3. **Lecture III: Multi-Asset Options**

In this lecture we will generalize the pricing methodology of Lecture II to multi-asset options. As a further generalization, we will allow drift, volatilities and correlations to be time-dependent and possibly stochastic. The resulting market-model is sometimes called a *generalized Black and Scholes market*. We will introduce the important notions of complete and incomplete markets. Roughly speaking, a market is complete if each risk can be traded. In complete markets each European pay-off can be replicated by some self-financing portfolio strategy. Pricing in complete markets is basically identical to pricing in the Black and Scholes model of lecture II, except that we now will be dealing with vectors of risky assets. By contrast, pricing in incomplete markets solely on the basis of absence of arbitrage is in general impossible, and one has either to go back to older ideas from Economy, like expected utility maximization, or introduce new ones, like mean variance hedging. This is a domain of ongoing research.

We consider a market consisting of $N+1$ assets with time-t prices $S_{0,t}, S_{1,t}, \cdots, S_{N,t}$. By convention, we usually take one of them, say $S_{0,t}$, to be the risk-less asset, typically a savings bond, and we will usually write B_t instead of $S_{0,t}$. We will suppose that prices are driven by K stochastic *risk-factors* $W_{1,t}, \cdots, W_{K,t}$, which will be modelled by K *independent* Brownian motions[7] and that the price evolution is given by a system of SDE's of the following form:

$$\text{(53)} \qquad \begin{cases} dS_{0,t} = r_t S_{0,t} dt \quad (\text{or } dB_t = r_t B_t dt) \\ dS_{j,t} = \mu_{j,t} S_{j,t} dt + S_{j,t} \sum_{k=1}^{K} \sigma_{jk,t} dW_{k,t}. \end{cases}$$

Here the coefficients $r_t, \mu_{j,t}$ and $\sigma_{jk,t}$ are supposed to be *adapted* in the sense that they are measurable with respect to the filtration generated by the riskfactors:

$$\text{(54)} \qquad \mathcal{F}_t^W = \sigma\left(W_{1,s}, \cdots, W_{K,s} : s \leq t\right),$$

that is, they only depend on (one or several of) the Brownian motions $W_{1,s}, \cdots, W_{K,s}$ for times $s \leq t$.

To shorten notations, we will often use vectors, whose components will not only be numbers, but can also be random variables or processes. Vectors which will be distinguished by underlined letters, for example

$$\underline{W}_t = (W_{1,t}, \cdots, W_{K,t}),$$

and

$$\underline{S}_t = (S_{0,t}, \cdots, S_{N,t}).$$

[7] all defined on the same, unspecified, probability space $(\Omega, \mathcal{F}, \mathbb{P})$

The dimension of the vector under consideration will usually be clear from the context. We also introduce the $N \times K$-matrix of the σ_{jk}'s:

$$\mathbf{\Sigma}_t = \begin{pmatrix} \sigma_{11,t} & \sigma_{12,t} & \cdots & \sigma_{1K,t} \\ \sigma_{21,t} & \sigma_{22,t} & \cdots & \sigma_{2K,t} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N1,t} & \sigma_{N2,t} & \cdots & \sigma_{NK,t} \end{pmatrix},$$

acting on K-vectors in the usual way, by left-multiplication of $\mathbf{\Sigma}_t$ with the vector written as a $K \times 1$-matrix (or column vector). The dot-product between two K-vectors $\underline{v}$ and $\underline{w}$ will be their inner product:

$$\underline{v} \cdot \underline{w} = v_1 w_1 + \cdots + v_K w_K,$$

or the matrix product of $\underline{v}$ written as a $1 \times K$ matrix with $\underline{u}$ written as a $K \times 1$ matrix.

The vector notation will allow us to write things down more compactly. For example, (53) can also be written as:

$$\frac{dS_{j,t}}{S_{j,t}} = \mu_{j,t} dt + (\mathbf{\Sigma}_t \, d\underline{W}_t)_j \,, \quad j = 1, \cdots, N, \tag{55}$$

the last term on the right being the j-t component of $\mathbf{\Sigma}_t$ applied to the vector $d\underline{W}_t = (dW_{1,t}, \cdots, dW_{K,t})$. At some point in the future we might become lazy, and simply denote vectors by letters, without underlining them. The context should then make clear whether we are talking about vectors or scalars (ordinary real numbers) .

A word of caution: $\mathbf{\Sigma}_t dt$ is *not* the (conditional) variance-covariance matrix of the infinitesimal returns $dS_{j,t}/S_{j,t}$. Indeed, writing $\mathbb{E}_t$ for conditional expectation with respect to $\mathcal{F}_t^W$, the latter is given by the $N \times N$-matrix $(V_{ij,t})_{i,j}$, given by

$$\begin{aligned} V_{ij,t} &= \mathbb{E}_t \left(\left(\frac{dS_{i,t}}{S_{i,t}} - \mu_{i,t} dt \right) \left(\frac{dS_{j,t}}{S_{j,t}} - \mu_{j,t} dt \right) \right) \\ &= \mathbb{E}_t \left(\sum_{l,k=1}^{K} \sigma_{il,t} \sigma_{jk,t} dW_{l,t} dW_{k,t} \right) \\ &= \sum_{k=1}^{K} \sigma_{ik,t} \sigma_{jk,t} dt, \end{aligned}$$

where we used that $\mathbb{E}(dW_{l,t} dW_{k,t}) = \delta_{lk}$ (the Kronecker-delta), as follows from the independence of $W_{l,t}$ and $W_{k,t}$ if $l \neq k$. Hence the conditional variance-covariance matrix over an infinitesimal time-window $[t, t + dt]$ equals

$$(\mathbf{\Sigma}_t \mathbf{\Sigma}_t') \ dt,$$

where $\mathbf{\Sigma}_t'$ is the transpose of $\mathbf{\Sigma}$.

Now suppose we want to price a European contingent claim written on the vector of assets $\underline{S}_T$. For example, we might want to price a simple call option on an portfolio with weights $\underline{a} = (a_1, \cdots, a_N) \in \mathbb{R}^N$, whose pay-off at maturity is:

$$\max\left(\underline{a} \cdot \underline{S}_T - E, 0\right) = \max\left(\sum_{j=1}^{N} a_j S_{j,T} - E, 0\right). \tag{56}$$

Such an option is also called a basket option, since it is written on a basket of assets. Similarly, we could consider an Asian call on a basket, with pay-off

$$\max\left(\frac{1}{T}\int_0^T \underline{a} \cdot \underline{S_t} dt - E,\ 0\right).$$

See the exercises for other examples of multi-asset options, like rainbow options and spread options.

Just as in Lecture II, a general European claim X at T is defined as an $\mathcal{F}_t^{\underline{W}}$-measurable random variable. In other words, X is only allowed to depend upon the the risk factor-history $\{\underline{W}_t : t \leq T\}$[8] . Following the same strategy as in Lecture II, we will try to price such a claim X by looking for:

- A replicating self-financing portfolio strategy for the claim or, equivalently, for the discounted claim.
- An equivalent probability measure $\mathbb{Q}$ under which discounted prices are martingales.

The discounting will be done using the savings bond B_t: $\widetilde{S}_{j,t} = S_{j,t}/B_t$, but we could in principle discount using any of the available assets: see the remarks at the end of the lecture on change of numéraire.

A trading strategy will be a vector-valued process $(\varphi_{0,t}, \varphi_{1,t}, \cdots, \varphi_{N,t})$, with $\varphi_{j,t}$ $\mathcal{F}_t^{\underline{W}}$-measurable or adapted; $\varphi_{j,t}$ is precisely the number of assets $S_{j,t}$ the investor intends to hold at time t. We will single out the number of bonds, for which we will write, as before,

$$\varphi_{0,t} = \psi_t,$$

and we will designate by

$$\underline{\varphi}_t = (\varphi_{1,t}, \cdots, \varphi_{N,t}),$$

[8] In practice, we might not be able to observe $\underline{W}_t$ directly, but only be able to observe the price-history $\{\underline{S}_t : t \leq T\}$. In such situations it would be more appropriate to define European claims as rvs which are measurable w.r.t. $\mathcal{F}_T^{\underline{S}} := \sigma - (\underline{S}_t : t \leq T)$. On the other hand, one can also easily imagine situations were risk-factors can be directly observed, and determine the final pay-off, e.g. weather-based securities involving market risk and temperature risk, or options on volatility in stochastic volatility models.

the vector of holdings of the risky assets $(S_{1,t}, \cdots, S_{N,t})$. The time-$t$ value of this portfolio strategy is simply

$$\begin{aligned} V_t = V_t(\underline{\varphi}, \psi) &= \psi_t B_t + \sum_{j=1}^{N} \varphi_{j,t} S_{j,t} \\ &= \psi_t B_t + \underline{\varphi}_t \cdot \underline{S}_t, \end{aligned}$$

and the strategy will be called *self-financing* if the change in value in an infinitesimal time-interval $[t, t+dt]$ is entirely due to capital gains (and losses):

$$\begin{aligned} dV_t &= \psi_t dB_t + \sum_{j=1}^{N} \varphi_{j,t} dS_{j,t} \\ &= \psi_t dB_t + \underline{\varphi}_t \cdot \underline{dS}_t. \end{aligned}$$

Similar to proposition 1.2, we then have:

Proposition 3.1. *A trading strategy* $(\psi_t, \underline{\varphi}_t)_t$ *is self-financing iff it is also self-financing in the discounted securities market* $(1, \underline{\widetilde{S}}_t)$*, where*

$$\underline{\widetilde{S}}_t := B_t^{-1} \underline{S}_t = \left(\frac{S_{1,t}}{B_t}, \cdots, \frac{S_{N,t}}{B_t} \right).$$

Explicitly, letting $\widetilde{V}_t = V_t / B_t = \psi_t + \underline{\varphi}_t \cdot \underline{\widetilde{S}}_t$*, then*

$$dV_t = \underline{\varphi}_t \cdot \underline{dS}_t + \psi_t dB_t.$$

if and only if

$$d\widetilde{V}_t = \underline{\varphi}_t \cdot \underline{d\widetilde{S}}_t,$$

The proof is completely analogous to before, and will be left as an exercise.

Definition 3.2. We will call an $\mathcal{F}_T^W$-measurable random variable X a *replicable* (or *attainable*) claim if there exists a self-financing portfolio-strategy $(\underline{\varphi}_t, \psi_t)$ such that

$$X = V_T(\underline{\varphi}, \psi). \tag{57}$$

Equivalent forms of (57) are, writing V_t for $V_t(\underline{\varphi}, \psi)$:

$$X = V_0 + \int_0^T \underline{\varphi}_t \cdot \underline{dS}_t + \psi_t dB_t,$$

since $V_T = V_0 + \int_0^T dV_t$. Similarly, after discounting, if we let $\widetilde{X} = X / B_T$ and assume as usual that $B_0 = 1$, then (57) is also equivalent to

$$\begin{aligned} \widetilde{X} &= V_0 + \int_0^T d\widetilde{V}_t \\ &= V_0(\underline{\varphi}, \psi) + \int_0^T \underline{\varphi}_t \cdot \underline{d\widetilde{S}}_t. \end{aligned}$$

More generally, for any $t < T$,

$$\widetilde{X} = \widetilde{V}_t + \int_t^T \underline{\varphi}_u \cdot d\underline{\widetilde{S}}_u. \tag{58}$$

The fact that we introduced the special term of replicable claim should make you suspect that, in general, not all $\mathcal{F}_T^W$-measurable claims are replicable. This is indeed the case, and constitutes the difference between a complete and an incomplete market.

Definition 3.3. The market-model (53) is called *complete* if every $\mathcal{F}_T^W$-measurable European claim at T is replicable by some self-financing portfolio strategy.

We will return later to the question of when a market (53) is complete. For the moment we will just assume we that are dealing with a replicable claim X. Then $V_0 = V_0(\underline{\varphi}, \psi)$ will have to be the fair price $\pi_0(X)$ of the claim at 0, since starting off with a capital of V_0 and continuously investing and reinvesting according to the strategy $(\underline{\varphi}, \psi)$ we will ultimately arrive at T with a portfolio whose value $V_T(\underline{\varphi}, \psi)$ coincides with that of X, whatever the future state of the world. More generally, and for the same reason, $\pi_t(X)$, the value of X at $t < T$ will have to be equal to $V_t(\underline{\varphi}, \psi)$.

Now suppose for a moment that, just as in the single asset case, we can find a new probability measure $\mathbb{Q}$ which is equivalent to $\mathbb{P}$, and with respect to which all discounted prices $\widetilde{S}_{j,t}$ are martingales. That is, if we put

$$\mathbb{E}_{\mathbb{Q},t}(\cdot) = \mathbb{E}_{\mathbb{Q}}(\cdot|\mathcal{F}_t^W),$$

then we should have that

$$\mathbb{E}_{\mathbb{Q},t}(d\widetilde{S}_{j,t}) = 0. \tag{59}$$

By a by now familiar argument this will yield that the stochastic integral in the right and side of (58) is a martingale. Indeed,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q},t}\left(\int_t^T \underline{\varphi}_u \cdot d\underline{\widetilde{S}}_u\right) &= \int_t^T \mathbb{E}_{\mathbb{Q},t}\left(\underline{\varphi}_u \cdot d\underline{\widetilde{S}}_u\right) \\
&= \int_t^T \mathbb{E}_{\mathbb{Q},t}\left(\mathbb{E}_{\mathbb{Q},u}\left(\underline{\varphi}_u \cdot d\underline{\widetilde{S}}_u\right)\right) \\
&= 0,
\end{aligned}$$

since $\mathbb{E}_{\mathbb{Q},u}\left(\underline{\varphi}_u \cdot d\underline{\widetilde{S}}_u\right) = \sum_j \varphi_{j,u}\ \mathbb{E}_{\mathbb{Q},u}(d\widetilde{S}_{j,u}) = 0$. Hence we see from (58) that

$$\frac{\pi_t(X)}{B_t} = \frac{V_t}{B_t} = \mathbb{E}_{\mathbb{Q},t}\left(\frac{X}{B_T}\right), \tag{60}$$

A probability measure $\mathbb{Q}$ such that (59) holds is called an *Equivalent Martingale Measure* or EMM. We have shown:

Proposition 3.4. *Let X be a replicable claim, and suppose that there exists an EMM $\mathbb{Q}$. Then the price of the claim X at $t \leq T$ is equal to*

$$\pi_t(X) = B_t \mathbb{E}_{\mathbb{Q},t}\left(\frac{X}{B_T}\right). \tag{61}$$

We can integrate the ODE $dB_t = r_t B_t dt$ path-wise (that is, for each fixed$\omega \in \Omega$) separatedly) to

$$B_t(\omega) = e^{\int_0^t r_s(\omega) ds}, \tag{62}$$

assuming as usual that $B_0 = 1$. Pulling in the B_t in (65) under the conditional expectation sign (why is this allowed?) and using (62) to evaluate B_t/B_T, we then find the following equivalent form of (61):

$$\pi_t(X) = \mathbb{E}_{\mathbb{Q},t}\left(e^{-\int_t^T r_s ds}\, X\right) = \mathbb{E}_{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}\, X | \mathcal{F}_t^{\underline{W}}\right). \tag{63}$$

If the interest rate is r_t is non-stochastic, but only depends on time t (in which case we will write $r(t)$ instead of r_t), we can pull out the discount factor from under the expectation sign:

$$\pi_t(X) = e^{-\int_t^T r(s) ds}\, \mathbb{E}_{\mathbb{Q},t}(X),$$

but not if r_t is stochastic! If $r_t = r$ is constant, the discount factor simple becomes $\exp(-r(T-t))$, as in Lecture II .

Remark 3.5. It can happen that a market (53) possesses more than one EMM: explicit examples are given in the exercises below. Potentiall, different EMM's could give different prices in (61). However, for replicable claims all these prices coincide. To make this hard, mathematically, one has to be a bit more precise about the kind of trading strategies $\underline{\varphi}$ which are allowed in for setting up the replicating self-financing strategy. A useful class is the class of so-called *tame strategies*, which essentially means that $\widetilde{V}_t(\varphi)$ should stay bounded from below by some, possibly large, negative constant: $V_t(\underline{\varphi}) \geq -L$ for *some* $L \geq 0$: see for example Musiela and Rutkowski, section 10.1. If we allow self-financing portfolio strategies with unlimited down-side, funny things can happen using such strategies: for example, arbitrage opportunities will always occur, by employing so-called 'doubling strategies'; see also below.

To complete the picture, we have to determine when Equivalent Martingale Measures exists, and how to compute with them. It is easily seen that the discounted prices $\widetilde{S}_{j,t}$ satisfy the system of SDE's

$$d\widetilde{S}_{j,t} = \widetilde{S}_{j,t}\left((\mu_{j,t} - r)dt + (\mathbf{\Sigma}_t \, \underline{dW}_t)_j\right). \tag{64}$$

As in the Lecture II, the idea is to find a new probability $\mathbb{Q}$, and a new vector of Brownian motions $\underline{\widehat{W}} = (\widehat{W}_{1,t}, \cdots, \widehat{W}_{K,t})$ w.r.t. $\mathbb{Q}$, such that the system (64) transforms into one having zero drift terms:

$$d\widetilde{S}_{j,t} = \widetilde{S}_{j,t}(\mathbf{\Sigma}_t \, \underline{d\widehat{W}_t})_j = \widetilde{S}_{j,t}\left(\sum_{k=1}^{K}\sigma_{jk,t}d\widehat{W}_{k,t}\right). \tag{65}$$

For in that case, we will have

$$\mathbb{E}_{\mathbb{Q},t}(d\tilde{S}_{j,t}) = \widetilde{S}_{j,t}\left(\sum_{k=1}^{K}\sigma_{jk,t}\,\mathbb{E}_{\mathbb{Q},t}(d\widehat{W}_{k,t})\right) = 0,$$

as desired.

We will produce $\widehat{W}_{j,t}$ and $\mathbb{Q}$ using the following multi-dimensional version of the Girsanov theorem (which can be guessed from theorem 1.11 by replacing ordinary products by dot-products at the appropriate places):

Theorem 3.6. *(multi-dimensional Girsanov theorem) Let* $\underline{\gamma}_t = (\gamma_{1,t}, \cdots, \gamma_{K,t})$ *be a vector of processes which are adapted to the Brownian filtration* $\mathcal{F}^{\underline{W}_t}$*, and which satisfies the following technical condition (Novikov's condition):*

$$\mathbb{E}_{\mathbb{P}}\left(e^{\int_0^t |\gamma_s(\omega)|^2 ds/2}\right) < \infty, \tag{66}$$

where $|\underline{\gamma}_t|^2 = \underline{\gamma}_t \cdot \underline{\gamma}_t = \sum_j \gamma_{j,t}^2$*. Define a new measure* $\mathbb{Q}$ *by:*

$$\mathbb{E}_{\mathbb{Q}}(X) = \mathbb{E}_{\mathbb{P}}\left(X \exp\left(-\int_0^t \underline{\gamma}_s \cdot \underline{dW}_s - \frac{1}{2}\int_0^t |\underline{\gamma}_s|^2 ds\right)\right).$$

Then $\mathbb{Q}$ *is a probability measure, equivalent to* $\mathbb{P}$*, and the process* $\underline{\widehat{W}}_t = (\widehat{W}_{1,t}, \cdots, \widehat{W}_{K,t})$*, defined by*

$$\widehat{W}_{j,t} = \int_0^t \gamma_{j,s} ds + W_{j,t},$$

or, equivalently, by

$$d\widehat{W}_{j,t} = \gamma_{j,t}dt + dW_{j,t}, \quad 1 \le j \le K, \tag{67}$$

is a Brownian motion with respect to $\mathbb{Q}$.

Let us now start hunting for an EMM. Substituting $dW_{j,t} = -\gamma_{j,t}dt + d\widehat{W}_{j,t}$ in equation (64), we find that

$$\frac{d\tilde{S}_{j,t}}{S_{j,t}} = \left(\mu_j - r_t - (\mathbf{\Sigma}_t\,\underline{\gamma}_t)_j\right)dt + \left(\mathbf{\Sigma}_t\, d\underline{\widehat{W}}_t\right)_j, \tag{68}$$

To kill the drift term, we try to find a process $\underline{\gamma}_t$ such that for each t, $\underline{\gamma}_t = (\gamma_{1,t}, \cdots, \gamma_{K,t})$ solves the system of linear equations

$$\mu_{j,t} - r_t - (\mathbf{\Sigma}_t\underline{\gamma}_t)_j = 0,$$

or

$$\mu_{j,t} - r_t = \sum_k \sigma_{jk,t}\gamma_{k,t}.$$

Introducing the N-component vector of 1's, $\underline{1} = (1, 1, \cdots, 1)$, we can write this system in vector notation as:

$$\mathbf{\Sigma}_t \cdot \underline{\gamma}_t = \underline{\mu}_t - r_t\underline{1}, \tag{69}$$

It turns out that such a system of equations does not always have a solution!

Example 3.7. Take $N = 2$, $K = 1$, and $\underline{\mu}_t = (\mu_1, \mu_2)$,

$$\mathbf{\Sigma}_t = \begin{pmatrix} \sigma \\ \sigma \end{pmatrix},$$

and $r_t = r$, all constant. Suppose moreover that

$$\mu_1 \neq \mu_2.$$

Since $K = 1$, $\underline{\gamma}_t$ will only have one component, and if (69) would be satisfied then there would exist, for each time t, a $\gamma_t \in \mathbb{R}$ such that:

$$\begin{cases} \mu_1 - r = \gamma_t\sigma, \\ \mu_2 - r = \gamma_t\sigma. \end{cases}$$

But this would clearly imply that $\mu_1 = \mu_2$, which is a contradiction.

So existence of EMMs is not always guaranteed. To get an idea of what this might mean, economically speaking, we play around a bit more with the example we have just given. The prices of the assets S_1 and S_2 will be given by:

$$S_{j,t} = S_{j,0}e^{(\mu_j - \sigma^2/2)t + \sigma W_t},$$

where W_t is the (common) risk factor which drives the two assets. Suppose that

$$\mu_1 > \mu_2,$$

and let us construct a portfolio, at time 0, by selling short one unit of S_2 (the slow-growing asset), and buying $S_{2,0}/S_{1,0}$ units of asset S_1 (the fast-growing one). The total (net) investment at time 0 is therefore 0. What is the portfolio's worth at a later time t? This is clearly given by:

$$\frac{S_{2,0}}{S_{1,0}}S_{1,t} - S_{2,t} = S_{2,0}\left(e^{\mu_1 t} - e^{\mu_2 t}\right)e^{-\sigma^2 t/2 + \sigma W_t}, \tag{70}$$

which is always ≥ 0, whatever W_t does in the future. So we have a portfolio which, with 0 investment, will always give a positive profit in the future. Such an investment opportunity is called an *arbitrage*, and for economic reasons, such arbitrage opportunities should not exists in well-functioning markets with frictionless trading (trading without transaction costs) and unrestricted short sales[9]. The idea is that if, at some point in time, such an arbitrage opportunity exist, investors which realize this will try to take advantage by constructing arbitrage portfolios as above, and this will lead, via the mechanism of supply and demand, to changes in the parameters which make the arbitrage opportunity vanish. In the example, if people construct an arbitrage portfolio as the one indicated, excess demand for S_1 will drive up its price, and therefore lower its mean return rate μ_1. Similarly, excess supply of S_2 will lower its price, and increase its mean return rate μ_2. This process will continue until $\mu_1 = \mu_2$ (or, in a realistic market, until $\mu_1 - \mu_2$ has become so small that the transaction costs involved in setting up the portfolio exceed the potential gains which could be made over a realistic time-windows T).

This is a more general phenomenon:

Proposition 3.8. *Suppose that there does* not *exist an EMM* $\mathbb{Q}$ *for (53). Then there exists an arbitrage opportunity.*

**Proof.* We sketch the basic idea, limiting ourselves to constant parameters σ_{jk}, μ_j and r. We will work with the discounted process $\underline{\tilde{S}}_t$ since, clearly, if an arbitrage opportunity exists with the discounted process, one will exist also for the original process. Non-existence of an EMM means that there does not exist a vector $\underline{\gamma}$ such that

$$\mathbf{\Sigma}\,\underline{\gamma} = \underline{\mu} - r\underline{1}.$$

This means that $\underline{\mu} - r\underline{1}$ is not in the image of $\mathbf{\Sigma}$:

$$\underline{\mu} - r\underline{1} \notin \mathrm{Im}(\mathbf{\Sigma}) := \{\mathbf{\Sigma}\underline{\gamma} : \underline{\gamma} \in \mathbb{R}^K\}.$$

By a general theorem of linear algebra,

$$\mathrm{Im}(\mathbf{\Sigma}) = \mathrm{Ker}(\mathbf{\Sigma}')^{\perp},$$

where the right hand side is the set of vectors orthogonal to the kernel of the transpose $\mathbf{\Sigma}' : \mathbb{R}^N \to \mathbb{R}^K$. It follows that the kernel of $\mathbf{\Sigma}'$ is non-zero, and that there exists a $\underline{w} = (w_1, \cdots, w_N)$ such that

$$\mathbf{\Sigma}'\,\underline{w} = 0, \quad \underline{w} \cdot (\underline{\mu} - r\underline{1}) \neq 0.$$

We may assume without loss of generality that

$$\underline{w} \cdot (\underline{\mu} - r\underline{1}) = \sum_j w_j(\mu_j - r) > 0, \tag{71}$$

for if not, simply replace $\underline{w}$ by $-\underline{w}$. We will now choose a portfolio-strategy by taking

$$\varphi_{j,t} = \frac{w_j}{\tilde{S}_{j,t}}, \quad 1 \leq j \leq N. \tag{72}$$

[9] Observe that, instead of selling short one S_2 we would have sold short a large number L of them, we could multiply the value of (70) by L and have in principle unlimited profits with a totally risk-less strategy.

To make the strategy self-financing, we buy/short-sell (or deposit/borrow) ψ_t bonds B_t, where

$$\psi_t = \int_0^t \underline{\varphi}_u \cdot \underline{d\tilde{S}}_u - \underline{\varphi}_t \cdot \underline{\tilde{S}}_t.$$

(For then the portfolio's total discounted value is

$$\begin{aligned} \tilde{V}_t(\phi, \psi) &= \underline{\varphi}_t \cdot \underline{\tilde{S}}_t + \psi_t \\ &= V_0 + \int_0^t \underline{\varphi}_u \cdot \underline{d\tilde{S}}_u, \end{aligned}$$

so that $d\tilde{V}_t = \sum_j \varphi_{j,t} d\tilde{S}_{j,t} = \sum_j \varphi_{j,t} d\tilde{S}_{j,t} + \psi_t d\tilde{B}_t,$, since $d\tilde{B}_t = d(1) = 0$. This is a general trick: any trading strategy in the risky assets can be made self-financing by using the risk-less asset to "balance the books", so to speak, by borrowing if there is a shortfall, and depositing in a savings account when there is a surplus.)

We furthermore pick ψ_0 so as to have an initial investment of 0: $V_0(\varphi, \psi) = 0$. We next compute the infinitesimal change in portfolio value, using (72):

$$\begin{aligned} d\tilde{V}_t &= \sum_j \varphi_{j,t} d\tilde{S}_{j,t} \\ &= \sum_j w_j(\mu_j - r)dt + \sum_{j,k} w_j \sigma_{jk} dW_{k,t} \\ &= \sum_j w_j(\mu_j - r)dt, \end{aligned}$$

since $\mathbf{\Sigma}' \cdot \underline{w} = 0$. Because of (71), we have a self-financing strategy with 0 initial investment and risk-less positive growth, which is an arbitrage opportunity. QED

Remark 3.9. There is a converse to this result: existence of an EMM implies absence of arbitrage, provided we give a sufficiently careful definition of arbitrage opportunities, to avoid so-called doubling strategies (which used to be called 'martingales' by the gamblers of previous centuries). For example, if we define an arbitrage to be a self-financing portfolio strategy $(\underline{\varphi}, \psi)$ with initial value $V_0(\underline{\varphi}, \psi) = 0$ such that, for some $T > 0$, $V_T(\underline{\varphi}, \psi)$ is non-negative and with non-zero probability even strictly positive, and which is also 'tame', in the sense that its losses over the time-window $[0, T]$ are assured to be bounded from below by some, possibly large, negative constant $-L$ ('no unlimited downside'; see also remark 3.5 above), then existence of an EMM implies that such arbitrage opportunities cannot exist, and vice-versa. This can be generalized: there is now a considerable body of mathematical literature devoted to proving the equivalence of absence of arbitrage and existence of an Equivalent Martingale Measures, in an abstract setting where prices are given by general semi-martingales, which include many more processes besides Brownian motions with drift (for example, Lévy-porcesses). A detailed discussion would take us too far afield, and we refer to the literature, for example Musiela and Rutkowski, section 10.1, and its references.

We summarize the discussion up to this point in the following proposition:

Proposition 3.10. *There exists an equivalent martingale measure for the generalized Black and Scoles market (53) if there exists a vector-valued process* $\underline{\gamma}_t$ *such that*

$$\mathbf{\Sigma}_t\, \underline{\gamma}_t = \underline{\mu}_t - r\underline{1}.$$

This is in particular the case if $K = N$ *and if* $\mathbf{\Sigma}_t$ *is an invertible matrix, for then*

$$\underline{\gamma}_t = \mathbf{\Sigma}_t^{-1}(\underline{\mu}_t - r\underline{1}),$$

and $\underline{\gamma}_t$ *then is in fact unique.*

Remark 3.11. As often, we're being slightly imprecise here: strictly speaking, the solution $\underline{\gamma}_t$ should also satisfy the boundedness condition (66), in order that $\mathbb{Q}$ be a bona fida probability measure (it might be possible that $\mathbb{Q}(\Omega) < 1$ otherwise). In the $K = N$ case with $\mathbf{\Sigma}_t$ invertible, this will be OK if $\mathbf{\Sigma}_t^{-1}$ and $\underline{\mu}_t$ remain bounded, for t's in bounded intervals $[.0, T]$. We will usually skip these mathematical niceties, but simply warn that there can be unpleasant surprises when these conditions are not met.

Let us now try to understand when a Black and Scholes market is going to be complete, assuming without loss of generality that we can find an EMM (for if not, we would be dealing with a market which as arbitrage opportunities, in which it is dangerous to do business). Take an arbitrary $\mathcal{F}_T^W$-measurable claim X with $\mathbb{E}(X^2) < \infty$. This will also be measurable with respect to the time-T filtration generated by the new Brownian motion $\widehat{W}_t$, and by a suitable generalization of the Martingale Representation theorem, we will be able to write $\widetilde{X}$ as a stochastic integral

$$\begin{aligned} \widetilde{X} &= X_0 + \int_0^T h_{1,t} d\widehat{W}_{1,t} + \cdots + h_{K,t} d\widehat{W}_{K,t} \\ &= X_0 + \int_0^T \underline{h}_t \cdot \underline{d\widehat{W}}_t, \end{aligned} \tag{73}$$

for some adapted vector process $\underline{h}_t$. Now to get from (73) to (58), we would want to replace the Brownian differentials $\underline{d\widehat{W}}_t$ by a linear combination of the differentials of the discounted price process $\underline{dS}_t$.

The only way in which we can do this is by inverting (65), that is, by going from

$$\frac{d\widetilde{S}_{j,t}}{\widetilde{S}_{j,t}} = \sum_k \sigma_{jk,t} d\widehat{W}_{k,t},$$

to

$$d\widehat{W}_{k,t} = \sum_l \eta_{kl,t} \frac{d\widetilde{S}_{l,t}}{\widetilde{S}_{l,t}}.$$

for suitable coefficients $\eta_{kl,t}$. In other words, writing s_j for $d\widetilde{S}_{j,t}/\widetilde{S}_{j,t}$ and v_k for $d\widehat{W}_{k,t}$, the system of linear equations

$$\begin{array}{ccc} s_1 & = & \sigma_{11,t}v_1 + \cdots + \sigma_{1K,t}v_K \\ \vdots & \vdots & \vdots \\ s_N & = & \sigma_{N1,t}v_1 + \cdots + \sigma_{NK,t}v_K \end{array}$$

should have at least one solution $(v_1, \cdots, v_K)$. A closer analysis shows that this is only possible if $N \geq K$, and if the K column vectors of $\mathbf{\Sigma}_t$ are all independent, that is, if the $N \times K$-matrix $\mathbf{\Sigma}_t$ as maximal rank, meaning K independent row vectors. (To see this, just imagine you do Gauss-elimination on the system). In particular, to have a complete market we should have at least as many assets N as there are risk-factors, K. Finally, if the rank of the system is maximal, K, then the solution will be unique. In this case, the solution to $\mathbf{\Sigma}_t \underline{\gamma}_t = \underline{\mu}_t - r_t \underline{1}$ will be unique also, and there will only one EMM. Summarizing, we have shown in this chapter:

Theorem 3.12. *Suppose that the generalized Black and Scholes market (53) admits no arbitrage. Then there exists an EMM $\mathbb{Q}$. Moreover, this market is complete if and only if $N \geq K$ and the $K \times N$ matrix $\mathbf{\Sigma}_t = (\sigma_{jk})_{1\leq j\leq N, 1\leq k\leq K}$ has maximal rank K. In this case there is a unique EMM $\mathbb{Q}$.*

If $N = K$, the condition on $\mathbf{\Sigma}_t$ simply becomes that $\mathbf{\Sigma}_t$ should be non-singular (that is, invertible), for all t.

In a complete market we can price any European claim by the formulas (61), (63).

Up till now we always discounted by dividing by the savings bond B_t. Another way of looking upon this is by saying that we measure wealth relative to one of the assets, B_t, so as to deal with dimensionless or 'pure' numbers. An interesting idea is to measure wealth relative to one of the other (risky) assets. Denoting B_t by $S_{0,t}$, so as to deal with it on the same footing as the others, we might 'discount' by dividing by $S_{i,t}$ for an arbitrary but fixed j, and consider

$$\widetilde{S}^i_{j,t} := \frac{S_{j,t}}{S_{i,t}}, \tag{74}$$

provided of course that with probability 1, $S_{i,t}$ is never 0. In financial parlance, we say that we are choosing $S_{i,t}$ as a *numéraire*. We can then redo the theory: self-financing strategies remain self-financing strategies, and under the conditions of proposition 3.10, Girsanov's theorem will give us a new probability measure $\mathbb{Q}^i$ under which (74) will be martingales. In a complete market, an arbitrary European claim will then be valued by the formula

$$\pi_t(X) = S_{i,t}\mathbb{E}_{\mathbb{Q}^i}\left(\frac{X}{S_{i,T}}|\mathcal{F}_t\right). \tag{75}$$

As we will see later, these ideas have important applications for valuing options in bond markets.

3.1. **Exercises.**

Exercise 3.13. Write down a detailed proof of proposition 3.1.

Exercise 3.14. One way to interpret (56) is as an ordinary call, written on the portfolio with holdings a_j in S_j. Will this immediately lead to a pricing formula? Can you think of circumstances under which it would?

Exercise 3.15. Suppose that the time-evolution of a market with N risky assets is given by (53), with constant coefficients $\mu_{j,t} = \mu_j$ and $\sigma_{jk,t} = \sigma_{jk}$. Using the multi-dimensional Ito formula, show that

$$\begin{aligned} S_{j,t} &= S_{j,0}\exp\left((\mu_j - \frac{1}{2}\sum_k \sigma_{jk}^2)t + \sum_k \sigma_{jk}W_k\right) \\ &= S_{j,0}\exp\left((\mu_j - (\frac{1}{2}\mathbf{\Sigma\Sigma}')_j)t + (\mathbf{\Sigma}\ \underline{W}_t)_j\right). \end{aligned}$$

Exercise 3.16. a) Assuming again that we are dealing with constant coefficients μ_j and σ_{jk}, and also with a complete market, the previous exercise, show that the time-t value of a basket call with pay-off $\max(\underline{a}\cdot \underline{S}_t - E, 0)$ is given by the multi-dimensional integral

$$\int_{\mathbb{R}^K} \max\left(\sum_{j=1}^N a_j S_{t,j} e^{(r-(\mathbf{\Sigma\Sigma}')_j)(T-t)+\sum_{k=1}^K \sigma_{jk}w_k}, 0\right)\ e^{-(w_1^2+\cdots+w_K^2)/2}\frac{dw}{(2\pi T)^{K/2}}.$$

b) Assume now that $K = N$. By making the change of variables $\underline{w} = \mathbf{\Sigma}^{-1}\underline{x}$, show that this price can also be written as

$$\int_{\mathbb{R}^K} \max(\sum_{j=1}^N a_j S_{t,j} e^{(r-(\mathbf{\Sigma\Sigma}')_j)(T-t)+x_j}, 0)\ e^{-\frac{1}{2}\underline{x}\cdot\mathbb{V}^{-1}\underline{x}}\frac{dx}{(2\pi)^{N/2}\sqrt{\det\ (\mathbb{V})}},$$

where $\mathbb{V}$ is the variance-covariance formula.

c) Suppose that $a_1 = 1, a_2 = \cdots = a_N = 0$. By manipulating the formula of part a), show that the value of the basket option will simply be given by the Black and Scholes formula for $S_{1,t}$, with volatility σ_1^2.

Another way to show this is by using that the evolution of $S_{1,t}$ can be written as

$$dS_{1,t} = rS_{1,t}dt + \sigma_1^2 S_{1,t}dZ_t,$$

for some conveniently defined *new* Brownian motion $(Z_t)_{t\geq 0}$, and $\sigma_1^2 := \sum_{j=1}^K \sigma_{jk}^2$; see the next exercise.

d) What will be the value of a basket of calls, with pay-off $a_1 \max(S_{1,T} - E_1, 0) + \cdots + a_N \max(S_{N,T} - E_N, 0)$?

e) Will the results of c) and d) still be true if the $\sigma_{jk,t}$ are not constant? Analyze the case were one or several of the σ_{1k} explicitly depend on $S_{2,t}$.

Exercise 3.17. Consider two risky assets $S_{1,t}$ and $S_{2,t}$ whose infinitesimal returns $dS_{1,t}/S_{1,t}$, $dS_{2,t}/S_{2,t}$ have constant correlation $\rho \in [-1, 1]$, and volatilities $\sigma_{1,t}^2$ and $\sigma_{2,t}^2$.

a) Show that the two assets can be modelled by

$$\begin{aligned} dS_{1,t} &= S_{1,t}\left(\mu_{1,t}dt + \sigma_{1,t}^2 dW_{1,t}\right) \\ dS_{2,t} &= S_{2,t}\left(\mu_{2,t}dt + \sigma_{2,t}^2(\rho dW_{1,t} + \sqrt{1-\rho^2}dW_{2,t}\right). \end{aligned}$$

b) Define a new process $(Z_{2,t})_{t\geq 0}$ by:

$$dZ_{2,t} = \rho dW_{1,t} + \sqrt{1-\rho^2}dW_{2,t},$$

or, more explicitly,

$$Z_{2,t} = \rho W_{1,t} + \sqrt{1-\rho^2}W_{2,t}.$$

Show that $(Z_{2,t})_{t\geq 0}$ is still a Brownian motion, which has constant correlation ρ with $dW_{1,t}$ in the sense that

$$dW_{1,t}\, dZ_{2,t} = \rho dt.$$

Writing $Z_{1,t} := W_{1,t}$, the market model can also be written in the alternative form:

$$\begin{aligned} dS_{1,t} &= S_{1,t}\left(\mu_{1,t}dt + \sigma_1^2 dZ_{1,t}\right) \\ dS_{2,t} &= S_{2,t}\left(\mu_{1,t}dt + \sigma_{1,t}^2 dZ_{2,t}\right) \\ dZ_{1,t}\, dZ_{2,t} &= \rho dt, \end{aligned}$$

which is sometimes more transparent. This generalizes in two directions: first of all, we can let the correlation be time-dependent and adapted, that is, replace ρ by an $\mathcal{F}_t^W$-measurable process ρ_t. In this case, to show that $dZ_{2,t} = \rho_t dW_{1,t} + \sqrt{1-\rho_t^2}dW_{2,t}$ is again a Brownian motion, we need a theorem of Paul Lévy, to the effect that *a driftless stochastic process X_t such that $\mathbb{E}_t(dX_t^2) = dt$ is necessarily a BM.* Secondly, we can generalize to arbitrary N:

c) Consider a market model (53) with $N = K$ (as many risk-factors as tradables). By defining suitable new correlated Brownian motions

$Z_{1,t}, \cdots Z_{N,t}$, show that the evolution of the risky assets can be written in the form

$$\begin{aligned} dS_{1,t} &= \mu_{1,t} S_{1,t} dt + S_{1,t} \sigma_{1,t} dZ_{1,t} \\ &\vdots \\ dS_{N,t} &= \mu_{N,t} S_{1,t} dt + S_{1,t} \sigma_{N,t} dZ_{1,t}, \end{aligned}$$

where the volatilities $\sigma_{j,t}^2$ and the correlations between the $dZ_{j,t}$'s are the following ones, defined using the variance-covariance matrix $\mathbb{V}_t = \mathbf{\Sigma}_t \mathbf{\Sigma}_t'$: if

$$\sigma_{j,t}^2 := \sum_k \sigma_{jk,t}^2,$$

then $dZ_{j,t} dZ_{k,t} = \rho_{jk,t} dt$, with

$$\rho_{jk,t} = V_{jk,t} / \sigma_{j,t} \sigma_{k,t}.$$

Exercise 3.18. Consider again two risky assets $S_{1,t}$ and $S_{2,t}$ with constant correlation ρ, and constant volatilities σ_1^2 and σ_2^2. Derive explicit integral formulas for the values of:

- a spread call $\max(S_{1,T} - S_{2,T} - E, 0)$.
- A rainbow option $\max(S_{1,T} - E_1, E_2 - S_{2,T}, 0)$ (constructed out of of a call and a put).

Discuss the special cases of a correlation ρ equal to ± 1.

Exercise 3.19. Markets can also be incomplete if $N \geq K$, and the conditions of theorem 3.12 are met, but we cannot, or are not allowed to, trade in all of the assets $S_{j,t}$. An example of such a situation is given by the class of *diffusion stochastic volatility models*:

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma_t S_t \, dW_{1,t}) \\ d\sigma_t &= a_t dt + b_t (\rho \, dW_{1,t} + \sqrt{1 - \rho^2} \, dW_{2,t}). \end{aligned} \quad (76)$$

Here we cannot directly trade in the 'volatility asset' σ_t. As a simple mean-reverting model, we take $a_t = \alpha(\theta - \sigma_t)$ and $b_t = \eta$ constant. Use Girsanov's theorem to determine all EMM $\mathbb{Q}$ which turn $\widetilde{S}_t = e^{-rt} S_t$ into a martingale.

Exercise 3.20. Fill in the details leading to the 'numeraire pricing formula' (75), including the existence and construction of $\mathbb{Q}^i$ using Girsanov's theorem, by adapting the arguments leading up to (61) and proposition 3.10.