Geometry

(Many slides adapted from Octavia Camps and Amitabh Varshney)

Much of material in Appendix A

Goals

- Represent points, lines and triangles as column vectors.
- Represent motion as matrices.
- Move geometric objects with matrix multiplication.
- Refresh memory about geometry and linear algebra

Vectors

- Ordered set of numbers: (1,2,3,4)
- Example: (*x*,*y*,*z*) coordinates of pt in space.

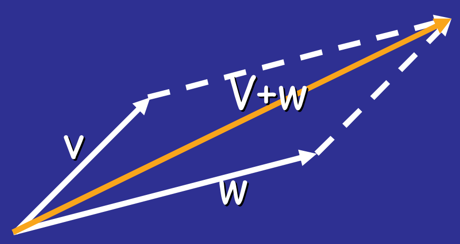
$$v = (x_1, x_2, \dots, x_n)$$

$$\|v\| = \sqrt{\sum_{i=1}^{n} x_i^2}$$

$$\text{If } \|v\| = 1, v \text{ is a unit vector}$$

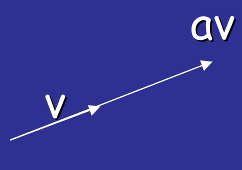
Vector Addition

$$\mathbf{v} + \mathbf{w} = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

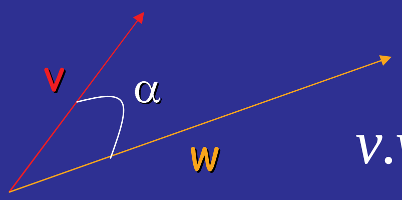


Scalar Product

$$a\mathbf{v} = a(x_1, x_2) = (ax_1, ax_2)$$



Inner (dot) Product



$$v.w = (x_1, x_2).(y_1, y_2) = x_1 y_1 + x_2.y_2$$

The inner product is a SCALAR!

$$v.w = (x_1, x_2).(y_1, y_2) = \|v\| \cdot \|w\| \cos\alpha$$

$$v.w = 0 \Leftrightarrow v \perp w$$

First, we note that if we scale a vector, we scale its inner product. That is, <sv,w> = s<v,w>. This follows pretty directly from the definition.

So from now on, we can assume that w, and maybe v are unit vector. Then, as an example, we can consider the case where w = (1,0). It follows from the definition of cosine that <v,w> = ||v||cos(alpha). We can also see that taking <v,(1,0)> and <v,(0,1)> produces the (x,y) coordinates of v. That is, if (1,0) and (0,1) are an orthonormal basis, taking inner products with them gives the coordinates of a point relative to that basis. This is why the inner product is so useful. We just have to show that this is true for any orthonormal basis, not just (1,0) and (0,1).

How do we prove these properties of the inner product? Let's start with the fact that orthogonal vectors have 0 inner product. Suppose one vector is (x,y), and WLOG x,y>0. Then, if we rotate that by 90 degrees counterclockwise, we'll get (y, -x). Rotating the vector is just like rotating the coordinate system in the opposite direction. And (x,y)*(y,-x) = xy – yx = 0.

Next, note that v*w = (v*w)/(||v||||w||) * ||v||||w|| This means that if we can show that when v and w are unit vectors v*w = cos alpha, then it will follow that in general v*w = ||v|| ||w|| cos alpha. So suppose v and w are unit vectors.

Next, note that if w1 + w2 = w, then v*w = v*(w1+w2) = v*w1 + v*w2. For any w, we can write it as the sum of w1+w2, where w1 is perpendicular to v, and w2 is in the same direction as v. So v*w1 = 0. v*w2 = ||w2||, since v*w2/||w2|| = 1. Then, if we just draw a picture, we can see that cos alpha = ||w2|| = v*w2 = v*w.

Points

Using these facts, we can represent points.
Note:

(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)

x = (x,y,z).(1,0,0)   y = (x,y,z).(0,1,0)

z = (x,y,z).(0,0,1)

Lines

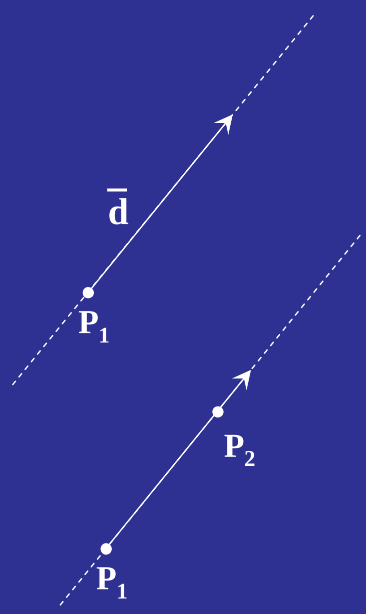
- Line: $y = mx + a$
- Line: sum of a point and a vector

  $\mathbf{P} = \mathbf{P_1} + \alpha\ \bar{\mathbf{d}}$

  (where $\bar{\mathbf{d}}$ is a column vector)
- Line: Affine sum of two points

  $\mathbf{P} = \alpha_1 \mathbf{P_1} + \alpha_2 \mathbf{P_2}$, where $\alpha_1 + \alpha_2 = 1$

  Line Segment: For $0 \leq \alpha_1, \alpha_2 \leq 1$, **P** lies between $\mathbf{P_1}$ and $\mathbf{P_2}$

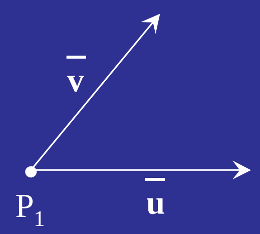


- Line: set of points equidistant from the origin in the direction of a unit vector. (a,b).(x,y) = -c.

Plane and Triangle

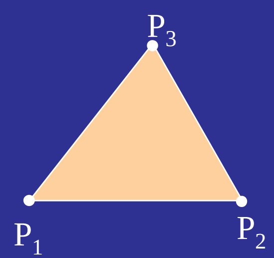
- Plane: sum of a point and two vectors

  $\mathbf{P} = \mathbf{P}_1 + \alpha\,\bar{\mathbf{u}} + \beta\,\bar{\mathbf{v}}$



  Plane: set of points equidistant from origin in direction of a vector.

- Triangle: Affine sum of three points with $\alpha_i \geq 0$



  $\mathbf{P} = \alpha_1\mathbf{P}_1 + \alpha_2\mathbf{P}_2 + \alpha_3\mathbf{P}_3,$

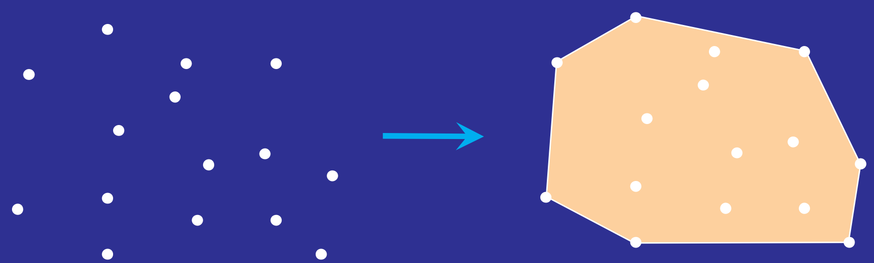
  where $\alpha_1 + \alpha_2 + \alpha_3 = 1$

  P lies between $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$

Generalizing ...

Affine Sum of arbitrary number of points: Convex Hull

$\mathbf{P} = \alpha_1\mathbf{P_1} + \alpha_2\mathbf{P_2} + \ldots + \alpha_n\mathbf{P_n}$, where $\alpha_1 + \alpha_2 + \ldots + \alpha_n = 1$ and $\alpha_i \geq 0$

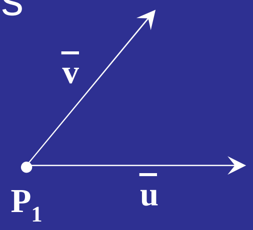


Normal of a Plane

Plane: sum of a point and two vectors

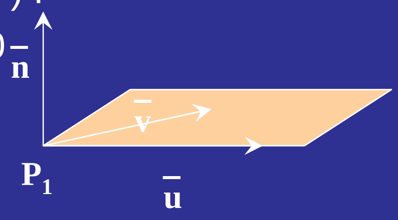
$\mathbf{P} = \mathbf{P}_1 + \alpha\,\bar{\mathbf{u}} + \beta\,\bar{\mathbf{v}}$

$\mathbf{P} - \mathbf{P}_1 = \alpha\,\bar{\mathbf{u}} + \beta\,\bar{\mathbf{v}}$



If $\bar{\mathbf{n}}$ is orthogonal to $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ ($\mathbf{n} = \bar{\mathbf{u}} \times \bar{\mathbf{v}}$) :

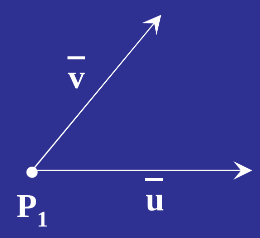
$\bar{\mathbf{n}}^{\mathbf{T}} \bullet (\mathbf{P} - \mathbf{P}_1) = \alpha\,\bar{\mathbf{n}}^{\mathbf{T}} \bullet \bar{\mathbf{u}} + \beta\,\bar{\mathbf{n}}^{\mathbf{T}} \bullet \bar{\mathbf{v}} = 0$



Implicit Equation of a Plane

$\bar{\mathbf{n}}^T \bullet (\mathbf{P} - \mathbf{P_1}) = 0$

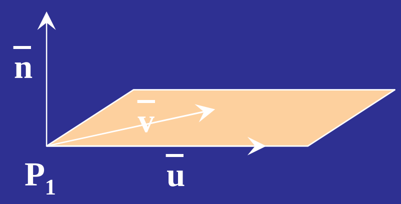
Let $\bar{\mathbf{n}} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ $\quad \mathbf{P} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\quad \mathbf{P_1} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$



Then, the equation of a plane becomes:

$a\,(x - x_1) + b\,(y - y_1) + c\,(z - z_1) = 0$

$a\,x + b\,y + c\,z + d = 0$

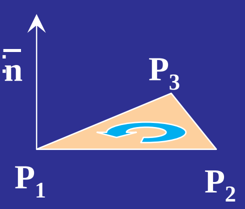


Thus, the coefficients of x, y, z in a plane equation define the normal.

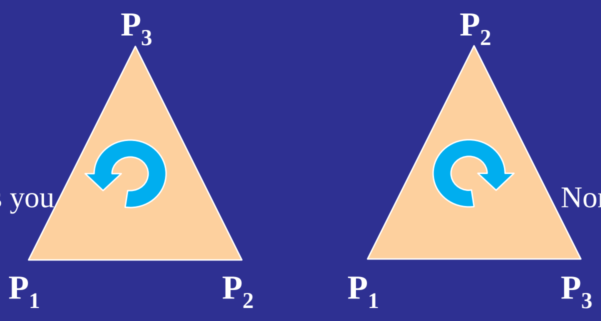
Normal of a Triangle

Normal of the plane containing the triangle (P_1, P_2 , P_3): $\bar{n}$

$$\bar{n} = (P_2 - P_1) \times (P_3 - P_1)$$



Normal pointing towards you



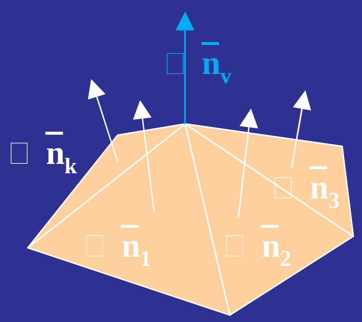
Normal pointing away from you

- Models constructed with consistent ordering of triangle vertices :
  - all clockwise or all counter-clockwise.
- Usually normals point out of the model.

Normal of a Vertex in a Mesh

$\bar{\mathbf{n}}_v = (\bar{\mathbf{n}}_1 + \bar{\mathbf{n}}_2 + \dots + \bar{\mathbf{n}}_k) / k = \sum \bar{\mathbf{n}}_i / k$

= average of adjacent triangle normals



or better:

$$\bar{\mathbf{n}}_v = \sum (A_i \, \bar{\mathbf{n}}_i) \Big/ \Big(k \sum (A_i) \Big)$$

= area-weighted average of adjacent triangle normals