Complex analysis in a nutshell.

Definition. A function f of one complex variable is said to be *differentiable at* $z_0 \in \mathbb{C}$ if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists and does not depend on the manner in which the variable $z \in \mathbb{C}$ approaches z_0.

Cauchy-Riemann equations. A function $f(z) = f(x, y) = u(x, y) + \mathrm{i}v(x, y)$ (with u and v the real and the imaginary parts of f respectively) is differentiable at $z_0 = x_0 + \mathrm{i}y_0$ if and only if it satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial v}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{at} \ \ (x_0, y_0).$$

Definition. A function f is *analytic* at z_0 if it is differentiable in a neighborhood of z_0.

Harmonic functions. Let D be a region in $\mathbb{R}^2$ identified with $\mathbb{C}$. A function $u : D \to \mathbb{R}$ is the real (or imaginary) part of an analytic function if and only if it is *harmonic,* i.e., if it satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Cauchy formulas. Let a function f be analytic in an open simply connected region D, let Γ be a simple closed curve contained entirely in D and traversed once counterclockwise, and let z_0 lie inside Γ. Then

$$\oint_\Gamma f(z)\,\mathrm{d}z = 0,$$

$$\oint_\Gamma \frac{f(z)\,\mathrm{d}z}{z - z_0} = 2\pi\mathrm{i} f(z_0),$$

$$\oint_\Gamma \frac{f(z)\,\mathrm{d}z}{(z - z_0)^{n+1}} = \frac{2\pi\mathrm{i}}{n!} f^{(n)}(z_0), \quad n \in \mathbb{N}.$$

Definition. A function analytic in $\mathbb{C}$ is called *entire.*

Zeros and poles. If a function f analytic in a neighborhood of a point z_0, vanishes at z_0, and is not identically zero, then $f(z) = (z - z_0)^k g(z)$ where $k \in \mathbb{N}$, g is another function analytic in a neighborhood of z_0, and $g(z_0) \neq 0$. In other words, zeroes of analytic functions are always isolated and of finite order.

Rouche's theorem. Suppose f and g are analytic in a disk $|z - z_0| \leq r$ and $|g(z)| < |f(z)|$ on the circle $|z - z_0| = r$. Then the functions $f + g$ and f have the same number of zeros (counting multiplicities) in $\{z : |z - z_0| < r\}$.

Maximum principle. If a function f is analytic in a disk $|z - z_0| \leq r$, the

$$|f(z)| \leq \max_{|\xi - z_0| = r} |f(\xi)| \quad \text{whenever} \ \ |z - z_0| < r,$$

with equality if and only if f is a constant. The maximum principle also holds for harmonic functions.

Liouville's theorem. If a function f is entire and bounded, then it is constant.

Singularities. If a function f is differentiable in a punctured neighborhood of z_0 but not at the point z_0 itself, then z_0 is an *isolated singularity* of f. There are three kinds of isolated singularities:

- *Removable* singularities: $f(z)$ is bounded as $z \to z_0$. Then f may be extended to a function analytic in a neighborhood of z_0. Example: $f(z) = \frac{\sin z}{z}$, $z_0 = 0$.
- *Poles:* There is a number $k \in \mathbb{N}$ such that the function $(z - z_0)^k f(z)$ has a removable singularity at z_0. The smallest integer k with that property is called the order of the pole. If k is the order of the pole, then necessarily

$$\lim_{z \to z_0} (z - z_0)^k f(z) \neq 0.$$

- *Essential* singularities: are those not of the two preceding kinds. If z_0 is an essential singularity of f, then, for any complex number $w \in \mathbb{C}$, a sequence (z_j) converging to z_0 can be found so that $\lim_{j \to \infty} f(z_j) = w$.

Taylor and Laurent series. If a function f is analytic in a disk $|z - z_0| \leq r$, then it expands into its Taylor series

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2}(z - z_0)^2 + \cdots + f^{(n)}(z_0)(z - z_0)^n + \cdots$$

The series is uniformly convergent in the disk $|z - z_0| \leq r$.

If a function f is analytic in an annular region $r \leq |z - z_0| \leq R$, then it expands into its Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n, \quad \text{where } c_n = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{f(z)\,\mathrm{d}z}{(z - z_0)^{n+1}}, \quad n \in \mathbb{Z},$$

where Γ is any curve lying in the annulus $r \leq |z - z_0| \leq R$ homeomorphic to a circle and traversed once in the counterclockwise direction.

Remark. A function f has a pole of order k at z_0 if and only if

$$c_{-k} \neq 0, \quad c_j = 0 \text{ for all } j < -k$$

in its Laurent expansion centered at z_0. An essential singularity gives rise to a Laurent series with an infinite negative part.

Definition. The *residue* of f at z_0, denoted by $\mathrm{Res} f(z_0)$, is the coefficient c_{-1} in its Laurent expansion centered at z_0 or, equivalently,

$$\mathrm{Res} f(z_0) = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma f(z)\,\mathrm{d}z,$$

where Γ is a closed curve enclosing z_0 and traversed once in the counterclockwise direction.

The residue theorem. If a function f is analytic in a simply connected domain D except for a finite number of isolated singularities and if a curve Γ is within D, then

$$\oint_\Gamma f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\sum_{n=1}^{N}\mathrm{Res}f(z_n)$$

where the z_n's are the singularities of f contained within C. (The curve is, as usual, traversed once counterclockwise.)

Finding residues. Here are several formulas which simplify finding residues.

- If f has a simple pole at z_0, then

$$\mathrm{Res}f(z_0) = \lim_{z\to z_0}(z-z_0)f(z).$$

- If f has a pole of order m at z_0, then

$$\mathrm{Res}f(z_0) = \lim_{z\to z_0}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)].$$

- If $f(z) = g(z)/h(z)$ where h has a simple zero at z_0 and g is analytic at z_0, then

$$\mathrm{Res}f(z_0) = \lim_{z\to z_0}\frac{g(z)}{h'(z)}.$$

Using the residue theorem. Integrals of the form

$$\int_0^{2\pi} f(\sin\theta,\cos\theta)\,\mathrm{d}\theta$$

reduce to integrals along the unit circle using the substitution

$$z = e^{\mathrm{i}\theta},\quad \mathrm{d}\theta = \frac{\mathrm{d}z}{\mathrm{i}z},\quad \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right),\quad \sin\theta = \frac{1}{2\mathrm{i}}\left(z-\frac{1}{z}\right).$$

Integrals along the entire real line

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

may be often converted to (limits of) contour integrals. The standard trick is to use one of the half circles $C_R^+ := \{Re^{\mathrm{i}\phi} : \phi\in[0,\pi]\}$ or $C_R^- := \{Re^{\mathrm{i}\phi} : \phi\in[0,-\pi]\}$ and the segment $[-R,R]$ and then let $R\to\infty$. This requires that the integral of f along $C_R^\pm$ tend to zero. Some integrals, e.g., those containing hyperbolic functions, require instead a rectangular contour with a segment parallel to $[-R,R]$ chosen so as to produce a multiple of the original integral while integrating along the new segment $[-R+\mathrm{i}c, R+\mathrm{i}c]$. Then one needs to make sure that contributions from the sides $[-R,-R+\mathrm{i}c]$, $[R,R+\mathrm{i}c]$ tend to zero. Fancier contours may be required in some cases.

Examples.

1. Evaluate
$$\int_0^{2\pi} e^{e^{\mathrm{i}\theta}} \mathrm{d}\theta.$$

2. Prove that the polynomial
$$p(z) = z^{47} - z^{23} + 2z^{11} - z^5 + 4z^2 + 1$$
has at least one root in the disk $|z| < 1$.

3. Suppose that f is analytic inside and on the unit circle $|z| = 1$ and satisfies $|f(z)| < 1$ for $|z| = 1$. Show that the equation $f(z) = z^3$ has exactly three solutions inside the unit circle.

4. How many zeroes does the function $f(z) = 3z^{100} - e^z$ have inside the unit circle? Are they distinct?

5. Evaluate
$$\int_{-\infty}^{\infty} \frac{\sin^2 x \,\mathrm{d}x}{x^2}.$$

6. Evaluate
$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(1 + x + x^2)^2}.$$

7. Evaluate
$$\int_{-\infty}^{\infty} \frac{x \sin x \,\mathrm{d}x}{x^2 + 4x + 20}.$$

8. Evaluate
$$\int_{-\infty}^{\infty} \frac{\cos(\pi x) \,\mathrm{d}x}{4x^2 - 1}.$$

9. Prove that
$$\int_0^{\infty} \frac{x \,\mathrm{d}x}{e^x - e^{-x}} = \frac{\pi^2}{8}.$$

10. Show that a positive harmonic function on $\mathbb{R}^2$ is necessarily constant.