

PID Control for the Two Degree of Freedom Helicopter

AAE364L

In this Lab we will design a PID controller to fly the helicopter with two degrees of freedom.

1 Review of PID controllers

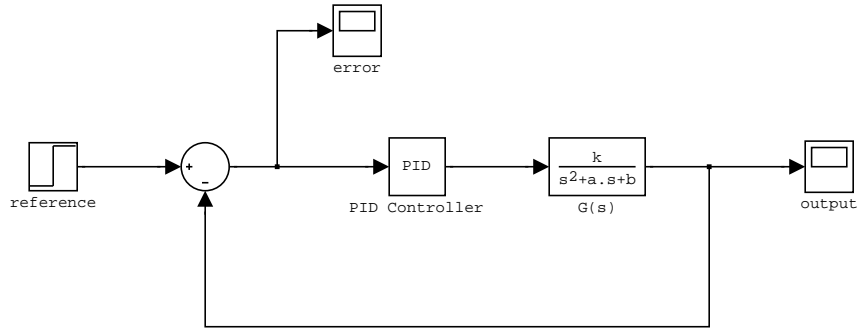


Figure 1: A PID control system.

In this section we will review the proportional, integral and derivative (PID) controller. Consider the closed loop system in Figure 1 where $G(s)$ is the open loop plant. In our helicopter problem, the transfer function is of the form $G(s) = \frac{k}{s^2 + as + b}$. However, for the moment assume that G is an arbitrary transfer function. The error is denoted by e and the output by y . Recall that a PID controller is a compensator of the form

$$C(s) = \gamma_p + \gamma_d s + \frac{\gamma_i}{s} = \frac{\gamma_d s^2 + \gamma_p s + \gamma_i}{s}. \quad (1)$$

The proportional gain is γ_p while γ_d is the derivative gain and γ_i is the integral gain. The closed loop transfer functions from the reference signal r to the output y , and from the reference signal r to the error e are respectively given by

$$\begin{aligned} \frac{Y(s)}{R(s)} &= \frac{C(s)G(s)}{1 + C(s)G(s)} = \frac{(\gamma_d s^2 + \gamma_p s + \gamma_i)G(s)}{s + (\gamma_d s^2 + \gamma_p s + \gamma_i)G(s)} \\ \frac{E(s)}{R(s)} &= \frac{1}{1 + C(s)G(s)} = \frac{s}{s + (\gamma_d s^2 + \gamma_p s + \gamma_i)G(s)}. \end{aligned} \quad (2)$$

Now assume that the gains γ_p , γ_i and γ_d are chosen such that the closed loop system is stable. In other words, assume that the gains γ_p , γ_i and γ_d are chosen such that all the zeros

of $1 + (\gamma_p + \gamma_d s + \frac{\gamma_i}{s})G(s)$ are contained in the open left hand plane $\{s \in \mathbb{C} : \Re s < 0\}$. If the reference signal $r(t) = r_0$ is a constant for all t and $\gamma_i \neq 0$, then the steady state output y_{ss} and the steady state error e_{ss} are respectively given by

$$y_{ss}(t) = \lim_{t \rightarrow \infty} y(t) = r_0 \quad \text{and} \quad e_{ss}(t) = \lim_{t \rightarrow \infty} e(t) = 0. \quad (3)$$

In other words, the steady state output $y_{ss}(t) = r_0$ and the steady state error $e_{ss}(t)$ equals zero.

2 Varying the PID gains

One way to design a PID controller is to vary the gains γ_p , γ_d and γ_i to arrive at a desired closed loop response. For example, the proportional gain γ_p acts like a spring. By increasing γ_p , in general, one decreases the rise time and increases the overshoot. The derivative gain γ_d acts like damping or a shock absorber. By increasing γ_d , in general, one decreases the overshoot and settling time. So one can vary γ_p , γ_d and γ_i in a simulation of the step response of the closed loop system to achieve the desired response. This method is demonstrated in the web page www.engin.umich.edu/group/ctm/. The following table is taken from this web page and shows what happens when one increases the gains γ_p , γ_d and γ_i in the PID controller.

Closed loop response	Rise time	Overshoot	Settling time	Steady state error
γ_p	decrease	increase	small change	decrease
γ_i	decrease	increase	increase	eliminate
γ_d	small change	decrease	decrease	small change

Table 1: The effect of increasing PID gains.

There are many procedures to design a PID controller for system of the form in Figure 1. However, our problem has saturation, and saturation can cause major problems in PID control design. The integral gain γ_i is a double-edged sword in PID design with saturation. The integral gain will eliminate the steady state error due to a step. However, in the presence of saturation a large integral gain can greatly increase the settling time and may even cause the system to become unstable. This problem is called *integral windup* and is even more pronounced when the open loop system G is unstable. Since one of our open loop systems for the helicopter has a pole on the imaginary axis, we have to be careful about integral windup.

3 Pole placement for PID controllers

In general one can choose the gains γ_p , γ_i and γ_d to place the poles of the closed loop system at any three distinct locations in the open left hand plane $\{s \in \mathbb{C} : \Re s < 0\}$. However, placing three poles does not guarantee that the closed loop system will be stable. If the McMillan of G is greater than or equal to three, then one does not have any control over the remaining poles of the closed loop system, and the closed loop system can still be unstable.

(The McMillan degree of a proper rational transfer function $G(s) = p(s)/q(s)$ is the degree of the denominator polynomial q . If $\gamma_i \neq 0$, then the McMillan degree of the closed loop system equals the McMillan degree of G plus one.) In other words, the PID compensator can only place three poles. If the closed loop system has more than three poles, then placing three poles does not necessarily guarantee that the closed loop system will be stable.

In our Helicopter problem, the pitch and yaw transfer functions are of the form:

$$G(s) = \frac{k}{s^2 + as + b}$$

where a , b and k are constants. In this case, the closed loop transfer functions in (2) are determined by

$$\begin{aligned} \frac{Y(s)}{R(s)} &= \frac{(\gamma_d s^2 + \gamma_p s + \gamma_i)k}{s^3 + (a + k\gamma_d)s^2 + (b + k\gamma_p)s + k\gamma_i} \\ \frac{E(s)}{R(s)} &= \frac{s^3 + as^2 + bs}{s^3 + (a + k\gamma_d)s^2 + (b + k\gamma_p)s + k\gamma_i}. \end{aligned} \quad (4)$$

Notice that the poles of both of these transfer functions are given by the zeros of the following polynomial:

$$d(s) = s^3 + (a + k\gamma_d)s^2 + (b + k\gamma_p)s + k\gamma_i. \quad (5)$$

So the closed loop system is stable if and only if the polynomial d is stable. (By definition a polynomial $d(s)$ is stable if all the the zeros of $d(s)$ are contained in the open left half plane $\{s \in \mathbb{C} : \Re s < 0\}$.) Moreover, one can choose the constants γ_i , γ_d and γ_p to place the poles of the closed loop system or the zeros of $d(s)$ at any three points $\{\lambda_1, \lambda_2, \lambda_3\}$ in the complex plane \mathbb{C} . To see this compute the constants α , β and γ such that

$$p(s) = (s - \lambda_1)(s - \lambda_2)(s - \lambda_3) = s^3 + \alpha s^2 + \beta s + \gamma. \quad (6)$$

(The MATLAB command is `poly([λ_1 λ_2 λ_3])`. Then $\{\lambda_1, \lambda_2, \lambda_3\}$ are the poles of $d(s)$ if and only if $d(s) = p(s)$, or equivalently, by comparing like coefficients

$$\alpha = a + k\gamma_d, \quad \beta = b + k\gamma_p \quad \text{and} \quad \gamma = \gamma_i k.$$

By solving for γ_i , γ_d and γ_p , we obtain

$$\begin{aligned} \gamma_d &= \frac{\alpha - a}{k}, \quad \gamma_p = \frac{\beta - b}{k} \quad \text{and} \quad \gamma_i = \frac{\gamma}{k} \\ p(s) &= (s - \lambda_1)(s - \lambda_2)(s - \lambda_3) = s^3 + \alpha s^2 + \beta s + \gamma \end{aligned} \quad (7)$$

By computing the PID gains according to equation (7), we guarantee that $\{\lambda_1, \lambda_2, \lambda_3\}$ are the zeros of the polynomial $d(s)$. In other words, $\{\lambda_1, \lambda_2, \lambda_3\}$ are the poles of the closed loop systems in equation (4).

An example. Consider the open loop transfer function given by

$$G(s) = \frac{2}{s^2 - 3s + 10}.$$

It is noted that G is unstable. (Recall that a quadratic polynomial of the form $p(s) = s^2 + as + b$ is stable if and only if $a > 0$ and $b > 0$.) Assume that one wants to place the poles of the closed loop system at $\{-1, -2, -4\}$. Then using the command `poly([-1, -2, -4])` in MATLAB, we arrive at

$$(s + 1)(s + 2)(s + 4) = s^3 + 7s^2 + 14s + 8.$$

So the gains in the PID controller which places the poles of the closed loop system at $\{-1, -2, -4\}$ are determined by

$$\gamma_d = \frac{7 + 3}{2} = 5, \quad \gamma_p = \frac{14 - 10}{2} = 2 \quad \text{and} \quad \gamma_i = \frac{8}{2} = 4.$$

4 The pitch and yaw transfer functions

The equations of motion for our two degree of freedom Helicopter are given by

$$\begin{aligned} J_p \ddot{\theta} + J_y \sin(\theta) \cos(\theta) \dot{\psi}^2 + mg(h \sin(\theta) + R_c \cos(\theta)) + c_p \dot{\theta} \\ = lk_{pp}v_p + k_{py}v_y \\ (J_y \cos(\theta)^2 + J_{shaft}) \ddot{\psi} - 2J_y \cos(\theta) \sin(\theta) \dot{\theta} \dot{\psi} + c_y \dot{\psi} \\ = lk_{yy}v_y \cos(\theta) + k_{yp}v_p \cos(\theta). \end{aligned} \tag{8}$$

The linearized equations of motion for the helicopter about level flight are determined by

$$\begin{aligned} J_p \ddot{\theta} + c_p \dot{\theta} + mgh\theta &= lk_{pp}\delta v_p + k_{py}\delta v_y \\ (J_p + J_{shaft})\ddot{\psi} + c_y \dot{\psi} &= lk_{yy}\delta v_y + k_{yp}\delta v_p \\ \delta v_p &= v_p - v_{pe} \\ \delta v_y &= v_y - v_{ye}. \end{aligned} \tag{9}$$

Recall that v_{pe} is the pitch equilibrium voltage while v_{ye} is the yaw equilibrium voltage. Now assume that the yaw angle for the helicopter is fixed and the voltage to the yaw motor is kept constant at v_{ye} , that is, $\delta v_y = 0$ and $\psi = \dot{\psi} = \ddot{\psi} = 0$. In this case, the linearized equations of motion reduce to

$$\begin{aligned} J_p \ddot{\theta} + c_p \dot{\theta} + mgh\theta &= lk_{pp}\delta v_p \\ \delta v_p &= v_p - v_{pe} \\ v_{pe} &= \frac{mgR_c lk_{yy}}{lk_{pp}lk_{yy} - k_{py}k_{yp}}. \end{aligned} \tag{10}$$

The expression of v_{pe} follows from the equilibrium solution to the first equation in (8) with θ , $\dot{\theta}$ and $\ddot{\theta}$ all equal to zero. The transfer function G_p from the voltage δv_p applied to the pitch motor to the pitch angle θ is given by

$$G_p(s) = \frac{\Theta(s)}{\delta V_p(s)} = \frac{lk_{pp}}{J_p s^2 + c_p s + mgh} = \frac{\frac{lk_{pp}}{J_p}}{s^2 + \frac{c_p}{J_p} s + \frac{mgh}{J_p}}. \tag{11}$$

As expected, $\Theta(s) = \mathcal{L}(\theta)$ is the Laplace transform of the pitch angle $\theta(t)$. Notice that the open loop transfer function G_p is stable.

Now assume that the pitch angle for the helicopter is fixed and the voltage to the pitch motor is kept constant at v_{pe} , that is, $\delta v_p = 0$ and $\theta = \dot{\theta} = \ddot{\theta} = 0$. In this case, the linearized equations of motion reduce to

$$\begin{aligned} (J_p + J_{shaft})\ddot{\psi} + c_y\dot{\psi} &= lk_{yy}\delta v_y \\ \delta v_y &= v_y - v_{ye} \\ v_{ye} &= \frac{-mgR_c k_{yp}}{lk_{pp}lk_{yy} - k_{py}k_{yp}}. \end{aligned} \quad (12)$$

The transfer function G_y from the voltage δv_y applied to the yaw motor to the yaw angle ψ is given by

$$G_y(s) = \frac{\Psi(s)}{\delta V_y(s)} = \frac{lk_{yy}}{(J_p + J_{shaft})s^2 + c_y s} = \frac{\frac{lk_{yy}}{J_p + J_{shaft}}}{s^2 + \frac{c_y}{J_p + J_{shaft}}s}. \quad (13)$$

As expected, $\Psi(s) = \mathcal{L}(\psi)$ is the Laplace transform of the yaw angle $\psi(t)$. Notice that $G_y(s)$ has one pole at zero and another pole at $-\frac{c_y}{J_p + J_{shaft}}$. Hence the open loop transfer function G_y is marginally stable.

5 Pole placement for the pitch and yaw controllers

5.1 The pitch controller

Consider a pitch PID compensator of the form:

$$C_p(s) = \gamma_{pp} + \gamma_{dp}s + \frac{\gamma_{ip}}{s}$$

where γ_{pp} , γ_{dp} and γ_{ip} are the corresponding PID pitch controller gains. In this setting, the closed loop system from a pitch reference signal r_p to the pitch angle θ is given by

$$\frac{\Theta}{R_p} = \frac{C_p G_p}{1 + C_p G_p} \quad \text{where} \quad \delta V_p = C_p(s)(R_p(s) - \Theta(s)).$$

Since $\delta v_p = v_p - v_{pe}$, in the time domain we obtain

$$v_p(t) = v_{pe} + \gamma_{pp}(r_p - \theta) + \gamma_{dp}(\dot{r}_p - \dot{\theta}) + \gamma_{ip} \int_0^t (r_p(\tau) - \theta(\tau))d\tau.$$

Design a PID controller such that the poles of the closed loop system $\frac{\Theta}{R_p} = \frac{C_p G_p}{1 + C_p G_p}$ are -1 and the other two poles have a damping ratio $\zeta = 0.7$ and natural frequency $\omega_n = 6$. Suppose we are able to fix the yaw angle constant without turning on the yaw motor, that is, $\psi = 0$ and $v_y = 0$, then the equation of motion for the nonlinear feedback system is given by

$$\begin{aligned} lk_{pp}v_p &= J_p\ddot{\theta} + c_p\dot{\theta} + mg(h \sin(\theta) + R_c \cos(\theta)) \\ v_p(t) &= v_{pe} + \gamma_{pp}(r_p - \theta) + \gamma_{dp}(\dot{r}_p - \dot{\theta}) + \gamma_{ip} \int_0^t (r_p(\tau) - \theta(\tau))d\tau \\ v_{pe} &= \frac{mg(h \sin(\theta_d) + R_c \cos(\theta_d))lk_{yy}}{lk_{pp}lk_{yy} - k_{py}k_{yp}}. \end{aligned}$$

Here v_{pe} is the front motor voltage required to maintain a desired pitch angle θ_d with $\dot{\theta} = 0$ and $\ddot{\theta} = 0$. In the level flight condition $\theta_d = 0$. In other words, $\theta_d = \theta_{ss}(t)$ is the steady state pitch angle.

5.2 The yaw controller

Consider a yaw PID compensator of the form:

$$C_y(s) = \gamma_{py} + \gamma_{dy}s + \frac{\gamma_{iy}}{s}.$$

In this setting, the closed loop system from a yaw reference signal r_y to the yaw angle ψ is determined by

$$\frac{\Psi}{R_y} = \frac{C_y G_y}{1 + C_y G_y} \quad \text{where} \quad V_y = C_y(s)(R_y(s) - \Psi(s)).$$

In the time domain, the yaw voltage is given by

$$v_y(t) = v_{ye} + \gamma_{py}(r_y - \psi) + \gamma_{dy}(\dot{r}_y - \dot{\psi}) + \gamma_{iy} \int_0^t (r_y(\tau) - \psi(\tau)) d\tau$$

Design a PID controller such that the poles of the closed loop system $\frac{\Psi}{R_y} = \frac{C_y G_y}{1 + C_y G_y}$ are $-1/2$ and $-1 \pm i$. Suppose we are able to fix the pitch angle constant at zero without turning on the pitch motor, that is, $\theta = 0$, $\dot{\theta} = 0$ and $v_p = 0$, then the equation of motion for the nonlinear feedback system with Coulomb friction is given by

$$\begin{aligned} l k_{yy} v_y &= (J_y \cos(\theta)^2 + J_{shaft}) \ddot{\psi} + c_y \dot{\psi} + F_{cy}(\dot{\psi}) \\ v_y(t) &= v_{ye} + \gamma_{py}(r_y - \psi) + \gamma_{dy}(\dot{r}_y - \dot{\psi}) + \gamma_{iy} \int_0^t (r_y(\tau) - \psi(\tau)) d\tau \\ v_{ye} &= \frac{-mg(h \sin(\theta_d) + R_c \cos(\theta_d)) k_{yp}}{l k_{pp} l k_{yy} - k_{py} k_{yp}}. \end{aligned}$$

Here $F_{cy}(\dot{\psi})$ is the force due to Coulomb friction. We assume that the size of the Coulomb friction in the yaw is much larger than in the pitch.

6 Combining the pitch and yaw controllers

Previously, you have designed a pitch PID controller C_p and a yaw PID controller C_y . In this section, we will combine our pitch and yaw PID controllers to fly the helicopter. The nonlinear equations of motion corresponding to our pitch and yaw PID controllers for the

helicopter are given by

$$\begin{aligned}
J_p \ddot{\theta} + J_y \sin(\theta) \cos(\theta) \dot{\psi}^2 + mg(h \sin(\theta) + R_c \cos(\theta)) + c_p \dot{\theta} \\
= lk_{pp} v_p + k_{py} v_y \\
(J_y \cos(\theta)^2 + J_{shaft}) \ddot{\psi} - 2J_y \cos(\theta) \sin(\theta) \dot{\theta} \dot{\psi} + c_y \dot{\psi} \\
= lk_{yy} v_y \cos(\theta) + k_{yp} v_p \cos(\theta) - F_{cy}(\dot{\psi}) \\
v_p(t) = v_{pe} + \gamma_{pp}(r_p - \theta) + \gamma_{dp}(\dot{r}_p - \dot{\theta}) + \gamma_{ip} \int_0^t (r_p(\tau) - \theta(\tau)) d\tau \\
v_y(t) = v_{ye} + \gamma_{py}(r_y - \psi) + \gamma_{dy}(\dot{r}_y - \dot{\psi}) + \gamma_{iy} \int_0^t (r_y(\tau) - \psi(\tau)) d\tau \\
v_{pe} = \frac{mg(h \sin(\theta_d) + R_c \cos(\theta_d))lk_{yy}}{lk_{pp}lk_{yy} - k_{py}k_{yp}} \\
v_{ye} = \frac{-mg(h \sin(\theta_d) + R_c \cos(\theta_d))k_{yp}}{lk_{pp}lk_{yy} - k_{py}k_{yp}}.
\end{aligned}$$

Now let us use MATLAB to simulate these controllers on the full nonlinear system. To simulate our pitch and yaw PID controllers to fly the helicopter, load the following files in MATLAB:

- `heli_model.mdl`
- `setup_heli_PID_parameters.m`

In the file: `setup_heli_PID_parameters.m`, we have provided the following numbers needed to run the Simulink file `heli_model.mdl`

$$\begin{aligned}
K_{pp} &= 0.185 * k_{pp}; & B_{-p} &= c_p; \\
K_{yy} &= 0.185 * k_{yy}; & B_{-y} &= c_y; \\
K_{yp} &= k_{yp}; & m_{heli} &= 1.326; \\
K_{py} &= k_{py}; & l_{cm} &= 0.0168 \\
J_{eq-p} &= J_p; & l &= 0.185; \\
J_{eq-y} &= J_p + 0.0039;
\end{aligned}$$

If you want to you can also enter the numbers that you obtained from our previous helicopter identification experiment in the MATLAB file:

`setup_heli_PID_parameters.m`, or simply use the numbers that we have provided. (If some of the parameters that you obtained in the helicopter identification experiment were wrong, then the PID controller that you simulate in the pre-lab might not work on the actual helicopter experiment. However, your parameters might be more accurate than our parameters. In this case, your PID controller will actually work better.) Now run the MATLAB file: `setup_heli_PID_parameters.m`. Then enter your numbers from the pitch and yaw PID controllers in the following matrix

$$Ki = \begin{bmatrix} \gamma_{pp} & 0 & \gamma_{dp} & 0 & \gamma_{ip} & 0 \\ 0 & \gamma_{py} & 0 & \gamma_{dy} & 0 & \gamma_{iy} \end{bmatrix}.$$

Run the Simulink file `heli_model.mdl`. This Simulink file tests how well your controller follows a square wave. Now use the rules in Table 1, to adjust the PID gains for both the pitch and yaw controllers to find a more suitable controller for the helicopter. Note that those rules are for single input single output system. In our case, we have two inputs and two outputs system, and thus, those rules may or may not apply. All the gains in your PID controller should be under 40. Bring your best controller gains to the Lab with you.

7 Pre-lab: Due at the beginning of the experiment

The following is due at the beginning of the experiment. You will not be allowed to perform the experiment if the pre-lab is not completed.

- (i) Compute the pitch PID compensator C_p for G_p such that the closed loop poles are either $\{-1, -2, -3\}$ or $\{-1, -3 \pm 3i\}$. You can chose either set of poles but not both.
- (ii) Compute the yaw PID compensator C_y for G_y such that the closed loop poles are $-1/2$ and have natural frequency $\omega_n = 1$ rad/s and a damping ratio of $\zeta = 0.7$. (*You may use any technique to achieve this. You may also refer to Appendix 12.6 on page 17 for help, using $\gamma_{iy} = 1$ initially*).
- (iii) Use the given Simulink and setup files to simulate the system. For a better performance, adjust these gains using Table 1 as a guideline.
- (iv) Bring the best gain that you achieve in your simulation, that is, bring

$$Ki = \begin{bmatrix} \gamma_{pp} & 0 & \gamma_{dp} & 0 & \gamma_{ip} & 0 \\ 0 & \gamma_{py} & 0 & \gamma_{dy} & 0 & \gamma_{iy} \end{bmatrix}.$$

*The absolute values of all of your gains must be **less than forty**.*

- (v) Bring plots of your simulation results.

8 Part (i): PID testing and experimental improvement

In this part you will test your PID controller from the pre-lab on the actual helicopter. The idea is that you will implement your PID controller and then modify the gains to improve the performance. You must select a PID controller that you experimentally determine works the best on the helicopter.

8.1 The lab steps to test and improve the PID controller

Caution: Depending on your controller, the closed loop system may be unstable. Anytime you click start be ready to hit the stop button if you notice any evidence of instability, that is, oscillations with increasing magnitude.

- (i) Open MATLAB and change the directory to `Desktop\AAE364L\Sec#\PID`
- (ii) Run the setup file `setup_heli_PID_parameters.m`

- (iii) Enter K_i corresponding to the PID designed in your pre-lab, that is, type $K_i = [\dots]$ and fill in the values from Step (iv) of the pre-lab.
- (iv) Open the Simulink model `heli_PID` and set the simulation time to 99 seconds. Select **Tools - External Mode Control Panel - Signal & Triggering**, and change the duration to 50,000.
- (v) Set the Joystick/Computer Selector value to 2.
- (vi) Open the **Scopes** subsystem and, from there, the **Pitch+Reference** scope. **In order to properly record your data, select Parameters. Deselect the "Limit to 5000 data points" check-box. Select Save data to workspace. Enter an appropriate variable name, in this case *theta*. Make sure the format is "Structure with time". Repeat with the Yaw+Reference scope (*psi*), and the Voltage scope (*voltage*).**
- (vii) Click **Quarc - Build**.
- (viii) In the **Computer Generated Input** block, set the gain of the **sine wave pitch** signal generator to 15 degrees and set all the other gains to 0.
- (ix) Select **Simulation - Connect to Target**, and **Quarc - Start**. Observe the pitch and yaw responses for at least 30 seconds and stop the simulation.
- (x) In the **Computer Generated Input** block, set the gain of the **sine wave yaw** signal generator to 30 degrees and set all the other gains to 0. Repeat Step (ix).
- (xi) Tune your PID gains, that is, enter a new K_i in MATLAB, and repeat Steps (viii)-(x).
- (xii) When you are pleased with your experimental PID controller, **save at least 90 seconds** of pitch, yaw, and voltage data and the gain matrix K_i in MATLAB via (**File - Save - Save as Mat file**).
Make sure to save data for *both* **sine wave pitch** (Step (viii)) and **sine wave yaw** (Step (x)) input signals.

9 Part (ii): Closed loop flight testing

In this part of the lab you can test your best PID controller by flying the helicopter with the joystick and trying to follow a path.

9.1 The lab steps to closed loop flight testing

- (i) Re-open the Simulink model `heli_PID.mdl` if it is closed.
- (ii) Change the Joystick/Computer Selector to 1.
- (iii) Open the **Pitch+Reference** scope, right-click on the *y*-axis, select **Axes Properties**, and set the fixed range to $[-30, 30]$. Repeat for the **Yaw+Reference** scope, and set the fixed range to $[-90, 90]$.

- (iv) Set the simulation time to 119 seconds. Select **Tools - External Mode Control Panel - Signal & Triggering**, and change the duration to 60,000. Click **Quarc - Build**.
- (v) In the **Computer Generated Input** block, set the gain of the **sine wave pitch** signal generator to 15 degrees and the other gains to 0.
- (vi) Select **Simulation - Connect to Target**, and **Quarc - Start**. The green line in the scopes represent the actual pitch and yaw of the helicopter. The red lines are the reference signals. You need to track the reference signal by flying the helicopter with the joystick.
- (vii) Repeat Steps (v) and (vi), but instead change the **sine wave yaw** gain to 30 degrees and the other gains to 0.
- (viii) Saving this data is not required.

10 Part(iii): Open loop flight

In this part of the experiment you will try to fly the helicopter with no control system in place. In other words, you will try to fly the helicopter ‘open loop’. As expected, it will be difficult to fly the helicopter with no control system.

10.1 The lab steps to open loop flight

- (i) Open the file **heli_Open_Loop.m** and **heli_Open_Loop.mdl** in MATLAB.
- (ii) Set the simulation time to 119 seconds. Select **Tools - External Mode Control Panel - Signal & Triggering**, and change the duration to 60,000. Click **Quarc - Build**.
- (iii) Open the **Pitch+Reference** scope, right-click on the *y*-axis, select **Axes Properties**, and set the fixed range to $[-30, 30]$. Repeat for the **Yaw+Reference** scope, and set the fixed range to $[-90, 90]$.
- (iv) In the **Computer Generated Input** block, set the gain of the **sine wave pitch** signal generator to 15 degrees and the other gains to 0.
- (v) Select **Simulation - Connect to Target**, and **Quarc - Start**. The green line in the scopes represent the actual pitch and yaw of the helicopter. The red lines are the reference signals. You need to track the reference by flying the helicopter with the joystick.
- (vi) Repeat Steps (iv) and (v), but instead change the **sine wave yaw** gain to 30 degrees and the other gains to 0.
- (vii) Saving data is not required.

11 The laboratory report

- (a) Hand in the plots from Part (i) for your best controller. For each input signal, generate three plots and put them on **the same page**. These plots must be (i) the pitch and pitch reference, (ii) the yaw and yaw reference, and (iii) the two voltages. All the plots must occupy two pages in total.
- (b) Analyze the voltages generated by the controller and determine if there was saturation. Discuss the effect of saturation in the helicopter response (whether you had saturation or not).
- (c) Write a section describing what is good and bad about the controller that you have designed. You may want to include settling time and overshoot in your discussion about why you think this is a good controller. Describe how you can improve this controller if you had more time. You can even hand in more plots to convince the reader that you have designed a good controller.
- (d) Discuss the need for the integrators in the pitch and yaw loops.
- (e) For the last part of the lab report, discuss what you learned about flying the helicopter with the joystick using the PID controller and open loop.

12 Appendix

**The following material is included for your reference only.
No task is required of you from this point on.**

12.1 Solution for equilibrium input voltages

Recall that the equations of motion of the helicopter model are given in (8). Let $\theta = \theta_d$ and $\psi = \psi_d$ be an equilibrium position of this system of equations. At equilibrium, (8) reduced to

$$\begin{bmatrix} mg(h \sin(\theta_d) + R_c \cos(\theta_d)) \\ 0 \end{bmatrix} = \begin{bmatrix} lk_{pp} & k_{py} \\ k_{yp} \cos(\theta_d) & lk_{yy} \cos(\theta_d) \end{bmatrix} \begin{bmatrix} v_{pe} \\ v_{ye} \end{bmatrix}. \quad (14)$$

We see from (14) that ψ_d is unspecified, and thus ψ_d can have any value. By solving (14), we arrive at

$$\begin{aligned} v_{pe} &= \frac{mg(h \sin(\theta_d) + R_c \cos(\theta_d))lk_{yy}}{lk_{pp}lk_{yy} - k_{py}k_{yp}} \\ v_{ye} &= \frac{-mg(h \sin(\theta_d) + R_c \cos(\theta_d))k_{yp}}{lk_{pp}lk_{yy} - k_{py}k_{yp}}. \end{aligned} \quad (15)$$

12.2 PROOF TO SHOW THAT γ_d AND γ_p ARE REAL.

Recall that the inverse of a 2×2 matrix is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ab - cd} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \quad (16)$$

Notice that for any complex number γ , we have $\gamma - \bar{\gamma} = 2i\Im\gamma$ where \Im denotes the imaginary part of a complex number. Using this we see that the solution to (29) is given by

$$\begin{aligned} \begin{bmatrix} \gamma_d \\ \gamma_p \end{bmatrix} &= -\frac{1}{|\lambda z|^2(\lambda - \bar{\lambda})} \begin{bmatrix} \bar{\lambda}\bar{z} & -\lambda z \\ -\bar{\lambda}^2\bar{z} & \lambda^2 z \end{bmatrix} \begin{bmatrix} \lambda + \gamma_i z \\ \bar{\lambda} + \gamma_i \bar{z} \end{bmatrix} \\ &= -\frac{1}{2i\Im\lambda|\lambda z|^2} \begin{bmatrix} |\lambda|^2\bar{z} + \bar{\lambda}\gamma_i|z|^2 - |\lambda|^2 z - \lambda\gamma_i|z|^2 \\ -|\lambda|^2\bar{\lambda}\bar{z} - \gamma_i\bar{\lambda}^2|z|^2 + |\lambda|^2\lambda z + \gamma_i\lambda^2|z|^2 \end{bmatrix} \\ &= -\frac{1}{2i\Im\lambda|\lambda z|^2} \begin{bmatrix} -2i|\lambda|^2\Im z - 2i\gamma_i|z|^2\Im\lambda \\ 2i|\lambda|^2\Im(\lambda z) + 2i\gamma_i|z|^2\Im(\lambda^2) \end{bmatrix} \\ &= \frac{1}{|\lambda z|^2\Im\lambda} \begin{bmatrix} |\lambda|^2\Im z + \gamma_i|z|^2\Im\lambda \\ -|\lambda|^2\Im(\lambda z) - \gamma_i|z|^2\Im(\lambda^2) \end{bmatrix}. \end{aligned}$$

This readily implies that

$$\begin{bmatrix} \gamma_d \\ \gamma_p \end{bmatrix} = \frac{1}{|\lambda z|^2\Im\lambda} \begin{bmatrix} |\lambda|^2\Im z + \gamma_i|z|^2\Im\lambda \\ -|\lambda|^2\Im(\lambda z) - \gamma_i|z|^2\Im(\lambda^2) \end{bmatrix}. \quad (17)$$

Therefore the gains γ_d and γ_p are real numbers. To compute γ_d and γ_p , one can use either (30) of (17).

12.3 Linear stability analysis

In this section, we will provide a stability analysis on what actually happens to the linearized equations of motion when one combines the PID pitch and yaw controllers. Recall that the linearized equations of motion about level flight are determined by

$$\begin{aligned}
J_p \ddot{\theta} + c_p \dot{\theta} + mgh\theta &= lk_{pp}\delta v_p + k_{py}\delta v_y \\
(J_p + J_{shaft})\ddot{\psi} + c_y \dot{\psi} &= lk_{yy}\delta v_y + k_{yp}\delta v_p \\
\delta v_p &= v_p - v_{pe} \\
\delta v_y &= v_y - v_{ye}.
\end{aligned} \tag{18}$$

By taking the Laplace transform with all the initial conditions set equal to zero, we obtain

$$\begin{aligned}
\begin{bmatrix} \Theta(s) \\ \Psi(s) \end{bmatrix} &= \begin{bmatrix} \frac{a_{11}}{d} & \frac{a_{12}}{d} \\ \frac{a_{21}}{q} & \frac{a_{22}}{q} \end{bmatrix} \begin{bmatrix} \delta V_p(s) \\ \delta V_y(s) \end{bmatrix} \\
d(s) &= s^2 + \frac{c_p}{J_p}s + \frac{mgh}{J_p} \\
q(s) &= s^2 + \frac{c_y}{J_p + J_{shaft}}s \\
a_{11} &= \frac{lk_{pp}}{J_p} \quad \text{and} \quad a_{12} = \frac{k_{py}}{J_p} \\
a_{21} &= \frac{k_{yp}}{J_p + J_{shaft}} \quad \text{and} \quad a_{22} = \frac{lk_{yy}}{J_p + J_{shaft}}.
\end{aligned} \tag{19}$$

It is emphasized, that in our helicopter problem a_{12} and a_{21} are very small compare to a_{11} and a_{22} . This plays an important role in our stability analysis of the linearized system. Our pitch and yaw PID controllers are of the form:

$$\begin{aligned}
\delta V_p(s) &= \frac{N_p(s)}{s} (R_p(s) - \Theta(s)) \\
N_p(s) &= \gamma_{dp}s^2 + \gamma_{pp}s + \gamma_{ip} \\
\delta V_y(s) &= \frac{N_y(s)}{s} (R_y(s) - \Psi(s)) \\
N_y(s) &= \gamma_{dy}s^2 + \gamma_{py}s + \gamma_{iy}.
\end{aligned} \tag{20}$$

As expected, $R_p(s) = \mathcal{L}r_p$ is the reference signal for the pitch, while $R_y(s) = \mathcal{L}r_y$ is the reference signal corresponding to the yaw. Moreover, $N_p(s)/s = C_p(s)$ is the PID controller for the pitch, and $N_y(s)/s = C_y(s)$ is the PID controller for the yaw. By substituting (20) into (19), we arrive at

$$\begin{bmatrix} \Theta(s) \\ \Psi(s) \end{bmatrix} = \begin{bmatrix} \frac{a_{11}}{d} & \frac{a_{12}}{d} \\ \frac{a_{21}}{q} & \frac{a_{22}}{q} \end{bmatrix} \begin{bmatrix} \frac{N_p}{s} & 0 \\ 0 & \frac{N_y}{s} \end{bmatrix} \begin{bmatrix} R_p(s) - \Theta(s) \\ R_y(s) - \Psi(s) \end{bmatrix}.$$

This readily implies that

$$\begin{bmatrix} 1 + \frac{a_{11}N_p}{s^d} & \frac{a_{12}N_y}{s^d} \\ \frac{a_{21}N_p}{sq} & 1 + \frac{a_{22}N_y}{sq} \end{bmatrix} \begin{bmatrix} \Theta(s) \\ \Psi(s) \end{bmatrix} = \begin{bmatrix} \frac{a_{11}N_p}{s^d} & \frac{a_{12}N_y}{s^d} \\ \frac{a_{21}N_p}{sq} & \frac{a_{22}N_y}{sq} \end{bmatrix} \begin{bmatrix} R_p(s) \\ R_y(s) \end{bmatrix}.$$

Multiplying the first row by sd , and multiplying the second row by sq , we arrive at

$$\begin{bmatrix} sd + a_{11}N_p & a_{12}N_y \\ a_{21}N_p & sq + a_{22}N_y \end{bmatrix} \begin{bmatrix} \Theta(s) \\ \Psi(s) \end{bmatrix} = \begin{bmatrix} a_{11}N_p & a_{12}N_y \\ a_{21}N_p & a_{22}N_y \end{bmatrix} \begin{bmatrix} R_p(s) \\ R_y(s) \end{bmatrix}.$$

Recall that the inverse of a 2×2 matrix is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Using this fact, we arrive at

$$\begin{aligned} \begin{bmatrix} \Theta \\ \Psi \end{bmatrix} &= \frac{1}{\xi(s)} \begin{bmatrix} (sq + a_{22}N_y) & -a_{12}N_y \\ -a_{21}N_p & (sd + a_{11}N_p) \end{bmatrix} \begin{bmatrix} a_{11}N_p & a_{12}N_y \\ a_{21}N_p & a_{22}N_y \end{bmatrix} \begin{bmatrix} R_p \\ R_y \end{bmatrix} \\ \xi(s) &= (sd + a_{11}N_p)(sq + a_{22}N_y) - a_{21}a_{12}N_pN_y. \end{aligned} \quad (21)$$

In other words, the input output equation in frequency domain is given by

$$\boxed{\begin{bmatrix} \Theta(s) \\ \Psi(s) \end{bmatrix} = F(s) \begin{bmatrix} R_p(s) \\ R_y(s) \end{bmatrix}} \quad (22)$$

where $F(s)$ is the 2×2 matrix transfer function from the vector reference signal $[r_p \ r_y]^tr$ to the vector output $[\theta \ \psi]^tr$ is given by

$$\boxed{F = \frac{1}{\xi(s)} \begin{bmatrix} (sq + a_{22}N_y) & -a_{12}N_y \\ -a_{21}N_p & (sd + a_{11}N_p) \end{bmatrix} \begin{bmatrix} a_{11}N_p & a_{12}N_y \\ a_{21}N_p & a_{22}N_y \end{bmatrix}} \quad (23)$$

while $\xi(s)$ is the characteristic equation of this system given by

$$\boxed{\xi(s) = (sd(s) + a_{11}N_p(s))(sq(s) + a_{22}N_y(s)) - a_{21}a_{12}N_p(s)N_y(s)} \quad (24)$$

It is noted that F is a 2×2 matrix whose entries are transfer function of McMillan degree of at most 6. In other words, the entries of F consists of proper rational functions such that the degree of the denominator is at most 6. Finally, it is noted that F is stable if and only if the sixth order polynomial $\xi(s)$ is stable.

Now let us return to our PID design procedure. Recall that we designed a pitch PID controller $C_p = N_p/s$ from the pitch reference signal r_p to the pitch angle θ for the open loop transfer function $G_p = a_{11}/d$. In this case, the closed loop transfer function is given by

$$\frac{\Theta(s)}{R_p} = \frac{C_p G_p}{1 + C_p G_p} = \frac{a_{11}N_p}{sd + a_{11}N_p}.$$

The poles of the closed loop transfer function $\frac{C_p G_p}{1 + C_p G_p}$ are the zeros of the third order polynomial $sd + a_{11}N_p$. So the closed loop transfer function $\frac{C_p G_p}{1 + C_p G_p}$ is stable if and only if the polynomial $sd + a_{11}N_p$ is stable. (Recall that a polynomial $p(s)$ is *stable* if all the zeros of $p(s)$ are contained in the open left half plane $\{s \in \mathbb{C} : \Re s < 0\}$.) In our design procedure

we placed the poles of $\frac{C_p G_p}{1+C_p G_p}$, or equivalently, the zeros of $sd + a_{11}N_p$ in the open left hand plane to meet a pitch performance objective.

Next we designed a yaw PID controller $C_y = N_y/s$ from the yaw reference signal r_y to the yaw angle ψ for the open loop transfer function $G_y = a_{22}/q$. In this case, the closed loop transfer function is given by

$$\frac{\Psi(s)}{R_y} = \frac{C_y G_y}{1 + C_y G_y} = \frac{a_{22}N_y}{sq + a_{22}N_y}.$$

The poles of the closed loop transfer function $\frac{C_y G_y}{1+C_y G_y}$ are the zeros of the third order polynomial $sq + a_{22}N_y$. In particular, the closed loop transfer function $\frac{C_y G_y}{1+C_y G_y}$ is stable if and only if the polynomial $sq + a_{22}N_y$ is stable. In our design procedure we placed the poles of $\frac{C_y G_y}{1+C_y G_y}$, or equivalently, the zeros of $sq + a_{22}N_y$ at certain points in the open left hand plane to meet a yaw performance objective.

Now observe that the poles of the closed loop transfer function F in (23) for the whole system are given by the zeros of

$$\xi(s) = (sd + a_{11}N_p)(sq + a_{22}N_y) - a_{21}a_{12}N_pN_y.$$

In our particular problem, $a_{21}a_{12}$ is very small. So the polynomial

$$\xi(s) \approx (sd + a_{11}N_p)(sq + a_{22}N_y).$$

In our design procedure, $(sd + a_{11}N_p)$ and $(sq + a_{22}N_y)$ are both stable polynomials. Because $a_{21}a_{12}$ is small, we hope that $\xi(s)$ is also stable. However, there is no guarantee that ξ is stable, or equivalently, the closed loop transfer function F is stable. In fact, it is well known that a small perturbation of the coefficients of a polynomial can have a drastic effect on the roots of that polynomial. In other words, a small perturbation can make a stable polynomial unstable. Putting these concerns aside for the moment, it turns out that our PID design procedure does not make our closed loop system unstable.

Now assume that the closed loop system is stable, that is, $\xi(s)$ is a stable polynomial. Now consider the reference pitch angle $r_p = \theta_d$ and reference yaw angle $r_y = \psi_d$. By employing the final value theorem in (21), we arrive at

$$\begin{aligned} \begin{bmatrix} \theta_\infty \\ \psi_\infty \end{bmatrix} &= \lim_{t \rightarrow \infty} \begin{bmatrix} \theta(t) \\ \psi(t) \end{bmatrix} = F(0) \begin{bmatrix} \theta_d \\ \psi_d \end{bmatrix} \\ &= \frac{1}{\xi(0)} \begin{bmatrix} a_{22}\gamma_{yi} & -a_{12}\gamma_{yi} \\ -a_{21}\gamma_{pi} & a_{11}\gamma_{pi} \end{bmatrix} \begin{bmatrix} a_{11}\gamma_{pi} & a_{12}\gamma_{yi} \\ a_{21}\gamma_{pi} & a_{22}\gamma_{yi} \end{bmatrix} \begin{bmatrix} \theta_d \\ \psi_d \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_d \\ \psi_d \end{bmatrix} = \begin{bmatrix} \theta_d \\ \psi_d \end{bmatrix}. \end{aligned}$$

In other words, the steady state pitch angle $\theta_{ss} = \theta_\infty = \theta_d$ and the steady state yaw angle $\psi_{ss} = \psi_\infty = \psi_d$. Therefore if the closed loop system is stable, then the PID control design will fly the helicopter to the desired pitch angle θ_d and desired yaw angle ψ_d .

Note: Following parts are only for graduate students, however, all students can do them with the permission of the TA.

12.4 Steady state error for a ramp with a PID controller

Consider the open loop transfer function G and the PID compensator

$$C(s) = \gamma_p + \gamma_d s + \frac{\gamma_i}{s}.$$

The transfer function from the reference signal $r(t)$ to the error $e(t)$ is given by

$$\frac{E}{R} = \frac{1}{1 + C(s)G(s)} = \frac{s}{s + (\gamma_d s^2 + \gamma_p s + \gamma_i)G(s)}.$$

Assume that γ_p , γ_d and γ_i are chosen such that the closed loop system is stable, or equivalently, all the zeros of $1 + C(s)G(s)$ are contained in $\{s \in \mathbb{C} : \Re s < 0\}$. Consider a ramp input $r(t) = \alpha t$ where α is a constant. Moreover, assume that $G(0) = G(s)|_{s=0}$ is nonzero. Then the steady state error $e_{ss}(t)$ due to this ramp is the constant function given by

$$e_{ss}(t) = \frac{\alpha}{\gamma_i G(0)} \quad (\text{where } r(t) = \alpha t). \quad (25)$$

PROOF. Since $R(s) = \alpha/s^2$ is the Laplace transform of $r(t) = \alpha t$, we have

$$E(s) = \frac{1}{(1 + C(s)G(s))} \frac{\alpha}{s^2} = \frac{\alpha}{(s + (\gamma_d s^2 + \gamma_p s + \gamma_i)G(s))s}.$$

According to the final value theorem

$$\begin{aligned} e_{ss}(t) &= \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) \\ &= \frac{\alpha}{s + (\gamma_d s^2 + \gamma_p s + \gamma_i)G(s)} \Big|_{s=0} = \frac{\alpha}{\gamma_i G(0)}. \end{aligned}$$

Therefore the steady state error is given by (25).

Remark. If $G(0) = 0$, then the steady state error $e_{ss}(t)$ is infinite.

12.5 Ramp specifications for a PID controller

As before, consider the open loop transfer function G and the PID compensator

$$C(s) = \gamma_p + \gamma_d s + \frac{\gamma_i}{s}.$$

Assume that $G(0) \neq 0$. A ramp error problem is to find a PID controller such that the following two conditions hold:

- (i) For a ramp $r(t) = \alpha t$, the corresponding steady state error $e_{ss}(t) = \beta$ where β is a specified constant.
- (ii) The closed loop system $\frac{CG}{1+CG}$ is stable.

SOLUTION. The specifications in Part (i), imply that

$$\gamma_i = \frac{\alpha}{\beta G(0)} \quad (\text{where } r(t) = \alpha t \text{ and } e_{ss}(t) = \beta). \quad (26)$$

Then for this specified γ_i , find γ_p and γ_d such that the closed loop system is stable. If there is no γ_p and γ_d such that the closed loop system is stable, then one has to change β in Part (i). If there is no γ_i , γ_p and γ_d such that the closed loop system is stable, then this design method fails.

To obtain γ_i in equation (26), recall that

$$e_{ss}(t) = \frac{\alpha}{\gamma_i G(0)} \quad (\text{where } r(t) = \alpha t).$$

The specifications in Part (i), means that $\beta = e_{ss}(t)$. In other words,

$$\beta = \frac{\alpha}{\gamma_i G(0)}.$$

Therefore the integral gain γ_i is given by (26).

12.6 Pole placement with γ_i specified.

As before, consider the closed loop system in Figure 1 where $G(s)$ is the open loop plant and $C(s) = \gamma_i/s + \gamma_d s + \gamma_p$ is a PID controller. Recall that the closed loop system is determined by

$$\begin{aligned} \frac{Y(s)}{R(s)} &= \frac{C(s)G(s)}{1 + C(s)G(s)} = \frac{(\gamma_d s^2 + \gamma_p s + \gamma_i)G(s)}{s + (\gamma_d s^2 + \gamma_p s + \gamma_i)G(s)} \\ \frac{E(s)}{R(s)} &= \frac{1}{1 + C(s)G(s)} = \frac{s}{s + (\gamma_d s^2 + \gamma_p s + \gamma_i)G(s)}. \end{aligned} \quad (27)$$

Moreover, let us assume that the integral gain γ_i is specified. Then our problem is to specify the proportional gain γ_p and derivative gain γ_d to place two poles of the closed loop corresponding to a specified damping ratio ζ and natural frequency ω_n . In other words, compute γ_p and γ_d such that the closed loop transfer function is of the form

$$\frac{Y(s)}{R(s)} = \frac{p(s)}{(s^2 + 2\zeta\omega_n s + \omega_n^2)q(s)}$$

where the polynomials p and $(s^2 + 2\zeta\omega_n s + \omega_n^2)q(s)$ have no common roots. This means that the poles of the closed loop system will contain the zeros of $s^2 + 2\zeta\omega_n s + \omega_n^2$. The idea behind this design procedure is that hopefully the other poles of the closed loop system are dominated by the roots of $s^2 + 2\zeta\omega_n s + \omega_n^2$, and the closed loop system behaves like a second order system whose denominator is $s^2 + 2\zeta\omega_n s + \omega_n^2$. In particular, one can choose, ζ and ω_n for a specified percent overshoot or settling time.

To proceed with our analysis, assume that ζ and ω_n are specified. To obtain the corresponding PID gains γ_d and γ_p , observe that

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = (s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2) = (s - \lambda)(s - \bar{\lambda})$$

where

$$\lambda = -\zeta\omega_n + i\omega_n\sqrt{1-\zeta^2} \quad \text{and} \quad \bar{\lambda} = -\zeta\omega_n - i\omega_n\sqrt{1-\zeta^2}. \quad (28)$$

In other words, λ and $\bar{\lambda}$ are the roots of $s^2 + 2\zeta\omega_n s + \omega_n^2$. So our problem is given a specified integral gain γ_i and two complex numbers λ and $\bar{\lambda}$ in $\{s \in \mathbb{C} : \Re(s) < 0\}$ where $\Im(\lambda) \neq 0$, then find the proportional gain γ_p and derivative gain γ_d such that λ and $\bar{\lambda}$ are contained in the poles of the closed loop system.

To solve this problem, set $z = G(\lambda) = G(s)|_{s=\lambda}$. In general, z is a complex number. Since $G(s)$ is a rational function consisting of polynomials with real coefficients, we see that $G(\bar{\lambda}) = \overline{G(\lambda)} = \bar{z}$. Notice that λ is a pole of the closed loop system if and only if λ is a zero of $s + (\gamma_d s^2 + \gamma_p s + \gamma_i)G(s)$, that is,

$$0 = [s + (\gamma_d s^2 + \gamma_p s + \gamma_i)G(s)]|_{s=\lambda} = \lambda + (\gamma_d \lambda^2 + \gamma_p \lambda + \gamma_i)z.$$

By taking the complex conjugate and using the fact that γ_i , γ_p and γ_d must be real, we arrive at

$$0 = \bar{\lambda} + (\gamma_d \bar{\lambda}^2 + \gamma_p \bar{\lambda} + \gamma_i)\bar{z}.$$

By converting the previous two equations to matrix form, we obtain

$$\begin{bmatrix} \lambda^2 z & \lambda z \\ \bar{\lambda}^2 \bar{z} & \bar{\lambda} \bar{z} \end{bmatrix} \begin{bmatrix} \gamma_d \\ \gamma_p \end{bmatrix} = - \begin{bmatrix} \lambda + \gamma_i z \\ \bar{\lambda} + \gamma_i \bar{z} \end{bmatrix}. \quad (29)$$

We claim that the solution to this system is real. It is emphasized that the gains γ_d and γ_p must be real or else we cannot implement our controller design. Therefore the gains that achieve a damping ratio of ζ and a natural frequency ω_n with specified integral gain γ_i are determined by

$$\begin{aligned} \begin{bmatrix} \gamma_d \\ \gamma_p \end{bmatrix} &= -\Re \left[\begin{bmatrix} \lambda^2 z & \lambda z \\ \bar{\lambda}^2 \bar{z} & \bar{\lambda} \bar{z} \end{bmatrix}^{-1} \begin{bmatrix} \lambda + \gamma_i z \\ \bar{\lambda} + \gamma_i \bar{z} \end{bmatrix} \right] \\ \lambda &= -\zeta\omega_n + i\omega_n\sqrt{1-\zeta^2} \\ z &= G(\lambda). \end{aligned} \quad (30)$$

Here we also computed the real part \Re of the final answer. We extracted the real part because MATLAB may introduce some small numerical errors. In other words, the numerical solution might contain a small imaginary part. So taking the real part eliminates these numerical errors. The proof to show that γ_d and γ_p are real is given in the appendix.

Remark. *It is emphasized that this design procedure does not guarantee that the closed loop system will be stable. It can happen that one can specify γ_i , ζ and ω_n . Then compute the appropriate gains γ_d and γ_p and the closed loop system $\frac{CG}{1+CG}$ is unstable. In this case, one has to modify γ_i , ζ and ω_n to try to ensure that the closed loop system is stable. If there are no γ_i , ζ and ω_n such that the closed loop system is stable, then this design method fails and one has implement other design techniques.*

An example. Consider the open loop transfer function given by

$$G(s) = \frac{2}{s^2 + 6s + 4}.$$

Assume that $\zeta = 0.7$, $\omega_n = 2$, and $\gamma_i = 4$. In this setting

$$\lambda = -\zeta\omega_n + i\omega_n\sqrt{1-\zeta^2} = -0.7 \times 2 + i2\sqrt{1-0.7^2} = -1.4 + 1.4283i.$$

In other words, $\lambda = -1.4 + 1.4283i$. Furthermore,

$$\begin{aligned} z &= G(\lambda) = \frac{2}{s^2 + 6s + 4} \Big|_{s=\lambda} = \frac{2}{\lambda^2 + 6\lambda + 4} \\ &= \frac{2}{(-1.4 + 1.4283i)^2 + 6(-1.4 + 1.4283i) + 4} \\ &= -0.2187 - 0.2232i. \end{aligned}$$

Thus $z = -0.2187 - 0.2232i$. By using MATLAB to compute γ_d and γ_p in (30), we arrive at

$$\gamma_d = -0.6 \quad \text{and} \quad \gamma_p = 2.8.$$

Therefore the corresponding PID compensator is given by

$$C(s) = \frac{-0.6s^2 + 2.8s + 4}{s}.$$

The closed loop system is given by

$$\frac{Y(s)}{R(s)} = \frac{CG}{1 + CG} = \frac{-1.2s^2 + 5.6s + 8}{s^3 + 4.8s^2 + 9.6s + 8}.$$

The poles of this system are -2 and $-1.4 \pm 1.4283i$. Hence the closed loop system is stable. Moreover, as expected λ and $\bar{\lambda}$ are contained in the poles for the closed loop system. So the poles for the closed loop system contain the damping ratio $\zeta = 0.7$ and natural frequency $\omega_n = 2$.

The MATLAB commands that we used to compute the closed loop transfer function are given by

- $C = \text{tf}([-0.6 \quad 2.8 \quad 4], [1 \quad 0])$
- $G = \text{tf}(2, [1 \quad 6 \quad 4])$
- $F = C * G / (1 + C * G)$
- $F = \text{minreal}(F)$.

Then F is the closed loop transfer function. The minreal command is used to eliminate the common poles and zeros in the MATLAB calculation of the closed loop transfer function F .

12.7 Ramp, damping ratio and natural frequency specifications for a PID controller

As before, consider the open loop transfer function G and the PID compensator

$$C(s) = \gamma_p + \gamma_d s + \frac{\gamma_i}{s}.$$

Assume that $G(0) \neq 0$. Find a PID controller such that the following conditions hold:

- (i) For a specified ramp reference input $r(t) = \alpha t$, the corresponding steady state error $e_{ss}(t) = \beta$, where β is a specified constant.
- (ii) The closed loop system $\frac{CG}{1+CG}$ has two poles corresponding to a specified damping ratio ζ and natural frequency ω_n .
- (iii) The closed loop system $\frac{CG}{1+CG}$ is stable.

SOLUTION. According to equation (26), the specifications in Part (i), imply that

$$\gamma_i = \frac{\alpha}{\beta G(0)} \quad (\text{where } r(t) = \alpha t \text{ and } e_{ss}(t) = \beta). \quad (31)$$

Then for this specified γ_i , the gains γ_p and γ_d are determined by equation (30). Finally, check to see if the closed loop system is stable. If the closed loop system not stable, then one has to change the steady state error β in Part (i) or ζ and ω_n in Part (ii), to see if the corresponding closed loop system is stable.

Remark. *Computing the constants γ_p and γ_d in this fashion does not guarantee that the closed loop system*

$$\frac{CG}{1+CG}$$

is stable. So you must check to see if the closed loop system $\frac{CG}{1+CG}$ is stable. If $\frac{CG}{1+CG}$ is not stable, then you have to keep changing γ_i , ζ and ω_n until $\frac{CG}{1+CG}$ is stable. If one cannot find a γ_i , γ_p and γ_d such that $\frac{CG}{1+CG}$ is stable, then this design method fails. In this case, one has to apply other control techniques.

An example. Consider the open loop transfer function given by

$$G(s) = \frac{2}{s^2 + 6s + 4}.$$

Assume that $r(t) = 2t$, and the corresponding steady state error $e_{ss}(t) = 1$, that is, $\alpha = 2$ and $\beta = 1$. Moreover, assume that $\zeta = 0.7$ and $\omega_n = 2$. Notice that $G(0) = 1/2$. So in this case, the integral gain

$$\gamma_i = \frac{\alpha}{\beta G(0)} = \frac{2}{1/2} = 4.$$

Hence $\gamma_i = 4$.

12.8 Ramp steady state error PID controller

As before, consider a pitch PID controller of the form,

$$C_p(s) = \gamma_{pp} + \gamma_{dp}s + \frac{\gamma_{ip}}{s}.$$

Compute an integral gain γ_{ip} such that the steady state error due to the ramp $r(t) = 2t$ is $e_{ss}(t) = \beta = 1$. The closed loop system from a pitch reference signal r_p to the pitch error $e = r_p - \theta$ is given by

$$\frac{E}{R_p} = \frac{1}{1 + C_p(s)G_p(s)} = \frac{s}{s + (\gamma_{dp}s^2 + \gamma_{pp}s + \gamma_{ip})G_p(s)}.$$

According to (31), the integral gain to achieve this steady state error is computed by

$$\gamma_{ip} = \frac{\alpha}{\beta G_p(0)}.$$

Since $e_{ss}(t) = 1$ and $\alpha = 2$, we arrive at

$$\gamma_{ip} = \frac{2}{G_p(0)}.$$

For this specified γ_{ip} , find the gains γ_{pp} and γ_{dp} such that the closed loop system has a pairs of poles corresponding to a damping ratio of $\zeta = 0.7$ and a natural frequency of $\omega_n = 5$. Check to see if the closed loop system is stable.

12.9 A PID controller for the yaw angle with $\gamma_{iy} = 1$.

As before, consider a yaw PID controller of the form:

$$C_y(s) = \gamma_{py} + \gamma_{dy}s + \frac{\gamma_{iy}}{s}.$$

Notice that $G_y(0) = \infty$. Hence according to the steady state error analysis in equation (31), any positive γ_{iy} we choose will result in zero steady state error due to a ramp. However, we can not choose $\gamma_{iy} = 0$ because the steady state error will not go to zero. So assume that the integral gain $\gamma_{iy} = 1$. The closed loop system from a reference signal r_y to the yaw error $e = r_y - \psi$ is determined by

$$\frac{E}{R_y} = \frac{1}{1 + C_y(s)G_y(s)} = \frac{s}{s + (\gamma_{dy}s^2 + \gamma_{py}s + 1)G_y(s)}.$$

For $\gamma_{iy} = 1$, find γ_{py} and γ_{dy} such that the the closed loop system has a damping ration of $\zeta = 0.7$ and a natural frequency of $\omega_n = 1$. Check to see if the closed loop system is stable. (In some applications, one chooses the natural frequency for the pitch five times larger than the natural frequency for the yaw.)

As before, run the file: `setup_heli_PID_parameters.m`. Then enter your numbers from the pitch and yaw PID controllers you just computed in the following matrix

$$Ki = \begin{bmatrix} \gamma_{pp} & 0 & \gamma_{dp} & 0 & \gamma_{ip} & 0 \\ 0 & \gamma_{py} & 0 & \gamma_{dy} & 0 & \gamma_{iy} \end{bmatrix}.$$

Run the Simulink file: `heli_model.mdl`. This Simulink file tests how well your controller follows a square wave. Now use the rules in Table 1, to adjust the PID gains for both the pitch and yaw controllers to find a better controller for the helicopter. Recall that those rules are for single input single output system, and they may or may not apply here. All the gains in your PID controller should be under 40. Bring this controller to the Lab with you.

12.10 Ramp specification pre-lab (optional)

- (i) Compute the pitch PID compensator C_p for G_p such that the ramp input of $r(t) = 2t$ degrees/sec yields the steady state error $e_{ss} = 1$ degree and the poles of the closed loop system $\frac{C_p G_p}{1 + C_p G_p}$ have natural frequency $\omega_n = 5$ rad/s and the damping ratio $\zeta = 0.7$.

- (ii) Compute the poles of the closed loop system $\frac{C_p G_p}{1+C_p G_p}$. Is this closed loop system stable?
- (iii) Now assume that the integral gain $\gamma_{iy} = 1$ for the yaw PID controller. Find γ_{py} and γ_{dy} such that the poles of the closed loop system $\frac{C_y G_y}{1+C_y G_y}$ have natural frequency $\omega_n = 1$ rad/s and the damping ratio $\zeta = 0.7$.
- (iv) Compute the poles of the closed loop system $\frac{C_y G_y}{1+C_y G_y}$. Is this closed loop system stable?
- (v) Using the Simulink and setup files provided, adjust these gains by the method in Table 1 to achieve a better performance.
- (vi) Bring the best gain that you achieve in your simulation, that is, bring

$$Ki = \begin{bmatrix} \gamma_{pp} & 0 & \gamma_{dp} & 0 & \gamma_{ip} & 0 \\ 0 & \gamma_{py} & 0 & \gamma_{dy} & 0 & \gamma_{iy} \end{bmatrix}.$$

It is emphasized that the absolute values of all of your gains must be less than forty.