Solutions to PHY2049 Exam 2 (Nov. 3, 2017)

Problem 1: *In figure a, both batteries have emf $\mathcal{E}$ = 1.2 V and the external resistance R is a variable resistor. Figure b gives the electric potentials V between the terminals of each battery as functions of R; Curve 1 corresponds to battery 1, and curve 2 corresponds to battery 2. The horizontal scale is set by* $R_s = 0.20\ \Omega$. *What is the internal resistance of battery 1?*

Problem 1 Solution: This problem was based on 27.14 from the text, which was assigned homework. Let us call the internal resistances r_1 and r_2. All the batteries and resistors are in series so the current is,

$$i = \frac{2\mathcal{E}}{R + r_1 + r_2} \; .$$

This means that the voltages across the terminals of the two batteries are,

$$V_1 = \mathcal{E} - ir_1 = \Big(\frac{R - r_1 + r_2}{R + r_1 + r_2}\Big)\mathcal{E} \; ,$$
$$V_2 = \mathcal{E} - ir_2 = \Big(\frac{R + r_1 - r_2}{R + r_1 + r_2}\Big)\mathcal{E} \; .$$

Note that at $R = \frac{1}{2}R_s$ we have $V_2 = 0$. This implies,

$$r_2 = \frac{1}{2}R_s + r_1 \; .$$

Substituting this into the expression for V_1, which is $\frac{1}{3}\mathcal{E}$ at $R = \frac{1}{2}R_s$, implies,

$$\frac{R_s}{R_s + 2r_1} = \frac{1}{3} \qquad \Longrightarrow \qquad r_1 = R_s \; .$$

Problem 2: *In the figure, the ideal batteries have emfs $\mathcal{E}_1$ = 10 V and $\mathcal{E}_2$ = 5 V, and the resistances are each* $4\ \Omega$. *What is the current in R_2?*

Problem 2 Solution: This problem was based on 27.30 from the text. Suppose we use i_1 to label the clockwise current in the left hand loop, and i_2 to label the counter-clockwise current in the right hand loop. Then Kirchhoff's Loop Law implies,

$$\begin{aligned} \mathcal{E} - Ri_1 - R(i_1 + i_2) &= 0 , \\ \frac{1}{2}\mathcal{E} - Ri_2 - R(i_1 + i_2) &= 0 . \end{aligned}$$

Adding the two equations together gives,

$$\frac{3}{2}\mathcal{E} - 3R(i_1 + i_2) = 0 \qquad \Longrightarrow \qquad R(i_1 + i_2) = \frac{1}{2}\mathcal{E} .$$

Substituting this into the second equation gives,

$$\frac{1}{2}\mathcal{E} - Ri_2 - \frac{1}{2}\mathcal{E} = 0 \qquad \Longrightarrow \qquad i_2 = 0 .$$

Problem 3: *In the figure, two batteries each with emf $\mathcal{E}$ = 12 V and an internal resistance $r = 0.30\ \Omega$ are connected in parallel across a resistance R. For what value of R is the dissipation rate in the resistor a maximum?*

Problem 3 Solution: This problem was based on 27.39 from the text, which was assigned homework. By symmetry, the current through each battery is the same; let us call it i. This means the current through R is $2i$, and Kirchhoff's loop law implies,

$$\mathcal{E} - 2iR - iR = 0 \qquad \Longrightarrow \qquad i = \frac{\mathcal{E}}{2R + r} .$$

The power dissipated in R is,

$$P = i^2 R = \frac{R\mathcal{E}^2}{(2R + r)^2} .$$

Differentiating with respect to R and setting the derivative to zero gives $R = \frac{1}{2}r$.

Problem 4: *In the figure, $R_1 = 10\ \mathrm{k}\Omega$, $R_2 = 15\ \mathrm{k}\Omega$, $C = 0.4\ \mu\mathrm{F}$, and the ideal battery has emf $\mathcal{E}$ = 20 V. First, the switch is closed a long time so that the steady state is reached. Then the switch is opened at time $t = 0$. What is the magnitude of the current in R_2 at time $t = 4$ ms?*

Problem 4 Solution: This problem was based on 27.65 from the text, which was assigned homework. First, consider the long term situation with the switch closed. No more current goes into the capacitor after it has fully charged, so the circuit essentially reduces to R_1 and R_2 in series. This means the current through R_2 and the voltage drop across it are,

$$i = \frac{\mathcal{E}}{R_1 + R_2} = 0.8 \text{ mA} \qquad \Longrightarrow \qquad V_2 = iR_2 = 12 \text{ V} .$$

This is also the voltage drop across the capacitor, hence it carries a charge of $Q_{\max} = 4.8\ \mu$C. When the switch is opened the circuit consists of just R_2 and C, and the charge on the capacitor and the current through R_2 are,

$$Q(t) = Q_{\max} e^{-t/R_2C} \qquad \Longrightarrow \qquad i(t) = -\frac{Q_{\max}}{R_2C} e^{-t/R_2C} .$$

The time constant is $\tau = R_2C = 6$ ms so the current at $t = 4$ ms is,

$$i(4 \text{ ms}) = -0.8 \text{ mA} \times e^{-2/3} \simeq -0.41 \text{ mA} .$$

Problem 5: *A beam of protons with a velocity of $v = 1.5 \times 10^6 \hat{j}$ m/s is sent into a region of uniform magnetic field of $B = 2.0 \times 10^{-3} \hat{i}$ T. What electric field in N/C is necessary (in magnitude and direction) such that the protons continue in a straight line without deflection by the magnetic field?*

Problem 5 Solution: The direction of the force caused by the B-field on the protons is $\hat{j} \times \hat{i} = -\hat{k}$. So the E-field has to point into the opposite direction. The magnitude has to be equal to E=vB=3000 N/C. So the answer is: $\vec{E} = 3 \times 10^3 \hat{k}$ N/C?

Problem 6: *An ion of charge +4e is sent into a region with constant magnetic field of magnitude B=0.25 T perpendicular to the plane of motion of the ion. The charge makes a U-turn. What is its mass in kg if it exits the region after a time $t = 7.8 \times 10^{-6} s$?*

Problem 6 Solution: The force caused by the magnetic field has to be equal to the centrifugal force:

$$m\frac{v^2}{r} = qvB \qquad \Rightarrow \qquad r = \frac{mv}{qB} \tag{1}$$

We still need to replace the velocity which we can get from the pathlength and the time:

$$\pi r = vt \qquad \Rightarrow \qquad v = \frac{\pi r}{t} \tag{2}$$

So we have

$$r = \frac{m\pi r}{qBt} \qquad \Rightarrow \qquad m = \frac{qBt}{\pi} = 4 \times 10^{-25}\text{kg} \tag{3}$$

Problem 7: *A particle with a negative charge propagates through a uniform magnetic field. At time t=0 it has a velocity of $\vec{v} = 3.0\hat{i}+2.0\hat{j}$ and experiences a force into the positive $\hat{k}$-direction. Which of the following directions could be correct for the magnetic field?*

Problem 7 Solution: The magnetic field component in the $\hat{k}$ direction has to be zero otherwise the force would not point into the $\hat{k}$ direction. Because the charge is negative, $\hat{v} \times \hat{B}$ should point into the $-\hat{k}$ direction. Now draw a vector showing $\vec{v}$ and go through the three remaining possibilities for the cross product. From the given possible solutions only a B-field that points into the $-\hat{j}$ direction creates a cross product with $\vec{v}$ that points into the $-\hat{k}$ direction.

Problem 8: *The figure shows a wood cylinder of mass $m = 0.25$ kg and length $L = 0.10$ m, with $N = 10$ turns of wire wrapped around it longitudinally, so that the wire coil contains the long central axis of the cylinder. The cylinder is released on a plane inclined at an angle θ to the horizontal, with the plane of the coil parallel to the incline plane. If there is a vertical magnetic field of magnitude 0.50 T, what is the least current i through the coil that keeps the cylinder from rolling down the plane? Note that the friction between the wood and the plane keeps the cylinder from sliding.*

Problem 8 Solution: This problem was based on 28.51 from the text, which was assigned homework. The magnetic dipole moment of the circuit is $\mu = N \times 2RL \times i$. The cylinder is subject to a gravitational torque, which pushes it to roll down, and a magnetic torque, which pushes it to roll up,

$$\tau_{\text{grav}} = mgR\sin(\theta) \qquad , \qquad \tau_{\text{mag}} = \mu B\sin(\theta) = 2NLiBR\sin(\theta) \ .$$

For the cylinder not to roll we must have $\tau_{\rm grav} = \tau_{\rm mag}$. Cancelling the factor of $R\sin(\theta)$ and solving for the current gives,

$$i = \frac{mg}{2NLB} = 2.45 \text{ A} .$$

Problem 9: *A current is set up in a wire loop consisting of a semicircle of radius 4.0 cm, a smaller concentric semicircle, and two radial straight lengths, all in the same plane. Figure a shows the arrangement but is not drawn to scale. The magnitude of the magnetic field produced at the center of curvature is* 47.25 μT. *The smaller semicircle is then flipped over (rotated) until the loop is again entirely in the same plane (figure b). The magnetic field produced at the (same) center now has magnitude* 15.75μT. *What is the radius of the smaller semicircle?*

Problem 9 Solution: This problem was based on 29.18 from the text, which was assigned homework. Let us call the current through the two circuits i, and the two radii $R_{\rm big} = 4$ cm and $R_{\rm small}$. The magnetic fields of the two circuits are,

$$B_a = \frac{\mu_0 i}{4}\Big[\frac{1}{R_{\rm small}} + \frac{1}{R_{\rm big}}\Big] \qquad , \qquad B_b = -\frac{\mu_0 i}{4}\Big[\frac{1}{R_{\rm small}} - \frac{1}{R_{\rm big}}\Big] .$$

Most of the unknown factors cancel out when we take the ratio of the absolute values,

$$\frac{|B_b|}{|B_a|} = \frac{\frac{1}{R_{\rm small}} - \frac{1}{R_{\rm big}}}{\frac{1}{R_{\rm small}} + \frac{1}{R_{\rm big}}} \equiv \rho .$$

Solving for $R_{\rm small}$ gives,

$$R_{\rm small} = \Big(\frac{1-\rho}{1+\rho}\Big) R_{\rm big} = 2 \text{ cm} .$$

Problem 10: *In the figure, five long parallel wires in an xy plane are separated by distance* $d = 50$ cm. *The currents into the page are* $i_1 = 2.0$ A, $i_3 = 0.25$ A, $i_4 = 4.0$ A, *and* $i_5 = 2.0$ A*; the current out of the page is* $i_2 = 4.0$ A. *What is the magnitude of the net force per unit length acting on wire 3 due to the currents in the other wires?*

Problem 10 Solution: This problem was based on 29.39 from the text, which was assigned homework. Recall that parallel currents attract, while opposite ones repel. Also the force per unit length of two currents i_a and i_b separated by a distance r is $\mu_0 i_a i_b/2\pi r$. Hence the force per until length on wire 3 is,

$$\frac{\Delta F}{\Delta \ell} = \frac{\mu_0 i_3}{2\pi d}\Big(-\frac{1}{2}i_1 + i_2 + i_4 + \frac{1}{2}i_5\Big) = 0.8\ \mu\text{N/m}\ .$$

Problem 11: *Each of the four wires in the figure carries a current of magnitude i either into or out of the page. What is the value of the line integral $\oint \vec{B}\cdot d\vec{s}$ along the line and in the direction shown in the figure?*

Problem 11 Solution: This problem was based on 29.45 from the text. This problem tests Ampere's law. Because the red circle is directed clockwise, the positive direction is *into* the page. This means that the two wires out of the page contribute $-2i$, whereas the wire into the page contributes $+i$. Hence $\mu_0 i_{\text{enc}} = -\mu_0 i$.

Problem 12: *In the figure, current $i = 56$ mA is set up in a loop having two radial lengths and two semicircles of radii $a = 5.7$ cm and $b = 9.4$ cm with a common center P. What is the magnitude and direction (into or out of the page) of the loop's magnetic dipole moment in units of 10^{-3} Am^2?*

Problem 12 Solution: This problem was based on 29.62 from the text, which was assigned homework. Because the current moves clockwise, the magnetic dipole is directed into the page. Its magnitude is,

$$\mu = Ai = \frac{\pi}{2}\Big[a^2 + b^2\Big]i \simeq 1.1 \times 10^{-3}\ \text{Am}^2\ .$$

Problem 13: *The figure shows two circular regions; region 1 of radius $r_1 =$ 15 cm and region 2 of radius $r_2 = 10$ cm. separated by 60 cm. The magnetic field in region 1 is 50mT coming out of the plane of the page, and that in region 2 is 30mT coming also out of the plane of the page. The magnitudes of both fields are decreasing at a rate of 5mT/s. Calculate the path integral $\oint \vec{E}\cdot d\vec{s}$ in mV along the path drawn in the figure; ignore the bending of the path at the crossing. Note: The direction of the path integral matters.*

Problem 13 Solution: The first step to solving this is to recall that the path integral over a closed path over the electric field is equal to the change in magnetic flux through the closed path. The second step is to remember that this depends on the direction of the change in flux but also on the direction through the loop:

$$\oint \vec{E} d\vec{s} = -\frac{d\Phi_B}{dt} = -A\frac{dB}{dt} = -\pi r^2 \frac{dB}{dt} \tag{4}$$

In this particular case, the left part will induce an E-field which 'rotates' counterclockwise around the decreasing magnetic flux. This is along the direction of integration and the result will be positive. In the right half, the E-field rotates also counterclockwise to the decreasing magntic flux which is against the direction of integration. So this part will be negative.

$$\oint \vec{E} d\vec{s} = \pi \frac{dB}{dt} \left(r_1^2 - r_2^2 \right) = 0.2\,\mathrm{mV} \tag{5}$$

Problem 14: *Current i_1 flows through the circular loop of wire in the center, as shown in the figure. The direction of the current is counterclockwise, when viewed from above. (The grey bars, which indicate the central axes of the loops, have been added to help you visualize the geometry of the problem.) If this current increases with time, what are the directions of the currents i_2, i_3, i_4 induced in the three adjacent circular loops of wire? Loop 3, whose current is i_3, is located above loop 1, whose current is i_1. In the answers, cw stands for clockwise, and ccw for counterclockwise, both when viewed from above or from the right, and 0 means no induced current.*

Problem 14 Solution: The current i_1 creates a magnetic dipole field in which the magnetic field lines go up from the first loop, up through the third loop and then bend around to go down through loop 4 and the benc again to close in loop 1. The lines will be tangential to loop 2. If the current in loop one increases, the flux through loop 3 increases and the induced current has to go in the opposite direction from i_1, so clockwise has to be the second answer. For the same reason the induced current in loop four has to be ccw. There is no flux through loop 2, so that loop doesn't care and shows no induced current. Correct answer: 0, cw, ccw

Problem 15: *The rod in the figure is moving at a constant speed in a direction perpendicular to a 0.60 T magnetic field, which is directed into the* *paper. The rod has a length of 1.6 m and has negligible electrical resistance. The rails also have negligible resistance. If the power dissipated by the light bulb is 4 W and its resistance is* $100\,\Omega$*, what is the speed of the rod in m/s?*

Problem 15 Solution: From the power dissipation and the resistance, we can get the induced emf. From the emf, we get the change in magnetic flux which is caused by the change in area. In math:

$$P = \frac{emf^2}{R} \qquad \Rightarrow \qquad emf = V = \sqrt{P \cdot R} = \sqrt{400}\mathrm{V} = 20\mathrm{V} \tag{6}$$

The change in magntic flux here is:

$$\frac{d\phi_B}{dt} = B\frac{dA}{dt} = B{\cdot}L{\cdot}v = emf \qquad \Rightarrow \qquad v = \frac{emf}{BL} = \frac{20\,\mathrm{V}}{0.6\,\mathrm{T}\cdot 1.6\,\mathrm{m}} = 21\,\mathrm{m/s} \tag{7}$$

Problem 16: *Suppose the emf of the battery in a serial RL circuit varies with time so that the current is given by* $i(t) = 0.5 \cdot t$ *where i is in A and t is in seconds. Assume* $R = 1.0\,\Omega$ *and* $L = 6\,H$*. What is the emf at* $t = 10\,s$*?*

Problem 16 Solution: KVL gives

$$emf = V(t) = Ri + L\frac{di}{dt} \tag{8}$$

with R, and L given and

$$i(10\,\mathrm{s}) = 5\,mA \qquad \frac{di}{dt} = 0.5\frac{\mathrm{A}}{\mathrm{s}} \tag{9}$$

we have:

$$V(10\,\mathrm{s}) = 1\Omega \cdot 5\mathrm{A} + 6 \cdot 0.5\frac{\mathrm{A}}{\mathrm{s}} = 8\,\mathrm{V} \tag{10}$$

Problem 17: *An LC circuit has a capacitance of* $10\,\mu F$ *and an inductance of 20 mH. At time* $t_0 = 0$ *the charge on the capacitor is* $20\,\mu C$ *and the current is 80 mA. The maximum possible current in mA is.*

Problem 17 Solution: The easiest way to find the solution is energy conservation. At time t_0, the energy is split between the cap and the inductor:

$$U_C = \frac{q^2}{2C} = \frac{(20 \times 10^{-6}\mathrm{C})^2}{20\mu\mathrm{F}} = 20\,\mu\mathrm{J} \qquad U_L = 0.5{\cdot}L{\cdot}i^2 = 10\,\mathrm{mH}{\cdot}(80\,\mathrm{mA})^2 = 64\,\mu\mathrm{J} \tag{11}$$

The total energy is then $84\,\mu$J which we can use to calculate the maximum current:

$$i_{max} = \sqrt{\frac{2U_{tot}}{L}} = \sqrt{\frac{168\,\mu\text{J}}{20\,\text{mH}}} = \sqrt{8.4 \times 10^{-3}}\text{A} = 92\,\text{mA} \tag{12}$$

Problem 18: *The amplitude of oscillations at frequency 10^6 Hz in an RLC-circuit decreases by a factor of two in 1 ms. If one doubles the resistance R, while keeping L and C the same, how long does it take now for the amplitude of oscillations to decrease by a factor of two?*

Problem 18 Solution: The damping constant is $\gamma = R/2L$. We double the damping, we damp twice as fast. So 0.5 ms is the correct answer.

Problem 19: *A 1.5 mH inductor in an oscillating LC circuit stores a maximum energy of $10\,\mu J$. If the resonance frequency of the circuit is 1 kHz, what is the maximum voltage in V across the capacitor?*

Problem 19 Solution: Energy is conserved. So at one point all the energy has to be in the cap. Once we know C, we can then get the voltage from the energy. To calculate C we use L and f_0:

$$f_0 = \frac{1}{2\pi\sqrt{LC}} \qquad \Rightarrow \qquad C = \frac{1}{4\pi^2 f_0^2 L} = 17\,\mu\text{F} \tag{13}$$

$$E_{Cap} = \frac{VQ}{2} = \frac{V^2C}{2} \qquad \Rightarrow \qquad V = \sqrt{\frac{2E_{Cap}}{C}} = \sqrt{\frac{2E_L}{C}} = 1.1\,\text{V} \tag{14}$$

Problem 20: *Consider an serial RLC circuit with $R = 10\,\Omega$, $L = 1.0\,H$, and $C = 1.0\ \mu\text{F}$. It is connected to a voltage source which produces $V_0 = 10\ V \sin\omega_0 t$ where ω_0 is the angular resonance frequency of the circuit. Find the amplitude of the voltage across the inductor.*

Problem 20 Solution: We are on resonance which means that the voltages across L and C are equal and opposite in sign and 90 degrees out of phase with the voltage across the resistor. This also means that the impedance is just R as ($\omega_0 L = 1/\omega_0 C$). So the current is:

$$i = \frac{V_0}{R} = I_0 \sin\omega_0 t \qquad I_0 = 1\,\text{A} \tag{15}$$

The voltage across the inductor is then:

$$V_L = L\frac{di}{dt} = L\omega_0 I_0 \cos\omega_0 t \tag{16}$$

As the angular resonance frequency is:

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad \Rightarrow \qquad V_L = \sqrt{\frac{L}{C}}\frac{V_0}{R}\cos\omega_0 t = 1000\,\mathrm{V}\cos\omega_0 t \tag{17}$$

So 1000 V is the correct answer.