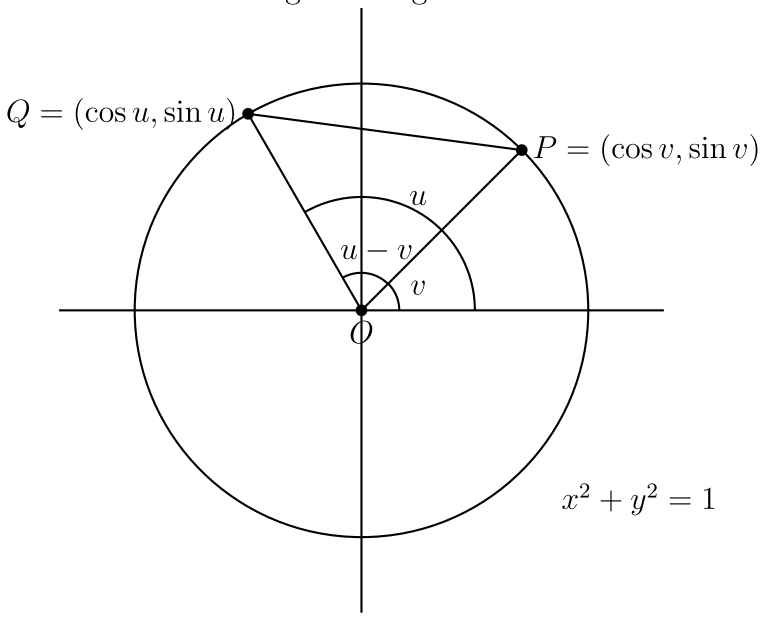
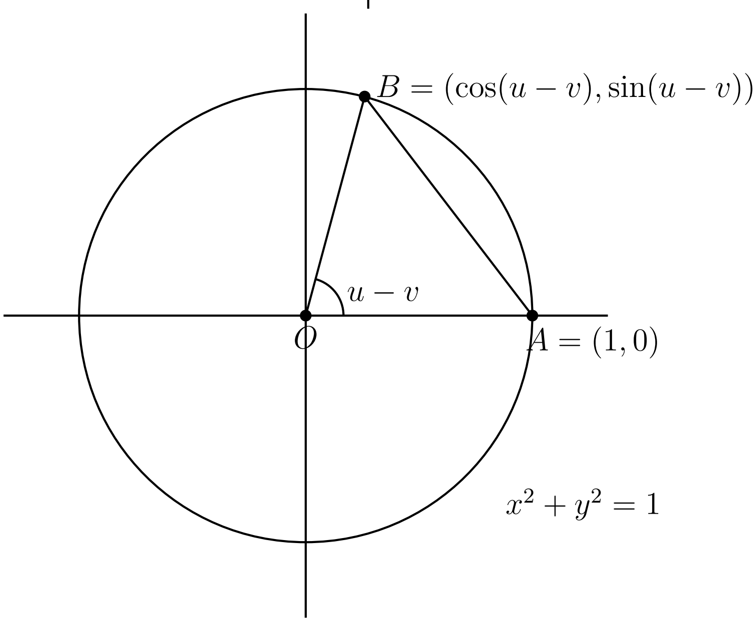
7.2 Sum and Difference Identities, 7.3 Double-Angle Formulas

Difference Formula for Cosine

Consider the following two diagrams:





The triangles POQ and AOB are **congruent**. Therefore,

$$d(P, Q) = d(A, B).$$

By squaring both sides, we have that

$$d(P, Q)^2 = d(A, B)^2.$$

We calculate both sides separately:

$$\begin{aligned}
d(P,Q)^2 &= (\cos u - \cos v)^2 + (\sin u - \sin v)^2 \\
&= \cos^2 u - 2\cos u \cos v + \cos^2 v + \sin^2 u - 2\sin u \sin v + \sin^2 v && \text{Expand the squares} \\
&= (\cos^2 u + \sin^2 u) + (\cos^2 v + \sin^2 v) - 2\cos u \cos v - 2\sin u \sin v && \text{Group terms} \\
&= 1 + 1 - 2\cos u \cos v - 2\sin u \sin v && \text{Pythagorean identity} \\
&= 2 - 2\cos u \cos v - 2\sin u \sin v.
\end{aligned}$$

$$\begin{aligned}
d(A,B)^2 &= (\cos(u-v) - 1)^2 + (\sin(u-v) - 0)^2 \\
&= \cos^2(u-v) - 2\cos(u-v) + 1 + \sin^2(u-v) && \text{Expand the squares} \\
&= 1 - 2\cos(u-v) + 1 && \text{Pythagorean identity} \\
&= 2 - 2\cos(u-v).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& d(A,B)^2 = d(P,Q)^2 \\
\implies\ & 2 - 2\cos(u-v) = 2 - 2\cos u \cos v - 2\sin u \sin v \\
\implies\ & -2\cos(u-v) = -2\cos u \cos v - 2\sin u \sin v && \text{Subtract 2 on both sides} \\
\implies\ & \cos(u-v) = \cos u \cos v + \sin u \sin v && \text{Divide by } -2 \text{ on both sides.}
\end{aligned}$$

Difference Formula for Cosine

Therefore, the **difference formula** for cosine is

$$\cos(u-v) = \cos u \cos v + \sin u \sin v.$$

Sum Formula for Cosine

$$\begin{aligned}
\cos(u+v) &= \cos(u - (-v)) \\
&= \cos u \cos(-v) + \sin u \sin(-v) && \text{Difference formula} \\
&= \cos u \cos v - \sin u \sin v && \text{Even-Odd Identities.}
\end{aligned}$$

Therefore, the **sum formula** for cosine is

$$\cos(u+v) = \cos u \cos v - \sin u \sin v.$$

Sum/Difference Formulas for Sine

Similarly, we can calculate for the sine function to find the **difference formula** for sine: $\sin(u - v) = \sin u \cos v - \cos u \sin v$, and the **sum formula** for sine: $\sin(u + v) = \sin u \cos v + \cos u \sin v$.

Double-Angle Formulas

We can use the sum formulas for sine and cosine to find the double-angle formulas:

$$\begin{aligned} \sin(2u) &= \sin(u + u) \\ &= \sin u \cos u + \cos u \sin u && \text{Sum formula} \\ &= 2 \sin u \cos u. \end{aligned}$$

$$\begin{aligned} \cos(2u) &= \cos(u + u) \\ &= \cos u \cos u - \sin u \sin u && \text{Sum formula} \\ &= \cos^2 u - \sin^2 u. \end{aligned}$$

$$\begin{aligned} \tan(2u) &= \tan(u + u) \\ &= \frac{\tan u + \tan u}{1 - \tan u \tan u} && \text{Sum formula} \\ &= \frac{2 \tan u}{1 - \tan^2 u}. \end{aligned}$$

Therefore, we have the **double-angle formulas**:

$$\sin(2u) = 2 \sin u \cos u;$$

$$\cos(2u) = \cos^2 u - \sin^2 u;$$

$$\tan(2u) = \frac{2 \tan u}{1 - \tan^2 u}.$$

Since $\sin^2 u + \cos^2 u = 1$, we also have that

$$\cos(2u) = 2 \cos^2 u - 1$$

and

$$\cos(2u) = 1 - 2 \sin^2 u.$$

Summary of Identities

i) Reciprocal Identities

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

ii) Quotient Identities

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

iii) Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \qquad 1 + \tan^2 x = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

iv) Even-Odd Identities

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x \qquad \tan(-x) = -\tan x$$

v) Sum/Difference Formulas

$$\cos(u - v) = \cos u \cos v + \sin u \sin v \qquad \cos(u + v) = \cos u \cos v - \sin u \sin v$$

$$\sin(u - v) = \sin u \cos v - \cos u \sin v \qquad \sin(u + v) = \sin u \cos v + \cos u \sin v$$

vi) Double-Angle Formulas

$$\sin(2u) = 2 \sin u \cos u \qquad \cos(2u) = \cos^2 u - \sin^2 u = 2\cos^2 u - 1$$

$$\tan(2u) = \frac{2 \tan u}{1 - \tan^2 u} \qquad \cos(2u) = \cos^2 u - \sin^2 u = 1 - 2\sin^2 u$$

Example: Find the exact value of

$$\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

Example: Find the exact value of

$$\sin\left(\frac{5\pi}{12}\right)$$

Example: Given that $\tan u = 5/12$, with u in quadrant I, and $\cos v = 3/5$, with v in quadrant IV, find $\sin(u - v)$ and $\cos(u + v)$.

Example: Given that $\sin u = -4/5$, with u in quadrant III, find $\sin(2u)$, $\cos(2u)$, and $\tan(2u)$.

7.2 Problems: 5, 6, 10, 12, 21

7.3 Problems: 5, 8, 24, 25

7.5 Solving Trigonometric Equations

A **trigonometric equation** is an equation involving one or more trigonometric functions. Like verifying trigonometric identites, there is no one method that will work for all problems, but usually a good strategy is to try to use trigonometric identities to simplify the equation, and then solve the equation using algebraic techniques.

Since trigonometric functions are periodic, trigonometric equations have an infinite number of solutions; however, we will usually restrict our search to solutions in the interval $[0, 2\pi)$.

Example: Find all solutions in the interval $[0, 2\pi)$ of the following equations:

i) $2\sin x - \sqrt{2} = 0$

ii) $\sin x + 1 = \sin(-x)$

iii) $3\tan^2 x - 1 = 0$

iv) $2\sin(3x) - \sqrt{2} = 0$

v) $\cos(2x) = \sin x$

vi) $\sin(2x) - \sin x = 0$

vii) $\cot x \cos^2 x = 2 \cot x$

viii) $2 \sin^2 x + 3 \cos x - 3 = 0$

7.5 Problems: 2, 6, 11, 15, 18, 20, 24, 25, 35, 38, 41, 42, 50