", 3, 8] // FullForm
Out[93]//FullForm= " ciao "
```

13. The definition looks very similar to that for Fibonacci numbers except that the first two values are strings and the operation is string concatenation (StringJoin).

[illegible]

There lots of interesting things that you can discover regarding Fibonacci words.

```
In[98]:= Table[StringCount[FibonacciWord[i], "1"], {i, 1, 15}]
Out[98]:= {0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377}
```

If you were generating very large Fibonacci words, it would be advisable to use dynamic programming as described in Section 5.3 to reduce the scope of the recursion.

14. Start by importing the example FASTA file containing the mitochondrial human genome sequence.

```
In[99]:= hsMito = First@Import["ExampleData/mitochondrion.fa.gz"];
```

Here is the NGrams function:

```
In[100]:= NGrams[alphabet : {__String}, n_Integer?Positive] :=  
  Flatten[Outer[StringJoin, Apply[Sequence, Table[alphabet, {n}]]]]
```

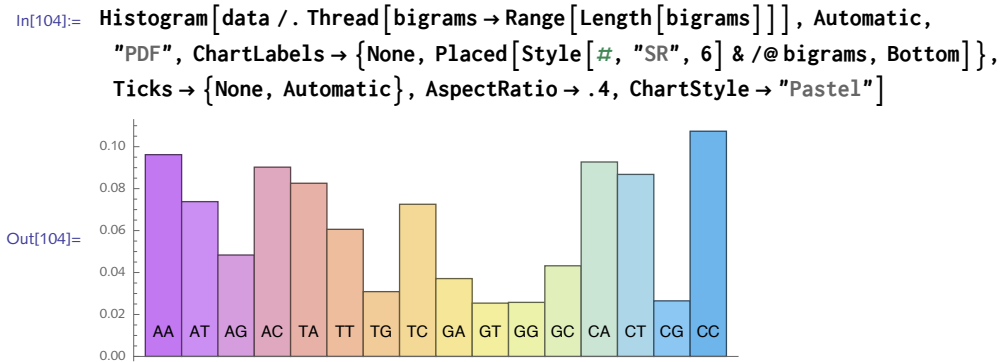
This gives all possible bigrams from the DNA alphabet.

```
In[101]:= bigrams = NGrams[{"A", "T", "G", "C"}, 2]  
Out[101]= {AA, AT, AG, AC, TA, TT, TG, TC, GA, GT, GG, GC, CA, CT, CG, CC}
```

This finds all the above bigrams in the sequence.

```
In[102]:= data = StringCases[hsMito, Alternatives@@bigrams, Overlaps -> True];  
In[103]:= RandomSample[data, 12]  
Out[103]= {TC, CC, GG, GG, AT, CC, AC, CT, CC, AC, AA, TC}
```

And here is the histogram plot.



### 7.3 String patterns: exercises

1. Given a list of words, some of which start with uppercase characters, convert them all to words in which the first character is lowercase. You can use the words in the dictionary as a good sample set.
2. Create a function `Palindromes[n]` that finds all palindromic words of length  $n$  in the dictionary. For example, *kayak* is a five-letter palindrome.

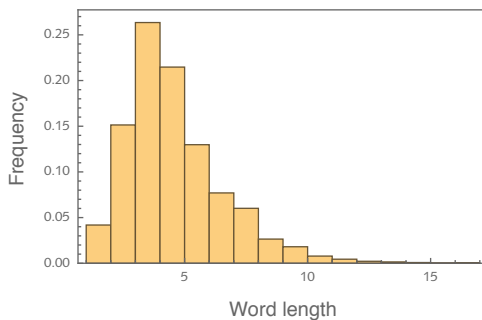
3. Modify the `listSort` function from Section 4.3 by creating another rule that can be used to sort lists of string characters.
4. Find the number of unique words in a body of text such as *Alice in Wonderland*. This text can be imported from `ExampleData`:

```
In[1]:= ExampleData[{"Text", "AliceInWonderland"}];
```

After splitting the text into words, convert all uppercase characters to lowercase so that you count words such as *hare* and *Hare* as the same word.

5. Another important task in computational linguistics is comparing the complexity of text-based materials such as newspapers or school texts. There are many measures that you might use: the length of the text, average sentence length, levels of reasoning required, word frequency, and many more. One metric is word length. Use the built-in function `TextWords` (new in *Mathematica* 10.1) to generate a histogram like that in Figure 7.3 showing the word-length distribution for a text.

FIGURE 7.3. Word-length distribution for *Alice in Wonderland*.



Then compare the word-length distribution for several different text sources such as those available in `ExampleData["Text"]` or on [gutenberg.org](http://gutenberg.org).

6. Repeat Exercise 5 but use sentence length instead of word length as the measure.
7. *Semordnilaps* ("palindromes" spelled backwards) are words that, when reversed, also spell a word. For example, *live* and *stressed* are semordnilaps because reversed, they still spell words: *evil* and *desserts*. Find all semordnilaps in the dictionary.

### 7.3 Solutions

1. We will work with a small sample of words from the dictionary.

```
In[1]:= SeedRandom[0];
        words = DictionaryLookup[];
        sample = RandomSample[words, 12]

Out[3]= {unaccustomed, Godhead, Jutland, dilated, observe, matchplay,
         wintergreen, clans, hobgoblin, rampantly, dis inheriting, performances}
```

StringReplace operates on any words that match the pattern and leave those that do not match unchanged.

```
In[4]:= StringReplace[sample, f_?UpperCaseQ ~~ r___ -> ToLowerCase[f] ~~ r]

Out[4]= {unaccustomed, godhead, jutland, dilated, observe, matchplay,
         wintergreen, clans, hobgoblin, rampantly, dis inheriting, performances}
```

But there is a function built in that does this directly.

```
In[5]:= ?Decapitalize
```

Decapitalize[string] yields a string in which the first character has been made lower case. >>

```
In[6]:= Decapitalize[sample]

Out[6]= {unaccustomed, godhead, jutland, dilated, observe, matchplay,
         wintergreen, clans, hobgoblin, rampantly, dis inheriting, performances}
```

2. You can do a dictionary lookup with a pattern that tests whether the word is palindromic. Then find all palindromic words of a given length. Note the need for BlankSequence (\_\_\_) as the simple pattern \_ would only find words consisting of one character.

```
In[7]:= Palindromes[len_Integer] :=
        DictionaryLookup[w___ /; (w == StringReverse[w] && StringLength[w] == len)]
```

We also add a rule to return all palindromes of any length.

```
In[8]:= Palindromes[] := DictionaryLookup[w___ /; (w == StringReverse[w])]

In[9]:= Palindromes[7]

Out[9]= {deified, repaper, reviver}

In[10]:= Palindromes[]

Out[10]= {a, aha, aka, bib, bob, boob, bub, CFC, civic, dad, deed, deified, did, dud, DVD,
          eke, ere, eve, ewe, eye, gag, gig, huh, I, kayak, kook, level, ma'am, madam, mam,
          MGM, minim, mom, mum, nan, non, noon, nun, oho, pap, peep, pep, pip, poop, pop,
          pup, radar, redder, refer, repaper, reviver, rotor, sagas, sees, seres, sexes,
          shahs, sis, solos, SOS, stats, stets, tat, tenet, TNT, toot, tot, tut, wow, WWW}
```



3. A simple change to the rule from Section 4.3 will allow for lists of string characters. Here we just add a rule to the original rule for sorting lists of numbers. As written, this only works for lists of lowercase letters.

```
In[11]:= listSort = {
  {x___, a_?StringQ, b_?StringQ, y___} =>
  {x, b, a, y} /; First[ToCharacterCode[b]] < First[ToCharacterCode[a]],
  {x___, a_?NumericQ, b_?NumericQ, y___} => {x, b, a, y} /; b < a
};
```

```
In[12]:= {82, 5, 29, 98, 98, 43, 90, 11, 52, 46, 38, 48} //. listSort
```

```
Out[12]:= {5, 11, 29, 38, 43, 46, 48, 52, 82, 90, 98, 98}
```

```
In[13]:= RandomSample@CharacterRange["a", "z"]
```

```
Out[13]:= {h, t, x, b, v, g, n, w, s, m, k, o, f, l, i, y, q, c, a, p, e, j, u, r, d, z}
```

```
In[14]:= strList = {"i", "w", "z", "p", "a", "s", "n", "l", "h", "q", "r", "d", "c",
  "f", "o", "y", "j", "u", "x", "t", "m", "b", "g", "e", "k", "v"};
```

```
In[15]:= strList //. listSort
```

```
Out[15]:= {a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z}
```

4. First import some sample text.

```
In[16]:= text = ExampleData[{"Text", "AliceInWonderland"}];
```

To split into words, use a similar construction to that in this section.

```
In[17]:= words = StringSplit[text, Characters[":;\n',.?/\-` *"] ..];
Short[words, 4]
```

```
Out[18]//Short= {I, DOWN, THE, RABBIT, HOLE, Alice, was, beginning, to, get, very,
  <<9949>>, as, well, she, might, what, a, wonderful, dream, it, had, been}
```

Get the total number of (nonunique) words.

```
In[19]:= Length[words]
```

```
Out[19]= 9971
```

Convert uppercase to lowercase.

```
In[20]:= lcwords = ToLowerCase[words];
Short[lcwords, 4]
```

```
Out[21]//Short= {i, down, the, rabbit, hole, alice, was, beginning, to, get, very,
  <<9949>>, as, well, she, might, what, a, wonderful, dream, it, had, been}
```

Finally, count the number of unique words.

```
In[22]:= DeleteDuplicates[lcwords] // Length
```

```
Out[22]= 1643
```

In fact, splitting words using a list of characters as we have done here is not terribly robust. A better approach uses regular expressions (introduced in Section 7.4):

```
In[23]:= words = StringSplit[text, RegularExpression["\\W+"]];
Length[words]
```

```
Out[24]= 9969
```

```
In[25]:= lcwords = StringReplace[words,
    RegularExpression["([A-Z])"] -> ToLowerCase["$1"]];
DeleteDuplicates[lcwords] // Length
```

```
Out[26]= 1528
```

Using TextWords, this can be done directly, although the count is a bit different due to slightly different assumptions between what we used in StringSplit and what TextWords assumes.

```
In[27]:= words = DeleteDuplicates@ToLowerCase[TextWords[text]];
Length[words]
```

```
Out[28]= 1549
```

5. We will work with the following text: *A Portrait of the Artist as a Young Man*, by James Joyce (available at [Project Gutenberg](#)). We use TextWords to get a list of the words and then use StringLength on these words to get the distribution of word lengths.

Some postprocessing with StringTake is needed to remove metadata at the beginning and at the end of the file.

```
In[29]:= joyce = StringTake[Import["http://www.gutenberg.org/cache/epub/4217/pg4217.txt",
    "Text"], 688 ;; -18843];
StringTake[joyce, {74, 164}]
```

```
Out[30]= as animum dimittit in artes."
Ovid, Metamorphoses, VIII., 18._
</p>
```

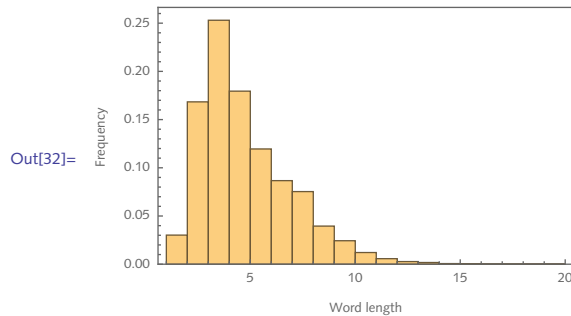
Chapter 1

Once upon

```

In[31]:= words = TextWords[joyce];
Histogram[StringLength[words], Automatic, "PDF", Frame → True,
  FrameLabel → {"Word length", "Frequency"},
  FrameTicks → {{Automatic, None}, {Automatic, None}}]

```

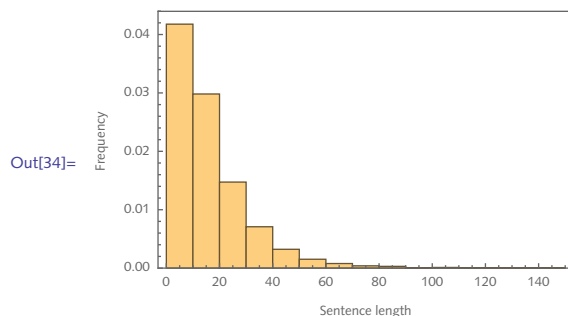


6. Here we use `TextSentences` to split the text into discrete sentences and then map `WordCount` over the list of sentences.

```

In[33]:= sentences = TextSentences[joyce];
In[34]:= Histogram[Map[WordCount, sentences], Automatic, "PDF",
  Frame → True, FrameLabel → {"Sentence length", "Frequency"},
  FrameTicks → {{Automatic, None}, {Automatic, None}}]

```



7. First, here is a list of all words in the dictionary.

```

In[35]:= words = DictionaryLookup[__];

```

And this reverses those words.

```

In[36]:= revwords = StringReverse[words];
RandomSample[revwords, 12]

```

```

Out[37]= {gniniatniam, ezilaitini, ecilprus, etadrohc, sriahchsup,
  aerdnA, srepod, dedrof, eimmiJ, decudorper, dnuorgwohs, straehteews}

```

Finally, here are all those reversed words that are in the dictionary.

```
In[38]:= Intersection[words, revwords]
```

```
Out[38]= {a, abut, agar, ah, aha, aka, am, animal, are, at, ate, auks, avid, bad, bag, ban,
bard, bat, bats, bed, bib, bin, bob, bod, bog, bonk, boob, boy, brag, bub, bud,
bun, buns, bur, burg, bus, but, buts, cam, CFC, civic, cod, dab, dad, dag,
dam, dart, deb, debut, decaf, decal, deed, deem, deep, deeps, deer, deffer,
deified, deliver, denier, denies, denim, deres, desserts, devil, dew, dial,
dialer, diaper, did, dim, diva, dob, doc, dog, don, doom, door, dos, drab,
draw, drawer, draws, dray, dual, dub, dud, DVD, edit, eel, eh, eke, em, emir,
emit, er, era, ere, ergo, eta, etas, eve, evil, eviler, ewe, eye, faced, fer,
fires, flog, flow, gab, gad, gag, gal, gals, gar, garb, gas, gel, gem, gig,
girt, gnat, gnus, gob, god, golf, got, grub, gulp, gulper, gum, gums, guns,
gut, ha, hap, he, ho, hoop, huh, I, it, jar, kayak, keel, keels, keep, knits,
knob, know, KO, kook, laced, lag, lager, laid, lair, lamina, lap, laud, lee,
leek, leer, leg, leper, level, lever, liar, lit, live, lived, loop, loops,
loot, looter, loots, lop, ma, ma'am, mac, macs, mad, madam, mam, maps, mar,
mart, mat, maws, may, me, meed, meet, meg, MGM, mid, mils, mined, minim, mom,
mood, moor, mop, mot, mu, mug, mum, nab, nan, nap, naps, net, new, nib, nip,
nips, nit, no, nod, non, noon, not, now, nub, nun, nus, nut, nuts, ogre, oh,
oho, OK, on, oohs, pacer, pah, pal, pals, pan, pans, pap, par, part, parts,
pas, pat, paws, pay, peed, peek, peels, peep, pees, pep, per, perts, pets, pin,
pins, pip, pis, pit, plug, pol, pols, pom, pooh, pool, pools, poop, pop, ports,
pot, pots, prat, pup, pus, radar, rag, raga, rail, raj, ram, rap, raps, rat,
rats, raw, re, rebut, recap, recaps, redder, redraw, reed, reel, ref, refer,
reffed, regal, reined, relaid, relive, remit, rennet, rep, repaid, repaper,
repel, replug, retool, retros, revel, reviled, revival, reward, rial, rime,
rood, room, rot, rotor, rub, sag, sagas, sap, saps, sate, saw, scam, seep,
sees, seined, sered, seres, serif, sexes, shahs, shoo, sip, sis, six, skua,
slag, slap, sleek, sleep, sleets, slim, sloop, sloops, slop, smart, smug, smut,
snap, snaps, snip, snips, snit, snoops, snot, snub, snug, sod, solos, sorter,
SOS, spacer, spam, span, spans, spar, spas, spat, spay, speed, spin, spins,
spit, spool, spools, spoons, sports, spot, spots, sprat, stab, star, stats,
steels, step, stets, stew, stink, stool, stop, stops, stows, strap, straw,
strep, stressed, strop, strops, stub, stun, sub, sun, sung, sup, swam, swap,
sward, sway, swot, swots, ta, tab, tam, tang, tap, taps, tar, tarp, tarps,
tat, teem, ten, tenet, tenner, ti, tide, til, time, timer, tin, tins, tip,
tips, TNT, tog, tom, ton, tons, tool, toot, top, tops, tor, tort, tot, tow,
tows, trad, tram, trams, trap, trig, trot, throw, tub, tuba, tubed, tuber, tug,
tums, tun, tut, um, war, ward, warder, warts, was, way, wed, wen, wets, wolf,
won, wonk, wort, wot, wow, WWW, xis, yam, yap, yaps, yard, yaw, yaws, yob}
```

A word of warning: if you tried to do this by comparing every word in the dictionary with every reversed word in the dictionary, the number of comparisons would be extremely large and bog down the computation.

```
In[39]:= TimeConstrained[
  DictionaryLookup[word__ /; MemberQ[words, StringReverse[word]]],
  20]
Out[39]= $Aborted
```

## 7.4 Regular expressions: exercises

1. Use regular expressions to count the occurrences of the vowels (*a, e, i, o, u*) in the following text:

```
In[1]:= text = "How many vowels do I have?";
```

2. Using regular expressions, find all words in the text *Alice in Wonderland* that contain the letter *q*. Then find all words that contain either *q* or *Q*. The text can be imported with the following:

```
In[2]:= words = TextWords[ExampleData[{"Text", "AliceInWonderland"}]];
Take[words, 11]
Out[3]= {I, DOWN, THE, RABBIT-HOLE, Alice, was, beginning, to, get, very, tired}
```

3. Rewrite the genomic example in Section 7.3 to use regular expressions instead of string patterns to find all occurrences of the sequence *AAanythingT*. Here is the example using general string patterns:

```
In[4]:= gene = GenomeData["IGHV357"];
StringCases[gene, "AA" ~~ _ ~~ "T"]
Out[5]= {AAGT, AAGT, AAAT, AAGT, AAAT, AAAT}
```

4. Rewrite the web page example in Section 7.3 to use regular expressions to find all phone numbers on the page, that is, expressions of the form *nnn-nnn-nnnn*. Modify accordingly for other web pages and phone numbers formatted for other regions.
5. Use a regular expression to find all words given by `DictionaryLookup` that consist only of the letters *a, e, i, o, u*, and *y* in any order with any number of repetitions of the letters.
6. Use regular expressions to rewrite the solution to Exercise 2 in Section 7.2 to find all words containing three double letter repeats in a row.
7. The basic rules for pluralizing words in the English language are roughly, as follows: if a noun ends in *ch, s, sh, j, x*, or *z*, it is made plural by adding *es* to the end. If the noun ends in *y* and is preceded by a consonant, replace the *y* with *ies*. If the word ends in *ium*, replace with *ia* ([The Chicago Manual of Style 2010](#)). Of course, there are many more rules and even more exceptions, but you can implement a basic set of rules to convert singular words to plural based on these rules and then try them out on the following list of words:

```
In[6]:= words = {"building", "finch", "fix", "ratio", "envy", "boy", "baby",
  "faculty", "honorarium"};
```

8. A common task in transcribing audio is cleaning up text, removing certain phrases such as *um*, *er*, and *so on*, and other tags that are used to make a note of some sort. For example, the

following transcription of a lecture from the University of Warwick, Centre for Applied Linguistics ([Base Corpus](#)) contains quite a few fragments that should be removed, including newline characters, parenthetical remarks, and nonwords. Use `StringReplace` with the appropriate rules to clean this text and then apply your code to a larger corpus.

```
In[7]:= text =
      "okay well er today we're er going to be carrying on with the er French
      \nRevolution you may have noticed i was sort of getting rather er
      enthusiastic \nand carried away at the end of the last one i was sort
      of almost er like i sort \nof started at the beginning about someone
      standing on a coffee table and s-, \nshouting to arms citizens as if i
      was going to sort of leap up on the desk and \nsay to arms let's storm
      the Rootes Social Building [laughter] or er let's go \nout arm in arm
      singing the Marseillaise or something er like that";
```

9. Modify the solution to Exercise 8 in Section 7.2 so that `StringTally` includes an option `IncludeCharacters` that, by default, has value `LetterCharacters`, which should mean that `StringTally` only tallies letters; that is, excludes non-letter characters from the tally. Other values for `IncludeCharacters` could be `All` (include all characters in the tally), `WordCharacters` (include only word characters), `PunctuationOnly` (include only punctuation).
10. In web searches and certain problems in natural language processing, it is often useful to filter out certain words prior to performing a search or processing of text to help with the performance of the algorithms. Words such as *and*, *the*, and *is* are commonly referred to as *stop words* for this purpose. Lists of stop words are almost always created manually based on the constraints of a particular application. Sample lists of stop words are also commonly available across the Internet. For our purposes here, we will use one such list included with the materials for this book.

```
In[8]:= stopwords = Rest@Import["StopWords.dat", "List"];
      RandomSample[stopwords, 12]

Out[9]:= {who'll, recently, ninety, be, no, taking, isn't, unless, what, soon, name, beside}
```

Using the above list of stop words, or any other that you are interested in, first filter some sample “search phrases” and then remove all stop words from a larger piece of text. If your function was called `FilterText`, it might work like this:

```
In[10]:= searchPhrases = {"Find my favorite phone", "How deep is the ocean?",
      "What is the meaning of life?";

In[11]:= Map[FilterText[#, stopwords] &, searchPhrases]

Out[11]:= {{Find, favorite, phone}, {deep, ocean}, {meaning, life}}
```

11. Modify the previous exercise so that the user can supply a list of punctuation in addition to the list of stop words to be used to filter the text.

## 7.4 Solutions

1. Here is the short piece of text.

```
In[1]:= text = "How many vowels do I have?";
```

The regular expression `[aeiou]` is matched by any of the vowels.

```
In[2]:= StringCount[text, RegularExpression["[aeiou]"]]
```

```
Out[2]= 7
```

2. First, get a list of the words in the text *Alice in Wonderland*.

```
In[3]:= words = TextWords[ExampleData[{"Text", "AliceInWonderland"}]];
```

The regular expression `".+q.+"` is matched by strings starting with some number (possibly zero) of characters, followed by an explicit `q`, followed by some number of characters. Flatten is needed to remove the empty lists that are returned for each nonmatch of the pattern.

```
In[4]:= Flatten@StringCases[words, RegularExpression[".*q.*"]]
```

```
Out[4]= {quite, quite, quite, queer, question, inquisitively, Conqueror,
         quiver, quite, quiet, quite, quite, quite, queer-looking, question,
         quite, Conqueror, conquest, question, question, quite, question, quite,
         quicker, quite, quite, quite, squeaking, quite, question, quietly,
         quite, croquet, croquet, question, quite, croquet, queer-shaped,
         quite, quietly, croquet, croquet, question, croquet-ground, croquet,
         quarrelling, quarrel, croqueting, croquet-ground, quarreling, quite, quite}
```

Simply replacing `q` with the alternative `(q | Q)` gets all occurrences of `q` either lowercase or uppercase.

```
In[5]:= Flatten@StringCases[words, RegularExpression[".*(q|Q).*"]]
```

```
Out[5]= {quite, quite, quite, queer, question, inquisitively, Conqueror, quiver, quite,
         quiet, quite, quite, quite, queer-looking, question, quite, Conqueror,
         conquest, question, question, quite, question, quite, quicker, Quick,
         quite, quite, quite, squeaking, quite, question, quietly, quite, Queen,
         croquet, Queen, croquet, question, quite, croquet, Queen, queer-shaped,
         quite, quietly, croquet, Queen, QUEEN'S, CROQUET, Queen, Queen, Queen,
         Queen, Queens, QUEEN, Queen, croquet, Queen, question, Queen, Queen's,
         Queen, croquet-ground, croquet, quarrelling, Queen, quarrel, Queen,
         croqueting, Queen, Queen, Queen, croquet-ground, Queen, quarreling, Queen,
         Queen, quite, Queen, Queen, quite, Queen, Queen, Queen, Queen, Queen}
```

3. The pattern used earlier in the chapter was `"AA" ~~ _ ~~ "T"`. In a regular expression, we want the character `A` repeated exactly once. Use the expression `"A{2,2}"` for this. The regular expression `"."` stands for any character.

```
In[6]:= gene = GenomeData["IGHV357"];
```

```
In[7]:= StringCases[ gene, RegularExpression["A{2,2}.T"] ]
Out[7]= {AAGT, AAGT, AAAT, AAGT, AAAT, AAAT}
```

4. First, read in the web page.

```
In[8]:= webpage = Import["http://www.wolfram.com/company/contact.cgi", "HTML"];
```

In the original example in Section 7.3, we used the pattern `NumberString`, to represent arbitrary strings of numbers. The regular expression `"\\d+"` accomplishes a similar thing but it will also match strings of numbers that may not be in a phone number format (try it!). Instead, use `"\\d{3}"` to match a list of exactly three digits, and so on.

```
In[9]:= StringCases[ webpage,
    RegularExpression["\\d{3}.\\d{3}.\\d{4}"] ] // DeleteDuplicates
Out[9]= {217-398-0700, 217-398-0747, 617-764-0094}
```

5. The first solution uses regular expressions. The second uses string patterns and alternatives.

```
In[10]:= DictionaryLookup[RegularExpression["[aeiouy]+"], IgnoreCase → True]
Out[10]= {a, aye, eye, I, IOU, oi, ya, ye, yea, yo, you}

In[11]:= DictionaryLookup[{"a" | "e" | "i" | "o" | "u" | "y"} .., IgnoreCase → True]
Out[11]= {a, aye, eye, I, IOU, oi, ya, ye, yea, yo, you}
```

6. This one is a bit tricky and requires the use of named string patterns. So in the following, `\\n` refers to the immediately preceding pattern `(.)`. It is important to name each pattern differently so they can be matched by different characters.

```
In[12]:= words = DictionaryLookup[];
In[13]:= StringCases[words, RegularExpression["\\w*(.)\\1(.)\\2(.)\\3\\w*"] ] //
    Flatten
Out[13]= {bookkeeper, bookkeepers, bookkeeping}
```

See the built-in tutorial [Regular Expressions](#) for more information on named string patterns.

7. Here is the short list of words with which we will work.

```
In[14]:= words = {"building", "finch", "fix", "ratio", "envy", "boy", "baby",
    "faculty", "honorarium"};
```

Using regular expressions, these rules encapsulate those given in the exercise.



```
In[15]:= rules = {
  (RegularExpression["(\\w+)x"] := "$1" ~~ "x" ~~ "es"),
  (RegularExpression["(\\w+) (ch)"] := "$1" ~~ "$2" ~~ "es"),
  (RegularExpression["(\\w+) ([aeiou]) (y)"] := "$1" ~~ "$2" ~~ "$3" ~~ "s"),
  (RegularExpression["(\\w+) (y)"] := "$1" ~~ "ies"),
  (RegularExpression["(\\w+) (i)um"] := "$1" ~~ "$2" ~~ "a"),
  (RegularExpression["(\\w+) (.)"] := "$1" ~~ "$2" ~~ "s")
};
```

```
In[16]:= StringReplace[words, rules]
```

```
Out[16]= {buildings, finches, fixes, ratios, envies, boys, babies, faculties, honoraria}
```

Of course, lots of exceptions exist:

```
In[17]:= StringReplace[{"man", "cattle"}, rules]
```

```
Out[17]= {mans, cattles}
```

Check against the built-in function `Pluralize`:

```
In[18]:= Map[Pluralize, words]
```

```
Pluralize::noplural : No valid English pluralization found for honorarium. >>
```

```
Out[18]= {buildings, finches, fixes, ratios, envies, boys, babies, faculties, honorarium}
```

8. We use a combination of string patterns and regular expressions to remove the various fragments. The regular expression `"\[.+\]"` matches strings that start with `[`, followed by an arbitrary number of characters, followed by `]`, followed by a space. Because brackets are used in regular expressions to denote sequences of characters, you need to escape them to refer to the explicit characters `[` or `]`.

```
In[19]:= text =
```

```
"okay well er today we're er going to be carrying on with the er
French \nRevolution you may have noticed i was sort of getting
rather er enthusiastic \nand carried away at the end of the
last one i was sort of almost er like i sort \nof started at
the beginning about someone standing on a coffee table and s-,
\nshouting to arms citizens as if i was going to sort of leap
up on the desk and \nsay to arms let's storm the Rootes Social
Building [laughter] or er let's go \nout arm in arm singing
the Marseillaise or something er like that";
```

```
In[20]:= StringReplace[text, {"\n" → "", " er" → "", " s-" → "",
    RegularExpression["\[.+\] " → ""}]
```

```
Out[20]= okay well today we're going to be carrying on with the French Revolution you
    may have noticed i was sort of getting rather enthusiastic and carried
    away at the end of the last one i was sort of almost like i sort of
    started at the beginning about someone standing on a coffee table
    and, shouting to arms citizens as if i was going to sort of leap up
    on the desk and say to arms let's storm the Rootes Social Building or
    let's go out arm in arm singing the Marseillaise or something like that
```

9. Start by setting the options framework.

```
In[21]:= Clear[StringTally]
```

```
In[22]:= Options[StringTally] = {IncludeCharacters → LetterCharacters};
```

Set up a message that will be returned if the user specifies a bad value for the IncludeCharacters option.

```
In[23]:= StringTally::badopt =
    "The option value `1` can only take on the following values: All,
    LetterCharacters, WordCharacters, PunctuationOnly.";
```

Here then is the rewritten function.

```
In[24]:= StringTally[txt_, OptionsPattern[]] :=
    Module[{ic = OptionValue[IncludeCharacters]},
    Tally[StringCases[txt,
        Which[
            ic === All, RegularExpression["."],
            ic === LetterCharacters, RegularExpression["[:alpha:]"],
            ic === WordCharacters, RegularExpression["\\w"],
            ic === PunctuationOnly, RegularExpression["\\W"],
            True, Message[StringTally::badopt, ic]
        ]]]]
```

Import Dostoyevsky's *Crime and Punishment* which has quite a few non-Ascii characters.

```
In[25]:= text = Import["http://www.gutenberg.org/ebooks/2554.txt.utf-8", "Text"];
```

```
In[26]:= StringTally[text, IncludeCharacters → WordCharacters] // Sort
Out[26]:= {{_, 506}, {0, 21}, {1, 66}, {2, 16}, {3, 13}, {4, 18}, {5, 25}, {6, 10}, {7, 8},
  {8, 19}, {9, 12}, {a, 71960}, {æ, 4}, {à, 2}, {ä, 2}, {A, 2086}, {b, 11365},
  {B, 919}, {c, 18602}, {ç, 1}, {C, 276}, {d, 38452}, {D, 762}, {e, 104695},
  {é, 17}, {è, 1}, {ê, 3}, {E, 331}, {f, 16891}, {F, 316}, {g, 17969}, {G, 433},
  {h, 54070}, {H, 1943}, {i, 56803}, {î, 1}, {ï, 222}, {I, 5846}, {j, 773},
  {J, 66}, {k, 9437}, {K, 323}, {l, 35271}, {L, 580}, {m, 21968}, {M, 504},
  {n, 62470}, {N, 686}, {o, 71737}, {ô, 4}, {O, 496}, {p, 12583}, {P, 1415},
  {q, 766}, {Q, 22}, {r, 46718}, {R, 1776}, {s, 51775}, {S, 1743}, {t, 78967},
  {T, 1864}, {u, 27687}, {ü, 1}, {U, 94}, {v, 11012}, {V, 100}, {w, 19588},
  {W, 1407}, {x, 1322}, {X, 9}, {y, 20389}, {Y, 899}, {z, 894}, {Z, 194}}
```

io. First read in some sample phrases to prototype.

```
In[27]:= searchPhrases = {"Find my favorite phone", "How deep is the ocean?",
  "What is the meaning of life?"};
```

There are several ways to approach this problem. We will break it up into two steps: first eliminating punctuation, then a sample set of stop words.

```
In[28]:= tmp = StringSplit["How deep is the ocean?", Characters[":,;.!? "]]
Out[28]:= {How, deep, is, the, ocean}

In[29]:= stopwords = {"how", "the", "is", "an"};

In[30]:= Apply[Alternatives, stopwords]
Out[30]:= how | the | is | an
```

Note the need for `WordBoundary` in what follows; otherwise, *ocean* would be split leaving *oce* because *an* is a stop word.

```
In[31]:= StringSplit[tmp, WordBoundary ~~ Apply[Alternatives, stopwords] ~~ WordBoundary,
  IgnoreCase → True] // Flatten
Out[31]:= {deep, ocean}

In[32]:= FilterText[str_String, stopwords_List] := Module[{tmp},
  tmp = StringSplit[str, Characters[":,;.!? "]];
  Flatten@StringSplit[tmp, WordBoundary ~~ Apply[Alternatives, stopwords] ~~
    WordBoundary, IgnoreCase → True]
]

In[33]:= stopwords = Rest@Import["StopWords.dat", "List"];

In[34]:= FilterText["What is the meaning of life?", stopwords]
Out[34]:= {meaning, life}
```

ii. A slight modification is needed to accept a list of punctuation.

```
In[35]:= Characters@StringJoin[{".", "?"}] // FullForm
```

```
Out[35]//FullForm= List[".", "?"]
```

First remove the punctuation.

```
In[36]:= tmp = StringSplit["What is the meaning of life?",
    Characters@StringJoin[{".", "?"}] ..]
```

```
Out[36]= {What is the meaning of life}
```

Split into words.

```
In[37]:= First@StringCases[tmp, RegularExpression["\\w+"]]
```

```
Out[37]= {What, is, the, meaning, of, life}
```

Remove stop words.

```
In[38]:= StringSplit[%, WordBoundary ~~ Apply[Alternatives, stopwords] ~~ WordBoundary]
```

```
Out[38]= {{What}, {}, {}, {meaning}, {}, {life}}
```

Put these pieces together in a reusable function.

```
In[39]:= FilterText[str_String, stopwords_List, punctuation_List] := Module[{tmp},
    tmp = StringSplit[str, Characters@StringJoin[punctuation] ..];
    Flatten@StringSplit[First@StringCases[tmp, RegularExpression["\\w+"]],
        WordBoundary ~~ Apply[Alternatives, stopwords] ~~ WordBoundary,
        IgnoreCase -> True]
]
```

```
In[40]:= FilterText["What is the meaning of life?", stopwords, {".", "?"}]
```

```
Out[40]= {meaning, life}
```

Try it out on a list of phrases.

```
In[41]:= Map[FilterText[#, stopwords, {".", "?"}] &, searchPhrases]
```

```
Out[41]= {{Find, favorite, phone}, {deep, ocean}, {meaning, life}}
```

## 7.5 Examples: exercises

- I. Rewrite the small auxiliary function `gcRatio` introduced in this section to eliminate the step in counting the AT content and instead use the length of the entire string in the denominator of the ratio. For large strings (length over  $10^5$ ), this could speed up the computation by a factor of two.

- Generalize the `RandomString` function to allow for a `Weights` option so that you can provide a weight for each character in the generated string. Include a rule to generate a message if the number of weights does not match the number of characters or if the sum of the weights does not equal one. For example:

```
In[1]:= RandomString[{ "A", "T", "C", "G" }, 30, Weights → {.1, .2, .3, .4}]
Out[1]= GTCCTGTGATTCGGTTCAGTAGCCCGCTT

In[2]:= RandomString[{ "A", "T", "C", "G" }, {5, 10}, Weights → {.1, .4, .4, .1}]
Out[2]= {CCTATCCTAG, CACCTACCCC, CCTCACTTCG, CCCTCCCCAC, TCCCTCTGTT}

In[3]:= RandomString[{ "A", "T", "C", "G" }, {5, 10}, Weights → {.1, .4}]
RandomString::badwt :
  The length of the list of weights must be the same as the length of the list of characters.
```

- Rewrite the function `Anagrams` developed in Section 7.2 without resorting to the use of `Permutations`. Consider using the `Sort` function to sort the characters. Note the difference in speed of the two approaches: one involving string functions and the other list functions that operate on lists of characters. Increase the efficiency of your search by only searching for words of the same length as your source word.
- Create a function that searches the built-in dictionary for words containing a specified substring. Set up an option to your function whose value specifies where in the string the substring occurs: start, middle, end, anywhere. For example:

```
In[4]:= FindWordsContaining["cite", WordPosition → "End"]
Out[4]= {anthracite, calcite, cite, excite, incite, Lucite, overexcite, plebiscite, recite}
```

- Using texts from several different sources, compute and then compare the number of punctuation characters per 1000 characters of text. `ExampleData["Text"]` gives a listing of many different texts that you can use.
- The function `stringPartition` was developed specifically to deal with genomic data where one often needs uniformly sized blocks to work with. Generalize `stringPartition` to fully accept the same argument structure as the built-in `Partition`.
- Rewrite the text-encoding example from Section 7.2 using `StringReplace` and regular expressions. First create an auxiliary function to encode a single character based on a key list of the form  $\{ \{p_i, c_i\}, \dots \}$ , where  $p_i$  is a plaintext character and  $c_i$  is its ciphertext encoding. For example, the pair  $\{z, a\}$  would indicate the character  $z$  in the plaintext will be encoded as an  $a$  in the ciphertext. Then create an encoding function `encode[str, key]` using regular expressions to encode any string `str` using the `key` made up of the plaintext/ciphertext character pairs.

8. Word collocation refers to expressions of two or more words that create a familiar phrase, such as *black coffee* or *sharp as a tack*. They are important in many linguistic applications: natural language translation and corpus research involving social phenomena, for example. In this exercise you will create functions for extracting pairs of words of a predetermined form involving parts of speech such as {*adjective, noun*}.

Start by creating some functions to preprocess your text: split the text into pairs of words and, for simplicity, convert all words to lowercase. Next, filter out words that are not contained in the dictionary. Then, find all remaining pairs that are of a certain form involving the parts of speech. This information is contained in `WordData`:

```
In[5]:= WordData["split", "PartsOfSpeech"]
Out[5]= {Noun, Adjective, Verb}
```

Finally, create a function `Collocation[expr, {PoS1, PoS2}]` that returns all pairs in `expr` that consist of the part of speech `PoS1` followed by the part of speech `PoS2`. For example:

```
In[6]:= sentence =
  "Alice was beginning to get very tired of sitting by her sister on
  the bank, and of having nothing to do. Once or twice she had
  peeped into the book her sister was reading, but ";

In[7]:= PreProcessString[sentence]
Out[7]= {{was, beginning}, {beginning, to}, {to, get}, {get, very},
  {very, tired}, {tired, of}, {of, sitting}, {sitting, by}, {by, her},
  {her, sister}, {sister, on}, {on, the}, {the, bank}, {bank, and},
  {and, of}, {of, having}, {having, nothing}, {nothing, to}, {to, do},
  {do, once}, {once, or}, {or, twice}, {twice, she}, {she, had},
  {had, peeped}, {peeped, into}, {into, the}, {the, book}, {book, her},
  {her, sister}, {sister, was}, {was, reading}, {reading, but}}

In[8]:= Collocation[%, {"Verb", "Noun"}]
Out[8]= {{was, beginning}, {having, nothing}, {was, reading}}
```

Both `PreProcessString` and `Collocation` are included in the packages that accompany this book, but you may need to create your own versions for your applications.

9. Generating a list of all built-in symbols and then searching for those that have a certain property is a not uncommon task. Examples include finding all built-in functions with the attribute `Listable`, or all functions that have the `StepMonitor` option (see Section 5.5). The full list of built-in functions includes symbols that should be omitted from such searches. Here we display the first and last four symbols in the `System`` context, that is, the built-in symbols.

```
In[9]:= Drop[Names["System`*"], 5 ;; -5]
Out[9]= {a, b, c, d, $VersionNumber, $WolframID, $WolframUUID, λ}
```

Use regular expressions to create a list of only those built-in functions that begin with a capital

letter. Then use that code to rewrite `FunctionsWithOption` (Exercise 17 in Section 5.5) so it only checks this smaller list of functions for options.

## 7.5 Solutions

- Here is the `gcRatio` function as written in the text.

```
In[1]:= gcRatio[seq_String] := Module[{gc, at},
  gc = StringCount[seq, "G" | "C"];
  at = StringCount[seq, "A" | "T"];
  N[gc / (gc + at)]
]
```

Instead of computing the occurrences of A or T, note that the denominator of the ratio is really the length of the entire string since DNA contains only G, C, A, and T. So instead, use `StringLength[seq]` there. This will make the code a bit shorter and slightly faster.

```
In[2]:= gcRatio[seq_String] := Module[{gc},
  gc = StringCount[seq, "G" | "C"];
  N[gc / StringLength[seq]]
]
```

```
In[3]:= seq = "GCCCTAAGGTGGCCAAGCAC"
```

```
Out[3]:= GCCCTAAGGTGGCCAAGCAC
```

```
In[4]:= gcRatio[seq]
```

```
Out[4]:= 0.65
```

- One rule is needed for one-dimensional output and another for multi-dimensional output.

```
In[5]:= ClearAll[RandomString]
```

```
In[6]:= Options[RandomString] = {Weights -> {}};
```

```
In[7]:= RandomString::badwt =
  "The length of the list of weights must be the same as the length
  of the list of characters.";
```

```
In[8]:= RandomString[{c_String}, n_Integer:1, OptionsPattern[]] :=
  Module[{wts = OptionValue[Weights]},
    Which[
      Length[wts] == 0, StringJoin[RandomChoice[{c}, n]],
      Length[wts] == Length[{c}], StringJoin[RandomChoice[wts -> {c}, n]],
      True, Message[RandomString::badwt]
    ]
  ]
```

```
In[9]:= RandomString[{c_String}, {n_Integer, len_Integer}, OptionsPattern[]] :=
Module[{wts = OptionValue[Weights]},
Which[
Length[wts] == 0, Map[StringJoin, RandomChoice[{c}, {n, len}]],
Length[wts] == Length[{c}],
Map[StringJoin, RandomChoice[wts -> {c}, {n, len}]],
True, Message[RandomString::badwt]
]]
```

```
In[10]:= RandomString[{"A", "C", "T"}]
```

```
Out[10]= T
```

```
In[11]:= RandomString[{"A", "C", "T"}, 10]
```

```
Out[11]= AATATCCTCT
```

```
In[12]:= RandomString[{"A", "C", "T"}, {4, 10}]
```

```
Out[12]= {CTCAAACTCC, TATTTCTAAA, TCATAACTTT, TCTCACAATC}
```

```
In[13]:= RandomString[{"A", "C", "T"}, {4, 10}, Weights -> {.2, .7, .1}]
```

```
Out[13]= {CCCCACCCCA, TACCCCTCCA, CACCCCATAC, CACTCCCACC}
```

Check that an incorrect list for the weights returns a message.

```
In[14]:= RandomString[{"A", "C", "T"}, {4, 10}, Weights -> {.2, .7}]
```

```
RandomString::badwt :
```

The length of the list of weights must be the same as the length of the list of characters.

- Two words are anagrams if they contain the same letters but in a different order. This function is fairly slow as it sorts and compares every word in the dictionary with the sorted characters of the input word.

```
In[15]:= Anagrams2[word_String] :=
Module[{chars = Sort[Characters[word]]},
DictionaryLookup[x_ /; Sort[Characters[x]] == chars]]
```

```
In[16]:= Anagrams2["parsley"] // Timing
```

```
Out[16]= {1.78331, {parleys, parsley, players, replays, sparely}}
```

You can speed things up a bit by only working with those words in the dictionary of the same length as the source word.

```
In[17]:= Anagrams3[word_String] := Module[{len = StringLength[word], words},
words = DictionaryLookup[w_ /; StringLength[w] == len];
Select[words, Sort[Characters[#]] == Sort[Characters[word]] &]
]
```



```
In[18]:= Anagrams3["parsley"] // Timing
Out[18]= {0.728484, {parleys, parsley, players, replays, sparely}}
```

In fact, you can speed this up a bit further by using regular expressions even though the construction of the regular expression in this case is a bit clumsy looking. The lesson here is that conditional string patterns tend to be slower.

```
In[19]:= Anagrams4[word_String] := Module[{len = StringLength[word], words},
  words = DictionaryLookup[RegularExpression["\\w{" <> ToString[len] <> "}"]];
  Select[words, Sort[Characters[#]] == Sort[Characters[word]] &]
]

In[20]:= Anagrams4["parsley"] // Timing
Out[20]= {0.083218, {parleys, parsley, players, replays, sparely}}
```

4. If the test substring is *cite*, here is how we would find all words that end in *cite*. Note the triple blank pattern to match any sequence of zero or more characters.

```
In[21]:= DictionaryLookup[___ ~~ "cite"]
Out[21]= {anthracite, calcite, cite, excite, incite, Lucite, overexcite, plebiscite, recite}
```

You could also use `StringEndsQ`, but it is quite a bit slower:

```
In[22]:= DictionaryLookup[w___ /; StringEndsQ[w, "cite"]]
Out[22]= {anthracite, calcite, cite, excite, incite, Lucite, overexcite, plebiscite, recite}
```

Here are all words that begin with *cite*.

```
In[23]:= DictionaryLookup["cite" ~~ ___]
Out[23]= {cite, cited, cites}

In[24]:= DictionaryLookup[w___ /; StringStartsQ[w, "cite"]]
Out[24]= {cite, cited, cites}
```

And this gives all words that have *cite* somewhere in them, at the beginning, middle, or end.

```
In[25]:= DictionaryLookup[___ ~~ "cite" ~~ ___]
Out[25]= {anthracite, calcite, cite, cited, cites, elicited, excite, excited, excitedly,
  excitement, excitements, exciter, exciters, excites, incite, incited,
  incitement, incitements, inciter, inciters, incites, Lucite, Lucites,
  overexcite, overexcited, overexcites, plebiscite, plebiscites, recite,
  recited, reciter, reciters, recites, solicited, unexcited, unsolicited}
```

```
In[26]:= DictionaryLookup[w__ /; StringContainsQ[w, "cite"]]
```

```
Out[26]= {anthracite, calcite, cite, cited, cites, elicited, excite, excited, excitedly,
excitement, excitements, exciter, excitors, excites, incite, incited,
incitement, incitements, inciter, incitors, incites, Lucite, Lucites,
overexcite, overexcited, overexcites, plebiscite, plebiscites, recite,
recited, reciter, recitors, recites, solicited, unexcited, unsolicited}
```

Using the double blank gives words that have *cite* in them but not beginning or ending with *cite*.

```
In[27]:= DictionaryLookup[__ ~~ "cite" ~~ __]
```

```
Out[27]= {elicited, excited, excitedly, excitement, excitements, exciter,
excitors, excites, incited, incitement, incitements, inciter, incitors,
incites, Lucites, overexcited, overexcites, plebiscites, recited,
reciter, recitors, recites, solicited, unexcited, unsolicited}
```

Let us put these pieces together in a reusable function `FindWordsContaining`. We will include one option, `WordPosition` that identifies where in the word the substring is expected to occur.

```
In[28]:= Options[FindWordsContaining] = {WordPosition -> "Start"};
```

Depending upon the value of the option `WordPosition`, Which directs which expression will be evaluated.

```
In[29]:= FindWordsContaining[str_String, OptionsPattern[]] :=
Module[{wp = OptionValue[WordPosition]},
Which[
wp == "Start", DictionaryLookup[str ~~ ____],
wp == "Middle", DictionaryLookup[__ ~~ str ~~ __],
wp == "End", DictionaryLookup[____ ~~ str],
wp == "Anywhere", DictionaryLookup[____ ~~ str ~~ ____]
]]
```

This could also be done with regular expressions. The pattern `"\\bcite.*\\b"` matches any string starting with a word boundary followed by the string *cite*, followed by characters repeated one or more times, followed by a word boundary.

```
In[30]:= DictionaryLookup[RegularExpression["\\bcite.*\\b"]]
```

```
Out[30]= {cite, cited, cites}
```

With suitable modifications to the above for the target string occurring in the middle, end, or anywhere, here is the rewritten function. Note the need for `StringJoin` here to properly pass the argument `str`, as a string, into the body of the regular expression.

```
In[31]:= Options[FindWordsContaining] = {WordPosition -> "Start"};
```

```
In[32]:= FindWordsContaining[str_String, OptionsPattern[]] :=
Module[{wp = OptionValue[WordPosition]},
  Which[
    wp == "Start",
    DictionaryLookup[RegularExpression[StringJoin["\\b", str, ".*\\b"]]],
    wp == "Middle",
    DictionaryLookup[RegularExpression[StringJoin["\\b.", str, ".*\\b"]]],
    wp == "End", DictionaryLookup[
      RegularExpression[StringJoin["\\b.*", str, "\\b"]]],
    wp == "Anywhere",
    DictionaryLookup[RegularExpression[StringJoin["\\b.*", str, ".*\\b"]]]
  ]]
```

```
In[33]:= FindWordsContaining["cite"]
```

```
Out[33]= {cite, cited, cites}
```

```
In[34]:= FindWordsContaining["cite", WordPosition → "End"]
```

```
Out[34]= {anthracite, calcite, cite, excite, incite, Lucite, overexcite, plebiscite, recite}
```

```
In[35]:= FindWordsContaining["cite", WordPosition → "Middle"]
```

```
Out[35]= {elicited, excited, excitedly, excitement, excitements, exciter,
exciters, excites, incited, incitement, incitements, inciter, inciters,
incites, Lucites, overexcited, overexcites, plebiscites, recited,
reciter, reciters, recites, solicited, unexcited, unsolicited}
```

```
In[36]:= FindWordsContaining["cite", WordPosition → "Anywhere"]
```

```
Out[36]= {anthracite, calcite, cite, cited, cites, elicited, excite, excited, excitedly,
excitement, excitements, exciter, exciters, excites, incite, incited,
incitement, incitements, inciter, inciters, incites, Lucite, Lucites,
overexcite, overexcited, overexcites, plebiscite, plebiscites, recite,
recited, reciter, reciters, recites, solicited, unexcited, unsolicited}
```

5. First read in a sample piece of text.

```
In[37]:= text = ExampleData[{"Text", "PrideAndPrejudice"}];
```

Check the length. Then partition into blocks consisting of 1000 characters each.

```
In[38]:= StringLength[text]
```

```
Out[38]= 682262
```

```
In[39]:= blocks = StringPartition[text, 1000];
```

Using regular expressions, we extract all characters from the first block that are not amongst A through z or o through 9 or whitespace.



```
In[53]:= keyRL3 =
  Transpose[{CharacterRange["a", "z"], RotateLeft[CharacterRange["a", "z"], 3]}]
Out[53]:= {{a, d}, {b, e}, {c, f}, {d, g}, {e, h}, {f, i}, {g, j}, {h, k},
  {i, l}, {j, m}, {k, n}, {l, o}, {m, p}, {n, q}, {o, r}, {p, s}, {q, t},
  {r, u}, {s, v}, {t, w}, {u, x}, {v, y}, {w, z}, {x, a}, {y, b}, {z, c}}
```

Next, encode a single character using a designated key.

```
In[54]:= encodeChar[char_String, key_List] := First@Cases[key, {char, next_} :> next]
In[55]:= encodeChar["z", keyRL3]
Out[55]:= c
```

Finally, here is the encoding function. Recall the "\$1" on the right-hand side of the rule refers to the first (and only in this case) regular expression on the left that is enclosed in parentheses.

```
In[56]:= encode[str_String, key_List] :=
  StringReplace[str, RegularExpression["([a-z])"] :> encodeChar["$1", key]]
```

The decoding uses the same key, but reverses the pairs.

```
In[57]:= decode[str_String, key_List] := encode[str, Map[Reverse, key]]
In[58]:= encode["squeamish ossifrage", keyRL3]
Out[58]:= vtxhdpvk rvvliudjh
In[59]:= decode[%, keyRL3]
Out[59]:= squeamish ossifrage
```

You might want to modify the encoding rule to deal with uppercase letters. One solution is simply to convert them to lowercase.

```
In[60]:= encode[str_String, key_List] :=
  StringReplace[ToLowerCase[str],
    RegularExpression["([a-z])"] :> encodeChar["$1", key]]
In[61]:= encode["Squeamish Ossifrage", keyRL3]
Out[61]:= vtxhdpvk rvvliudjh
```

8. Here is a sample sentence.

```
In[62]:= sentence =
  "Alice was beginning to get very tired of sitting by her sister on
  the bank, and of having nothing to do. Once or twice she had
  peeped into the book her sister was reading, but ";
```

Split into words.

```

In[63]:= words = TextWords[sentence]
Out[63]= {Alice, was, beginning, to, get, very, tired, of, sitting, by, her,
        sister, on, the, bank, and, of, having, nothing, to, do, Once, or, twice,
        she, had, peeped, into, the, book, her, sister, was, reading, but}

In[64]:= Clear[NGrams]
In[65]:= NGrams[words : {__String}, len_ : 2] := Partition[words, len, 1]
In[66]:= bigrams = NGrams[words]
Out[66]= {{Alice, was}, {was, beginning}, {beginning, to}, {to, get}, {get, very},
        {very, tired}, {tired, of}, {of, sitting}, {sitting, by}, {by, her}, {her, sister},
        {sister, on}, {on, the}, {the, bank}, {bank, and}, {and, of}, {of, having},
        {having, nothing}, {nothing, to}, {to, do}, {do, Once}, {Once, or}, {or, twice},
        {twice, she}, {she, had}, {had, peeped}, {peeped, into}, {into, the}, {the, book},
        {book, her}, {her, sister}, {sister, was}, {was, reading}, {reading, but}}

```

The collocation grabs parts of speech data from WordData.

```

In[67]:= Collocation[lis_List, {PoS1_String, PoS2_String}] :=
        Cases[lis, {p_ /; MemberQ[WordData[p, "PartsOfSpeech"], PoS1],
        q_ /; MemberQ[WordData[q, "PartsOfSpeech"], PoS2]}]
In[68]:= Collocation[bigrams, {"Noun", "Adjective"}]
Out[68]= {{get, very}, {sister, on}}

```

Try it on a larger text.

```

In[69]:= text = ExampleData[{"Text", "GettysburgAddress"}]
Out[69]= Four score and seven years ago, our fathers brought forth upon this continent a new
        nation: conceived in liberty, and dedicated to the proposition that all men are
        created equal. Now we are engaged in a great civil war...testing whether that nation,
        or any nation so conceived and so dedicated. . . can long endure. We are met on a
        great battlefield of that war. We have come to dedicate a portion of that field as
        a final resting place for those who here gave their lives that this nation might
        live. It is altogether fitting and proper that we should do this. But, in a larger
        sense, we cannot dedicate...we cannot consecrate... we cannot hallow this ground.
        The brave men, living and dead, who struggled here have consecrated it, far above
        our poor power to add or detract. The world will little note, nor long remember,
        what we say here, but it can never forget what they did here. It is for us the
        living, rather, to be dedicated here to the unfinished work which they who fought
        here have thus far so nobly advanced. It is rather for us to be here dedicated to
        the great task remaining before us...that from these honored dead we take increased
        devotion to that cause for which they gave the last full measure of devotion...
        that we here highly resolve that these dead shall not have died in vain...that
        this nation, under God, shall have a new birth of freedom...and that government
        of the people, by the people, for the people, shall not perish from this earth.

```

```
In[70]:= bigrams = NGrams[TextWords[text], 2]
```

```
Out[70]= {{Four, score}, {score, and}, {and, seven}, {seven, years}, {years, ago}, {ago, our},
{our, fathers}, {fathers, brought}, {brought, forth}, {forth, upon}, {upon, this},
{this, continent}, {continent, a}, {a, new}, {new, nation}, {nation, conceived},
{conceived, in}, {in, liberty}, {liberty, and}, {and, dedicated}, {dedicated, to},
{to, the}, {the, proposition}, {proposition, that}, {that, all}, {all, men},
{men, are}, {are, created}, {created, equal}, {equal, Now}, {Now, we}, {we, are},
{are, engaged}, {engaged, in}, {in, a}, {a, great}, {great, civil}, {civil, war},
{war, testing}, {testing, whether}, {whether, that}, {that, nation}, {nation, or},
{or, any}, {any, nation}, {nation, so}, {so, conceived}, {conceived, and}, {and, so},
{so, dedicated}, {dedicated, can}, {can, long}, {long, endure}, {endure, We}, {We, are},
{are, met}, {met, on}, {on, a}, {a, great}, {great, battlefield}, {battlefield, of},
{of, that}, {that, war}, {war, We}, {We, have}, {have, come}, {come, to}, {to, dedicate},
{dedicate, a}, {a, portion}, {portion, of}, {of, that}, {that, field}, {field, as},
{as, a}, {a, final}, {final, resting}, {resting, place}, {place, for}, {for, those},
{those, who}, {who, here}, {here, gave}, {gave, their}, {their, lives}, {lives, that},
{that, this}, {this, nation}, {nation, might}, {might, live}, {live, It}, {It, is},
{is, altogether}, {altogether, fitting}, {fitting, and}, {and, proper}, {proper, that},
{that, we}, {we, should}, {should, do}, {do, this}, {this, But}, {But, in}, {in, a},
{a, larger}, {larger, sense}, {sense, we}, {we, cannot}, {cannot, dedicate},
{dedicate, we}, {we, cannot}, {cannot, consecrate}, {consecrate, we}, {we, cannot},
{cannot, hallow}, {hallow, this}, {this, ground}, {ground, The}, {The, brave},
{brave, men}, {men, living}, {living, and}, {and, dead}, {dead, who}, {who, struggled},
{struggled, here}, {here, have}, {have, consecrated}, {consecrated, it}, {it, far},
{far, above}, {above, our}, {our, poor}, {poor, power}, {power, to}, {to, add},
{add, or}, {or, detract}, {detract, The}, {The, world}, {world, will}, {will, little},
{little, note}, {note, nor}, {nor, long}, {long, remember}, {remember, what},
{what, we}, {we, say}, {say, here}, {here, but}, {but, it}, {it, can}, {can, never},
{never, forget}, {forget, what}, {what, they}, {they, did}, {did, here}, {here, It},
{It, is}, {is, for}, {for, us}, {us, the}, {the, living}, {living, rather}, {rather, to},
{to, be}, {be, dedicated}, {dedicated, here}, {here, to}, {to, the}, {the, unfinished},
{unfinished, work}, {work, which}, {which, they}, {they, who}, {who, fought},
{fought, here}, {here, have}, {have, thus}, {thus, far}, {far, so}, {so, nobly},
{nobly, advanced}, {advanced, It}, {It, is}, {is, rather}, {rather, for}, {for, us},
{us, to}, {to, be}, {be, here}, {here, dedicated}, {dedicated, to}, {to, the}, {the, great},
{great, task}, {task, remaining}, {remaining, before}, {before, us}, {us, that},
{that, from}, {from, these}, {these, honored}, {honored, dead}, {dead, we}, {we, take},
{take, increased}, {increased, devotion}, {devotion, to}, {to, that}, {that, cause},
{cause, for}, {for, which}, {which, they}, {they, gave}, {gave, the}, {the, last},
{last, full}, {full, measure}, {measure, of}, {of, devotion}, {devotion, that},
{that, we}, {we, here}, {here, highly}, {highly, resolve}, {resolve, that}, {that, these},
{these, dead}, {dead, shall}, {shall, not}, {not, have}, {have, died}, {died, in},
{in, vain}, {vain, that}, {that, this}, {this, nation}, {nation, under}, {under, God},
{God, shall}, {shall, have}, {have, a}, {a, new}, {new, birth}, {birth, of}, {of, freedom},
{freedom, and}, {and, that}, {that, government}, {government, of}, {of, the}, {the, people},
{people, by}, {by, the}, {the, people}, {people, for}, {for, the}, {the, people},
{people, shall}, {shall, not}, {not, perish}, {perish, from}, {from, this}, {this, earth}}
```

```
In[71]:= Collocation[bigrams, {"Adjective", "Noun"}]
```

```
Out[71]= {{seven, years}, {continent, a}, {new, nation}, {in, liberty}, {all, men},
{engaged, in}, {in, a}, {civil, war}, {any, nation}, {dedicated, can}, {on, a},
{great, battlefield}, {in, a}, {larger, sense}, {brave, men}, {here, have},
{far, above}, {poor, power}, {world, will}, {little, note}, {dedicated, here},
{unfinished, work}, {here, have}, {far, so}, {great, task}, {honored, dead},
{increased, devotion}, {last, full}, {full, measure}, {under, God}, {new, birth}}
```

9. To find symbols in the list of built-in functions that start with a capital letter, use the following regular expression: `RegularExpression["^ [[:upper:]] \\w+"]`. This will be matched by a string that starts with one of the letters A through Z, followed by any sequence of characters. The caret `^` is used to denote the beginning of the string. We will use `StringCases` on the full list of built-in symbols with this regular expression as the pattern to match against.

```
In[72]:= StringCases[Names["System`*"], RegularExpression["^ [[:upper:]] \\w+"]];
```

This list will need to be flattened to delete the empty lists where a match did not occur. Here then is the rewritten function from Exercise 17, Section 5.5.

```
In[73]:= FunctionsWithOption[opt_Symbol] := Module[{names, lis},
  names = Flatten@StringCases[Names["System`*"],
    RegularExpression["^ [[:upper:]] \\w+"]];
  lis = DeleteCases[names, f_ /; Options[Symbol[f]] === {}];
  Select[lis, MemberQ[Options[Symbol[#]], opt, 2] &]]
```

```
In[74]:= FunctionsWithOption[StepMonitor]
```

```
Out[74]= {FindArgMax, FindArgMin, FindFit, FindMaximum, FindMaxValue,
FindMinimum, FindMinValue, FindRoot, NArgMax, NArgMin, NDSolve,
NDSolveValue, NMaximize, NMaxValue, NMinimize, NMinValue,
NonlinearModelFit, NRroots, ParametricNDSolve, ParametricNDSolveValue}
```



---

## 8

# Graphics and visualization

### 8.1 The graphics language: exercises

1. Create a color wheel by coloring successive sectors of a disk according to the Hue directive.
2. Construct a graphic containing a circle, a triangle, and a rectangle. Your graphic should include an identifying label for each object.
3. Create a three-dimensional graphic that includes six Cuboid graphics primitives, randomly placed in the unit cube. Add an opacity directive to make them transparent.
4. Create a graphic consisting of a cube together with a rotation of  $45^\circ$  about the vertical axis through the center of that cube. Then create a dynamically rotating cube using Manipulate or Animate.
5. Create a three-dimensional graphic consisting of twenty-four random points in the unit cube with every pair of points connected by a line. Add directives to make the points red and large and the lines gray and transparent.
6. Construct a graphic that consists of 500 points randomly distributed about the origin with standard deviation 1. Then, set the points to have random radii between 0.01 and 0.1 and colored randomly according to a Hue function.
7. Create a random walk on the binary digits of  $\pi$ . For a one-dimensional walk, use RealDigits[num, 2] to get the base 2 digits and then convert each 0 to -1 so that you have a vector of  $\pm 1$ s for the step directions; then use Accumulate.

For the two-dimensional walk, use Partition to pair up digits and then use an appropriate transformation to have the four pairs,  $\{0, 0\}$ ,  $\{0, 1\}$ ,  $\{1, 0\}$ , and  $\{1, 1\}$  map to the compass directions; then use Accumulate. See [Bailey et al. \(2012\)](#) for more on visualizing digits of  $\pi$ .

8. Create a graphic that represents the solution to the following algebraic problem that appeared in the *Calculus&Mathematica* courseware (Porta, Davis, and Uhl 1994). Find the positive numbers  $r$  such that the following system has exactly one solution in  $x$  and  $y$ :

$$\begin{aligned}(x-1)^2 + (y-1)^2 &= 2 \\ (x+3)^2 + (y-4)^2 &= r^2\end{aligned}$$

Once you have found the right number  $r$ , plot the resulting circles on the same axes, plotting the first circle with solid lines and the two solutions with dashed lines.

9. Create a graphic of the sine function over the interval  $(0, 2\pi)$  that displays vertical lines at each point calculated by the `Plot` function to produce the curve.
10. Bundle up the code fragments for the visualization of the triangle centroid into a function `CentroidPlot` that takes a `Triangle` graphics primitive as its argument and returns a graphic similar to that in this section. Set up your function to inherit options from `Graphics`.

Add a check to `CentroidPlot` to return a message if the three vertices of the triangle are collinear.

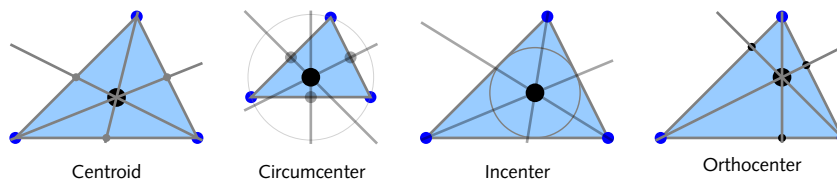
```
In[1]:= pts = {{0, 0}, {1, 1}, {2, 2}};
CentroidPlot[Triangle[pts]]
```

CentroidPlot::collinpts :

The points {{0, 0}, {1, 1}, {2, 2}} are collinear, giving a degenerate triangle.

11. The centroid of a triangle is only one kind of triangle center. The *circumcenter* is located at the intersection of the perpendicular bisectors of the sides of the triangle and is also the center of the circle passing through the vertices of the triangle, the circumcircle. The *incenter* is located at the intersection of the angle bisectors and is the center of the largest circle inside the triangle. The *orthocenter* is located at the intersection of the altitudes of the triangle (Kimberling 1994).

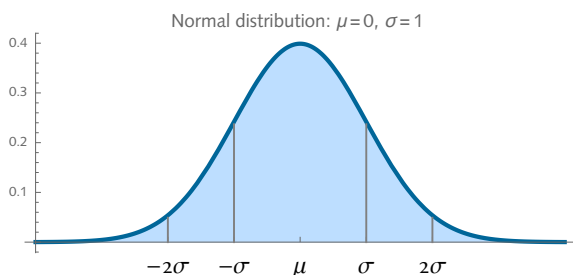
FIGURE 8.1. Triangle centers.



Create a graphic for each of these centers similar to that created for the centroid and the medians.

12. Using options to the `Plot` function, create a plot showing the probability density function (PDF) of a normal distribution together with vertical lines at the first and second standard deviations. Your plot should look something like that in Figure 8.2 for a normal distribution with  $\mu = 0$  and  $\sigma = 1$ .

FIGURE 8.2. PDF of normal distribution with standard deviation lines.



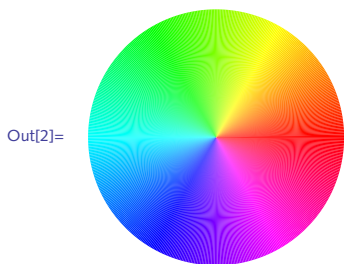
## 8.1 Solutions

1. The color wheel can be generated by mapping the Hue directive over successive sectors of a disk. Note that the argument to Hue must be scaled so that it falls within the range zero to one.

```
In[1]:= colorWheel[n_] :=
  Graphics[{Hue[Rescale[#, {0, 2 π}]], Disk[{0, 0}, 1, {#, # + n}]} &] /@
  Range[0, 2 π, n], ImageSize → Small]
```

Here is a color wheel created from 256 separate sectors (hues).

```
In[2]:= colorWheel[ $\frac{\pi}{256}$ ]
```



2. Here is the circle graphic primitive together with a text label.

```
In[3]:= circ = Circle[{.5, .5}, .5];
In[4]:= ctext = Text[Style["Circle", 8], {.5, -.2}];
```

This generates the graphics primitive for the triangle and its text label.

```
In[5]:= tri = Triangle[{{-2, 0}, {-1, 1}, {0, 0}}];
In[6]:= ttext = Text[Style["Triangle", 8], {-1, -.2}];
```

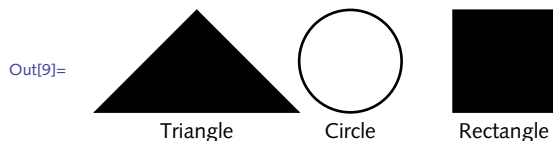
Here is the rectangle and label.

```
In[7]:= rect = Rectangle[{1.5, 0}, {2.5, 1}]
Out[7]= Rectangle[{1.5, 0}, {2.5, 1}]
```

```
In[8]:= rtext = Text[Style["Rectangle", 8], {2, -0.2}];
```

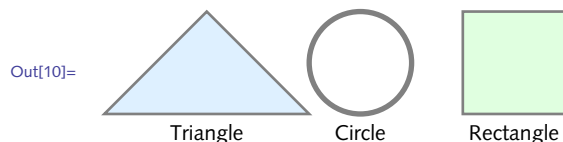
Finally, this displays each of these graphics elements all together.

```
In[9]:= Graphics[{tri, ttext, circ, ctext, rect, rtext}]
```



Add some directives to each primitive.

```
In[10]:= Graphics[{
  {EdgeForm[Gray], LightBlue, tri}, ttext,
  {Thick, Gray, circ}, ctext,
  {EdgeForm[Gray], LightGreen, rect}, rtext}]
```



3. Cuboid takes a list of three numbers as the coordinates of its lower-left corner. This maps the object across two such lists.

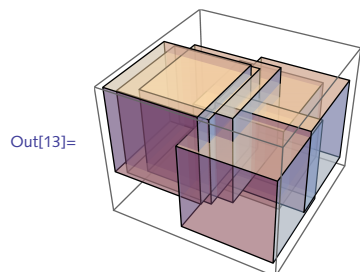
```
In[11]:= Map[Cuboid, RandomReal[1, {2, 3}]]
```

```
Out[11]= {Cuboid[{0.159256, 0.199787, 0.744115}], Cuboid[{0.629273, 0.77004, 0.902954}]}
```

Below is a list of six cuboids and the resulting graphic. Notice the large amount of overlap of the cubes. You can reduce the large overlap by specifying minimum *and* maximum values of the cuboid.

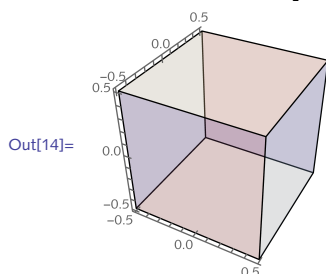
```
In[12]:= cubes = Map[Cuboid, RandomReal[1, {6, 3}]];
```

```
In[13]:= Graphics3D[{Opacity[.5], cubes}]
```



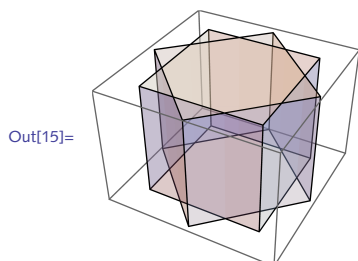
4. Start by creating a unit cube centered on the origin. An opacity directive adds transparency.

```
In[14]:= Graphics3D[{Opacity[.25], Cuboid[{-0.5, -0.5, -0.5}]}, Boxed → False,
  Axes → Automatic]
```



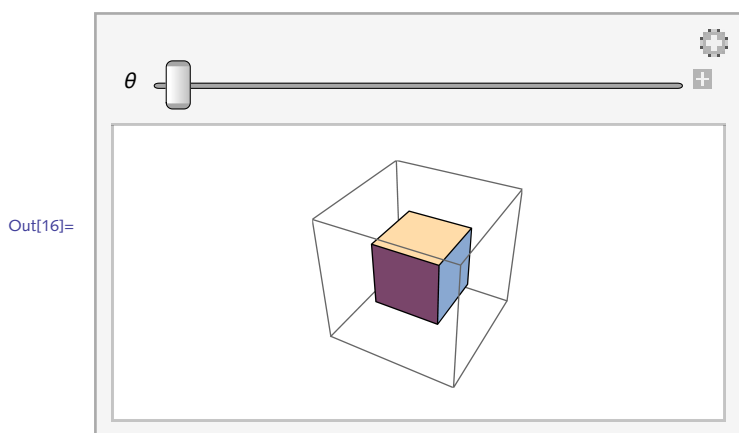
Next rotate  $45^\circ$ . Note the third argument of `Rotate` used to specify the axis about which the rotation should occur.

```
In[15]:= Graphics3D[{Opacity[.25], Cuboid[{-0.5, -0.5, -0.5}],
  Rotate[Cuboid[{-0.5, -0.5, -0.5}], 45°, {0, 0, 1}]}]
```



Here is the dynamic version. The angle  $\theta$  is the parameter that is manipulated here. The explicit `PlotRange` option is necessary to prevent its re-computation for every new angle of rotation.

```
In[16]:= Manipulate[
  Graphics3D[Rotate[Cuboid[{-0.5, -0.5, -0.5}],  $\theta$ , {0, 0, 1}], PlotRange → 1],
  { $\theta$ , 0, 2 \pi}]
```



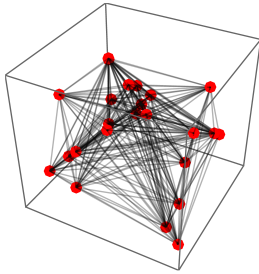
5. Start by creating the set of points.

```
In[17]:= pts = RandomReal[{0, 1}, {24, 3}];
```

To connect all pairs of points, use `Subsets[pts, {2}]`.

```
In[18]:= Graphics3D[{
  {Opacity[.3], Line[Subsets[pts, {2}]}},
  {Red, PointSize[Medium], Point[pts]}}
```

Out[18]=

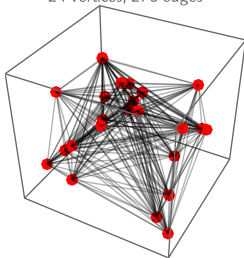


To add a label to the plot, use `StringForm` to slot some values into the string. The binomial  $\binom{n}{2}$  gives the number of pairs of points.

```
In[19]:= Graphics3D[{
  {Opacity[.3], Line[Subsets[pts, {2}]}},
  {Red, PointSize[Medium], Point[pts]}},
PlotLabel → StringForm["`1` vertices, `2` edges", Length[pts],
  Binomial[Length[pts], 2]]]
```

24 vertices, 276 edges

Out[19]=



6. First we create the `Point` graphics primitives using a normal distribution with mean zero and standard deviation one.

```
In[20]:= randomcoords := Point[RandomVariate[NormalDistribution[0, 1], {1, 2}]]
```

This creates the point sizes according to the specification given in the statement of the problem.

```
In[21]:= randomsize := PointSize[RandomReal[{.01, .1}]]
```

This creates a random color that will be used for each primitive.

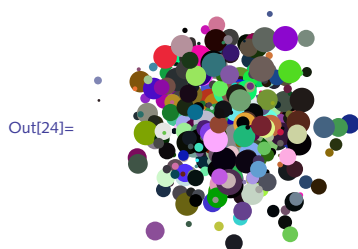
```
In[22]:= randomcolor := RandomColor[ColorSpace → "Hue"]
```

Here then are 500 points. (You may find it instructive to look at just one of these points.)

```
In[23]:= pts = Table[{randomcolor, randomsize, randomcoords}, {500}];
```

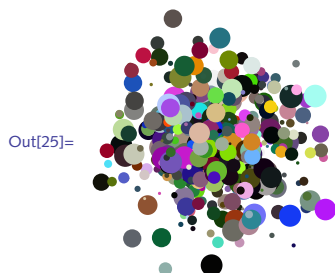
And here is the graphic.

```
In[24]:= Graphics[pts]
```



Alternatively, you could do it all in one step using MapThread.

```
In[25]:= With[{n = 500},
Graphics[MapThread[{#1, PointSize[#2], Point[#3]} &,
  {RandomColor[n, ColorSpace -> "Hue"], RandomReal[ {.01, .1}, n],
   RandomVariate[NormalDistribution[0, 1], {n, 2}]}]
]]
```



7. Here are the binary digits of  $\pi$ . First is used to get only the digits from RealDigits.

```
In[26]:= First[RealDigits[N[Pi, 12], 2]]
```

```
Out[26]= {1, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1,
0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0}
```

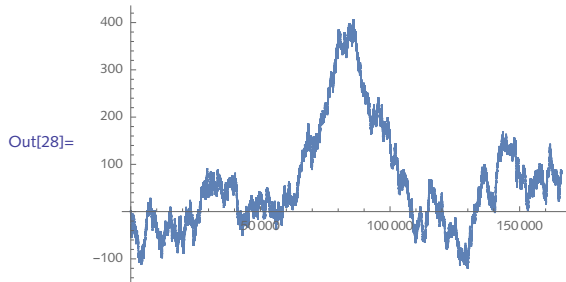
Convert zeros to negative ones.

```
In[27]:= 2 % - 1
```

```
Out[27]= {1, 1, -1, -1, 1, -1, -1, 1, -1, -1, -1, -1, 1, 1, 1, 1, 1, -1, 1, 1,
-1, 1, -1, 1, -1, 1, -1, -1, -1, 1, -1, -1, -1, 1, -1, -1, -1, 1, -1}
```

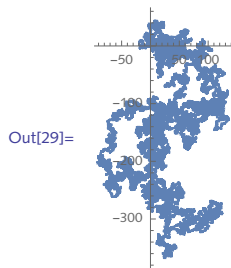
Here then is a plot for the first fifty thousand digits.

```
In[28]:= ListLinePlot[
  With[{digits = 50000},
    Accumulate[2 First[RealDigits[N[Pi, digits], 2]] - 1]
  ]]
```



For the two-dimensional case, use `Partition` to pair up the binary digits, then a transformation rule to convert them to compass directions.

```
In[29]:= With[{digs = First[RealDigits[N[Pi, 50000], 2]]},
  ListLinePlot[
    Accumulate[Partition[digs, 2, 2] /. {{0, 0} -> {-1, 0}, {1, 1} -> {0, -1}}],
    AspectRatio -> Automatic]]
```



8. The algebraic solution is given by the following steps. First solve the equations for  $x$  and  $y$ .

```
In[30]:= Clear[x, y, r]
```

```
In[31]:= soln = Solve[{(x - 1)^2 + (y - 1)^2 == 2, (x + 3)^2 + (y - 4)^2 == r^2}, {x, y}]
```

```
Out[31]= { {x -> 1/50 (-58 + 4 r^2 - 3 Sqrt[-529 + 54 r^2 - r^4]),
  y -> 1/50 (131 - 3 r^2 - 4 Sqrt[-529 + 54 r^2 - r^4]) },
  {x -> 1/50 (-58 + 4 r^2 + 3 Sqrt[-529 + 54 r^2 - r^4]),
  y -> 1/50 (131 - 3 r^2 + 4 Sqrt[-529 + 54 r^2 - r^4]) } }
```

Then find those values of  $r$  for which the  $x$  and  $y$  coordinates are identical.



```
In[32]:= Solve[{
  (x /. soln[[1]]) == (x /. soln[[2]]),
  (y /. soln[[1]]) == (y /. soln[[2]])},
  r]
Out[32]= {{r -> -5 - Sqrt[2]}, {r -> 5 - Sqrt[2]}, {r -> -5 + Sqrt[2]}, {r -> 5 + Sqrt[2]}}
```

Here then are those values of  $r$  that are positive.

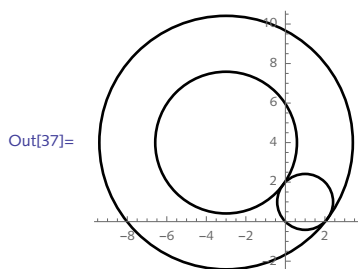
```
In[33]:= Cases[%, {r -> _?Positive}]
Out[33]= {{r -> 5 - Sqrt[2]}, {r -> 5 + Sqrt[2]}}
```

To display the solution, we will plot the first circle with solid lines and the two solutions with dashed lines together in one graphic. Here is the first circle centered at  $(1, 1)$ .

```
In[34]:= circ = Circle[{1, 1}, Sqrt[2]];
```

Here are the circles that represent the solution to the problem.

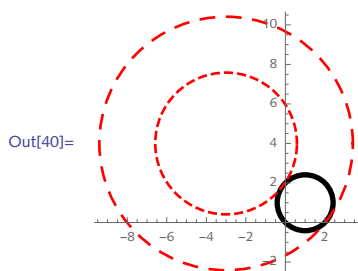
```
In[35]:= r1 = 5 - Sqrt[2];
r2 = 5 + Sqrt[2];
In[37]:= Graphics[{circ, Circle[{-3, 4}, r1], Circle[{-3, 4}, r2]}, Axes -> Automatic]
```



We wanted to display the solutions (two circles) using dashed lines. The graphics directive `Dashing[{x, y}]` directs all subsequent lines to be plotted as dashed, alternating the dash  $x$  units and the space  $y$  units. We use it as a graphics directive on the two circles `c1` and `c2`. The circles inherit only those directives in whose scope they appear.

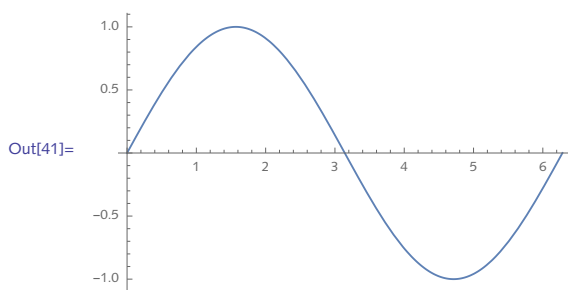
```
In[38]:= dashc1 = {Red, Dashing[{.025, .025}], Circle[{-3, 4}, r1]};
dashc2 = {Red, Dashing[{.05, .05}], Circle[{-3, 4}, r2]};
```

```
In[40]:= Graphics[{
  {Thick, circ},
  dashc1, dashc2}, Axes → Automatic]
```



9. Here is a plot of the sine function.

```
In[41]:= sinplot = Plot[Sin[x], {x, 0, 2 π}]
```



Using pattern matching, here are the coordinates that were used to construct the curve.

```
In[42]:= Short[coords = Cases[sinplot, Line[{x__}] :> x, Infinity], 2]
```

```
Out[42]//Short= {{1.28228 × 10-7, 1.28228 × 10-7}, {<<22>>, <<21>>}, <<427>>, {<<1>>}, {<<1>>}}
```

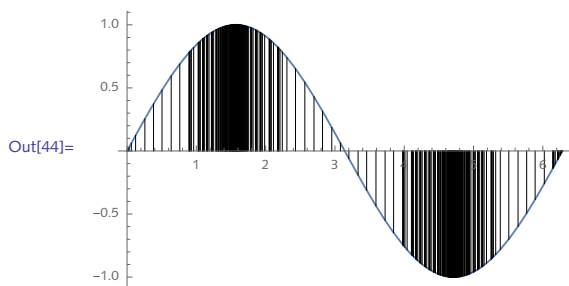
Create vertical lines from each coordinate.

```
In[43]:= Short[lines = Map[Line[{#[[1]], 0}, #] &, coords], 2]
```

```
Out[43]//Short= {<<1>>}
```

Here then is the final graphic.

```
In[44]:= Show[sinplot, Graphics[{Thickness[.001], lines}]]
```



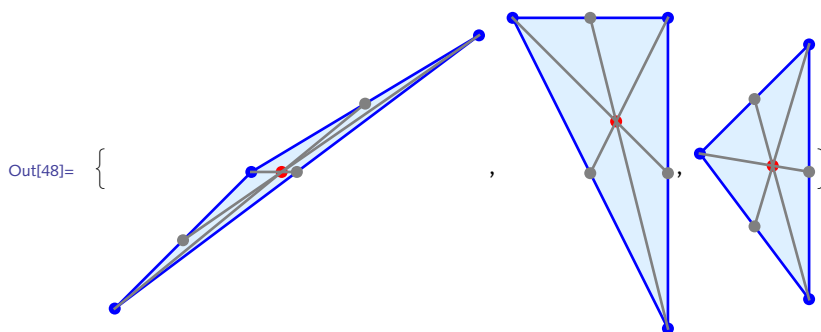
10. First, here is the code for the triangle medians from Section 5.5.

```
In[45]:= TriangleMedians[Triangle[{p1_, p2_, p3_}]] := Module[{midpts},
  midpts = (#1 + #2) / 2 & @@@ Subsets[{p1, p2, p3}, {2}];
  MapThread[Line[{#1, #2}] &, {{p3, p2, p1}, midpts}]]
```

Second, this bundles up the code fragments from Section 8.1 into a reusable function. We have added options inherited from Graphics.

```
In[46]:= CentroidPlot[Triangle[{p1_, p2_, p3_}], opts : OptionsPattern[Graphics]] :=
  Graphics[{
    {LightBlue, EdgeForm[Blue], Triangle[{p1, p2, p3}]},
    {Blue, PointSize[Medium], Point[{p1, p2, p3}]},
    {Red, PointSize[Medium], Point@RegionCentroid[Triangle[{p1, p2, p3}]]},
    {Gray, TriangleMedians[Triangle[{p1, p2, p3}]], PointSize[Medium],
     Point[(#1 + #2) / 2 & @@@ Subsets[{p1, p2, p3}, {2}]]},
    }, opts]
```

```
In[47]:= SeedRandom[15];
Table[CentroidPlot[Triangle[RandomInteger[{-5, 5}, {3, 2}]]], {3}]
```



To deal with the pathological situation where the points are collinear, first create a message that will be issued under these conditions.

```
In[49]:= CentroidPlot::collinpts =
  "The points `1' are collinear, giving a degenerate triangle.";
```

Three points are collinear if and only if the area of the triangle determined by those points is zero. The built-in `Area` function returns an error message in this situation:

```
In[50]:= pts = {{0, 0}, {1, 1}, {2, 2}};
          Area[Triangle[pts]]
          Area::reg : Triangle[{{0, 0}, {1, 1}, {2, 2}}] is not a correctly specified region. >>

Out[51]:= Area[Triangle[{{0, 0}, {1, 1}, {2, 2}}]]
```

We could trap for a non-numeric value returned by `Area`, but we will take another, more direct approach. The area of the triangle determined by three points is given by the following determinant:

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

In fact, we have already implemented this in the function `SignedArea` from Section 4.3.

```
In[52]:= SignedArea[Triangle[{v1_, v2_, v3_}]] :=
          1/2 Det[{v1, v2, v3} /. {x_, y_} -> {x, y, 1}]

In[53]:= SignedArea[Triangle[pts]]
Out[53]:= 0
```

Here then is the code to trap for this condition.

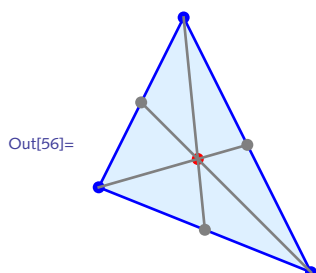
```
In[104]:= If[SignedArea[pts] == 0, Message[CentroidPlot::collinpts, pts],
            else_body_of_function]

In[54]:= CentroidPlot[Triangle[{p1_, p2_, p3_}], opts : OptionsPattern[Graphics]] :=
          If[SignedArea[Triangle[{p1, p2, p3}]] == 0,
            Message[CentroidPlot::collinpts, {p1, p2, p3}],

            Graphics[{
              {LightBlue, EdgeForm[Blue], Triangle[{p1, p2, p3}]},
              {Blue, PointSize[Medium], Point[{p1, p2, p3}]},
              {Red, PointSize[Medium], Point@RegionCentroid[Triangle[{p1, p2, p3}]]},
              {Gray, TriangleMedians[Triangle[{p1, p2, p3}]], PointSize[Medium],
               Point[(#1 + #2) / 2 & @@@ Subsets[{p1, p2, p3}, {2}]]}
            }, opts]
          ]

In[55]:= CentroidPlot[Triangle[pts]]
CentroidPlot::collinpts :
  The points {{0, 0}, {1, 1}, {2, 2}} are collinear, giving a degenerate triangle.
```

```
In[56]:= CentroidPlot[Triangle[RandomInteger[{-5, 5}, {3, 2}]]]
```



```
In[57]:=
```

- II. Start with the circumcenter which is the center of the circumscribing circle. It is located at the intersection of the perpendicular bisectors of the sides. To find the line from the center, perpendicular to a side, use `RegionNearest` and `InfiniteLine`. First, get the center.

```
In[58]:= {p1, p2, p3} = {{-1, 0}, {0, 2}, {1, 0}};
center = First@Circumsphere[{p1, p2, p3}]
```

```
Out[59]= {0, 3/4}
```

Here are the lines through each pair of vertices:

```
In[60]:= lines = InfiniteLine /@ Subsets[{p1, p2, p3}, {2}]
```

```
Out[60]= {InfiniteLine[{{-1, 0}, {0, 2}}],
          InfiniteLine[{{-1, 0}, {1, 0}}], InfiniteLine[{{0, 2}, {1, 0}}]}
```

And this gives the point on each line closest to the center.

```
In[61]:= pbs = Map[RegionNearest[#, center] &, lines]
```

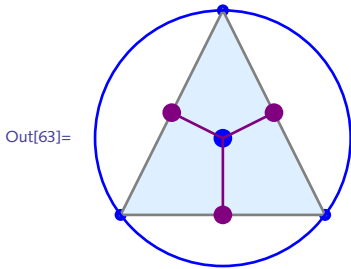
```
Out[61]= {{-1/2, 1}, {0, 0}, {1/2, 1}}
```

Finally, we want a line from each of these points to the center.

```
In[62]:= perpLines = Map[Line[{center, #}] &, Map[RegionNearest[#, center] &, lines]]
```

```
Out[62]= {Line[{{0, 3/4}, {-1/2, 1}}], Line[{{0, 3/4}, {0, 0}}], Line[{{0, 3/4}, {1/2, 1}}]}
```

```
In[63]:= Graphics[{
  {Blue, PointSize[Medium], Point[{p1, p2, p3}]},
  {EdgeForm[Gray], LightBlue, Triangle[{p1, p2, p3}]},
  {Blue, Circumsphere[{p1, p2, p3}]},
  {Blue, PointSize[Large], Point@center},
  {Purple, perpLines, PointSize[Large], Point@pbs}
}]
```



Next, compute the orthocenter. Using `InfiniteLine` will make it so that we can draw on the computational geometry machinery to do our computations.

```
In[64]:= {p1, p2, p3} = {{-1, 0}, {0, 2}, {1, 0}};
```

```
In[65]:= lines = Map[InfiniteLine, Subsets[{p1, p2, p3}, {2}]]
```

```
Out[65]= {InfiniteLine[{{-1, 0}, {0, 2}}],
  InfiniteLine[{{-1, 0}, {1, 0}}], InfiniteLine[{{0, 2}, {1, 0}}]}
```

The altitude is the line drawn from a vertex to the opposite side, perpendicular to that side. Note that we need to reverse the list  $\{p1, p2, p3\}$  so that the proper line is paired with the proper point.

```
In[66]:= altPts = MapThread[RegionNearest[#1, #2] &, {lines, Reverse[{p1, p2, p3}]}]
```

```
Out[66]= {{-3/5, 4/5}, {0, 0}, {3/5, 4/5}}
```

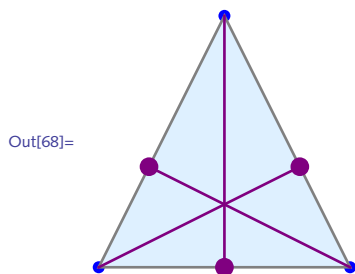
We need lines from each vertex to the corresponding point in `altPts`.

```
In[67]:= altLines = MapThread[Line[{#1, #2}] &, {Reverse@{p1, p2, p3}, altPts}]
```

```
Out[67]= {Line[{{1, 0}, {-3/5, 4/5}}], Line[{{0, 2}, {0, 0}}], Line[{{-1, 0}, {3/5, 4/5}}]}
```

Here then is the graphic showing the triangle in gray, its vertices in blue, and the altitude lines and points in purple.

```
In[68]:= Graphics[{
  {Blue, PointSize[Medium], Point[{p1, p2, p3}]},
  {EdgeForm[Gray], LightBlue, Triangle[{p1, p2, p3}]},
  {Purple, PointSize[Large], Point@altPts, altLines}
}]
```



Finally, the incenter, the center of which is the largest circle that fits entirely inside the triangle. For the formula for the incenter, we will need a function to compute the perimeter of the triangle. This one comes essentially from Section 4.3.

```
In[69]:= Perimeter[Triangle[pts : {p1_, p2_, p3_}]] :=
  RegionMeasure[Line[pts /. {p_, pn_} -> {p1, pn, p1}]]

In[70]:= inCenter[pts : {p1_, p2_, p3_}] :=
  Module[{a, b, c, p = Perimeter[Triangle[pts]]},
    {a, b, c} = {Norm[p2 - p3], Norm[p1 - p3], Norm[p2 - p1]};
    {(a p1[[1]] + b p2[[1]] + c p3[[1]])/p,
     (a p1[[2]] + b p2[[2]] + c p3[[2]])/p}]
```

Once you have the center, the radius will be the shortest distance from the center to one of the sides.

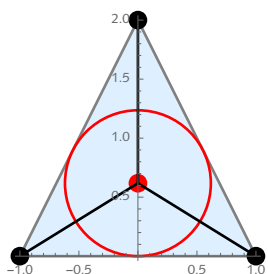
```
In[71]:= inRadius[pts : {pt1_, pt2_, pt3_}] :=
  EuclideanDistance[RegionNearest[InfiniteLine[pt1, pt3], inCenter[pts]],
    inCenter[pts]]
```

```

In[72]:= {p1, p2, p3} = {{-1, 0}, {0, 2}, {1, 0}};
pts = {p1, p2, p3};
Graphics[{
  {EdgeForm[Gray], LightBlue, Triangle[pts]},
  {PointSize[Large], Point[pts]},
  {Red, Circle[inCenter@pts, inRadius[pts]], PointSize[Large],
   Point[inCenter@pts]},
  {Map[Line[{inCenter@pts, #}] &, pts]}
}, Axes → True]

```

Out[74]=



12. First set the distribution and compute the mean and standard deviation.

```

In[75]:=  $\mathcal{D}$  = NormalDistribution[0, 1];
 $\sigma$  = StandardDeviation[ $\mathcal{D}$ ];
 $\mu$  = Mean[ $\mathcal{D}$ ];

```

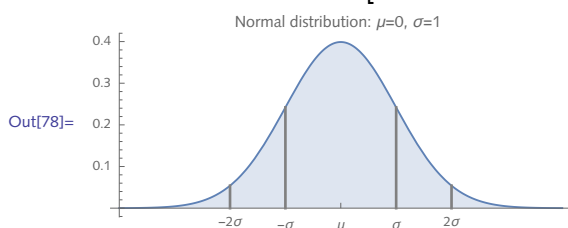
Next we manually construct four vertical lines at the standard deviations going from the horizontal axis to the pdf curve.



```

In[78]:= Plot[PDF[ $\mathcal{D}$ , x], {x, -4, 4},
  Filling → Axis,
  Epilog →
    {Gray, Line[{{ $\mu + \sigma$ , 0}, { $\mu + \sigma$ , PDF[ $\mathcal{D}$ ,  $\mu + \sigma$ ]}}},
      {{ $\mu - \sigma$ , 0}, { $\mu - \sigma$ , PDF[ $\mathcal{D}$ ,  $\mu - \sigma$ ]}}},
      {{ $\mu + 2\sigma$ , 0}, { $\mu + 2\sigma$ , PDF[ $\mathcal{D}$ ,  $\mu + 2\sigma$ ]}}},
      {{ $\mu - 2\sigma$ , 0}, { $\mu - 2\sigma$ , PDF[ $\mathcal{D}$ ,  $\mu - 2\sigma$ ]}}}],
  AxesOrigin → {-4, 0},
  Ticks → {{-2  $\sigma$ , "-2 $\sigma$ "}, {- $\sigma$ , "- $\sigma$ "}, { $\mu$ , " $\mu$ "}, { $\sigma$ , " $\sigma$ "}, {2  $\sigma$ , "2 $\sigma$ "},
    Automatic},
  AspectRatio → 0.4,
  PlotLabel → StringForm["Normal distribution:  $\mu$ =`1`,  $\sigma$ =`2`",  $\mu$ ,  $\sigma$ ]]

```



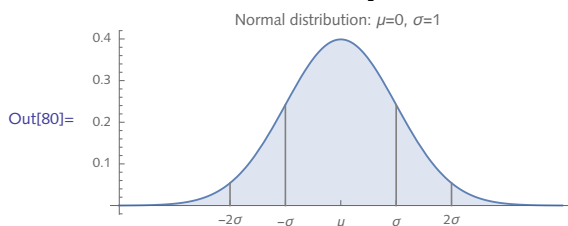
And here is a little utility function to make the code a bit more readable and easier to use.

```

In[79]:= sdLine[ $\mathcal{D}$ _,  $\mu$ _,  $\sigma$ _] :=
  Line[{{ $\mu + \sigma$ , 0}, { $\mu + \sigma$ , PDF[ $\mathcal{D}$ ,  $\mu + \sigma$ ]}}},
    {{ $\mu - \sigma$ , 0}, { $\mu - \sigma$ , PDF[ $\mathcal{D}$ ,  $\mu - \sigma$ ]}}}]

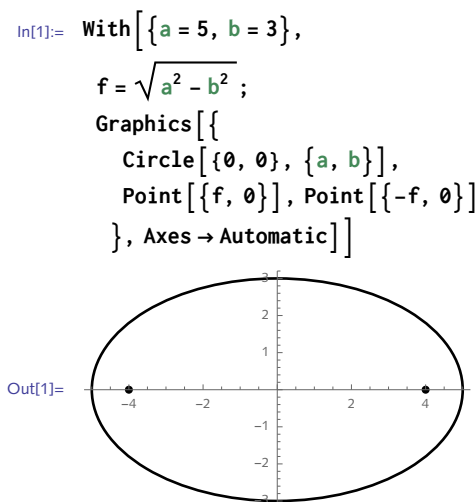
In[80]:= Plot[PDF[ $\mathcal{D}$ , x], {x, -4, 4},
  Filling → Axis,
  Epilog → {Gray, Thickness[.0035], sdLine[ $\mathcal{D}$ ,  $\mu$ ,  $\sigma$ ], sdLine[ $\mathcal{D}$ ,  $\mu$ , 2  $\sigma$ ]},
  AxesOrigin → {-4, 0},
  Ticks → {{-2  $\sigma$ , "-2 $\sigma$ "}, {- $\sigma$ , "- $\sigma$ "}, { $\mu$ , " $\mu$ "}, { $\sigma$ , " $\sigma$ "}, {2  $\sigma$ , "2 $\sigma$ "},
    Automatic},
  AspectRatio → 0.4,
  PlotLabel → StringForm["Normal distribution:  $\mu$ =`1`,  $\sigma$ =`2`",  $\mu$ ,  $\sigma$ ]]

```



## 8.2 Dynamic graphics: exercises

1. In the Manipulate examples in this section, the parameters were controlled by sliders. Moving the slider changes the value of the parameter and any expression dependent upon that parameter inside the Manipulate expression. Sometimes you want to choose values for a parameter from a list of discrete values. A setter bar is a convenient control object for this. One way to set it is to use a different syntax for the parameter list:  $\{param, \{val_1, val_2, \dots, val_n\}\}$  will cause Manipulate to automatically use a setter bar instead of a slider. Create a Manipulate object showing plots of sin, cos, or tan, each selectable from a setter bar.
2. Modify the above exercise to use a popup menu to choose the function to plot. You can explicitly set the control by using `ControlType → "PopupMenu"`.
3. Here is a graphic showing an ellipsoid together with its two foci:



Turn this into a dynamic graphic by making the semi-major and semi-minor axes lengths  $a$  and  $b$  dynamic which, upon updating, will cause the ellipse to change shape. Some thought will be needed to properly deal with the situation  $b > a$ .

4. Create a dynamic interface that displays various structure diagrams and space-filling plots of the amino acids. A list of the amino acids is given by

```
In[2]:= ChemicalData["AminoAcids"]
```

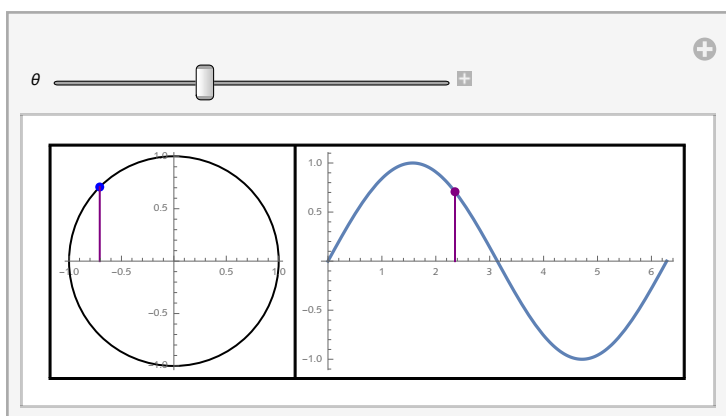
```
Out[2]= {glycine, L-alanine, L-serine, L-proline, L-valine, L-threonine, L-cysteine, L-isoleucine,
  L-leucine, L-asparagine, L-aspartic acid, L-glutamine, L-lysine, L-glutamic acid,
  L-methionine, L-histidine, L-phenylalanine, L-arginine, L-tyrosine, L-tryptophan}
```

The diagrams and plots that should be included are built into `ChemicalData`:

```
In[3]:= StringCases[ChemicalData["Properties"], __ ~~ "Diagram" | (__ ~~ "Plot")] //
Flatten
Out[3]= {BlackStructureDiagram, CHBlackStructureDiagram, CHColorStructureDiagram,
ColorStructureDiagram, MoleculePlot, SpaceFillingMoleculePlot}
```

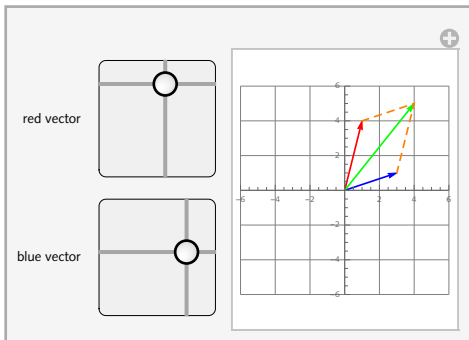
- Using the code developed in Section 8.1 for plotting the centroid of a triangle, create a dynamic interface that displays the triangle, the medians (lines from each vertex to the midpoint of the opposite side), and the triangle vertices as locators.
- In the 1920s and 1930s the artist Marcel Duchamp created what he termed *rotoreliefs*, spinning concentric circles giving a three-dimensional illusion of depth ([Duchamp 1926](#)). Create your own rotoreliefs: start with several concentric circles of different radii, then vary their centers around a path given by another circle, and animate.
- Create a plot of  $\sin(\theta)$  side-by-side with a circle and a dynamic point that moves along the curve and the circle as  $\theta$  varies from 0 to  $2\pi$  (Figure 8.3).

FIGURE 8.3. Dynamic visualization of  $\sin$  function.



- Take one of the two-dimensional random walk programs developed elsewhere in this book (Sections 6.3 and 10.3) and create an animation displaying successive steps of the random walk.
- Looking forward to Chapter 10, where we develop a full application for computing and visualizing random walks, create a dynamic interface that displays random walks, adding controls to select the number of steps from a pulldown menu, the dimension from a setter bar, and a checkbox to turn on and off lattice walks.
- Create a visualization of two-dimensional vector addition (Figure 8.4). The interface should include either a 2D slider for two vectors in the plane or locators to change the position of each vector; the display should show the two vectors as well as their vector sum. Extend the solution to three dimensions. (The solution of this vector interface is due to Harry Calkins of Wolfram Research.)

FIGURE 8.4. Dynamic visualization of vector arithmetic.



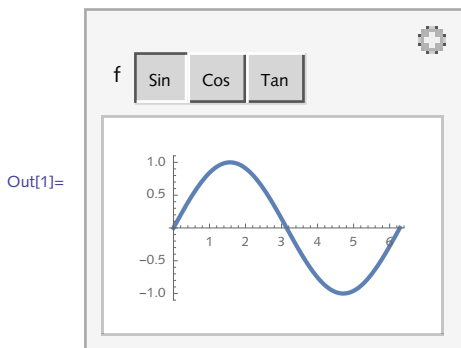
11. Create a dynamic interface consisting of a locator constrained to the unit circle. Check the documentation on `Locator` for information on constraining the movement of locators.
12. Create a dynamic interface that displays twenty random points in the unit square whose locations are randomized each time you click your mouse on the graphic. Add a checkbox to toggle the display of the shortest path (`FindShortestTour`) through the points (look up `EventHandler` and `MouseClicked` in the documentation).

A suggested addition would be to add a control to change the number of points that are used but take care to keep the total number of points manageable (see the note on Traveling Salesman problems at the end of the chapter).

## 8.2 Solutions

1. Giving the parameter list in the form  $\{param, \{val_1, val_2, val_3\}\}$  automatically causes `Manipulate` to use a setter bar as the control type.

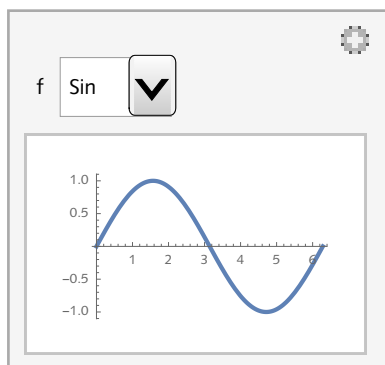
```
In[1]:= Manipulate[
  Plot[f[x], {x, 0, 2 π}],
  {f, {Sin, Cos, Tan}}
```



2. This explicitly sets the control type to a popup menu.

```
In[2]:= Manipulate[
  Plot[f[x], {x, 0, 2 π}],
  {f, {Sin, Cos, Tan}, ControlType → PopupMenu}]
```

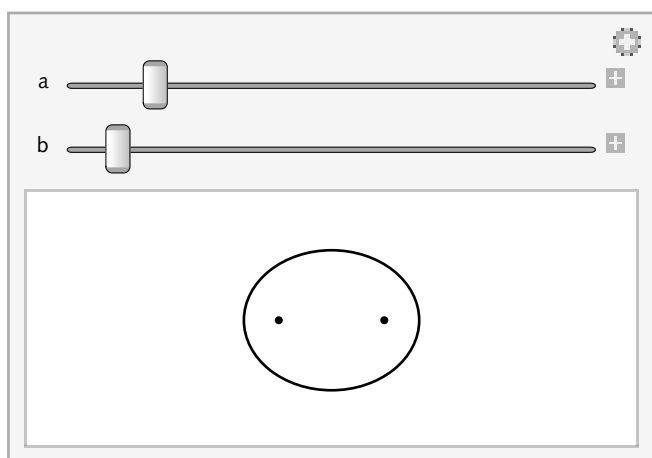
Out[2]=



3. If the parameter  $b$  is greater than  $a$ , you will need to switch the foci to the vertical axis. This is done inside the If statement before the graphics are given.

```
In[3]:= Manipulate[
  f = If[a^2 - b^2 ≥ 0, {√Abs[a^2 - b^2], 0}, {0, √Abs[a^2 - b^2]}];
  Graphics[{
    Circle[{0, 0}, {a, b}],
    Point[f], Point[-f]
  }, PlotRange → Max[a, b] + .1],
  {a, 1, 5}, {b, 1, 5}]
```

Out[3]=



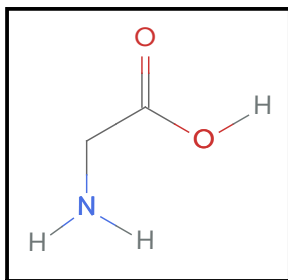
4. We will put this together in two parts: first create a function to display any amino acid using one of the various diagrams; then pour it into a Manipulate. Note, this function is dependent upon ChemicalData to create the displays. ChemicalData returns an Entity, hence the two rules, one for the chemical name as a string and the other as an Entity. Alternatively, you could modify it to use your own visualizations, such as the space-filling plots in Section 8.4.

```
In[4]:= AminoAcidPlot[aa_String, diagram_ : "ColorStructureDiagram"] :=  
        Labeled[Framed[ChemicalData[aa, diagram], ImageSize -> All],  
        ChemicalData[aa, "Name"], LabelStyle -> Directive["Menu", 9]]
```

```
In[5]:= AminoAcidPlot[aa_Entity, diagram_ : "ColorStructureDiagram"] :=  
        Labeled[Framed[ChemicalData[aa, diagram], ImageSize -> All],  
        ChemicalData[aa, "Name"], LabelStyle -> Directive["Menu", 9]]
```

```
In[6]:= AminoAcidPlot["Glycine"]
```

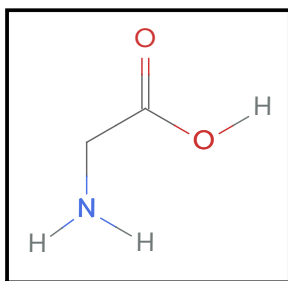
Out[6]=



glycine

```
In[7]:= AminoAcidPlot[ glycine (chemical) ]
```

Out[7]=

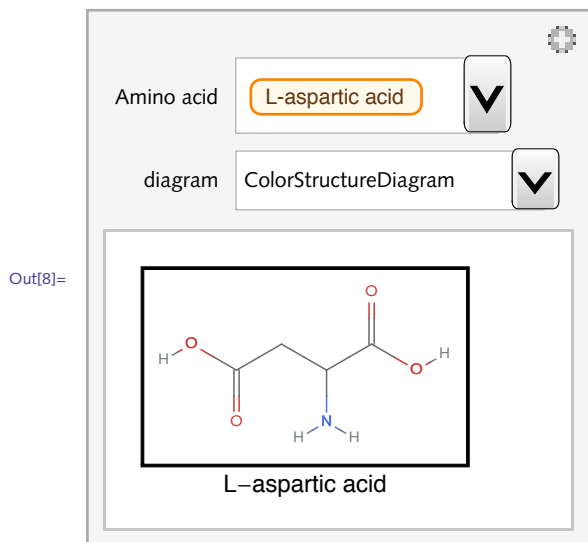


glycine

```

In[8]:= Manipulate[
  AminoAcidPlot[aminoacid, diagram],
  {{aminoacid, "LAlanine", "Amino acid"}, aa},
  {diagram, {"StructureDiagram", "CHColorStructureDiagram",
    "CHStructureDiagram", "ColorStructureDiagram", "MoleculePlot",
    "SpaceFillingMoleculePlot"}},
  Initialization -> {aa = ChemicalData["AminoAcids"]}, SynchronousUpdating -> False,
  SaveDefinitions -> True]

```

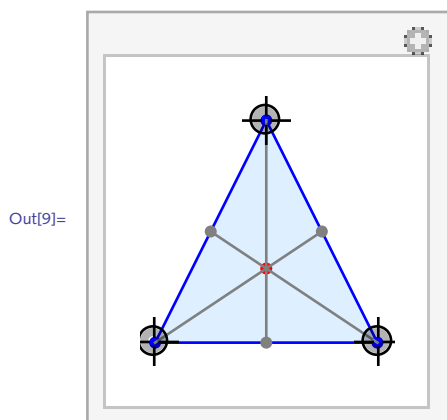


5. Borrowing the CentroidPlot from Exercise 10 from Section 8.1, here is the Manipulate.

```

In[9]:= Manipulate[CentroidPlot[Triangle[pts], PlotRange -> {-0.2, 2.2}],
  {{pts, {{-1, 0}, {0, 2}, {1, 0}}}, Locator}, SaveDefinitions -> True]

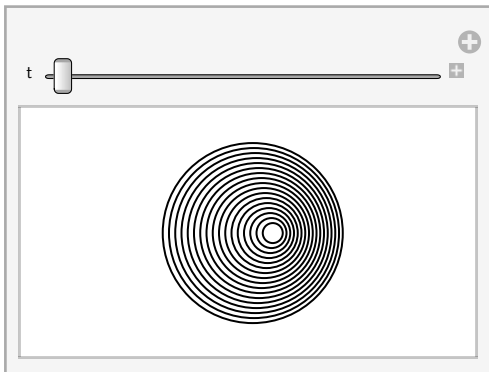
```



6. Modify the radii and the centers to get different effects. Try using transparent disks instead of circles.

```
In[10]:= Manipulate[
  Graphics[
    Table[Circle[r/4 {Cos[t], Sin[t]}, 1.1 - r], {r, .2, 1, .05}],
    PlotRange -> 1],
  {t, 0, 2 π, .1}]
```

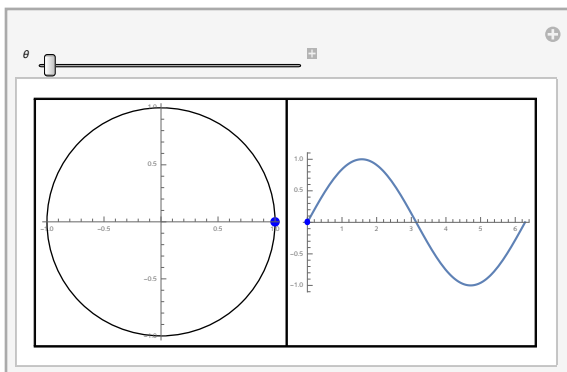
Out[10]=



7. Putting the two graphics pieces (Graphics [...] and Plot [...]) in a grid gives you finer control over their placement and formatting.

```
In[11]:= Manipulate[
  Grid[
    {{Graphics[{Circle[], Blue, PointSize[.04], Point[{Cos[θ], Sin[θ]}]},
      Axes -> True], Plot[Sin[x], {x, 0, 2 π},
      Epilog -> {Blue, Line[{θ, 0}, {θ, Sin[θ]}]}, PointSize[.025],
      Point[{θ, Sin[θ]}]}]}, Frame -> All], {θ, 0, 2 π}]
```

Out[11]=



8. First load the package that contains the random walk code. You could use your own implementation as well.

```
In[12]:= << EPM`RandomWalks`
```



Create a 1000-step, two-dimensional, lattice walk.

```
In[13]:= rw = RandomWalk[1000, Dimension → 2, LatticeWalk → True];
```

This is a basic start. Take is used to display successive increments. Note the need for the `i` in the parameter list to insure that steps only take on integer values.

```
In[14]:= Animate[
  Graphics[Line[Take[rw, n]]],
  {n, 2, Length[rw], 1}, AnimationRunning → False, SaveDefinitions → True]
```

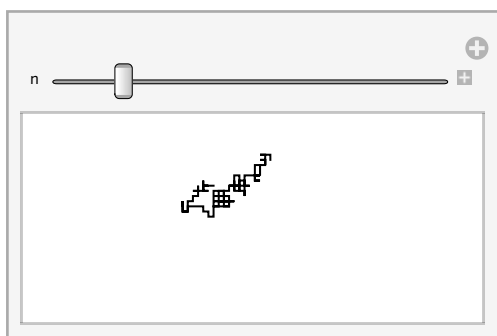
Out[14]=



The output above suffers from the fact that the display jumps around a lot as *Mathematica* tries to figure out a sensible plot range for *each* frame. Instead, we should fix the plot range for *all* frames to avoid this jumpiness. This is done in the definitions for `xran` and `yrans` in the Initialization below.

```
In[15]:= Manipulate[
  Graphics[Line[Take[rw, n]], PlotRange → {xran, yrans}],
  {n, 2, Length[rw], 1},
  Initialization → {
    rw = RandomWalk[1000, Dimension → 2, LatticeWalk → True];
    {xran, yrans} = Map[{Min[##], Max[##]} &, Transpose[rw]],
    SaveDefinitions → True]
```

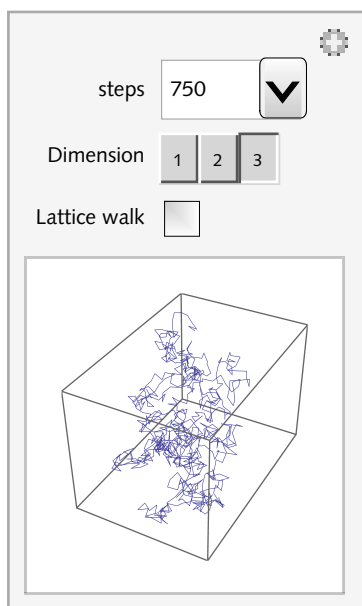
Out[15]=



9. Using the programs developed in Section 10.3, here is the code, including a pulldown menu for the steps parameter, a setter bar for the dimension parameter, and a checkbox for the lattice parameter.

```
In[16]:= Manipulate[
  ShowWalk@RandomWalk[steps, Dimension → dim, LatticeWalk → latticeQ],
  {steps, {100, 250, 500, 750, 1000, 10000}},
  {{dim, 1, "Dimension"}, {1, 2, 3}},
  {{latticeQ, True, "Lattice walk"}, {True, False}},
  Initialization → Needs["EPM`RandomWalks`"], SaveDefinitions → True]
```

Out[16]=

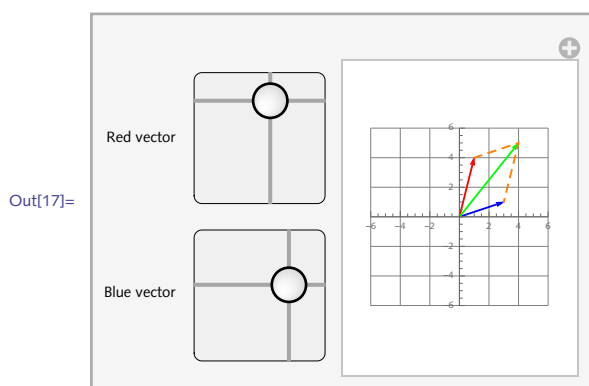


10. Here is the solution using Slider2D. Using Locator instead is left for the reader.

```

In[17]:= Manipulate[
  Graphics[{
    Red, Arrow[{{0, 0}, pt1}],
    Blue, Arrow[{{0, 0}, pt2}],
    Green, Arrow[{{0, 0}, pt1 + pt2}],
    Dashed, Orange, Line[{pt1, pt1 + pt2, pt2}],
    PlotRange -> 6, Axes -> True, GridLines -> Automatic],
  {{pt1, {1, 4}}, "Red vector"}, {-5, -5}, {5, 5}},
  {{pt2, {3, 1}}, "Blue vector"}, {-5, -5}, {5, 5}},
  ControlPlacement -> Left]

```

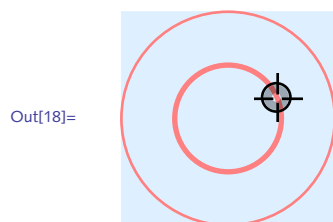


- II. The key here is to use a two-argument form of `Dynamic`, where the second argument gives the constraint on the parameter.

```

In[18]:= DynamicModule[{pt = {0, .5}},
  Graphics[{
    LightBlue, Rectangle[{-1, -1}, {1, 1}],
    Pink, Circle[], Thick, Circle[{0, 0}, .5],
    Locator[Dynamic[pt, (pt = .5 Normalize[#]) &]]
  }, PlotRange -> 1.25]]

```



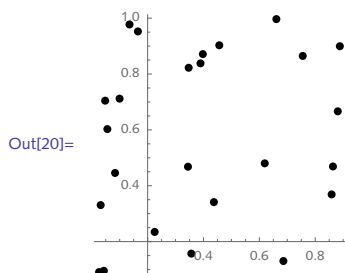
12. First, create a static version of the problem; we use `GraphicsComplex` to display the points and the tour.

```

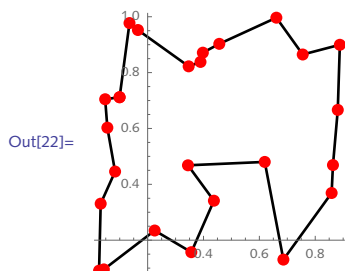
In[19]:= pts = RandomReal[1, {25, 2}];

```

```
In[20]:= Graphics[GraphicsComplex[pts, Point@Range[Length[pts]]], Axes → Automatic]
```



```
In[21]:= tour = Last[FindShortestTour[pts]];
Graphics[GraphicsComplex[pts,
  {Line[tour], Red, PointSize[Medium], Point[tour]}], Axes → Automatic]
```



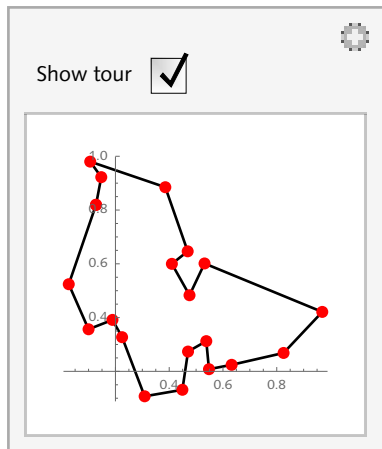
Here is the dynamic interface using `EventHandler` to choose a new set of random points with each mouse click.

```

In[23]:= Manipulate[
  DynamicModule[{pts = RandomReal[1, {20, 2}], tour},
    tour = Dynamic[Last[FindShortestTour[pts]]];
    EventHandler[
      Dynamic[
        Graphics[GraphicsComplex[pts,
          If[Not[showtour], Point@Range[Length[pts]],
            {Line[tour], Red, PointSize[Medium], Point[tour]}]],
        Axes → Automatic]],
      {"MouseClicked" => (pts = RandomReal[1, {20, 2}])}
    ],
    {{showtour, False, "Show tour"}, {True, False}},
    ContentSize → All]

```

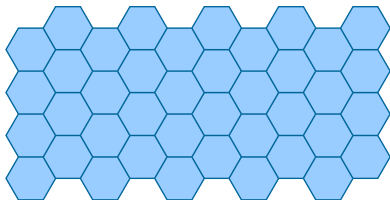
Out[23]=



### 8.3 Efficient structures: exercises

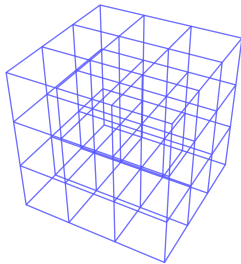
1. Create a hexagonal grid of polygons like that in Figure 8.5. First create the grid by performing appropriate translations using `Translate` or the geometric transformation `TranslationTransform`. Compare this approach with a multi-polygon approach.

FIGURE 8.5. Two-dimensional hexagonal lattice.



2. Take the example visualizing a 500 000-step random walk at the beginning of this section and replicate the output using `GraphicsComplex` instead of `ListLinePlot`. Compare the running times for each as the number of steps increases.
3. The `ShowWalk` function discussed in Section 4.1 for displaying random walks uses `ListLinePlot` to display the data. As mentioned in this section, `ListLinePlot` will get bogged down for large numbers of points. Using the solution to the previous exercise, create a new version for both the two- and three-dimensional cases of `ShowWalk` that uses `GraphicsComplex` instead; then test the new implementation against the one developed in Chapter 4.
4. Create a graphic consisting of a three-dimensional lattice, that is, lines passing through the integer coordinates in 3-space (Figure 8.6). Compare approaches that use multi-lines as opposed to those that do not.

FIGURE 8.6. *Three-dimensional integer lattice.*



5. A common problem in computational geometry is finding the boundary of a given set of points in the plane. One way to think about the boundary is to imagine the points as nails in a board and then to stretch a rubber band around all the nails. The stretched rubber band lies on a convex polygon commonly called the *convex hull* of the point set.

Create a function `ConvexHullPlot` for visualizing the convex hull together with the points on the interior of the convex hull. Your function should inherit options for `Graphics`. The built-in function `ConvexHullMesh` can be used to generate the hull polygon:

```
In[1]:= pts = RandomReal[1, {28, 2}];
        ℳ = ConvexHullMesh[pts]
```

Out[2]=



The zero-dimensional objects in the mesh are points and the one-dimensional objects are lines (two-dimensional objects would be polygons). These are the points and lines on the convex hull.

```

In[3]:= MeshPrimitives[R, 0];
Take[%, 2]

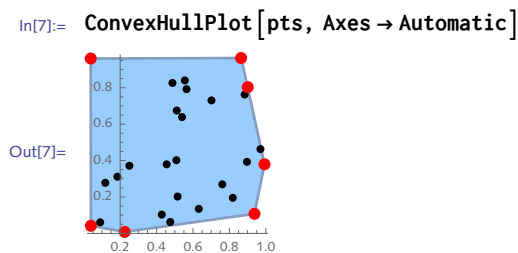
Out[4]= {Point[{0.992746, 0.379391}], Point[{0.0394433, 0.0418121}]}

In[5]:= MeshPrimitives[R, 1];
Take[%, 2]

Out[6]= {Line[{0.0383976, 0.960283}, {0.0394433, 0.0418121}]},
Line[{0.0394433, 0.0418121}, {0.225102, 0.00814885}]}

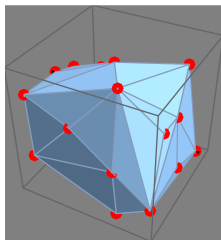
```

Your function should generate output similar to the following:



- Modify ConvexHullPlot from Exercise 5 to accept an option Dimension. With default value of two, ConvexHullPlot should produce output like that in Exercise 5. For Dimension  $\rightarrow 3$ , it should generate the convex hull together with large red points at each vertex of the hull (Figure 8.7).

FIGURE 8.7. Three-dimensional convex hull with hull points highlighted.



- Extend Exercise 7 from Section 8.1 to random walks on the base- $n$  digits of  $\pi$ . For example, in base 3, a 1 corresponds to an angle of  $120^\circ$  from the current position, 2 corresponds to  $240^\circ$ , and 0 to  $360^\circ$ . In base 4 the step angles will be multiples of  $90^\circ$  and in general, for base  $n$ , the step angles will be multiples of  $360^\circ/n$ . Use GraphicsComplex to visualize the walks. Include a color function that depends on the length of the walk.

## 8.3 Solutions

- Here is the implementation using TranslationTransform.

```

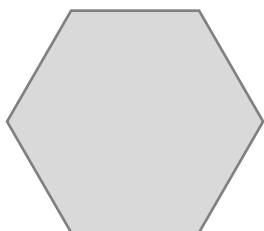
In[1]:= Clear[vertices, n,  $\alpha$ ]

```

```
In[2]:= vertices[n_] := Table[{Cos[ $\frac{2\pi\alpha}{n}$ ], Sin[ $\frac{2\pi\alpha}{n}$ ]}, { $\alpha$ , 0, n}]
```

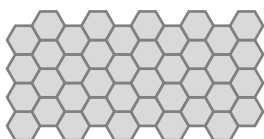
```
In[3]:= hexagon = Polygon[vertices[6]];
Graphics[{EdgeForm[Gray], LightGray, hexagon}]
```

Out[3]=



```
In[4]:= Graphics[{
  EdgeForm[Gray], LightGray,
  Table[GeometricTransformation[hexagon,
    TranslationTransform[{ $3i + \frac{3}{4}((-1)^j + 1)$ ,  $\frac{\sqrt{3}j}{2}$ }]
  ], {i, 5}, {j, 8}]
}]
```

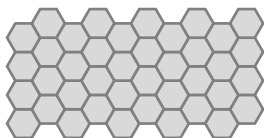
Out[4]=



Or use Translate directly.

```
In[5]:= gr1 = Graphics[{
  EdgeForm[Gray], LightGray,
  Table[Translate[hexagon, { $3i + \frac{3}{4}((-1)^j + 1)$ ,  $\frac{\sqrt{3}j}{2}$ }], {i, 5}, {j, 8}]
}]
```

Out[5]=



This implementation contains one Polygon per hexagon.

```
In[6]:= Count[gr1, _Polygon, Infinity]
```

Out[6]= 40



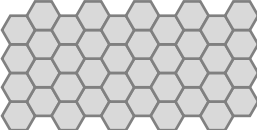
Now use multi-polygons. The following version of hexagon is defined so that it can take a pair of translation coordinates. Note also the need to flatten the table of vertices so that Polygon can be applied to the correct list structure.

```
In[7]:= Clear[hexagon];

hexagon[{x_, y_}] := Table[{Cos[ $\frac{2\pi i}{6}$ ] + x, Sin[ $\frac{2\pi i}{6}$ ] + y}, {i, 1, 6}]

In[9]:= gr2 =
Graphics[{EdgeForm[Gray], LightGray,
Polygon[Flatten[Table[hexagon[{ $3i + \frac{3}{4}((-1)^j + 1), \frac{\sqrt{3}j}{2}$ }], {i, 5}, {j, 8}],
1]]]}]

Out[9]=
```



```
In[10]:= Count[gr2, _Polygon, Infinity]
```

```
Out[10]= 1
```

- Here is the usage message for GraphicsComplex.

```
In[11]:= ?GraphicsComplex
```

GraphicsComplex[{ $pt_1, pt_2, \dots$ }, data] represents a graphics complex in which coordinates given as integers  $i$  in graphics primitives in data are taken to be  $pt_i$ . >>

The first argument to GraphicsComplex is a list of coordinate points, such as the output from RandomWalk. The second argument is a set of graphics primitives indexed by the positions of the points in the list of coordinates. The walk itself gives the coordinates that will be used as the first argument to GraphicsComplex.

```
In[12]:= walk = RandomWalk[ $5 \times 10^5$ ];
```

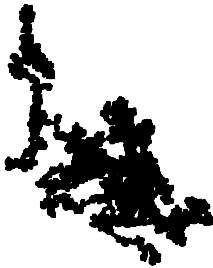
```
In[13]:= Short[walk, 2]
```

```
Out[13]/Short= {{0, 1}, {0, 2}, {0, 3}, {0, 2}, <<499 992>>,
{-447, 892}, {-446, 892}, {-445, 892}, {-445, 891}}
```

The second argument a list of primitives with the coordinate indices as arguments. We want to connect the first point to the second, to the third, etc., with a line. That would look like Line[{1, 2, 3, ...}]. The length of the list is given by Length[walk].

```
In[14]:= Graphics[GraphicsComplex[walk, {
    Line@Range[Length[walk]]
}]]
```

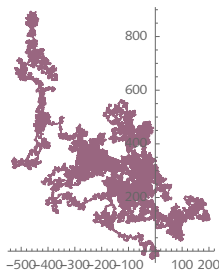
Out[14]=



You could add axes and some directives and options to modify the plot.

```
In[15]:= Graphics[GraphicsComplex[walk, {
    RGBColor[.6, .4, .5], Thickness[Tiny], Line@Range[Length[walk]]
}], Axes → Automatic] // Timing
```

Out[15]= {0.001048,

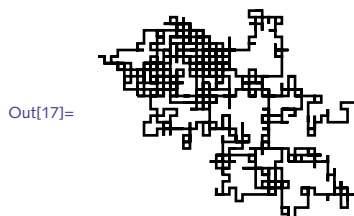


The time to compute (and display) this object is several orders of magnitude faster than that with `ListLinePlot`. So why is `ListLinePlot` so slow with data of this size? It is designed to be as general as possible and to deal with many extraordinary situations. This generality comes at a cost. It is very fast for moderately sized data sets, but for large sets, another approach is preferable.

3. First, using the same argument structure as that in `ShowWalk` (Chapter 10), we give `GraphicsComplex` the list of coordinates followed by `Line` wrapped around a list of the point indices 1 through  $n$  (the length of the walk).

```
In[16]:= ShowWalkGC[coords : {{_, _} ..}] := Graphics[
    GraphicsComplex[coords, Line[Range[Length[coords]]]]]
```

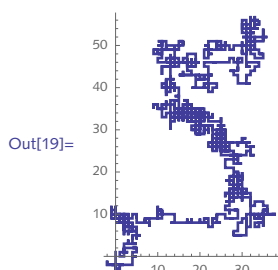
```
In[17]:= ShowWalkGC[RandomWalk[1500, Dimension → 2]]
```



Add options and color the path.

```
In[18]:= Clear[ShowWalkGC];
ShowWalkGC[coords : {{_, _} ..}, opts : OptionsPattern[Graphics]] := Graphics[
  GraphicsComplex[coords,
    {ColorData[1][1], Line[Range[Length[coords]]]}],
  opts,
  Axes → True, AspectRatio → Automatic]
```

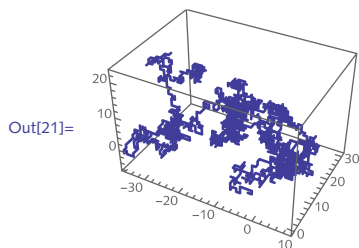
```
In[19]:= ShowWalkGC[RandomWalk[1500, Dimension → 2]]
```



And similarly for the three-dimensional case:

```
In[20]:= ShowWalkGC[coords : {{_, _, _} ..}, opts : OptionsPattern[Graphics3D]] :=
  Graphics3D[GraphicsComplex[coords,
    {ColorData[1][1], AbsoluteThickness[1], Line[Range[Length[coords]]]}],
  opts, Axes → True, AspectRatio → Automatic]
```

```
In[21]:= ShowWalkGC[RandomWalk[2500, Dimension → 3]]
```

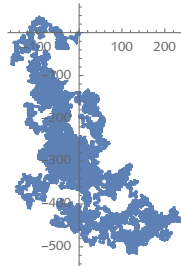


Finally, some timing comparisons between this implementation with GraphicsComplex and

the versions that used basic graphics primitives and built-in functions.

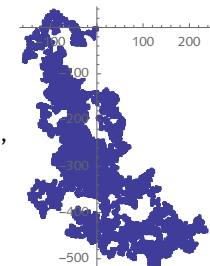
```
In[22]:= walk = RandomWalk[250000];
Timing[ShowWalk[walk, AspectRatio → Automatic]]
```

```
Out[23]= {4.28677, }
```



```
In[24]:= Timing[ShowWalkGC[walk]]
```

```
Out[24]= {0.017871, }
```



4. One approach to creating the lattice is to manually specify the coordinates for the lines and then map the `Line` primitive across these coordinates. We will work with a small lattice.

```
In[25]:= xmin = 0; xmax = 3;
ymin = 0; ymax = 3;
zmin = 0; zmax = 3;
Table[{ {x, ymin, zmin}, {x, ymax, zmin}}, {x, xmin, xmax}]
```

```
Out[28]= {{{0, 0, 0}, {0, 3, 0}}, {{1, 0, 0}, {1, 3, 0}},
          {{2, 0, 0}, {2, 3, 0}}, {{3, 0, 0}, {3, 3, 0}}}
```

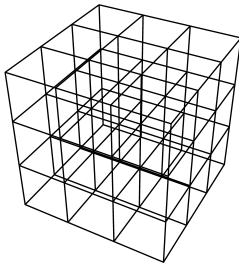
Here are the three grids.

```
In[29]:= gridX = Table[{ {xmin, y, z}, {xmax, y, z}}, {y, ymin, ymax}, {z, zmin, zmax}];
gridY = Table[{ {x, ymin, z}, {x, ymax, z}}, {x, xmin, xmax}, {z, zmin, zmax}];
gridZ = Table[{ {x, y, zmin}, {x, y, zmax}}, {x, xmin, xmax}, {y, ymin, ymax}];
```

Finally, map `Line` across these grids and display as a `Graphics3D` object.

```
In[32]:= gr1 = Graphics3D[{
    Map[Line, gridX, {2}],
    Map[Line, gridY, {2}],
    Map[Line, gridZ, {2}]
}, Boxed -> False]
```

Out[32]=



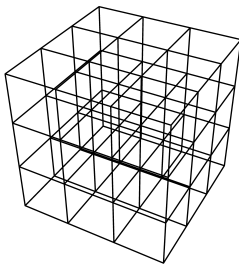
```
In[33]:= Count[gr1, _Line, Infinity]
```

Out[33]= 48

Using multi-lines reduces the number of Line objects substantially.

```
In[34]:= gr2 = Graphics3D[{
    Map[Line, gridX],
    Map[Line, gridY],
    Map[Line, gridZ]
}, Boxed -> False]
```

Out[34]=



```
In[35]:= Count[gr2, _Line, Infinity]
```

Out[35]= 12

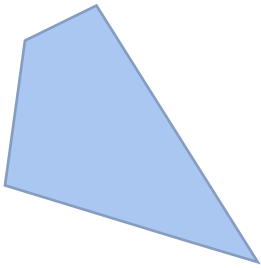
5. First, generate some random points in the plane.

```
In[36]:= pts = RandomReal[1, {8, 2}];
```

Next, generate the mesh using a built-in function.

```
In[37]:=  $\mathcal{R}$  = ConvexHullMesh[pts]
```

```
Out[37]=
```



Quite a bit of information is embedded in this `BoundaryMeshRegion`. For example, it is easy to extract the points on the boundary. They are mesh primitives of dimension zero.

```
In[38]:= Head[ $\mathcal{R}$ ]
```

```
Out[38]= BoundaryMeshRegion
```

```
In[39]:= MeshPrimitives[ $\mathcal{R}$ , 0]
```

```
Out[39]= {Point[{0.643154, 0.220953}],  
          Point[{0.357669, 0.997291}], Point[{0.0543152, 0.400786}],  
          Point[{0.120583, 0.881017}], Point[{0.892886, 0.146246}]}
```

Similarly, lines are mesh primitives of dimension one.

```
In[40]:= MeshPrimitives[ $\mathcal{R}$ , 1]
```

```
Out[40]= {Line[{0.0543152, 0.400786}, {0.643154, 0.220953}],  
          Line[{0.643154, 0.220953}, {0.892886, 0.146246}],  
          Line[{0.892886, 0.146246}, {0.357669, 0.997291}],  
          Line[{0.357669, 0.997291}, {0.120583, 0.881017}],  
          Line[{0.120583, 0.881017}, {0.0543152, 0.400786}]}
```

In fact, the polygon itself is a mesh primitive of dimension two.

```
In[41]:= MeshPrimitives[ $\mathcal{R}$ , 2]
```

```
Out[41]= {Polygon[{0.0543152, 0.400786}, {0.643154, 0.220953},  
                  {0.892886, 0.146246}, {0.357669, 0.997291}, {0.120583, 0.881017}]}
```

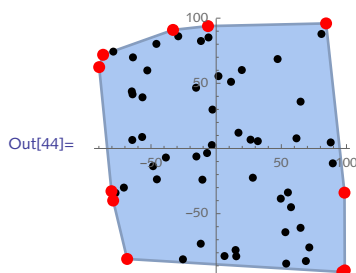
So the pieces of the mesh that we want to show are the region itself (bounded polygon, the original points, and the points on the boundary. We will highlight the points on the boundary by making them large and red. And we will pass options from `Graphics` on to the function.

```
In[42]:= ConvexHullPlot[pts_, opts : OptionsPattern[Graphics]] :=  
  Module[{ $\mathcal{R}$ },  $\mathcal{R}$  = ConvexHullMesh[pts];  
    Show[ $\mathcal{R}$ , Graphics[{Point[pts], Red, PointSize[Medium], MeshPrimitives[ $\mathcal{R}$ , 0]}],  
          opts]]
```

Try it out with a larger point set.

```
In[43]:= pts = RandomReal[{-100, 100}, {60, 2}];
```

```
In[44]:= ConvexHullPlot[pts, Axes → Automatic, ImageSize → Small]
```



6. First, set up the options framework.

```
In[45]:= Options[ConvexHullPlot] = Join[{Dimension → 2}, Options[Graphics],
Options[Graphics3D]];
```

Here is a message that will be issued if the Dimension option is set to anything other than 2 or 3.

```
In[46]:= ConvexHullPlot::baddim =
"The value `1` of the Dimension option should be either 2 or 3";
```

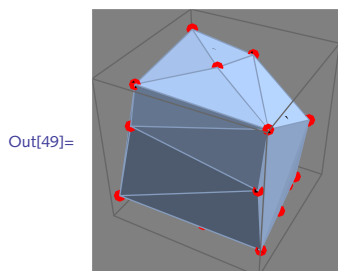
```
In[47]:= ConvexHullPlot[pts_, opts : OptionsPattern[]] :=
Module[{R, dim = OptionValue[Dimension]},
R = ConvexHullMesh[pts];
Which[
dim == 2,
Show[
{R, Graphics[{Point[pts], Red, PointSize[Medium],
MeshPrimitives[R, 0]}]},
FilterRules[{opts}, Options[Graphics]]],
dim == 3,
Show[
{R, Graphics3D[{Point[pts], Red, PointSize[Medium],
MeshPrimitives[R, 0]}]},
FilterRules[{opts}, Options[Graphics3D]]],
True, Message[ConvexHullPlot::baddim, dim]
]
]
```

Here is a larger point set.

```
In[48]:= pts = RandomReal[1, {40, 3}];
```

Exercise some options.

```
In[49]:= ConvexHullPlot[pts, Dimension → 3, Boxed → True, Background → Gray]
```



Giving a bad value for the Dimension option should produce an error message.

```
In[50]:= ConvexHullPlot[pts, Dimension → 4]
```

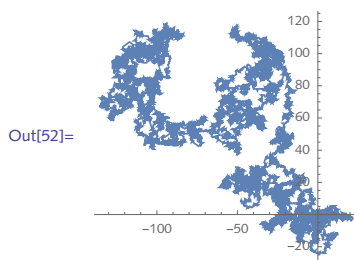
ConvexHullPlot::baddim : The value 4 of the Dimension option should be either 2 or 3

7. Here is the random walk on the digits of  $\pi$  in bases given by the second argument.

```
In[51]:= RandomWalkPi[d_, base_ /; base > 2] := Module[{digits, angles, rules},
  digits = First[RealDigits[N[ $\pi$ , d], base]];
  angles = Rest@Range[0., 2  $\pi$ , 2  $\pi$  / (base)];
  rules = MapThread[#1 → #2 &, {Range[0, base - 1], angles}];
  Accumulate[Map[{Cos[#], Sin[#]} &, digits /. rules]]
]
```

Using ListPlot, here is a quick visualization on base 5 digits:

```
In[52]:= ListLinePlot[RandomWalkPi[10000, 5], AspectRatio → Automatic]
```



Here is the GraphicsComplex.

```
In[53]:= walk = RandomWalkPi[10000, 5];
len = Length[walk];
```



```
In[55]:= Graphics[GraphicsComplex[walk, {AbsoluteThickness[.2], Line[Range[1en]]}],
  AspectRatio -> Automatic]
```

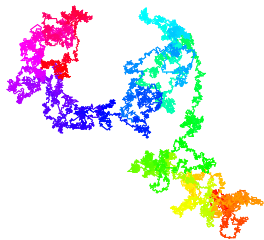
Out[55]=



And here it is with color mapped to the distance from the origin.

```
In[56]:= Graphics[GraphicsComplex[walk,
  Map[{Hue[ $\frac{\#[[1]]}{1en}$ ], AbsoluteThickness[.25], Line[#]} &,
  Partition[Range[2, 1en], 2, 1]]], AspectRatio -> Automatic]
```

Out[56]=



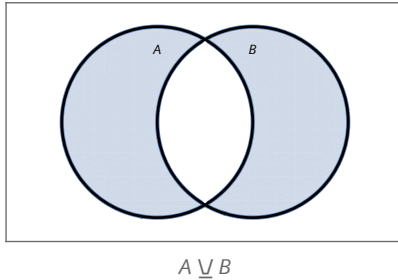
## 8.4 Examples: exercises

1. Create a function `ComplexListPlot` that plots a list of complex numbers in the complex plane using `ListPlot`. Set initial options so that the `PlotStyle` is red, the `PointSize` is a little larger than the default, and the horizontal and vertical axes are labeled “Re” and “Im,” respectively. Set up options to `ComplexListPlot` that are inherited from `ListPlot`.
2. Create a function `ComplexRootPlot` that plots the complex zeros of a polynomial in the plane. Use your implementation of `ComplexListPlot` that you developed in the previous exercise.
3. Use `Mesh` in a manner similar to its use in the `RootPlot` function to highlight the intersection of two surfaces, say  $\sin(2x - \cos(y))$  and  $\sin(x - \cos(2y))$ . You may need to increase the value of `MaxRecursion` to get the sampling just right.
4. The version of `VennDiagram` developed in this section used `Graphics` to create the circles and then combined them with `RegionPlot` using `Show`. Modify `VennDiagram` so that the circles are created entirely inside `RegionPlot`.

5. Create a new rule for `VennDiagram` that takes a logical expression as its first argument instead of a logical function. For example, your function should be able to handle input such as the following:

In[1]:= `VennDiagram[A  $\cup$  B, {A, B}]`

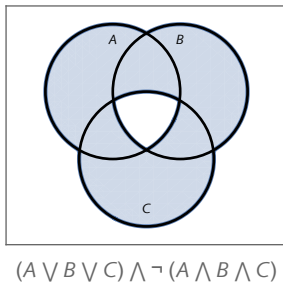
Out[1]=



6. Extend the previous exercise to a Venn diagram on three sets, using logical expressions as the first argument.

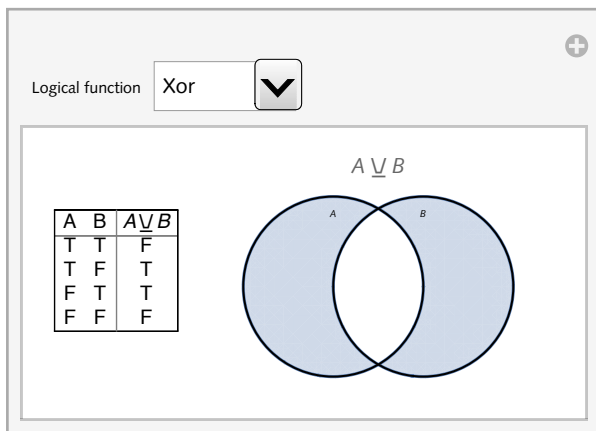
In[2]:= `VennDiagram[(A  $\cup$  B  $\cup$  C) &&  $\neg$  (A  $\cap$  B  $\cap$  C), {A, B, C}]`

Out[2]=



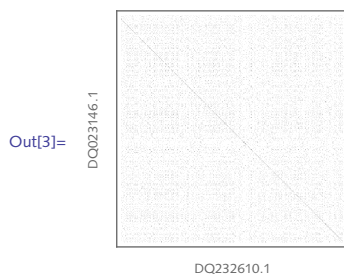
7. Modify the dynamic Venn diagram created in this section to also display a truth table like that in Figure 8.8. Include the truth table side-by-side with the Venn diagram. `TruthTable` was developed in Exercise 3 in Section 6.3.

FIGURE 8.8. Dynamic visualization of logical expressions.



8. The DotPlot function developed in this section uses a fixed window size, meaning that it only colors a dot black if a string of length one matches a string of length one in the two sequences under comparison. Add a `WindowSize` option to `DotPlot` that allows you to set the length of the sequences to match – you will likely need `stringPartition` developed in Section 7.5. Finally, set up `DotPlot` to inherit the options from `ArrayPlot`.

```
In[3]:= DotPlot[seq2, seq1, WindowSize → 3, FrameLabel → {"DQ023146.1", "DQ232610.1"}]
```



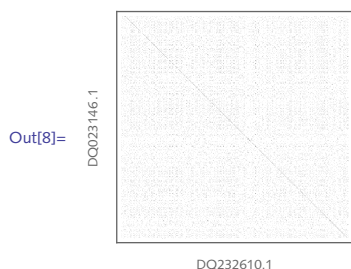
9. When making dot plots like in the previous exercise, if you knew you were *always* working with FASTA files, you could automate both the extraction of the frame labels from the FASTA accession ids and their insertion in the `FrameLabel` option of `ArrayPlot`.

```
In[4]:= Import["H5N1ChickenDQ023146.1.fasta", {"FASTA", "Accession"}]
```

```
Out[4]:= {DQ023146.1}
```

Create a version of `DotPlot` that accepts two FASTA files as input and has the same options structure as in the previous exercise.

```
In[8]:= DotPlot["H5N1ChickenDQ023146.1.fasta", "H5N1DuckDQ232610.1.fasta",
  WindowSize -> 3]
```



10. Modify the `Manipulate` expression animating the hypocycloid so that the plot range deals with the case where the radius of the inner circle is larger than the radius of the outer circle.
11. An *epicycloid* is a curve generated by tracing out a fixed point on a circle rolling around the outside of a second circle. The parametric formula for an epicycloid is similar to that for the hypocycloid:

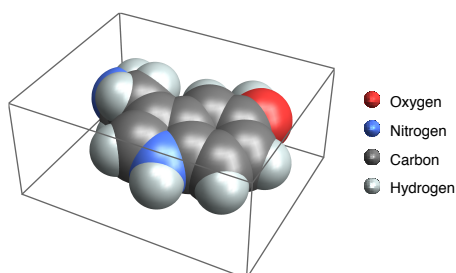
$$x = (a + b) \cos(\theta) - b \cos\left(\frac{a+b}{b} \theta\right),$$

$$y = (a + b) \sin(\theta) - b \sin\left(\frac{a+b}{b} \theta\right).$$

Create a dynamic animation of the epicycloid similar to that for the hypocycloid in this section.

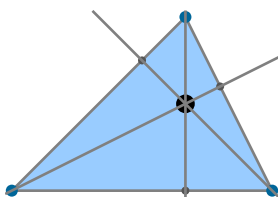
12. Modify `PathPlot` so that it inherits options from `Graphics` as well as having its own option, `PathClosed`, that can take on values of `True` or `False` and closes the path accordingly by appending the first point to the end of the list of coordinate points.
13. Modify `SimplePath` so that the point with the smallest  $x$ -coordinate of the list of data is chosen as the base point; repeat but with the largest  $y$ -coordinate; then try ordering the points about the polar angle each makes with the centroid of the set of points.
14. There are conditions under which the program `SimplePath` will occasionally fail (think collinear points). Experiment by repeatedly computing `SimplePath` for a set of ten integer-valued points until you see the failure. Determine the conditions that must be imposed for the program to work consistently.
15. Modify the `ChemicalSpaceFillingPlot` function to add legends that give identifying information for each atomic element in the plot (Figure 8.9). Consider using `Legended` and `SwatchLegend`.

FIGURE 8.9. Space-filling plot of serotonin (with legends).



16. Create a dynamic interface similar to the triangle circumcenter example in this section but instead compute the *orthocenter*, which is located at the intersection of the three altitudes of the triangle (Figure 8.10). The *altitude* of a triangle is a line through a vertex perpendicular to the opposite side.

FIGURE 8.10. Triangle orthocenter at the intersection of the altitudes.



17. Leonhard Euler in 1765 showed that for any triangle, the centroid, circumcenter, and orthocenter are collinear. In fact, the line that passes through these triangle centers also passes through several other notable points such as the incenter, the nine-point center, the de Longchamps point, and others. And, remarkably, when you change the shape of the triangle, the relative distances between the centers is unchanged.

Construct an interface to display a triangle with dynamic vertices together with the triangle's centroid, circumcenter, orthocenter, and the Euler line. Give distinct colors to each of the four sets: incenter and angle bisectors, medians and centroid, perpendicular bisectors and circumcenter and circumcircle, altitudes and orthocenter. Add a legend that identifies the objects by color.

18. One way to get a sense of the extent of data, such as a two-dimensional random walk, is to superimpose the eigenvectors of a certain tensor over a line plot of the walk. This tensor, called the radius of gyration tensor  $\mathcal{T}$ , is discussed in Section 6.3. For a given walk, the eigenvectors of  $\mathcal{T}$  point in the direction of the greatest and smallest spans of the walk, while the eigenvalues of  $\mathcal{T}$  give a measure of how elongated the walk is in the directions pointed by the corresponding eigenvectors.

Create a function `EigenvectorPlot[walk,  $\mathcal{T}$ , opts]` that takes a two-dimensional list *walk*, the radius of gyration tensor  $\mathcal{T}$  of that walk, and generates a visualization of the walk using

ListLinePlot together with the eigenvector/value lines as described above.

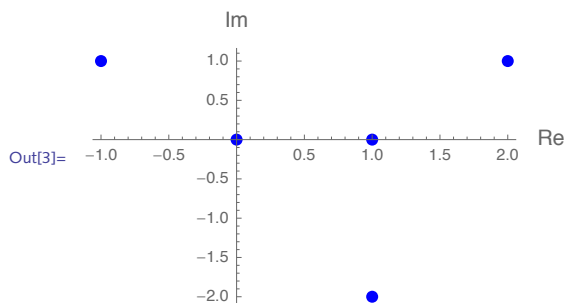
## 8.4 Solutions

- The function `ComplexListPlot` plots a list of numbers in the complex plane – the real part is identified with the horizontal axis and the imaginary part is identified with the vertical axis. Start by setting the options for `ComplexListPlot` to inherit those for `ListPlot`.

```
In[1]:= Options[ComplexListPlot] = Options[ListPlot];
In[2]:= ComplexListPlot[points_, opts : OptionsPattern[]] :=
  ListPlot[Map[{Re[#], Im[#]} &, points], opts,
    PlotStyle -> {Red, PointSize[.025]},
    AxesLabel -> {Style["Re", 8], Style["Im", 8]},
    LabelStyle -> Directive["Menu", 6]]
```

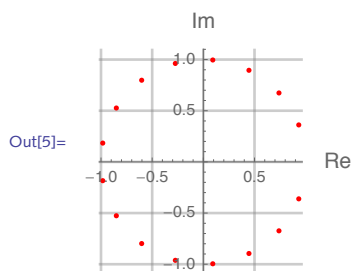
This plots four complex numbers in the plane and uses some options, inherited from `ListPlot`.

```
In[3]:= ComplexListPlot[{-1 + I, 2 + I, 1 - 2 I, 0, 1},
  PlotStyle -> {Blue, PointSize[Medium]}]
```



- The function `ComplexRootPlot` takes a polynomial, solves for its roots, and then uses `ComplexListPlot` from the previous exercise to plot these roots in the complex plane.

```
In[4]:= ComplexRootPlot[poly_, z_, opts : OptionsPattern[]] :=
  ComplexListPlot[z /. NSolve[poly == 0, z], opts, AspectRatio -> Automatic]
In[5]:= ComplexRootPlot[Cyclotomic[17, z], z, GridLines -> Automatic]
```



- Use `PlotStyle` to highlight the two different surfaces and `MeshStyle` and `Mesh` to highlight their intersection.

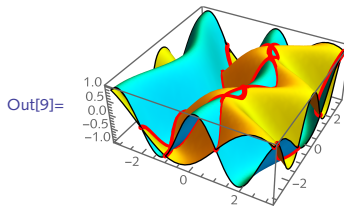
```

In[6]:= Clear[f, x, y, g]

In[7]:= f[x_, y_] := Sin[2 x - Cos[y]];
        g[x_, y_] := Sin[x - Cos[2 y]];

In[9]:= Plot3D[{f[x, y], g[x, y]}, {x, -π, π}, {y, -π, π}, Mesh → {{0.}},
            MaxRecursion → 4, MeshFunctions → {f[#1, #2] - g[#1, #2] &},
            MeshStyle → {Thick, Red}, PlotStyle → {Cyan, Yellow}]

```



4. We can display both the circle regions as well as the logical regions using `RegionPlot`. Three changes are needed to the function definition given in the text: the first argument to `RegionPlot` should now be a list with the regions (the two circles) added; a `PlotStyle` options should be given that specifies the circle interiors in white and the region colored blue; the text identifying the sets/circles needs to be added using the `Epilog` option. Also note that we have added the `MaxRecursion` option to `RegionPlot` to increase the resolution a bit to avoid some jaggedness at the intersection of the two circles (try reducing it to see the problem otherwise).

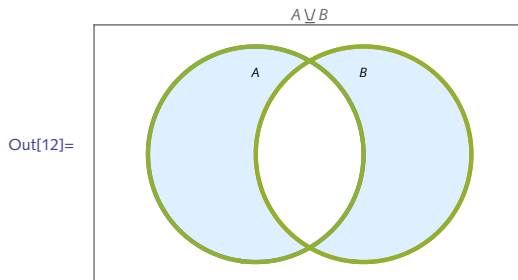
```

In[10]:= Clear[VennDiagram]

In[11]:= VennDiagram[f_, vars : {A_, B_}] :=
Module[{regions, x, y, c1 = {-0.5, 0}, c2 = {0.5, 0}},
  regions = Apply[(x - #1)^2 + (y - #2)^2 < 1 &, {c1, c2}, {1}];
  RegionPlot[{regions, Apply[f, regions]}, {x, -2, 2}, {y, -2, 2},
    Frame → True, FrameTicks → None, PlotLabel → f @@ vars, AspectRatio → Automatic,
    PlotRange → {{-2, 2}, {-1.2, 1.2}},
    PlotStyle → {White, White, LightBlue},
    MaxRecursion → 4,
    Epilog → {Text[First[vars], {-0.5, .75}], Text[Last[vars], {0.5, .75}]}]
]

In[12]:= VennDiagram[Xor, {A, B}]

```



5. The key to turning the logical expression into a function that can be applied to the regions for RegionPlot is to create a pure function. Here is a prototype expression and set of variables.

```
In[13]:= expr = (a ∨ b) && ¬ (a ∧ b);
vars = {a, b};
```

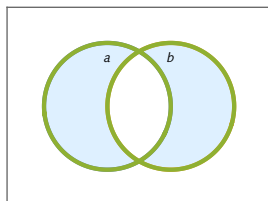
This substitutes the threaded rules into the expression where a is replaced with #1, and b is replaced with #2.

```
expr /. Thread[vars → {#1, #2}] &
```

Here then is the function.

```
In[15]:= Clear[VennDiagram]
In[16]:= VennDiagram[expr_, vars : {__} /; Length[vars] == 2] :=
Module[{c1, c2, regions, x, y, f},
{c1, c2} = {{-1/2, 0}, {1/2, 0}};
regions = (x - #1)^2 + (y - #2)^2 < 1 & @@@ {c1, c2};
f = expr /. Thread[vars → {#1, #2}] &;
RegionPlot[{regions, f @ regions}, {x, -2, 2}, {y, -3/2, 3/2},
AspectRatio → Automatic, FrameTicks → None,
FrameLabel → TraditionalForm[expr],
PlotStyle → {White, White, LightBlue},
MaxRecursion → 4,
Epilog → {Text[vars[[1]], {-1/2, 3/4}], Text[vars[[2]], {1/2, 3/4}]}
]]
In[17]:= VennDiagram[(a ∨ b) && ¬ (a ∧ b), {a, b}]
```

Out[17]=



$(a \vee b) \wedge \neg (a \wedge b)$

6. To start, here are the centers and the regions for three circles (sets).

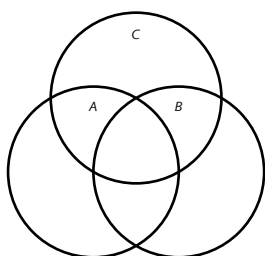
```
In[18]:= {c1, c2, c3} = {{-1/2, 0}, {1/2, 0}, {0, √3/2}};
regions = (x - #1)^2 + (y - #2)^2 < 1 & @@@ {c1, c2, c3}
```

```
Out[18]= { (1/2 + x)^2 + y^2 < 1, (-1/2 + x)^2 + y^2 < 1, x^2 + (-√3/2 + y)^2 < 1 }
```



```
In[19]:= Graphics[{Circle[c1], Circle[c2], Circle[c3],
  Text[A, {-1/2, .75}], Text[B, {1/2, .75}], Text[C, {0, 1.6}]}]
```

Out[19]=



We repeat what was done in the previous exercise but instead use an expression in three variables.

```
In[20]:= expr = (a ∨ b ∨ c) && ¬ (a ∧ b ∧ c);
vars = {a, b, c};
```

This substitutes the threaded rules into the expression where  $a$  is replaced with  $\#1$ ,  $b$  is replaced with  $\#2$ , and so on.

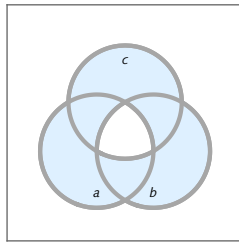
```
expr /. Thread[vars → {#1, #2, #3}] &
```

Here then is the function. It has the same name as above but will only be called if the variable list is of length three.

```
In[22]:= VennDiagram[expr_, vars : {__} /; Length[vars] == 3] :=
Module[{c1, c2, c3, regions, x, y, f},
  {c1, c2, c3} = {{-1/2, 0}, {1/2, 0}, {0,  $\sqrt{3}/2$ }};
  regions = (x - #1)^2 + (y - #2)^2 < 1 & @@@ {c1, c2, c3};
  f = expr /. Thread[vars → {#1, #2, #3}] &;
  RegionPlot[{regions, f @ regions}, {x, -2, 2}, {y, -3/2, 5/2},
    AspectRatio → Automatic, FrameTicks → None,
    FrameLabel → TraditionalForm[expr],
    PlotStyle → {White, White, White, LightBlue}, BoundaryStyle → GrayLevel[.65],
    Epilog → {Text[vars[[1]], {-1/2, -.75}], Text[vars[[2]], {1/2, -.75}],
      Text[vars[[3]], {0, 1.6}]}
  ]]
```

```
In[23]:= VennDiagram[(a ∨ b ∨ c) && ¬(a ∧ b ∧ c), {a, b, c}]
```

Out[23]=

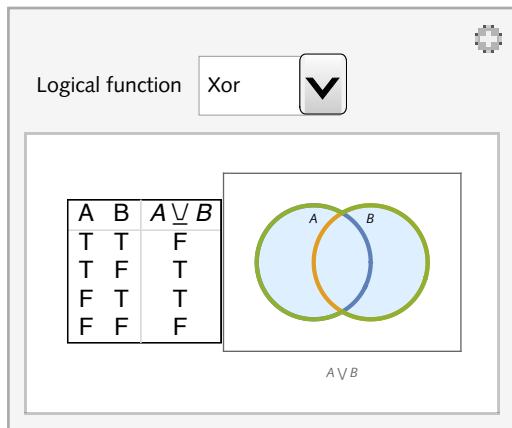


$$(a \vee b \vee c) \wedge \neg(a \wedge b \wedge c)$$

7. We will use the code for the TruthTable function from Exercise 3 in Section 6.3 together with the VennDiagram function from this section, using Row.

```
In[24]:= Manipulate[Row[{
  TruthTable[f[A, B], {A, B}],
  VennDiagram[A ∨ B, {A, B}]
}], {{f, Xor, "Logical function"}, {And, Or, Xor, Implies, Nand, Nor}},
SaveDefinitions → True]
```

Out[24]=



8. First, import two sequences.

```
In[25]:= seq1 = First@Import["H5N1ChickenDQ023146.1.fasta"];
seq2 = First@Import["H5N1DuckDQ232610.1.fasta"];
```

To set the window size, we need to partition the strings. stringPartition was defined in Section 7.5 and is available in the EPM packages.

```
In[27]:= << EPM*
```

This sets a window size of 4, meaning that strings of length 4 will be compared in what follows.

```
In[28]:= With[{windowSize = 4},
  Take[stringPartition[seq1, windowSize, 1, 1, {}], 12]
]
```

```
Out[28]:= {ACAT, CATC, ATCA, TCAT, CATG, ATGG, TGGC, GGCT, GCTT, CTTC, TTCT, TCTC}
```

First set up DotPlot to inherit options from ArrayPlot and to have one new option, WindowSize with a default value of one.

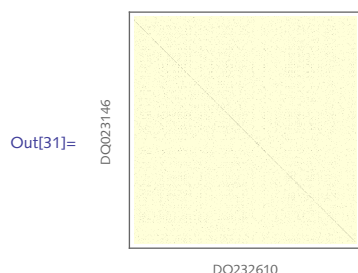
```
In[29]:= Options[DotPlot] = Join[{WindowSize → 1}, Options[ArrayPlot]];
```

The rest of the code is similar to what was developed in the chapter but instead of passing the two sequences whole to ArrayPlot, we pass the two partitioned sequences. We have also set the option Frame to True, which can be overridden by setting that option explicitly when you use DotPlot.

```
In[30]:= DotPlot[p1_, p2_, opts : OptionsPattern[]] :=
  Module[{w = OptionValue[WindowSize]},
    ArrayPlot[Outer[Boole[#1 == #2] &, stringPartition[p1, w, 1, 1, {}],
      stringPartition[p2, w, 1, 1, {}]],
      FilterRules[{opts}, Options@ArrayPlot], Frame → True]]
```

The noise in the earlier plots when we were comparing every nucleotide with every other across the two sequences is clearly cleaned up.

```
In[31]:= DotPlot[seq2, seq1, WindowSize → 4, FrameLabel → {"DQ023146", "DQ232610"},
  ColorRules → {1 → Black, 0 → LightYellow}, ImageSize → Small]
```



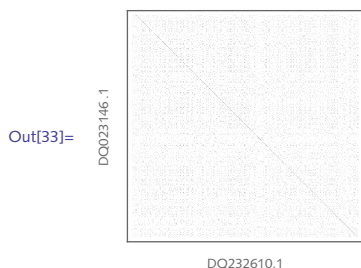
9. We just need to add a few lines grabbing the sequences and the accession numbers.

```

In[32]:= DotPlot[file1_, file2_, opts:OptionsPattern[]] :=
Module[{w = OptionValue[WindowSize], p1, p2, lab1, lab2},
  p1 = First@Import[file1, {"FASTA", "Sequence"}];
  p2 = First@Import[file2, {"FASTA", "Sequence"}];
  lab1 = First@Import[file1, {"FASTA", "Accession"}];
  lab2 = First@Import[file2, {"FASTA", "Accession"}];
  ArrayPlot[Outer[Boole[#1 == #2] &,
    stringPartition[p1, w, 1, 1, {}],
    stringPartition[p2, w, 1, 1, {}]
  ], FilterRules[{opts}, Options@ArrayPlot], Frame → True,
  FrameLabel → {lab1, lab2}]]

In[33]:= DotPlot["H5N1ChickenDQ023146.1.fasta", "H5N1DuckDQ232610.1.fasta",
  WindowSize → 3, ImageSize → Small]

```



10. The problem with the two radii is really only one of getting the plot range correct for the two situations. This can be done most simply with an If statement as the value of the PlotRange option.

```

In[34]:= Hypocycloid[{a_, b_}, θ_] :=
  {(a - b) Cos[θ] + b Cos[θ  $\frac{a-b}{b}$ ], (a - b) Sin[θ] - b Sin[θ  $\frac{a-b}{b}$ ]}

In[35]:= HypocycloidPlot[R_, r_, θ_] := Module[{center},
  center[th_, R1_, r2_] := (R1 - r2) {Cos[th], Sin[th]};

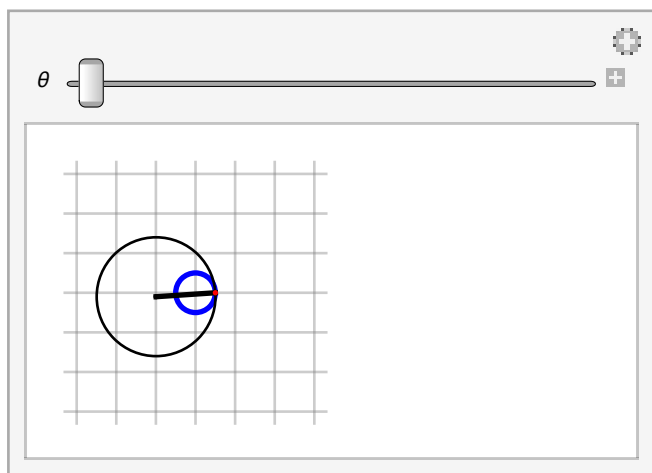
  Show[{
    ParametricPlot[Hypocycloid[{R, r}, t], {t, 0, θ}, PlotStyle → Red,
      Axes → None],

    Graphics[{
      {Blue, Thick, Circle[{0, 0}, R]}, {Circle[center[θ, R, r], r]},
      {PointSize[.02], Point[center[θ, R, r]]},
      {Thick, Line[{center[θ, R, r], Hypocycloid[{R, r}, θ]}]},
      {Red, PointSize[.02], Point[Hypocycloid[{R, r}, θ]}}
    ]
  }, PlotRange → If[r < R, 2 R, 2 r], GridLines → Automatic, ImageSize → Medium]]

```

```
In[36]:= Manipulate[HypocycloidPlot[1, 3,  $\theta$ ], { $\theta$ , 0.1, 6  $\pi$ }, SaveDefinitions → True]
```

Out[36]=

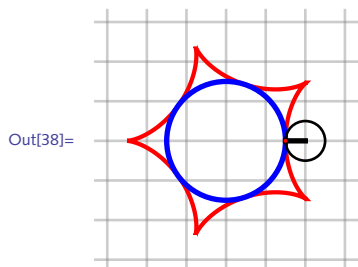


- II. Just a few modifications to the code for the hypocycloid are needed: use the formula for the epicycloid; change the center of the rotating circle so that its radius is  $R + r$ , not  $R - r$ ; and modify the plot range.

```
In[37]:= EpicycloidPlot[R_, r_,  $\theta$ _] := Module[{epicycloid, center},
  epicycloid[{a_, b_}, t_] :=
    {(a + b) Cos[t] - b Cos[t  $\frac{a+b}{b}$ ], (a + b) Sin[t] + b Sin[t  $\frac{a+b}{b}$ ]};
  center[th_, R1_, r2_] := (R1 + r2) {Cos[th], Sin[th]};
  Show[{
    ParametricPlot[epicycloid[{R, r}, t], {t, 0,  $\theta$ }, PlotStyle → Red,
      Axes → None, ImageSize → Small],
    Graphics[{
      {Blue, Thick, Circle[{0, 0}, R]},
      {Circle[center[ $\theta$ , R, r], r]},
      {PointSize[.015], Point[center[ $\theta$ , R, r]]},
      {Thick, Line[{center[ $\theta$ , R, r], epicycloid[{R, r},  $\theta$ ]}]},
      {Red, PointSize[.015], Point[epicycloid[{R, r},  $\theta$ ]]}
    ]}], PlotRange → 1.5 (R + r), GridLines → Automatic]
```

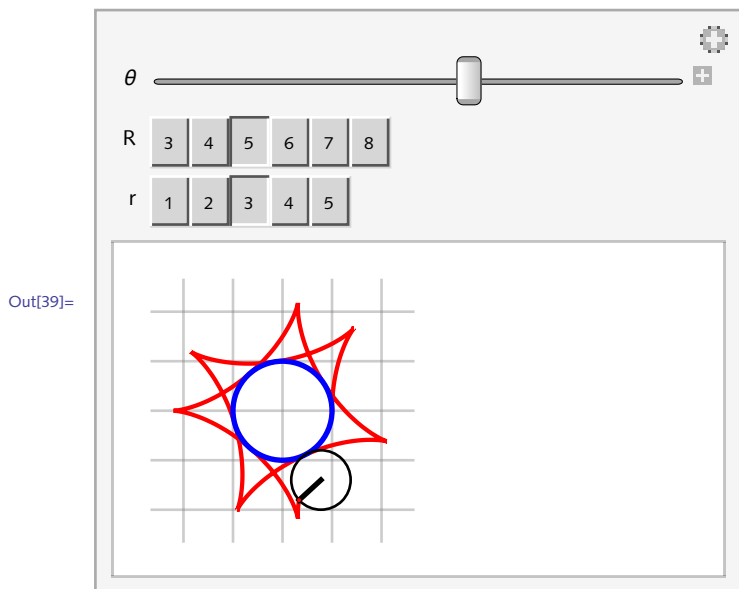
First, create a static image.

```
In[38]:= EpicycloidPlot[3, 1, 2  $\pi$ ]
```



And here is the dynamic version.

```
In[39]:= Manipulate[EpicycloidPlot[R, r,  $\theta$ ],  
  { $\theta$ , 0 + 0.01, 2 Denominator[(R - r) / r]  $\pi$ },  
  {R, {3, 4, 5, 6, 7, 8}, Setter}, {r, {1, 2, 3, 4, 5}, Setter},  
  SaveDefinitions -> True]
```



12. First, set up the options structure.

```
In[40]:= Options[PathPlot] = Join[{ClosedPath -> True}, Options[Graphics]];
```

Make two changes to the original PathPlot: add an If statement that checks the value of ClosedPath and if True, appends the first point to the end of the list; if False, it leaves the coordinate list as is. The second change is to filter those options that are specific to Graphics and insert them in the appropriate place.

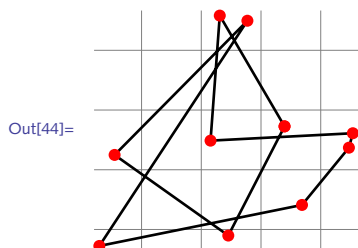
```

In[41]:= PathPlot[lis_List, opts : OptionsPattern[]] :=
  Module[{coords = lis}, If[OptionValue[ClosedPath],
    coords = coords /. {a_, b_} -> {a, b, a};
    Graphics[{Line[coords], PointSize[Medium], Red, Point[coords]},
    FilterRules[{opts}, Options[Graphics]]]]

In[42]:= SeedRandom[424];
coords = RandomReal[1, {10, 2}];

In[44]:= PathPlot[coords, ClosedPath -> True, GridLines -> Automatic]

```



13. A simple change to the program SimplePath chooses the base point with the largest y-coordinate.

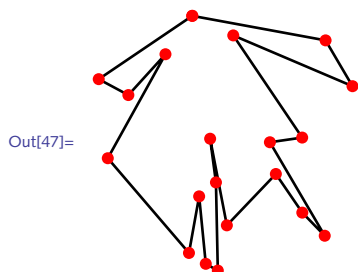
```

In[45]:= SimplePath3[lis_] := Module[{base, angle, sorted},
  base = Last[SortBy[lis, Last]];
  angle[a_, b_] := ArcTan @@ (b - a);
  sorted = Sort[Complement[lis, {base}], angle[base, #1] <= angle[base, #2] &];
  Join[{base}, sorted]]

In[46]:= pts = RandomReal[1, {20, 2}];

In[47]:= PathPlot[SimplePath3[pts]]

```



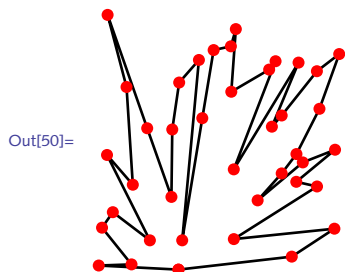
14. Choosing a base point randomly and then sorting according to the arc tangent could cause a number of things to go wrong with the algorithm. The default branch cut for ArcTan gives values between  $-\pi/2$  and  $\pi/2$  (think about why this could occasionally cause the algorithm in the text to fail). By choosing the base point so that it lies at some extreme of the diameter of the set of points, the polar angle algorithm given in the text will work consistently. If you choose the base point so that it is lowest and left-most, then all the angles will be in the range  $(0, \pi]$ .

```

In[48]:= SimplePath1[lis_List] := Module[{base, angle, sorted},
  base = First[SortBy[lis, Last]];
  angle[a_, b_] := ArcTan @@ (b - a);
  sorted = Sort[Complement[lis, {base}], angle[base, #1] ≤ angle[base, #2] &];
  Join[{base}, sorted]

In[49]:= pts = RandomReal[1, {40, 2}];
In[50]:= PathPlot[SimplePath1[pts]]

```



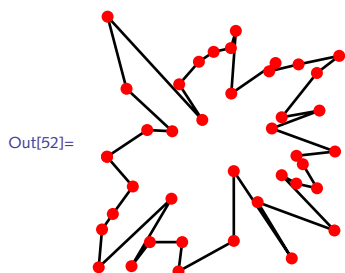
And here is a solution based on ordering the points according to the polar angle each makes with the centroid of the set of points.

```

In[51]:= SimplePathCM[lis_] := Module[{centroid, angle},
  centroid = RegionCentroid[Polygon@lis];
  angle[a_, b_] := Apply[ArcTan, b - a];
  Sort[lis, angle[centroid, #1] ≤ angle[centroid, #2] &]]

In[52]:= PathPlot[SimplePathCM[pts]]

```



15. Using `Legended` and `SwatchLegend`, you can add identifying information for each atomic element.



```

In[53]:= ChemicalSpaceFillingPlot[file_String] :=
Module[{elements, pos, radii, names, colors, labels},

  {pos, elements} =
    First /@ Import[file, {{"VertexCoordinates", "VertexTypes"}}];

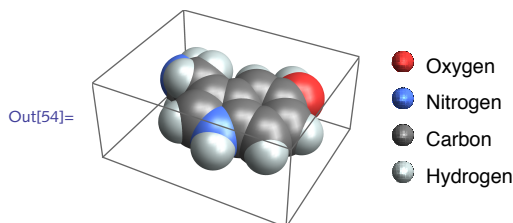
  radii = QuantityMagnitude@Map[ElementData[#, "VanDerWaalsRadius"] &,
    elements];

  labels = DeleteDuplicates[elements];
  names = ElementData[#, "StandardName"] & /@ labels;
  colors = labels /. ColorData["Atoms", "ColorRules"];

  Legended[
    Graphics3D[{Specularity[White, 50],
      MapThread[{ColorData["Atoms", #1], Sphere[#2, #3]} &,
        {elements, pos, radii}]
    }, Lighting -> "Neutral"],
    SwatchLegend[colors, names, LegendMarkers -> "SphereBubble",
      LabelStyle -> Directive[FontSize -> 7.5]]
  ]
]

In[54]:= ChemicalSpaceFillingPlot["5hydroxytryptamine.sdf"]

```

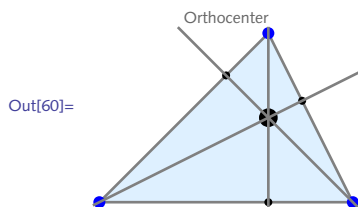


16. First, here is a static image of the triangle with orthocenter and altitudes.

```

In[55]:= {p1, p2, p3} = {{-1, 0}, {1, 2}, {2, 0}};
lines = InfiniteLine /@ Subsets[{p1, p2, p3}, {2}];
altPts = MapThread[RegionNearest[#1, #2] &, {lines, Reverse[{p1, p2, p3}]}];
altLines = MapThread[Line[{#1, #2}] &, {altPts, Reverse[{p1, p2, p3}]}];
iLines = Apply[InfiniteLine, altLines, {1}];
center = {x, y} /. First[Solve[{x, y} ∈ #1 &] /@ iLines, {x, y}];
Graphics[{ {LightBlue, EdgeForm[Gray], Triangle[{p1, p2, p3}]},
  {Blue, PointSize[Medium], Point[{p1, p2, p3}]}, Point[altPts],
  PointSize[Large], Point[center], {Gray, iLines}}, PlotLabel → "Orthocenter"]

```

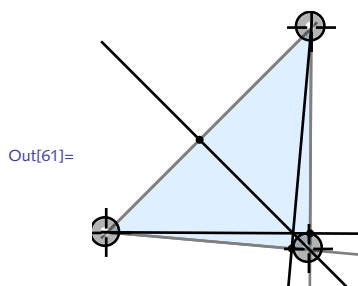


And here is the dynamic version. Note the orthocenter will be located outside the triangle if the triangle is not acute.

```

In[61]:= DynamicModule[{pts = 5 {{-1, 0}, {1, 2}, {2, 0}}},
  LocatorPane[Dynamic[pts, Appearance → Tiny],
    Dynamic[Graphics[{ {EdgeForm[Gray], LightBlue, Triangle[pts]},
      {Gray, InfiniteLine /@ Subsets[pts, {2}]},
      {Point[MapThread[RegionNearest[#1, #2] &,
        {InfiniteLine /@ Subsets[pts, {2}], Reverse[pts]}]}],
      MapThread[InfiniteLine[{#1, #2}] &,
        {MapThread[RegionNearest[#1, #2] &,
          {InfiniteLine /@ Subsets[pts, {2}], Reverse[pts]}],
          Reverse[pts]}]}]}]]]

```



17. (\* solution to appear \*)
18. Start with a two-dimensional random walk.

```

In[62]:= << EPM`

```

```

In[63]:= << EPM`RandomWalks`

```

```
In[64]:= walk = RandomWalk[10000, Dimension → 2, LatticeWalk → False];
        T = RadiusOfGyrationTensor[walk]
```

```
Out[65]= {{1689.48, 1477.75}, {1477.75, 1495.96}}
```

The eigenvectors of the radius of gyration tensor,  $\mathcal{T}$ , point in the directions of greatest and smallest spans of the walk. The eigenvalues give a measure of how elongated the walk is in these directions. This can be seen by creating lines along each eigenvector of a length proportional to the corresponding eigenvalues. In the computation below, the slope of the line is given by the  $y$ -coordinate of the eigenvector divided by the corresponding  $x$ -coordinate.

```
In[66]:= {λ1, λ2} = Eigenvalues[T]
```

```
Out[66]= {3073.63, 111.809}
```

```
In[67]:= {{v1x, v1y}, {v2x, v2y}} = Eigenvectors[T]
```

```
Out[67]= {{-0.729841, -0.683617}, {0.683617, -0.729841}}
```

```
In[68]:= {cmx, cmy} = Mean[walk]
```

```
Out[68]= {-36.9605, -31.5695}
```

```
In[69]:= ev1 =  $\frac{v1y}{v1x}$  (x - cmx) + cmy // Expand
```

```
Out[69]= 3.05005 + 0.936665 x
```

```
In[70]:= ev2 =  $\frac{v2y}{v2x}$  (x - cmx) + cmy // Expand
```

```
Out[70]= -71.0292 - 1.06762 x
```

Putting all these pieces together, we create the function `EigenvectorPlot` that returns a plot of the original data set together with plots of the orthogonal lines and puts a large red point at their intersection, the center of mass.

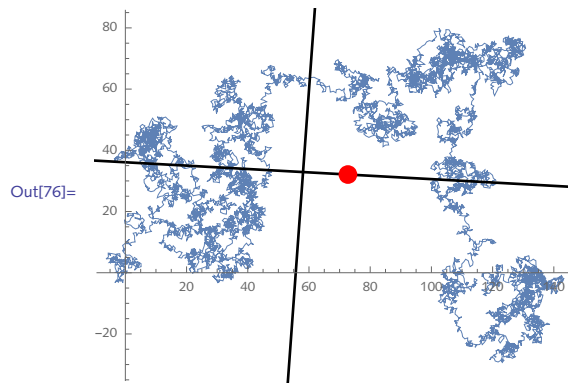
```
In[71]:= EigenvectorPlot[data : {{_, _} ..}, tensor_,
        opts : OptionsPattern[ListLinePlot]] :=
Module[{T = tensor, cmx, cmy, λ1, λ2, v1x, v1y, v2x, v2y},
  {λ1, λ2} = Eigenvalues[T];
  {cmx, cmy} = Mean[data];
  {{v1x, v1y}, {v2x, v2y}} = Eigenvectors[T];
  Show[{ListLinePlot[data, opts, PlotStyle → Thin],
    Graphics[{Line[{cmx - λ1, cmy -  $\frac{v1y \lambda1}{v1x}$ }, {cmx + λ1, cmy +  $\frac{v1y \lambda1}{v1x}$ }}],
      Line[{cmx - λ2, cmy -  $\frac{v2y \lambda2}{v2x}$ }, {cmx + λ2, cmy +  $\frac{v2y \lambda2}{v2x}$ }}],
      PointSize[Large], Red, Point[Mean[data]]}], AspectRatio → Automatic]]
```

```
In[72]:= << EPM`RandomWalks`
```

```
In[73]:= SeedRandom[4];  
walk = RandomWalk[10000, Dimension → 2, LatticeWalk → False];  
 $\tau$  = RadiusOfGyrationTensor[walk]
```

```
Out[75]= {{1977.94, -67.2011}, {-67.2011, 761.669}}
```

```
In[76]:= EigenvectorPlot[walk,  $\tau$ ]
```



---

## 9 Program optimization

### 9.1 Efficient programs: exercises

1. One of the problems with measuring the time it takes to complete a computation is that computers are often busy doing many things simultaneously: checking mail, running system scripts in the background, and so on. To give a more accurate measure of the time spent on a computational task, create a function `AverageTiming` that runs several trials and then averages the results. Set up the function to return both the result and the timing, similar to `Timing` and `AbsoluteTiming`. Be careful that your function does not evaluate its input before it is passed into the body of the function.
2. The  $n$ th triangular number is defined as the sum of the integers 1 through  $n$ . They are so named because they can be represented visually by arranging rows of dots in a triangular manner (Figure 9.1). Program several different approaches to computing triangular numbers and compare their efficiency.

FIGURE 9.1. Pictorial representation of the first five triangular numbers.



3. Recall the `Anagrams` function in Section 7.2 that used `Select` to find words in the dictionary consisting of a permutation of the letters of a given word. Here is another implementation, converting the word to a list of characters, getting all permutations of that list of characters, joining the characters in each sublist, and then checking against actual words in the dictionary.

```
In[1]:= Anagrams1[word_String] := Module[{words},  
  words = Map[StringJoin, Permutations[Characters[word]]];  
  DictionaryLookup[x_ /; MemberQ[words, x]]]
```

```
In[2]:= Timing[Anagrams1["alerts"]]
Out[2]= {16.3502, {alerts, alters, salter, staler}}
```

And here is an implementation that uses Alternatives instead of the conditional pattern above:

```
In[3]:= Anagrams2[word_String] := Module[{chars = Characters[word], words},
  words = Map[StringJoin, Permutations[chars]];
  DictionaryLookup[Alternatives @@ words]
]
In[4]:= Timing[Anagrams2["alerts"]]
Out[4]= {0.940914, {alerts, alters, salter, staler}}
```

The following implementation from Chapter 7 uses regular expressions and Select but only checks words in the dictionary of the same length as the test word:

```
In[5]:= Anagrams3[word_String] := Module[{len = StringLength[word], words},
  words = DictionaryLookup[RegularExpression["\\w{" <> ToString[len] <> "}"]];
  Select[words, Sort[Characters[#]] == Sort[Characters[word]] &]
]
In[6]:= Timing[Anagrams3["alerts"]]
Out[6]= {0.052693, {alerts, alters, salter, staler}}
```

Determine what is causing the sharp differences in timing between these three implementations.

- Several different implementations of the Hamming distance computation were given in Section 5.6; some run much faster than others. For example, the version with bit operators runs about one-and-a-half orders of magnitude faster than the version using Count and MapThread. Determine what is causing these differences.

```
In[7]:= HammingDistance1[lis1_, lis2_] :=
  Count[MapThread[SameQ, {lis1, lis2}], False]
In[8]:= HammingDistance2[lis1_, lis2_] := Total[BitXor[lis1, lis2]]
In[9]:= sig1 = RandomInteger[1, {10^6}];
  sig2 = RandomInteger[1, {10^6}];
In[11]:= Timing[HammingDistance1[sig1, sig2]]
Out[11]= {0.325111, 499258}
In[12]:= Timing[HammingDistance2[sig1, sig2]]
Out[12]= {0.012189, 499258}
```

5. Consider the computation of the diameter of a set of points in  $d$ -dimensional space,  $\mathbb{R}^d$ , as was done in Exercise 11, Section 5.1.

```
In[13]:= PointsetDiameter[pts_List] :=
         Max[Apply[EuclideanDistance, Subsets[pts, {2}], {1}]]
```

This function suffers from the fact that computing subsets is computationally expensive. Computing pairs of subsets typically is  $O(n^2)$  and so the time to do this computation will grow quadratically with the size of the point set. Beyond about 10 000 points, the time is substantial.

```
In[14]:= pts = RandomReal[1, {5000, 2}];
PointsetDiameter[pts] // Timing

Out[15]= {21.2414, 1.39259}
```

Try to speed up the computation of the diameter by using some computational geometry. In particular, note that the two points contained in the diameter must lie on the convex hull of the point set. Use this observation to substantially reduce the number of subsets that are computed (see [O'Rourke 1998](#)).

6. Searching for numbers which are both square and palindromic can be done by using the two predicate functions developed earlier, `SquareNumberQ` (Exercise 4, Section 2.4) and `PalindromeQ` (Section 1.1). For example, using these functions, the following finds all square palindromic numbers below  $10^6$ :

```
In[16]:= With[{n = 10^6},
             Select[Select[Range[n], SquareNumberQ], PalindromeQ]] // Timing

Out[16]= {10.1161, {1, 4, 9, 121, 484, 676, 10201,
                  12321, 14641, 40804, 44944, 69696, 94249, 698896}}
```

Somewhat surprisingly, checking for palindromes first and then finding square numbers amongst those, is about three to four times faster.

```
In[17]:= With[{n = 10^6},
             Select[Select[Range[n], PalindromeQ], SquareNumberQ]] // Timing

Out[17]= {3.79878, {1, 4, 9, 121, 484, 676, 10201,
                  12321, 14641, 40804, 44944, 69696, 94249, 698896}}
```

Determine why this is so.

7. Consider the Monte Carlo approach to approximating  $\pi$  discussed in several places in this book. One way to perform the simulation is to create a large vector of random points in the square and count the number of such points that lie within the circle  $x^2 + y^2 \leq 1$ . Here are two approaches, the first using `Apply` at level one, the second using `Map`:

```

In[18]:= pts = RandomReal[{-1, 1}, {10^6, 2}];
4. Total[Apply[Boole[#1^2 + #2^2 ≤ 1] &, pts, {1}]]/Length[pts] //
Timing
Out[19]= {3.31052, 3.1408}

In[20]:= 4. Total[Map[Boole[First[#]^2 + Last[#]^2 ≤ 1] &, pts]]/Length[pts] //
Timing
Out[20]= {0.27429, 3.1408}

```

Determine why the approach with Apply at level one is over an order of magnitude slower.

## 9.1 Solutions

- i. First, we set things up so that AverageTiming has the HoldAll attribute. This way its argument, the expression to be measured, does not evaluate before it is used inside the body of the AverageTiming function itself.

```

In[1]:= SetAttributes[AverageTiming, HoldAll]
In[2]:= AverageTiming[expr_, trials_] :=
Mean[Table[First[AbsoluteTiming[expr]], {trials}]]

```

As a simple test, here we compute the time needed to invert a large matrix.

```

In[3]:= mat = RandomReal[1, {1000, 1000}];
AbsoluteTiming[Inverse[mat];]
Out[4]= {0.047831, Null}

```

And for five trials, the average time is given by the following.

```

In[5]:= AverageTiming[Inverse[mat], 5]
Out[5]= 0.0421236

```

For a compound expression, you could either enclose the subexpressions in a list or separate them with semicolons.

```

In[6]:= AverageTiming[{
mat.mat,
Inverse[mat],
Det[mat]
}, 5]
Out[6]= 0.0694276

```



```
In[7]:= AverageTiming[
  mat.mat;
  Inverse[mat];
  Det[mat];,
  5]
```

```
Out[7]= 0.068383
```

Collect the results of the Table and pull out the parts needed – the timings and the result.

```
In[8]:= SetAttributes[AverageTiming, HoldAll]
```

```
In[9]:= AverageTiming[expr_, trials_] := Module[{lis},
  lis = Table[AbsoluteTiming[expr], {trials}];
  {Mean[lis[[All, 1]]], lis[[1, 2]]}
]
```

```
In[10]:= AverageTiming[FactorInteger[50! + 1], 5]
```

```
Out[10]= {0.739426, {{149, 1}, {3989, 1}, {74195127103, 1},
  {6854870037011, 1}, {100612041036938568804690996722352077, 1}}}
```

2. A first attempt, using a brute force approach, is to total the list  $\{1, 2, \dots, n\}$  for each  $n$ .

```
In[11]:= TriangularNumber[n_] := Total[Range[n]]
```

```
In[12]:= Table[TriangularNumber[i], {i, 1, 100}]
```

```
Out[12]= {1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190,
  210, 231, 253, 276, 300, 325, 351, 378, 406, 435, 465, 496, 528, 561, 595,
  630, 666, 703, 741, 780, 820, 861, 903, 946, 990, 1035, 1081, 1128, 1176,
  1225, 1275, 1326, 1378, 1431, 1485, 1540, 1596, 1653, 1711, 1770, 1830, 1891,
  1953, 2016, 2080, 2145, 2211, 2278, 2346, 2415, 2485, 2556, 2628, 2701, 2775,
  2850, 2926, 3003, 3081, 3160, 3240, 3321, 3403, 3486, 3570, 3655, 3741, 3828,
  3916, 4005, 4095, 4186, 4278, 4371, 4465, 4560, 4656, 4753, 4851, 4950, 5050}
```

```
In[13]:= Timing[TriangularNumber[107]]
```

```
Out[13]= {0.067392, 50000005000000}
```

A second approach uses iteration. As might be expected, this is the slowest of the approaches here.

```
In[14]:= TriangularNumber2[n_] := Fold[#1 + #2 &, 0, Range[n]]
```

```
In[15]:= Timing[TriangularNumber2[107]]
```

```
Out[15]= {1.40948, 50000005000000}
```

This is a situation where a little mathematical knowledge goes a long way. The  $n$ th triangular numbers is just the following binomial coefficient:  $\binom{n+1}{2}$ .

```
In[16]:= TriangularNumber3[n_] := Binomial[n + 1, 2]
```

```
In[17]:= Timing[TriangularNumber3[107]]
```

```
Out[17]= {0.000015, 50 000 005 000 000}
```

3. Here are the three anagrams functions from the exercise.

```
In[18]:= Anagrams1[word_String] := Module[{chars = Characters[word], words},
  words = Map[StringJoin, Permutations[chars]];
  DictionaryLookup[x__ /; MemberQ[words, x]]]
```

```
In[19]:= Anagrams2[word_String] := Module[{chars = Characters[word], words},
  words = Map[StringJoin, Permutations[chars]];
  DictionaryLookup[Alternatives @@ words]
]
```

```
In[20]:= Anagrams3[word_String] := Module[{len = StringLength[word], words},
  words = DictionaryLookup[RegularExpression["\\w{" <> ToString[len] <> "}"]];
  Select[words, Sort[Characters[#]] == Sort[Characters[word]] &]
]
```

Anagrams1 is the slowest. Here are the two main computations in that function. Almost the entire computational time is spent in the pattern matching second step below. This is because every word in the dictionary is compared with the list words. This brute force approach is clearly going to be very expensive.

```
In[21]:= With[{chars = Characters["alerts"]},
  Timing[words = Map[StringJoin, Permutations[chars]]];
]
```

```
Out[21]= {0.000828, Null}
```

```
In[22]:= Timing[DictionaryLookup[x__ /; MemberQ[words, x]]];
]
```

```
Out[22]= {16.3338, Null}
```

Anagrams2, by comparison is much faster as every word in the dictionary is not being compared with the list words.

```
In[23]:= Timing[DictionaryLookup[Alternatives @@ words]];]
```

```
Out[23]= {1.11167, Null}
```

But Anagrams3 is the fastest. First, it is only checking words of the same length as the test word "alerts". That list is nine times smaller than the entire list of words in the dictionary. But the biggest speed improvement comes from the fact that it is not using the pattern matcher so intensively but instead is sorting lists of characters, which, for short lists, is quite fast.

```
In[24]:= Timing[Anagrams3["alerts"]];]
```

```
Out[24]= {0.048765, Null}
```

```

In[25]:= Length[DictionaryLookup[w_ /; StringLength[w] == StringLength["alerts"]]]
Out[25]= 10 549

In[26]:= Length[DictionaryLookup[]]
Out[26]= 92 518

```

4. The first implementation essentially performs a transpose of the two lists, wrapping SameQ around each corresponding pair of numbers. It then does a pattern match (Count) to determine which expressions of the form SameQ[ $expr_1$ ,  $expr_2$ ] return False.

```

In[27]:= HammingDistance1[lis1_, lis2_] :=
  Count[MapThread[SameQ, {lis1, lis2}], False]

In[28]:= HammingDistance2[lis1_, lis2_] := Total[BitXor[lis1, lis2]]

In[29]:= sig1 = RandomInteger[1, {10^6}];
In[30]:= sig2 = RandomInteger[1, {10^6}];

```

In this case, it is the threading that is expensive rather than the pattern matching with Count.

```

In[31]:= res = MapThread[SameQ, {sig1, sig2}]; // Timing
Out[31]= {0.283291, Null}

In[32]:= Count[res, False] // Timing
Out[32]= {0.035301, 500 001}

```

The reason the threading is expensive can be seen by turning on the packing message as discussed in this section.

```

In[33]:= On["Packing"]

In[34]:= res = MapThread[SameQ, {sig1, sig2}];
Developer`FromPackedArray::punpack1 : Unpacking array with dimensions {1000000}. >>

Developer`FromPackedArray::punpack1 : Unpacking array with dimensions {1000000}. >>

```

The other factors contributing to the significant timing differences have to do with the fact that BitXor has the Listable attribute. MapThread does not. And so, BitXor can take advantage of specialized (compiled) codes internally to speed up its computations.

```

In[35]:= Attributes[BitXor]
Out[35]= {Flat, Listable, OneIdentity, Orderless, Protected}

In[36]:= Attributes[MapThread]
Out[36]= {Protected}

In[37]:= Timing[temp = BitXor[sig1, sig2];]
Out[37]= {0.011554, Null}

```

And finally, compute the number of ones using `Total` which is extremely fast at adding lists of numbers.

```
In[38]:= Timing[Total[temp];]
```

```
Out[38]= {0.002223, Null}
```

Return the packed array messaging to its default value.

```
In[39]:= Off["Packing"]
```

5. The key observation here is to note that the two points that make up the diameter of a set of points necessarily lie on the convex hull of that set of points.

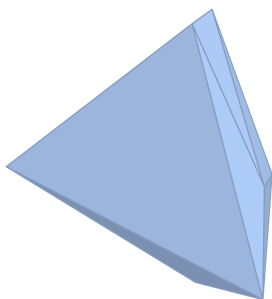
First, create some sample points in 3-space.

```
In[40]:= pts = RandomReal[1, {12, 3}];
```

This computes the convex hull.

```
In[41]:= R = ConvexHullMesh[pts]
```

```
Out[41]=
```



This gives the coordinates of the points on the hull.

```
In[42]:= MeshCoordinates[R]
```

```
Out[42]= {{0.943138, 0.120452, 0.295862},
           {0.00281024, 0.0372368, 0.55261}, {0.323796, 0.539883, 0.176544},
           {0.629912, 0.484183, 0.198963}, {0.404037, 0.786687, 0.905667},
           {0.747504, 0.550974, 0.384268}, {0.715909, 0.0785744, 0.302986},
           {0.648593, 0.864685, 0.186089}, {0.45307, 0.541464, 0.981177}}
```

So, instead of taking the subsets of the entire set of points, only take subsets from this list of mesh coordinates. Here then is the function from Exercise 11, Section 5.1, with this modification.

```
In[43]:= PointsetDiameterCH[pts_] := Module[{R},
  R = ConvexHullMesh[pts];
  Max@Apply[EuclideanDistance, Subsets[MeshCoordinates[R], {2}], {1}]
]
```

For timing comparisons, let's take a large point set.

```
In[44]:= pts = RandomReal[1, {2500, 3}];
```

First, here is the computation using Subsets.

```
In[45]:= PointsetDiameter[pts_List] :=
         Max[Apply[EuclideanDistance, Subsets[pts, {2}], {1}]]

In[46]:= Timing[PointsetDiameter[pts]]
Out[46]= {3.41869, 1.62249}
```

PointsetDiameterCH is substantially faster than the approach above using all subsets.

```
In[47]:= Timing[PointsetDiameterCH[pts]]
Out[47]= {0.041134, 1.62249}
```

The one disadvantage to this approach is that the computation of the convex hull mesh is only valid for dimensions one through three.

- One way to analyze the difference in this problem is to check how long it takes each of these predicates to pick out numbers from one to one million.

```
In[48]:= Count[Range[10^6], p_?PalindromeQ] // Timing
Out[48]= {0.37367, 0}

In[49]:= Count[Range[10^6], p_?SquareNumberQ] // Timing
Out[49]= {0.385763, 0}
```

Clearly SquareNumberQ is slow relative to PalindromeQ for checking the same number of integers, so making it only check 1998 numbers rather than one million is what helps.

- The mystery here is not clear until you look turn on the packed array messaging as described in the text.

```
In[50]:= On["Packing"]

In[51]:= pts = RandomReal[{-1, 1}, {10^6, 2}];
         4. Total[Boole[#1^2 + #2^2 ≤ 1] & @@@ pts] / Length[pts] // Timing
Developer`FromPackedArray::punpack1: Unpacking array with dimensions {1000000, 2}. >>

Out[52]= {2.47989, 3.14176}

In[53]:= pts = RandomReal[{-1, 1}, {10^6, 2}];
         4. Total[Boole[First[#]^2 + Last[#]^2 ≤ 1] & /@ pts] / Length[pts] //
           Timing
Out[54]= {0.282033, 3.14466}
```

Apply at level one (@@@), unpacks packed arrays and this step causes the significant slowdown in this computation.

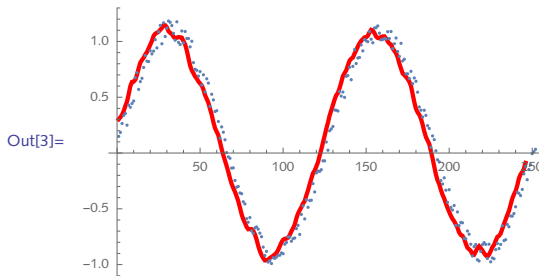
Reset the system option.

```
In[55]:= Off["Packing"];
```

## 9.2 Parallel processing: exercises

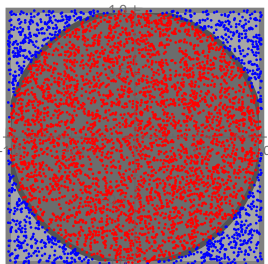
- Many different methods can be used to smooth a noisy signal. Depending upon the nature of the data (periodic, for example) and the nature of the noise, some smoothing methods are more appropriate than others. Given a noisy signal, compare a variety of smoothing methods in parallel by displaying the original signal together with each smoothed version. For example, this displays the original signal together with an eight-term weighted moving average:

```
In[1]:= signal = Table[Sin[t] + RandomReal[0.2], {t, -2 π, 2 π, 0.05}];
In[2]:= ma = MovingAverage[signal, {1, 2, 6, 8, 6, 2, 1} / 8];
ListPlot[{signal, ma}, Joined → {False, True}, PlotStyle → {Automatic, Red}]
```



Example smoothers to consider include moving averages with different numbers of terms and weights, a convolution with a Gaussian kernel, a lowpass filter, and any others you might be familiar with (wavelets, for example).

- The search for perfect numbers programmed in Exercise 6 in Section 5.1 gets bogged down for searches of more than one million numbers. Try to speed it up by considering the range of numbers searched, the built-in functions used, and the possibility of doing the computation in parallel.
- In the eighteenth century, Leonhard Euler proved that all even perfect numbers must be of the form  $2^{p-1} (2^p - 1)$  for  $2^p - 1$  prime and  $p$  a prime number. (No one has yet proved that any odd perfect numbers exist.) Use this fact to find all even perfect numbers for  $p < 10^4$ .
- A common task in many areas of computational linguistics is comparing certain features of a text across a broad corpus. One such comparison is counting the occurrence of a certain word across numerous texts. This is a good problem for parallel computation. Use the parallel tools to import and count the occurrence of a word, say *history*, across four different texts. Gutenberg.org is a good source for importing entire texts, but any available source could be used.
- Monte Carlo simulations are computations that use random sampling to approximate a numerical result. One of the classical examples is the approximation to  $\pi$ . The idea is to generate a large number of random numbers in a square and compute the proportion that lie within the inscribed circle (Figure 9.2). The approximation to  $\pi$  is four times this proportion. This method converges quite slowly, so a large number of points and averaging many trials is needed to get better approximations.

FIGURE 9.2. Monte Carlo simulation for approximating  $\pi$ .

Use `RandomReal` to create points in a square, then count points inside the inscribed disk using two different implementations – one with a `Do` loop and another using the computational geometry machinery (`RegionMember` in particular). Compare the efficiency of these two implementations.

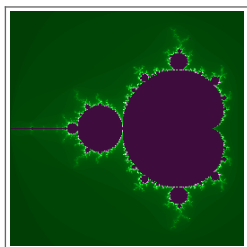
6. The following code can be used to create a plot of the Mandelbrot set. It uses `Table` to compute the value for each point in the complex plane on a small grid. We have deliberately chosen a relatively coarse grid ( $n = 100$ ) as this is an intensive and time-consuming computation. The last argument to `NestWhileList`, 250 here, sets a limit on the number of iterations that can be performed for each input. Increase the resolution of the graphic by running the computation of the table of points in parallel.

```
In[4]:= Mandelbrot[c_] := Length[NestWhileList[#^2 + c &, 0, Abs[#] < 2 &, 1, 250]]
```

```
In[5]:= data = With[{n = 100}, Table[Mandelbrot[x + I y], {y, -1.3, 1.3, 1/n},
    {x, -2, 0.6, 1/n}]];
```

```
In[6]:= ArrayPlot[data, ColorFunction -> "GreenPinkTones"]
```

Out[6]=



## 9.2 Solutions

1. Here is a noisy signal to work with.

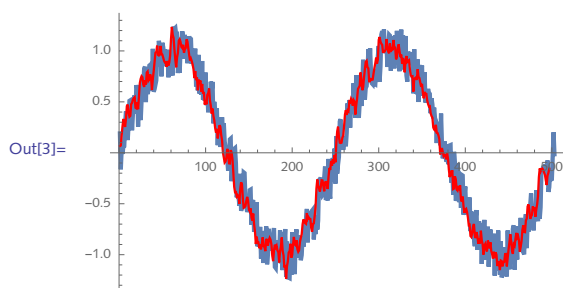
```
In[1]:= signal = Table[Sin[t] + RandomReal[{-0.25, 0.25}], {t, 0, 4 Pi, 0.025}];
```

Here is an eight-term moving average.

```

In[2]:= ma8 = MovingAverage[signal, {1, -2, 6, 8, 6, 2, 1} / 8];
ListLinePlot[{signal, ma8}, PlotStyle -> {Automatic, {Red, Thickness[.005]}}]

```



A six-term moving average:

```

In[4]:= ma = MovingAverage[signal, {1, 1, 2, 1, 1} / 6];

```

A low-pass filter:

```

In[5]:= LowpassFilter[signal, .5, 31];

```

And a convolution with a Gaussian kernel:

```

In[6]:= kernel[n_?OddQ] := Table[Exp[-k^2/n^2], {k, -Floor[n/2], Floor[n/2]}]

```

```

In[7]:= ListConvolve[kernel[17], signal];

```



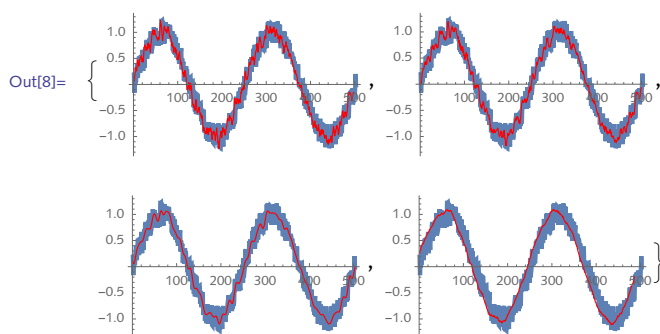
```

In[8]:= ParallelTable[
  ListLinePlot[{signal, comp},
    ImageSize → Tiny,
    PlotStyle → {Automatic, {Red, Thickness[.005]}}],
  {comp, {
    MovingAverage[signal, {1, -2, 6, 8, 6, 2, 1} / 8],
    MovingAverage[signal, {1, 1, 2, 1, 1} / 6],
    LowpassFilter[signal, .5, 31],

    kernel[n_?OddQ] := Table[ $\frac{\text{Exp}\left[-\frac{k^2}{n^2}\right]}{\sqrt{2\pi}}$ , {k, -Floor[ $\frac{n}{2}$ ], Floor[ $\frac{n}{2}$ ]}];

    ListConvolve[kernel[17] / 6, signal]}]}
]

```



2. Here is the definition from Exercise 6 in Section 5.1.

```

In[9]:= PerfectSearch[n_] := Module[{perfectQ},
  perfectQ[k_] := Total[Divisors[k]] == 2 k;
  Select[Range[n], perfectQ]
]

```

```

In[10]:= PerfectSearch[106] // Timing

```

```

Out[10]= {8.41595, {6, 28, 496, 8128}}

```

We can give a speed boost by using `DivisorSigma[1, k]` which gives the sum of the divisors of  $k$ .

```

In[11]:= PerfectSearch2[n_] := Module[{perfectQ},
  perfectQ[k_] := DivisorSigma[1, k] == 2 k;
  Select[Range[n], perfectQ]
]

```

```

In[12]:= PerfectSearch2[106] // Timing

```

```

Out[12]= {4.72435, {6, 28, 496, 8128}}

```

Reduce the number of sheer computations by using the, as yet, unproven conjecture that there

are no odd perfect numbers (confirmed for  $n < 10^{300}$ ). Also use a pure function for the predicate test.

```
In[13]:= PerfectSearch3[n_] := Select[Range[2, n, 2], (DivisorSigma[1, #] == 2 # &)]
```

```
In[14]:= PerfectSearch3[10^6] // Timing
```

```
Out[14]:= {2.35038, {6, 28, 496, 8128}}
```

```
In[15]:= PerfectSearchParallel[n_] :=  
Parallelize[Select[Range[2, n, 2], (DivisorSigma[1, #] == 2 # &)]]
```

```
In[16]:= PerfectSearchParallel[10^6] // Timing
```

```
Out[16]:= {0.143826, {6, 28, 496, 8128}}
```

- First we find those values of  $p$  for which  $2^p - 1$  is prime. This first step is quite compute-intensive; fortunately, it parallelizes well.

```
In[17]:= primes = Parallelize[Select[Range[10000], PrimeQ[2^# - 1] &]]
```

```
Out[17]:= {2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127,  
521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941}
```

So for each of the above values of the list `primes`,  $2^{p-1} (2^p - 1)$  will be even perfect numbers (thanks to Euler).

```
In[18]:= perfectLis = Map[2^{#-1} (2^# - 1) &, primes];
```

And finally, a check.

```
In[19]:= perfectQ[j_] := Total[Divisors[j]] == 2 j;
```

```
In[20]:= Map[perfectQ, perfectLis]
```

```
Out[20]:= {True, True, True, True, True, True, True, True, True, True, True,  
True, True, True, True, True, True, True, True, True, True}
```

These are very large numbers indeed.

```
In[21]:= 2^{#-1} (2^# - 1) &[9941] // N
```

```
Out[21]:= 5.988854963873362 × 10^{5984}
```

```
In[22]:= CloseKernels[]
```

```
Out[22]:= {KernelObject[1, local, <defunct>], KernelObject[2, local, <defunct>],  
KernelObject[3, local, <defunct>], KernelObject[4, local, <defunct>]}
```

- Start by importing four texts, James Joyce's *Ulysses*, Hermann Hesse's *Siddhartha*, Emily Brontë's *Wuthering Heights*, and Virginia Woolf's *Jacob's Room*:

```
In[23]:= joyce = Import["http://www.gutenberg.org/ebooks/4300.txt.utf-8", "Text"];
```

```
In[24]:= hesse = Import["http://www.gutenberg.org/ebooks/2500.txt.utf-8", "Text"];
```

```
In[25]:= bronte = Import["http://www.gutenberg.org/ebooks/768.txt.utf-8", "Text"];
```

```
In[26]:= wolff = Import["http://www.gutenberg.org/ebooks/5670.txt.utf-8", "Text"];
```

Here is the count for the word *history* in the James Joyce text, accounting for possible capitalization.

```
In[27]:= StringCount[joyce, "history" | "History"]
```

```
Out[27]= 30
```

And here is the computation across all four texts done in parallel.

```
In[28]:= LaunchKernels[];
```

```
In[29]:= ParallelTable[StringCount[text, "history" | "History"],
  {text, {joyce, hesse, bronte, wolff}}]
```

```
Out[29]= {30, 0, 9, 10}
```

```
In[30]:= CloseKernels[];
```

5. First, here is the implementation using a Do loop.

```
In[31]:= PiApprox[trials_] := Module[{in = 0, pt, pi, error},
  pt := RandomReal[1, {2}];
  Do[If[Total[pt2] ≤ 1, in++], {trials};
  pi = N[4 in / trials];
  error = Abs[ $\pi$  - pi];
  {pi, error}]
```

```
In[32]:= TableForm[Table[Join[{10i}, PiApprox[10i]], {i, 1, 6}],
  TableHeadings → {None, {"Trials", " $\pi$  approx", "Error"}}]
```

```
Out[32]//TableForm=
```

	Trials	$\pi$ approx	Error
	10	2.8	0.341593
	100	2.96	0.181593
	1000	3.148	0.00640735
	10 000	3.1312	0.0103927
	100 000	3.14068	0.000912654
	1 000 000	3.14122	0.000376654

Next, an implementation using RegionMember. Start by creating a disk and square. It is important to use Disk and not Circle in order to then be able to use the computational geometry function RegionMember – Circle is a region, but a one-dimensional region consisting of the curve itself only.

```
In[33]:= RegionDimension@Circle[{0, 0}, 1]
```

```
Out[33]= 1
```

```
In[34]:= RegionDimension@Disk[{0, 0}, 1]
```

```
Out[34]= 2
```

```
In[35]:= D = Disk[.5, .5], .5];
  P = Polygon[{{0, 0}, {1, 0}, {1, 1}, {0, 1}, {0, 0}}];
```

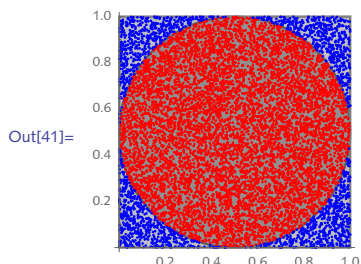
Here is the computation for 10 000 points.

```
In[37]:= pts = RandomReal[{0, 1}, {104, 2}];
in = Select[pts, RegionMember[D, #] &];
out = Complement[pts, in];
4 N[Length[in]/Length[pts], 20]
```

```
Out[40]= 3.122800000000000000000000
```

Here is an image.

```
In[41]:= Graphics[{
  {Opacity[.25], EdgeForm[{Thickness[.01], Gray}], D, P},
  PointSize[Tiny], Red, Point[in], Blue, Point[out]
}, Axes → True, ImageSize → Small]
```



The first implementation is much faster given that it is purely arithmetic.

```
In[42]:= Timing[PiApprox[104]]
```

```
Out[42]= {0.037764, {3.1388, 0.00279265}}
```

```
In[43]:= PiMonteCarlo[n_, prec_ : $MachinePrecision] := Module[{D, P, pts, in},
  D = Disk[{.5, .5}, .5];
  P = Polygon[{{0, 0}, {1, 0}, {1, 1}, {0, 1}, {0, 0}}];
  pts = RandomReal[{0, 1}, {n, 2}];
  in = Select[pts, RegionMember[D, #] &];
  4 N[Length[in]/Length[pts], prec]
]
```

```
In[44]:= Timing[PiMonteCarlo[104]]
```

```
Out[44]= {2.10258, 3.1136000000000000}
```

But we can speed up this second implementation by using a one-argument form of `RegionMember` to create a `RegionMemberFunction` object as described in Section 9.1.

```

In[45]:= PiMonteCarlo2[n_, prec_:$MachinePrecision] := Module[{D, P, pts, in, rm, pi},
  D = Disk[{.5, .5}, .5];
  P = Polygon[{{0, 0}, {1, 0}, {1, 1}, {0, 1}, {0, 0}}];
  pts = RandomReal[{0, 1}, {n, 2}];
  rm = RegionMember[D];
  in = Select[pts, rm[#] &];
  pi = 4 N[Length[in]/Length[pts], prec];
  {pi, Abs[ $\pi$  - pi]}
]

```

Let's parallelize. First, launch the kernels and distribute the definition of PiMonteCarlo2.

```

In[46]:= LaunchKernels[];
In[47]:= DistributeDefinitions[PiMonteCarlo2]
Out[47]:= {PiMonteCarlo2, D, P, pts, in}

```

Run twenty-four simulations and then average the results.

```

In[48]:= ParallelTable[PiMonteCarlo2[106], {24}]
Out[48]:= {{3.141060000000000, 0.000532653589793}, {3.141440000000000, 0.000152653589793},
  {3.138152000000000, 0.003440653589793}, {3.143144000000000, 0.001551346410207},
  {3.140664000000000, 0.000928653589793}, {3.142020000000000, 0.000427346410207},
  {3.141864000000000, 0.000271346410207}, {3.143704000000000, 0.002111346410207},
  {3.143156000000000, 0.001563346410207}, {3.139856000000000, 0.001736653589793},
  {3.139976000000000, 0.001616653589793}, {3.141120000000000, 0.000472653589793},
  {3.141684000000000, 0.000091346410207}, {3.140172000000000, 0.001420653589793},
  {3.144920000000000, 0.003327346410207}, {3.138968000000000, 0.002624653589793},
  {3.141128000000000, 0.000464653589793}, {3.139412000000000, 0.002180653589793},
  {3.138632000000000, 0.002960653589793}, {3.142496000000000, 0.000903346410207},
  {3.142932000000000, 0.001339346410207}, {3.138740000000000, 0.002852653589793},
  {3.142288000000000, 0.000695346410207}, {3.141844000000000, 0.000251346410207}}
In[49]:= Mean[%]
Out[49]:= {3.141223833333333, 0.001413221132483}

```

As an aside, you could try parallelizing the implementation with the Do loop by using ParallelDo or Parallelize, but you will need to share the variable in across subkernels using SetSharedVariable. The cost of communication between kernels to keep track of and increment this counter is quite expensive and hence any gains in running this function in parallel will be erased.

```

In[50]:= CloseKernels[];

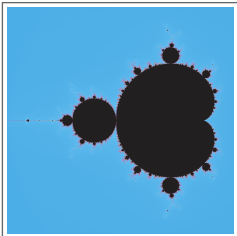
```

- Only two changes are required to run this in parallel – distribute the definition for Mandelbrot and change Table to ParallelTable. To increase the resolution the grid now has many more divisions in each direction ( $n = 400$ ).

```

In[51]:= Mandelbrot[c_] := Length[NestWhileList[#^2 + c &, 0, Abs[#] < 2 &, 1, 250]]
In[52]:= LaunchKernels[]
Out[52]:= {KernelObject[13, local], KernelObject[14, local],
           KernelObject[15, local], KernelObject[16, local]}

In[53]:= DistributeDefinitions[Mandelbrot]
Out[53]:= {Mandelbrot}

In[54]:= data = With[{n = 400}, ParallelTable[Mandelbrot[x + i y], {y, -1.3, 1.3,  $\frac{1}{n}$ },
           {x, -2, 0.6,  $\frac{1}{n}$ }}];
In[55]:= ArrayPlot[data, ColorFunction -> "CMYKColors"]
Out[55]= 
In[56]:= CloseKernels[];

```

### 9.3 Compiling: exercises

1. Create a compiled function that computes the distance to the origin of a two-dimensional point. Then compare it to some of the built-in functions such as `Norm` and `EuclideanDistance` to compute the distances for a large set of points. If you have a C compiler installed on your computer, use the `Compile` option `CompilationTarget -> "C"` and compare the results.
2. Modify the previous exercise under the assumption that complex numbers are given as input to your compiled function.
3. Padé approximants are rational functions that are often used to approximate functions whose Taylor series does not converge. For example, the Taylor series for  $\ln(1+x)$  has poor convergence on  $0 \leq x \leq 1$ . Here is its third-order Padé approximant:

```

In[1]:= PadéApproximant[Log[1 + x], {x, 0, 3}]
Out[1]= 
$$\frac{x + x^2 + \frac{11x^3}{60}}{1 + \frac{3x}{2} + \frac{3x^2}{5} + \frac{x^3}{20}}$$


```

Create a compiled function that computes the above expression for  $x$  a real number and then evaluate a range of values from zero to two and make a discrete plot of the differences between

the approximated values and  $\text{Log}[1 + x]$ .

4. Many other iteration functions can be used for the Julia set computation. Experiment with some other functions such as  $c \sin(z)$ ,  $c e^z$ , or Gaston Julia's original function:

$$z^4 + z^3 / (z - 1) + z^2 / (z^3 + 4z^2 + 5) + c.$$

For these functions, you will have to adjust the test to determine if a point is unbounded upon iteration. Try  $(\text{Abs}[\text{Im}[z]] > 50 \ \& )$ .

## 9.3 Solutions

- i. First, create a test point with which to work.

```
In[1]:= pt = RandomReal[1, {2}]
```

```
Out[1]= {0.380097, 0.592515}
```

The following does not quite work because the default pattern is expected to be a flat expression.

```
In[2]:= distReal = Compile[{{p, _Real}}, Sqrt[First[p]^2 + Last[p]^2],
    RuntimeAttributes -> {Listable}, Parallelization -> True]
```

Compile::part:

Part specification p[[1]] cannot be compiled since the argument is not a tensor of sufficient rank. Evaluation will use the uncompiled function. >>

Compile::part:

Part specification p[[1]] cannot be compiled since the argument is not a tensor of sufficient rank. Evaluation will use the uncompiled function. >>

```
Out[2]= CompiledFunction[ Argument count: 1
Argument types: {_Real}]
```

Give a third argument to the pattern specification to deal with this:  $\{p, \text{\_Real}, 1\}$ .

```
In[3]:= ArrayDepth[pt]
```

```
Out[3]= 1
```

```
In[4]:= distReal = Compile[{{p, _Real, 1}}, Sqrt[First[p]^2 + Last[p]^2],
    RuntimeAttributes -> {Listable}, Parallelization -> True]
```

```
Out[4]= CompiledFunction[ Argument count: 1
Argument types: {{_Real, 1}}]
```

```
In[5]:= distReal[pt]
```

```
Out[5]= 0.703951
```

Check it against the built-in function:

```
In[6]:= Norm[pt]
Out[6]= 0.703951
```

Check that it threads properly over a list of points.

```
In[7]:= pts = RandomReal[1, {3, 2}]
Out[7]= {{0.73132, 0.692825}, {0.00172384, 0.878972}, {0.0588752, 0.367734}}

In[8]:= distReal[pts]
Out[8]= {1.00739, 0.878974, 0.372417}
```

Norm does not have the Listable attribute so it must be mapped over the list.

```
In[9]:= Map[Norm, pts]
Out[9]= {1.00739, 0.878974, 0.372417}

In[10]:= distReal[pts] == Map[Norm, pts]
Out[10]= True
```

Now scale up the size of the list of points and check efficiency.

```
In[11]:= pts = RandomReal[1, {10^6, 2}];
In[12]:= AbsoluteTiming[distReal[pts];]
Out[12]= {0.050415, Null}

In[13]:= AbsoluteTiming[Map[Norm, pts];]
Out[13]= {0.087031, Null}

In[14]:= distReal[pts] == Map[Norm, pts]
Out[14]= True
```

Compiling to C (assuming you have a C compiler installed), speeds things up even more.

```
In[15]:= distReal = Compile[{{p, _Real, 1}}, Sqrt[First[p]^2 + Last[p]^2],
  RuntimeAttributes -> {Listable}, Parallelization -> True, CompilationTarget -> "C"]
```

Compile::nogen : A library could not be generated from the compiled function. >>

```
Out[15]= CompiledFunction[ Argument count: 1
Argument types: {{_Real, 1}}]
```

You can squeeze a little more speed out of these functions by using Part instead of First and Last.



```
In[16]:= distReal2 = Compile[{{p, _Real, 1}}, Sqrt[p[[1]]^2 + p[[2]]^2],
  RuntimeAttributes -> {Listable}, Parallelization -> True, CompilationTarget -> "C"]
```

Compile::nogen : A library could not be generated from the compiled function. >>

```
Out[16]= CompiledFunction[ Argument count: 1
Argument types: {{_Real, 1}}]
```

```
In[17]:= AbsoluteTiming[distReal2[pts];]
```

```
Out[17]= {0.07329, Null}
```

As an aside, the mean distance to the origin for random points in the unit square approaches the following, asymptotically.

```
In[18]:= NIntegrate[Sqrt[x^2 + y^2], {x, 0, 1}, {y, 0, 1}]
```

```
Out[18]= 0.765196
```

```
In[19]:= Mean@distReal[pts]
```

```
Out[19]= 0.764885
```

2. We need to make just three slight modifications to the code from the previous exercise: remove the rank specification; specify Complex as the type; extract the real and imaginary parts to do the norm computation.

```
In[20]:= distComplex = Compile[{{z, _Complex}}, Sqrt[Re[z]^2 + Im[z]^2],
  RuntimeAttributes -> {Listable}, Parallelization -> True]
```

```
Out[20]= CompiledFunction[ Argument count: 1
Argument types: {_Complex}]
```

```
In[21]:= pts = RandomComplex[1, {3}]
```

```
Out[21]= {0.707041 + 0. i, 0.0774789 + 0. i, 0.366904 + 0. i}
```

```
In[22]:= distComplex[pts]
```

```
Out[22]= {0.707041, 0.0774789, 0.366904}
```

```
In[23]:= distComplex[pts] == Map[Norm, pts]
```

```
Out[23]= True
```

3. Here is the Padé approximant for  $\text{Log}[1 + x]$  about zero, or order three.

```
In[24]:= PadeApproximant[Log[1 + x], {x, 0, 3}]
```

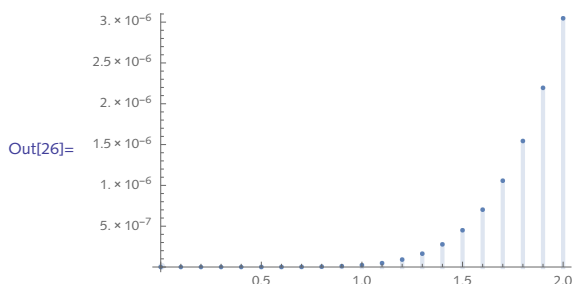
$$\text{Out[24]} = \frac{x + x^2 + \frac{11x^3}{60}}{1 + \frac{3x}{2} + \frac{3x^2}{5} + \frac{x^3}{20}}$$

Create a compiled function that computes the above expression for  $x$  a real number and then evaluates a range of values from 0 to 2 and makes a discrete plot of the differences between the approximated values and  $\text{Log}[1 + x]$ .

```
In[25]:= logC = Compile[{{x, _Real}},  $\frac{x + x^2 + \frac{11x^3}{60}}{1 + \frac{3x}{2} + \frac{3x^2}{5} + \frac{x^3}{20}}$ ]
```

```
Out[25]= CompiledFunction[ Argument count: 1  
Argument types: {_Real}]
```

```
In[26]:= DiscretePlot[Log[1 + x] - logC[x], {x, 0, 2, .1}, PlotRange -> All]
```

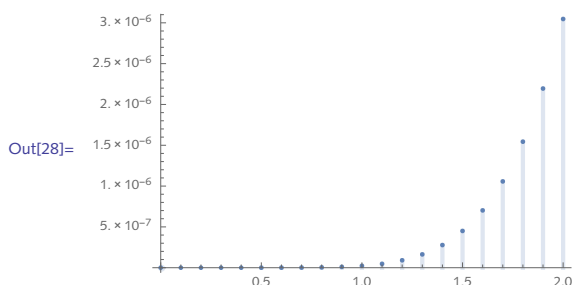


Here is a higher order approximant. Note the need to evaluate PadeApproximant before the compilation.

```
In[27]:= logC = Compile[{{x, _Real}}, Evaluate[PadeApproximant[Log[1 + x], {x, 0, 5}]]]
```

```
Out[27]= CompiledFunction[ Argument count: 1  
Argument types: {_Real}]
```

```
In[28]:= DiscretePlot[Log[1 + x] - logC[x], {x, 0, 2, .1}, PlotRange -> All]
```

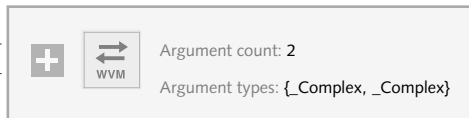


4. Here is the computation for the iteration function  $c \sin(z)$  using  $c = 1 + 0.4i$ .

```
In[29]:= cJulia2 = Compile[{{z, _Complex}, {c, _Complex}}, Module[{cnt = 1},
    FixedPoint[(cnt++;
    c Sin[#]) &, z, 100, SameTest -> (Abs[Im[#2]] > 50 &)];
    cnt], CompilationTarget -> "C", RuntimeAttributes -> {Listable},
    Parallelization -> True, "RuntimeOptions" -> "Speed"]
```

Compile::nogen : A library could not be generated from the compiled function. >>

Out[29]= CompiledFunction[

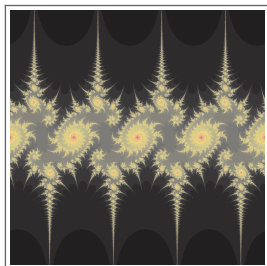


```
In[30]:= LaunchKernels[]
```

```
Out[30]= {KernelObject[17, local], KernelObject[18, local],
    KernelObject[19, local], KernelObject[20, local]}
```

```
In[31]:= With[{res = 100},
    ArrayPlot[ParallelTable[-cJulia2[x + y I, 1 + 0.4 I], {y, -2 π, 2 π, 1/res},
    {x, -2 π, 2 π, 1/res}], ColorFunction -> ColorData["CMYKColors"]]]
```

Out[31]=



```
In[32]:= CloseKernels[];
```



---

## 10 Packages

### 10.3 Working with packages: exercises

- I. The following set of exercises will walk you through the creation of a package `Collatz`, a package of functions for performing various operations related to the Collatz problem that we investigated earlier (Exercises 5 and 6 of Section 4.1, Exercise 11 of Section 5.3, and Exercise 10 of Section 5.4). Recall that the Collatz function, for any integer  $n$ , returns  $3n + 1$  for odd  $n$ , and  $n/2$  for even  $n$ . The (as yet unproven) Collatz conjecture is the statement that, for any initial positive integer  $n$ , the iterates of the Collatz function always reach the cycle  $4, 2, 1, \dots$ . Start by creating an auxiliary function `collatz[n]` that returns  $3n + 1$  for  $n$  odd and  $n/2$  for  $n$  even.
  - a. Create the function `CollatzSequence[n]` that lists the iterates of the auxiliary function `collatz[n]`. Here is some sample output of the `CollatzSequence` function:

```
In[1]:= CollatzSequence[7]
```

```
Out[1]= {7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1}
```

- b. Create a usage message for `CollatzSequence` and warning messages for each of the following situations:
    - `noint`: the argument to `CollatzSequence` is not a positive integer
    - `argx`: `CollatzSequence` was called with the wrong number of arguments.
  - c. Modify the definition of `CollatzSequence` that you created in part a. above so that it does some error trapping and issues the appropriate warning message that you created in part b.
  - d. Finally, put all the pieces together and write a package `Collatz`` that includes the appropriate `BeginPackage` and `Begin` statements, usage messages, warning messages, and function definitions. Make `CollatzSequence` a public function and `collatz` a private function. Put your package in a directory where *Mathematica* can find it on its search path and then test it to see that it returns correct output such as in the examples below.

```
In[2]:= Quit[];
```

```
In[1]:= << EPM`Collatz`
```

```
In[2]:= ?CollatzSequence
```

CollatzSequence[n] computes the sequence of Collatz iterates starting with initial value  $n$ . The sequence terminates as soon as it reaches the value 1.

Here are various cases in which CollatzSequence is given bad input:

```
In[3]:= CollatzSequence[-5]
```

CollatzSequence::notint : The argument, -5, to CollatzSequence must be a positive integer.

```
In[4]:= CollatzSequence[{a, b}]
```

CollatzSequence::notint : The argument, {a, b}, to CollatzSequence must be a positive integer.

And this computes the sequence for starting value 27:

```
In[5]:= CollatzSequence[27]
```

```
Out[5]= {27, 82, 41, 124, 62, 31, 94, 47, 142, 71, 214, 107, 322, 161, 484, 242, 121, 364,
182, 91, 274, 137, 412, 206, 103, 310, 155, 466, 233, 700, 350, 175, 526, 263,
790, 395, 1186, 593, 1780, 890, 445, 1336, 668, 334, 167, 502, 251, 754, 377,
1132, 566, 283, 850, 425, 1276, 638, 319, 958, 479, 1438, 719, 2158, 1079, 3238,
1619, 4858, 2429, 7288, 3644, 1822, 911, 2734, 1367, 4102, 2051, 6154, 3077,
9232, 4616, 2308, 1154, 577, 1732, 866, 433, 1300, 650, 325, 976, 488, 244, 122,
61, 184, 92, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1}
```

2. Take the StemPlot function developed in Section 6.2 and create a package around it. Include a usage message and appropriate warning messages that are issued when bad input is supplied.

### 10.3 Solutions

- i. Here are the definitions for the auxiliary collatz function.

```
In[1]:= collatz[n_?EvenQ] := n/2
```

```
In[2]:= collatz[n_?OddQ] := 3 n + 1
```

- a. This is essentially the definition given in the solution to Exercise 5 from Section 6.2.

```
In[3]:= CollatzSequence[n_] := NestWhileList[collatz, n, # != 1 &]
```

```
In[4]:= CollatzSequence[7]
```

```
Out[4]= {7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1}
```

- b. First we write the usage message for CollatzSequence, our public function. Notice that we write no usage message for the private collatz function.

```
In[5]:= CollatzSequence::usage =
    "CollatzSequence[n] computes the sequence of Collatz iterates
    starting with initial value n. The sequence terminates as soon
    as it reaches the value 1.";
```

Here is the warning message that will be issued whenever CollatzSequence is passed an argument that is not a positive integer.

```
In[6]:= CollatzSequence::notint =
    "First argument, '1', to CollatzSequence must be a positive integer.";
```

- c. Here is the modified definition which now issues the warning message created above whenever the argument  $n$  is not a positive integer.

```
In[7]:= CollatzSequence[n_] := If[IntegerQ[n] && n ≥ 0, NestWhileList[collatz, n, # ≠ 1 &],
    Message[CollatzSequence::notint, n]]
```

The following case covers the situation when CollatzSequence is passed two or more arguments. Note that it uses the built-in `argx` message, which is issued whenever built-in functions are passed the wrong number of arguments.

```
In[8]:= CollatzSequence[_, args_] /;
    Message[CollatzSequence::argx, CollatzSequence, Length[{args}] + 1] :=
    Null
```

- d. The package begins by giving usage messages for every exported function. The functions to be exported are *mentioned* here – *before* the subcontext `Private`` is entered – so that the symbol `CollatzSequence` has context `Collatz``. Notice that `collatz` is *not* mentioned here and hence will not be accessible to the user of this package.

```
In[9]:= Quit[]
```

```
In[1]:= BeginPackage["EPM`Collatz`"];
```

```
In[2]:= CollatzSequence::usage =
    "CollatzSequence[n] computes the sequence of Collatz iterates
    starting with initial value n. The sequence terminates as soon
    as it reaches the value 1.";
```

```
In[3]:= CollatzSequence::notint =
    "First argument, '1', to CollatzSequence must be a positive integer.";
```

A new context `EPM`Collatz`Private`` is then begun *within* `EPM`Collatz`. All the definitions of this package are given within this new context. The context `EPM`Collatz`CollatzSequence` is defined within the `System`` context. The context of `collatz`, on the other hand, is `EPM`Collatz`Private``.

```
In[4]:= Begin["`Private`"];
```

```
In[5]:= collatz[n_?EvenQ] := n / 2
```

```
In[6]:= collatz[n_?OddQ] := 3 n + 1
```

```

In[7]:= CollatzSequence[n_] := If[IntegerQ[n] && n ≥ 0, NestWhileList[collatz, n, # ≠ 1 &],
    Message[CollatzSequence::notint, n]]

In[8]:= CollatzSequence[_, args_] /;
    Message[CollatzSequence::argx, CollatzSequence, Length[{args}] + 1] :=
    Null

In[9]:= End[];

In[10]:= EndPackage[]

```

After the `End[]` and `EndPackage[]` functions are evaluated, `$Context` and `$ContextPath` revert to whatever they were before, except that `EPM`Collatz`` is added to `$ContextPath`. Users can refer to `CollatzSequence` using its short name, but they can only refer to the auxiliary function `collatz` by its full name. The intent is to discourage clients from using `collatz` at all, and doing so should definitely be avoided, since the author of the package may change or remove auxiliary definitions at a later time.

- Here is the code for the `StemPlots` package.

```

In[11]:= BeginPackage["EPM`StemPlots`"]

Out[11]= EPM`StemPlots`

In[12]:= StemPlot::usage =
    "StemPlot[data] returns a stem plot of the discrete data, data.";

In[13]:= StemPlot::badarg = "The first argument to StemPlot must be a list of numbers.";

In[14]:= Options[StemPlot] = Options[ListPlot];

In[15]:= Begin["`Private`"]

Out[15]= EPM`StemPlots`Private`

In[16]:= StemPlot[lis_, opts: OptionsPattern[]] :=
    If[MatchQ[lis, {__?NumericQ}],
        ListPlot[lis, opts, Filling → Axis],
        Message[StemPlot::badarg]
    ]

In[17]:= End[]

Out[17]= EPM`StemPlots`Private`

In[18]:= EndPackage[]

```

After saving in an appropriate location, this loads the package:

```
In[19]:= << EPM`StemPlots`
```

Check the usage message:

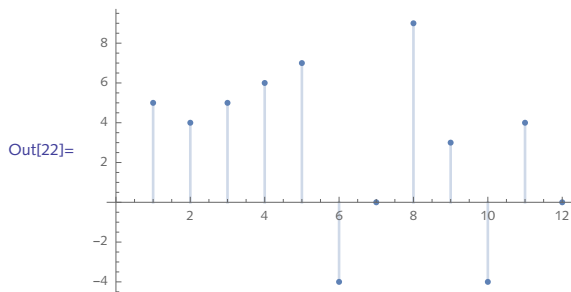


In[20]:= **?StemPlot**

StemPlot[*data*] returns a stem plot of the discrete data, *data*.

And try some sample input:

In[21]:= **data = RandomInteger[{-5, 10}, 12];**  
**StemPlot[data]**



In[23]:= **StemPlot[data,**  
**PlotStyle → Directive[PointSize[Medium]], FillingStyle → Red**  
**]**

